Recent Developments at Small-x

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Outline

- Introduction
  - Classical fields, Wilson lines
  - Non-linear small-x evolution
- Higher-order corrections to BFKL, BK, JIMWLK equations
- Particle production, di-hadron correlations, long-range rapidity correlations
- Connections to TMDs and Spin Physics

Introduction

The main principle

- Saturation physics is based on the existence of a large internal transverse momentum scale Q_S which grows with both decreasing Bjorken x and with increasing nuclear atomic number A

$$Q_s^2 \sim A^{1/3} \left(\frac{1}{x}\right)^{\lambda}$$

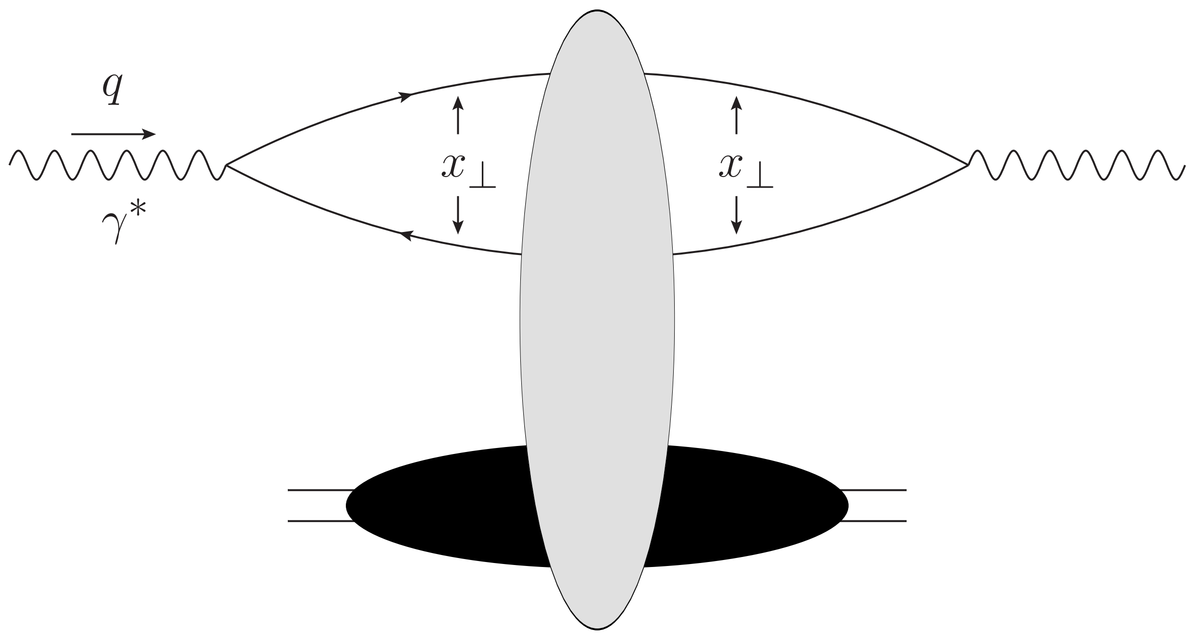
such that

$$\boxed{\alpha_S = \alpha_S\left(Q_S\right) << 1}$$

and we can use perturbation theory to calculate total cross sections, particle spectra and multiplicities, correlations, etc, from first principles.

Dipole picture of DIS

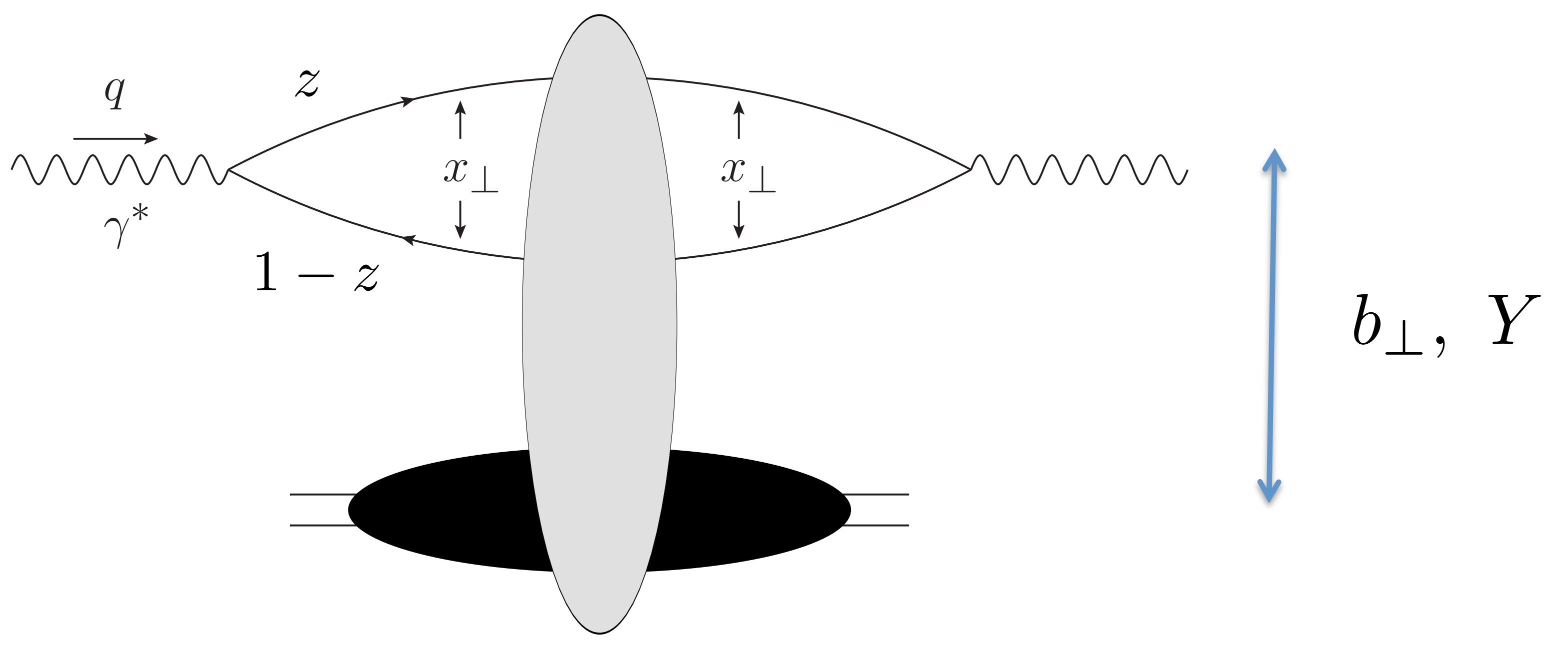
- In the dipole picture of DIS the virtual photon splits into a quark-antiquark pair, which then interacts with the target.
- The total DIS cross section and structure functions are calculated via:



Dipole Amplitude

- The total DIS cross section is expressed in terms of the (Im part of the) forward quark dipole amplitude N:

$$\sigma_{tot}^{\gamma^* A} = \int \frac{d^2 x_\perp}{2\pi} \, d^2 b_\perp \int_0^1 \frac{dz}{z\,(1-z)} \, |\Psi^{\gamma^* \to q\bar{q}}(\vec{x}_\perp, z)|^2 \; N(\vec{x}_\perp, \vec{b}_\perp, Y)$$



Dipole Amplitude

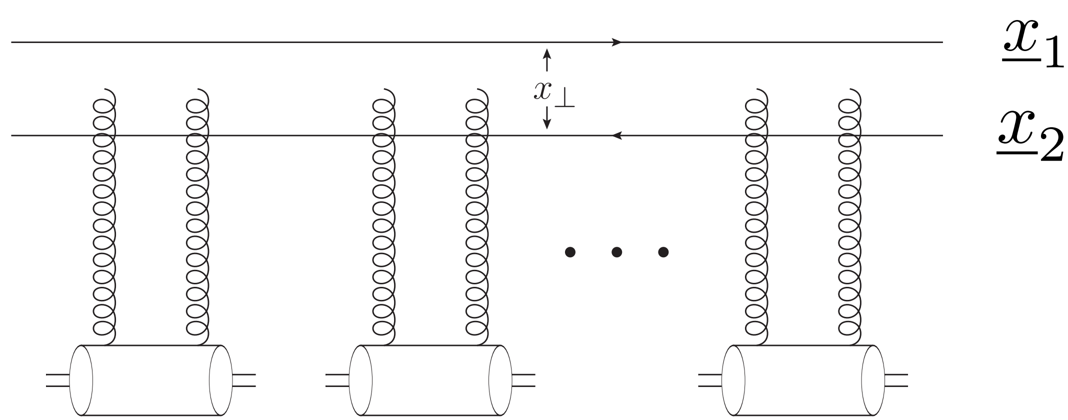
- The quark dipole amplitude is defined by

$$N(\underline{x}_1, \underline{x}_2) = 1 - \frac{1}{N_c} \left\langle \mathrm{tr} \left[V(\underline{x}_1) \, V^\dagger(\underline{x}_2) \right] \right\rangle$$

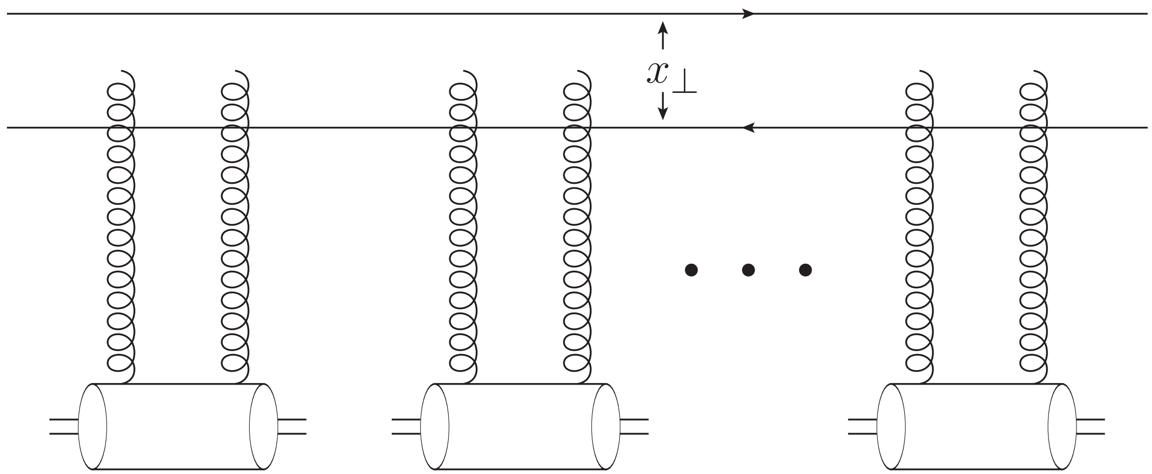
- Here we use the Wilson lines along the light-cone direction

$$V(\underline{x}) = \mathrm{P} \exp \left[i \, g \int\limits_{-\infty}^{\infty} dx^+ \, A^-(x^+, x^- = 0, \underline{x}) \right]$$

- In the classical Glauber-Mueller/McLerran-Venugopalan approach the dipole amplitude resums multiple rescatterings:

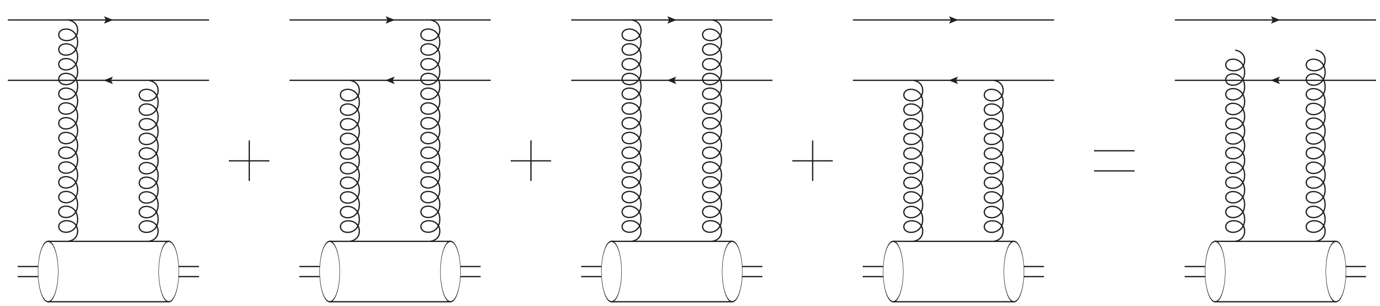


Quasi-classical dipole amplitude



A.H. Mueller, '90

Lowest-order interaction with each nucleon – two gluon exchange – the same resummation parameter as in the MV model: $\alpha_s^2\, A^{1/3}$

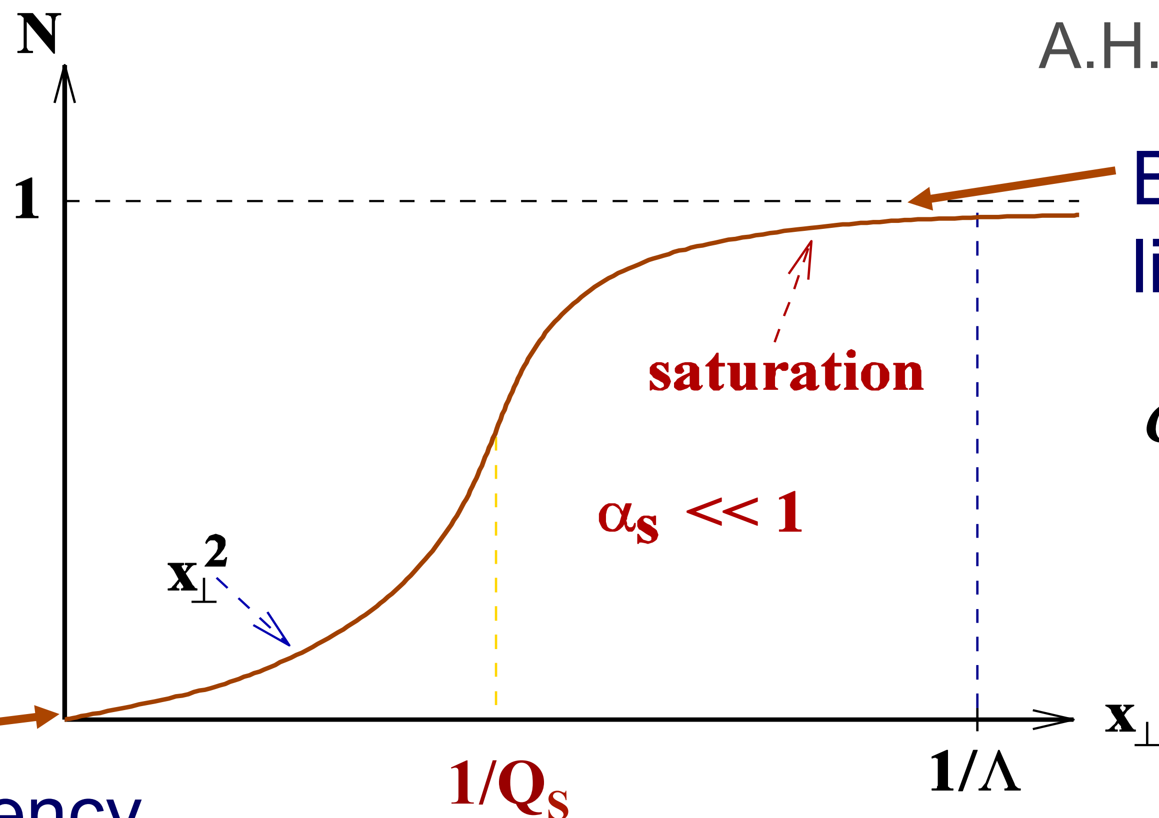


DIS in the Classical Approximation

The dipole-nucleus amplitude in the classical approximation is

$$N(x_\perp, Y) = 1 - \exp\left[-\frac{x_\perp^2\, Q_s^2}{4}\,\ln\frac{1}{x_\perp\,\Lambda}\right]$$

A.H. Mueller, '90



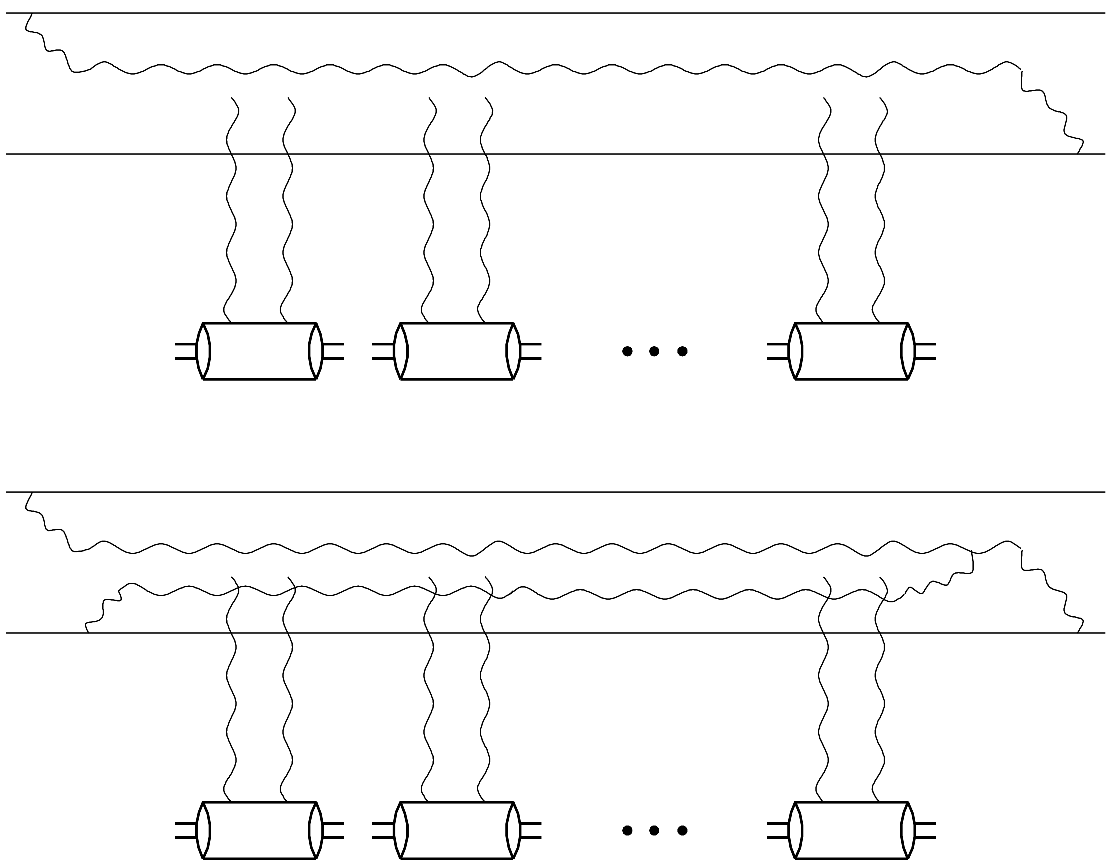
Black disk limit,

$$\sigma_{tot} < 2\pi R^2$$

Color transparency

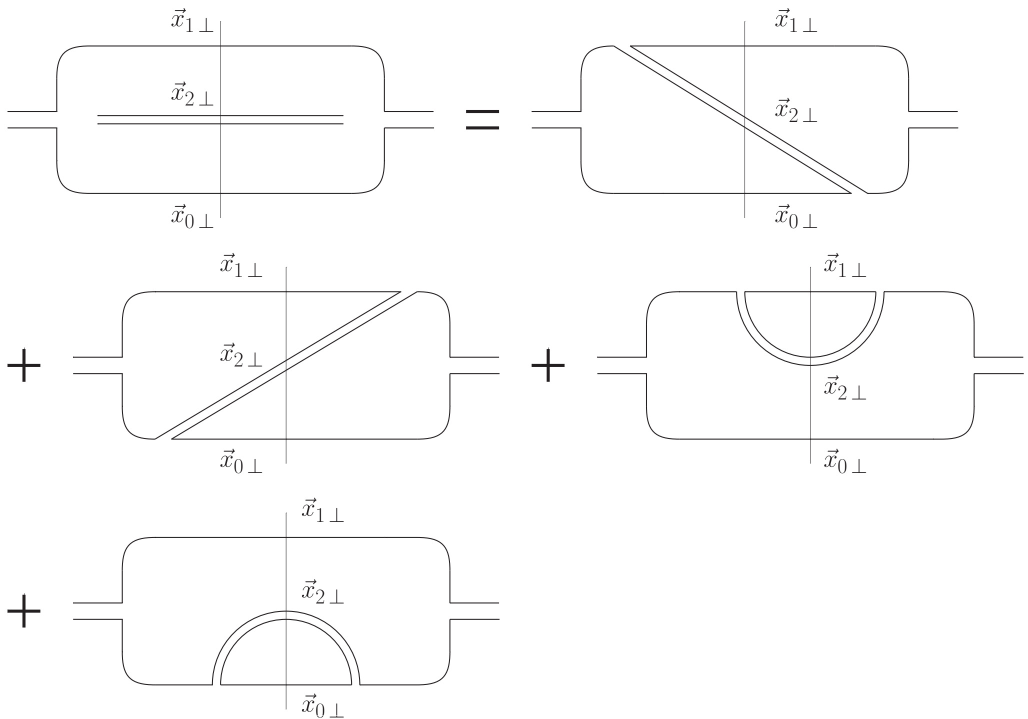
Dipole Amplitude

- The energy dependence comes in through nonlinear small-x BK/JIMWLK evolution, which resums the long-lived s-channel gluon corrections:



$$\alpha_s \ln \frac{1}{x} \sim \alpha_s Y \sim 1$$

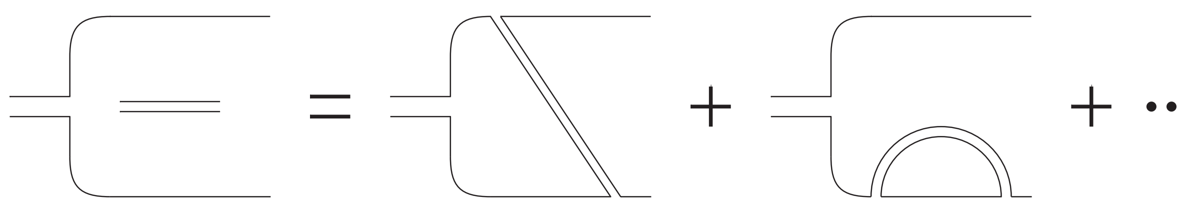
Notation (Large-N_C)



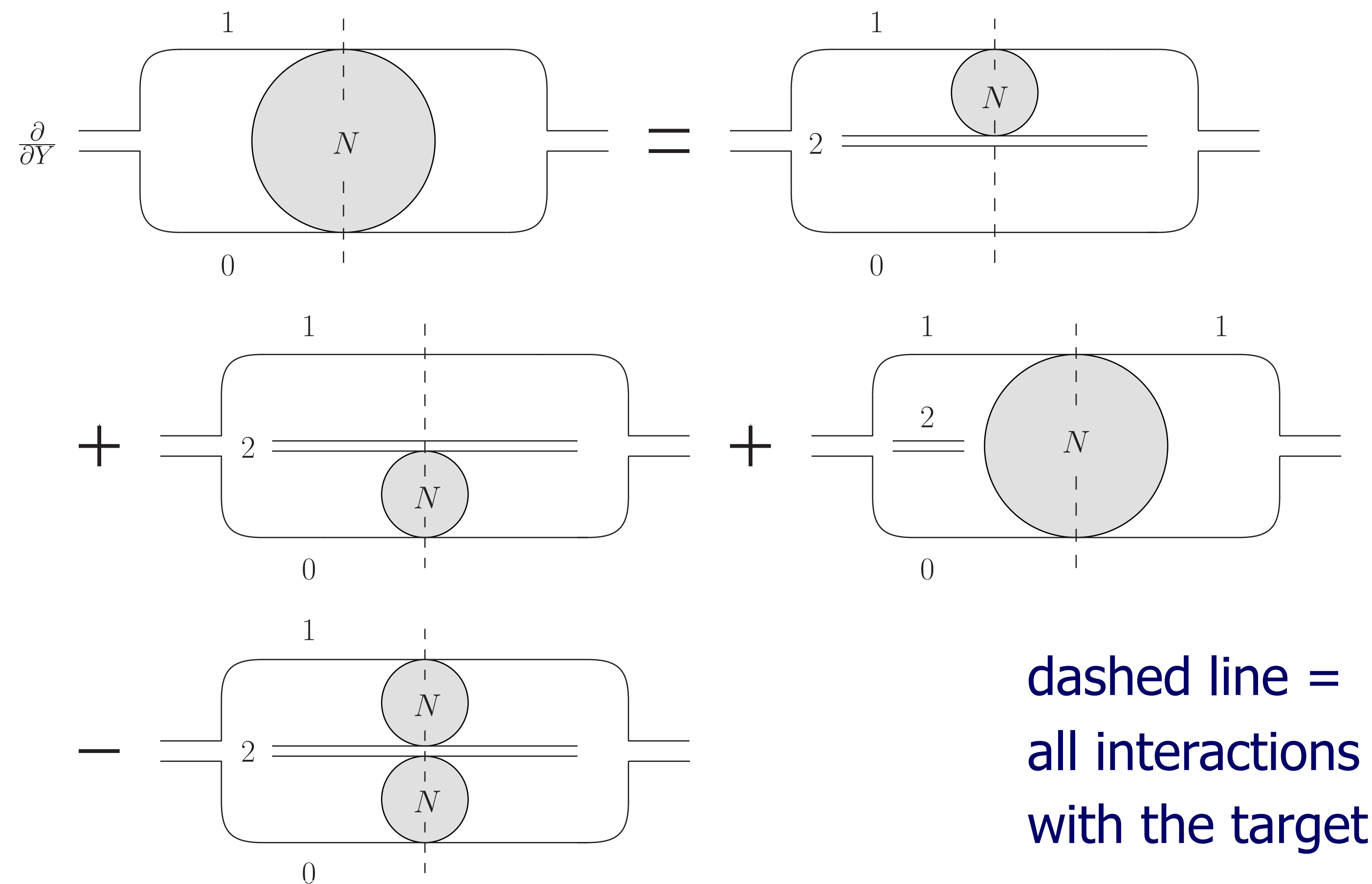
Real emissions in the amplitude squared

(dashed line – all Glauber-Mueller exchanges at light-cone time =0)

Virtual corrections in the amplitude (wave function)



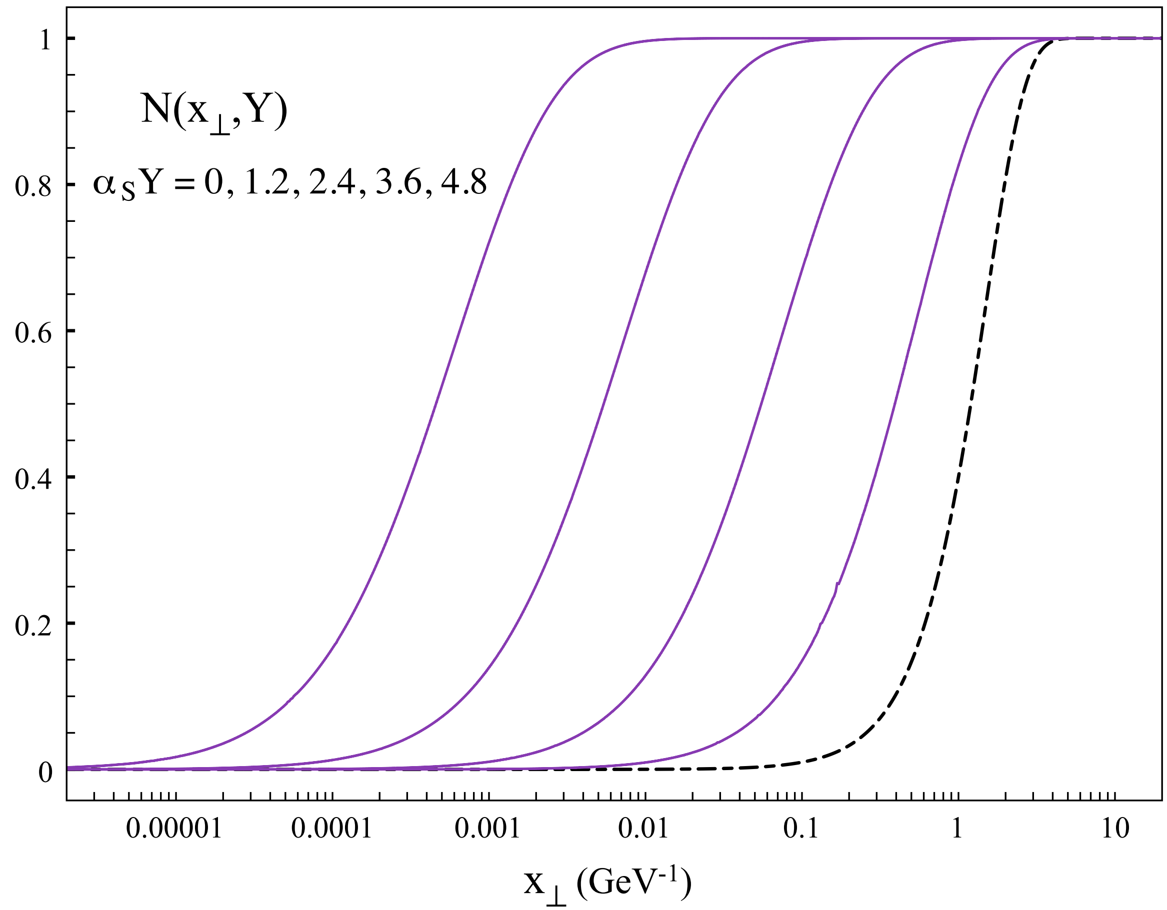
Nonlinear evolution at large N_c



$$\partial_Y N_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \frac{x_{01}^2}{x_{02}^2 x_{21}^2} \left[N_{\mathbf{x}_0,\mathbf{x}_2}(Y) + N_{\mathbf{x}_2,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_2}(Y)\, N_{\mathbf{x}_2,\mathbf{x}_1}(Y)\right]$$

Balitsky ‘96, Yu.K. ‘99

Solution of BK equation



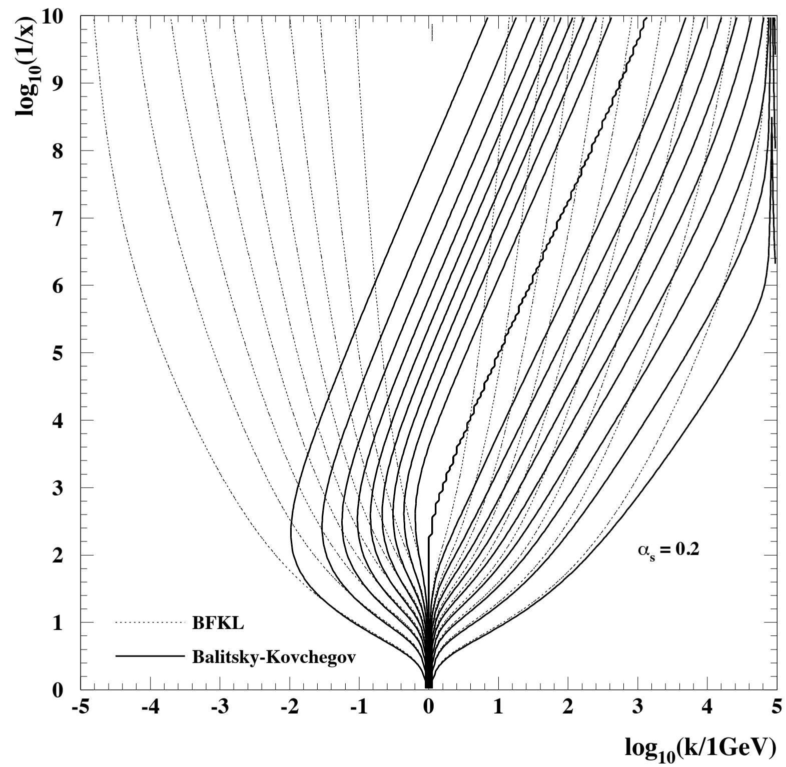
numerical solution by J. Albacete

BK Solution

- Preserves the black disk limit, N<1 always.

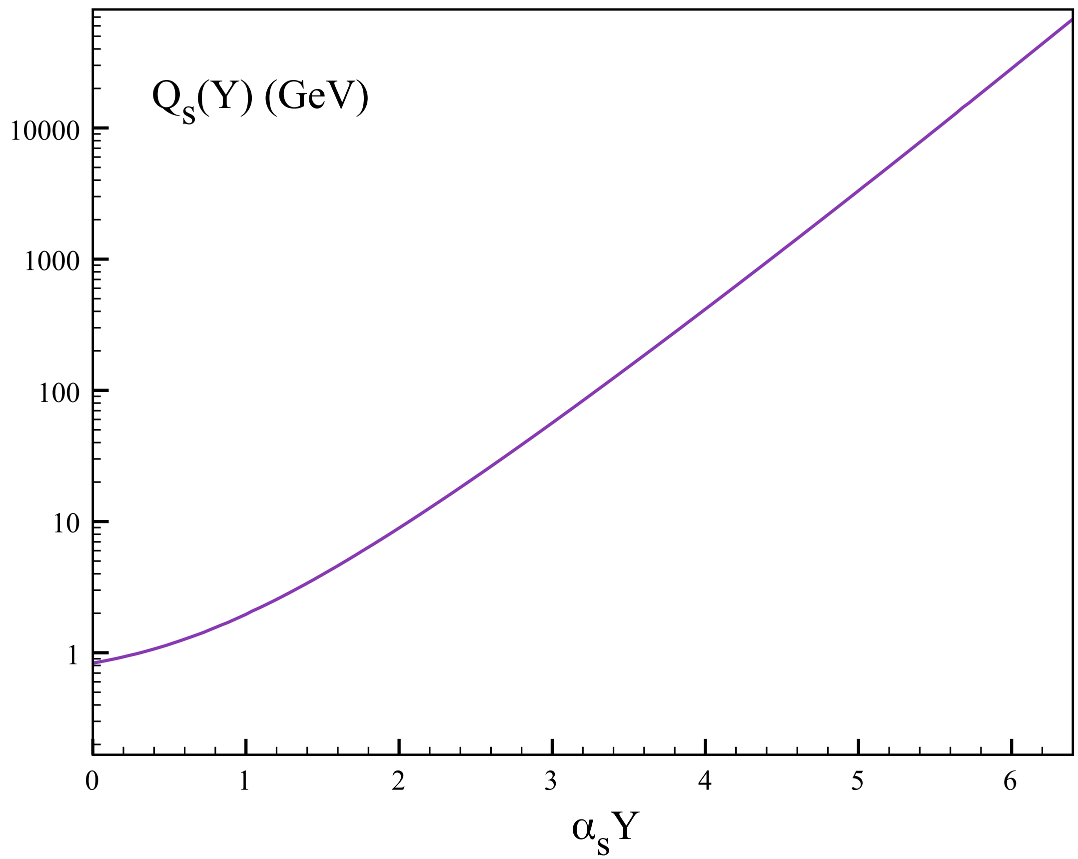
$$\sigma^{q\bar{q}A} = 2\int d^2b\, N(x_\perp, b_\perp, Y)$$

- Avoids the IR problem of BFKL evolution due to the saturation scale screening the IR:



Golec-Biernat, Motyka, Stasto '02

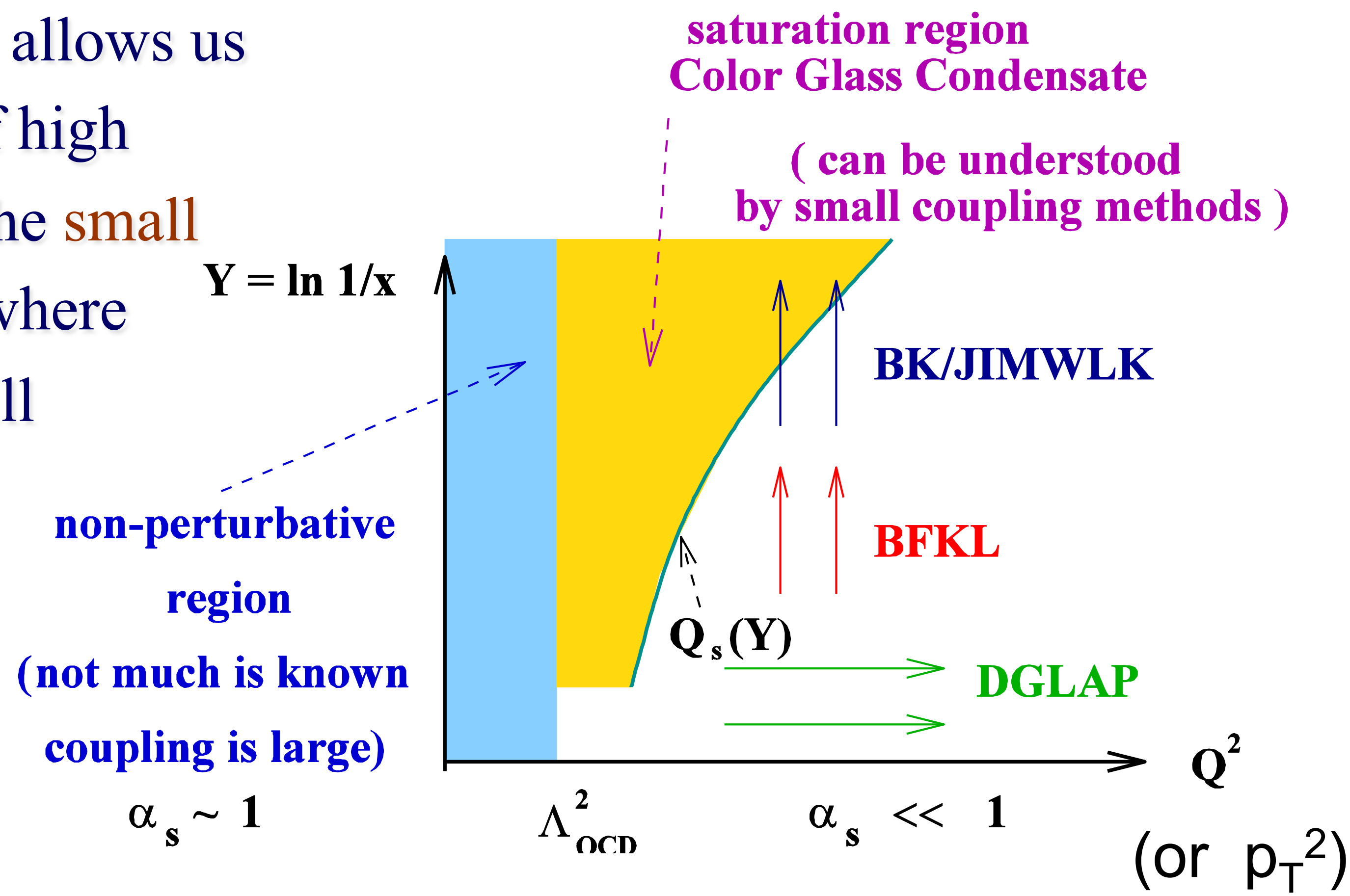
Saturation scale



numerical solution by J. Albacete

Map of High Energy QCD

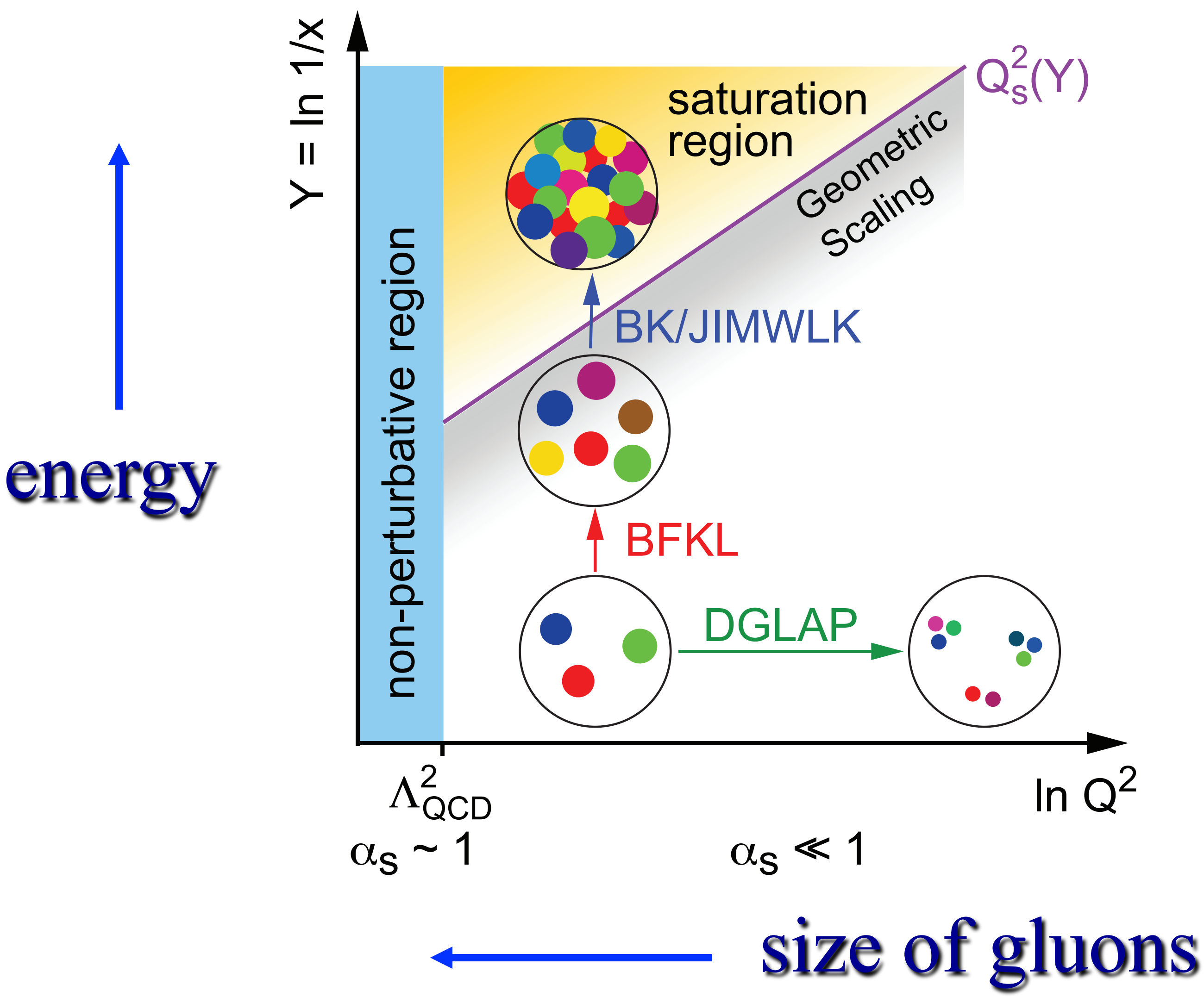
Saturation physics allows us to study regions of high parton density in the small coupling regime, where calculations are still under control!



Transition to saturation region is characterized by the saturation scale

$$Q_S^2 \sim A^{1/3} \left(\frac{1}{x} \right)^{\lambda}$$

Map of High Energy QCD



Going Beyond Large N_C: JIMWLK

To do calculations beyond the large-N_C limit on has to use a functional integro-differential equation written by Iancu, Jalilian-Marian, Kovner, Leonidov, McLerran and Weigert (JIMWLK):

$$\frac{\partial Z}{\partial Y} = \alpha_s \left\{ \frac{1}{2} \frac{\delta^2}{\delta\rho(u)\, \delta\rho(v)} \left[Z\, \chi(u,v) \right] - \frac{\delta}{\delta\rho(u)} \left[Z\, \sigma(u) \right] \right\}$$

where the functional Z[ρ] can then be used for obtaining wave function-averaged observables (like Wilson loops for DIS):

$$\langle O \rangle = \int D\rho\, Z[\rho]\, O[\rho]$$

Going Beyond Large N_C: JIMWLK

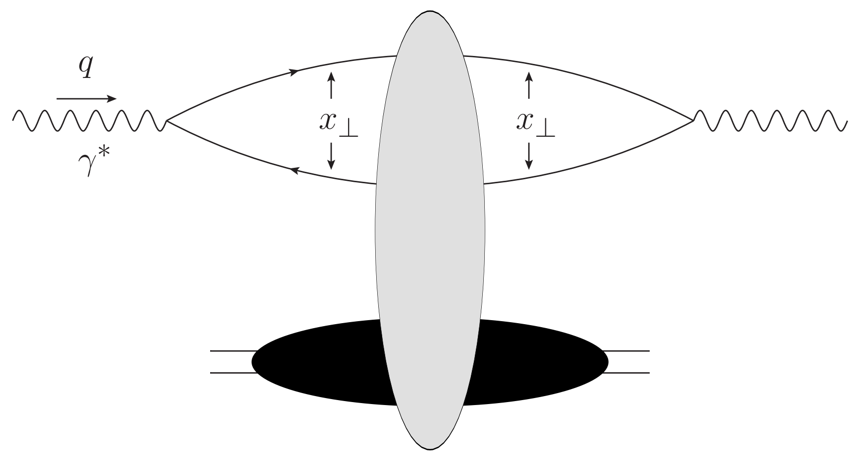
- The JIMWLK equation has been solved on the lattice by Rummukainen and H. Weigert '04

- For the dipole amplitude $N(x_0,x_1, Y)$, the **relative** corrections to the large-N_C limit BK equation are **< 0.001 !** Not the naïve $1/N_C^2 \sim 0.1$! (For realistic rapidities/ energies.)

- The reason for that is dynamical, and is largely due to saturation effects suppressing the bulk of the potential $1/N_C^2$ corrections (Yu.K., J. Kuokkanen, K. Rummukainen, H. Weigert, '08).

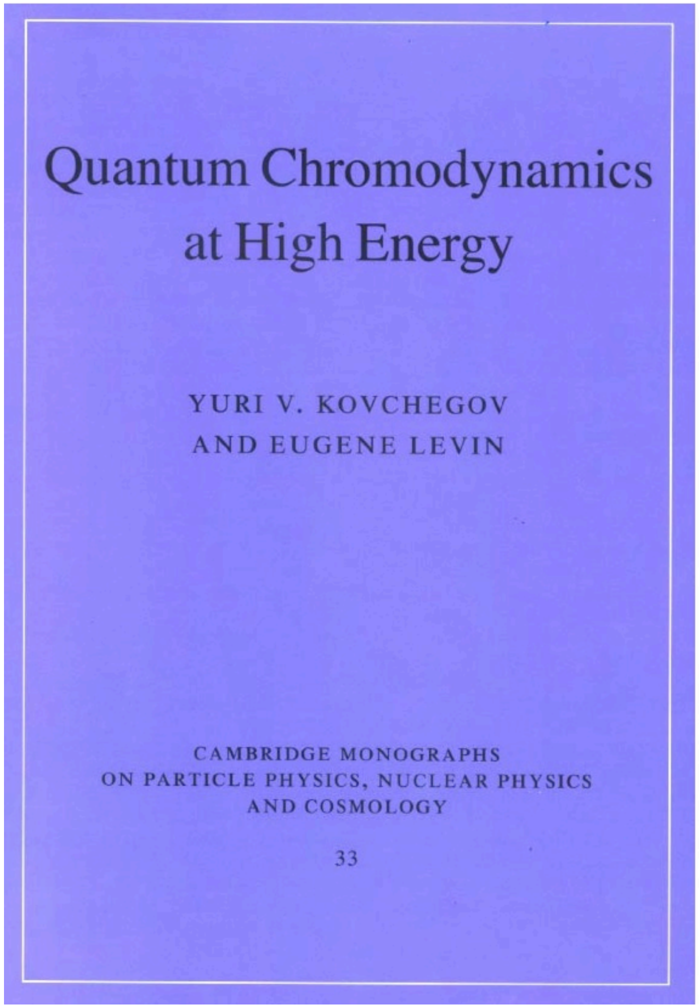
Dipole Amplitude and Other Operators

- Dipole scattering amplitude is a universal degree of freedom in saturation physics.
- It describes the total DIS cross section and structure functions:



- It also describes single inclusive quark and gluon production cross section in DIS and in p+A collisions.
- Works for diffraction in DIS and p+A.
- For correlations need also quadrupoles (J.Jalilian-Marian, Yu.K. '04; Dominguez et al '11) and other Wilson line operators.

A reference



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Higher-Order Corrections to Small-x Evolution

Non-linear evolution: why do we need higher-order corrections?

- Theoretically nothing is wrong with LO BK/JIMWLK: it preserves unitarity (black disk limit), prevents the IR catastrophe.
- Phenomenologically there is a problem though: LO BFKL intercept is way too large (compared to 0.2-0.3 needed to describe experiment)

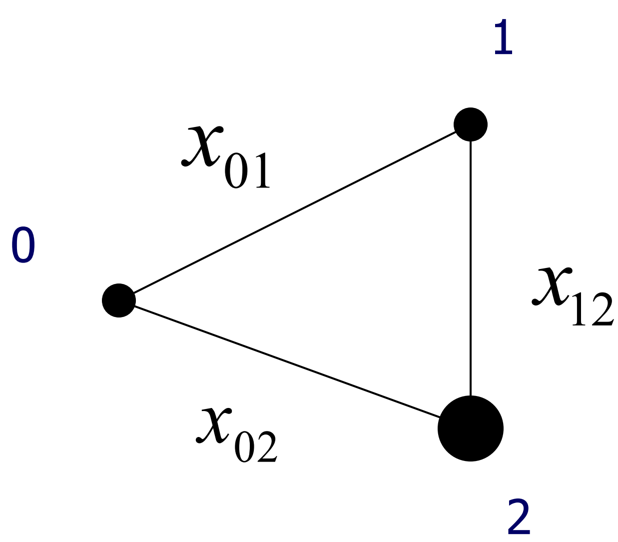
$$\alpha_P - 1 = 2.77\,\frac{\alpha_s\,N_c}{\pi}\;\approx\;0.79$$

- Seems like we need higher-order corrections to describe the data.
- First let's try to determine the scale of the coupling.

A. Running Coupling

What Sets the Scale for the Running Coupling?

$$\frac{\partial N(x_0,x_1,Y)}{\partial Y} = \frac{\alpha_S N_C}{2\pi^2}\int d^2x_2 \, \frac{x_{01}^2}{x_{02}^2\, x_{12}^2}$$
$$\times [N(x_0,x_2,Y) + N(x_2,x_1,Y) - N(x_0,x_1,Y) - N(x_0,x_2,Y)\, N(x_2,x_1,Y)]$$



transverse plane

What Sets the Scale for the Running Coupling?

$$\frac{\partial N(x_0, x_1, Y)}{\partial Y} = \frac{\alpha_S N_C}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{02}^2 x_{12}^2}$$
$$\times [N(x_0, x_2, Y) + N(x_2, x_1, Y) - N(x_0, x_1, Y) - N(x_0, x_2, Y) N(x_2, x_1, Y)]$$

$\alpha_S(???)$

In order to perform consistent calculations it is important to know the scale of the running coupling constant in the evolution equation.

There are three possible scales – the sizes of the "parent" dipole and "daughter" dipoles x_{01}, x_{21}, x_{20}. Which one is it?

Preview

- The answer is that the running coupling corrections come in as a **"triumvirate"** of couplings (H. Weigert, Yu. K. ' 06; I. Balitsky, '06):

$$\alpha_\mu \Rightarrow \frac{\alpha_s(\ldots)\,\alpha_s(\ldots)}{\alpha_s(\ldots)}$$

cf. M. Braun ' 94, E. Levin '94

- The scales of three couplings are somewhat involved.

BLM Prescription

To set the scale of the coupling constant we calculate the $\alpha_s N_f$ corrections to BK/JIMWLK evolution kernel to <u>all orders</u>.

We then would complete N_f to the QCD beta-function

$$\beta_2 = \frac{11 N_C - 2 N_f}{12\pi}$$

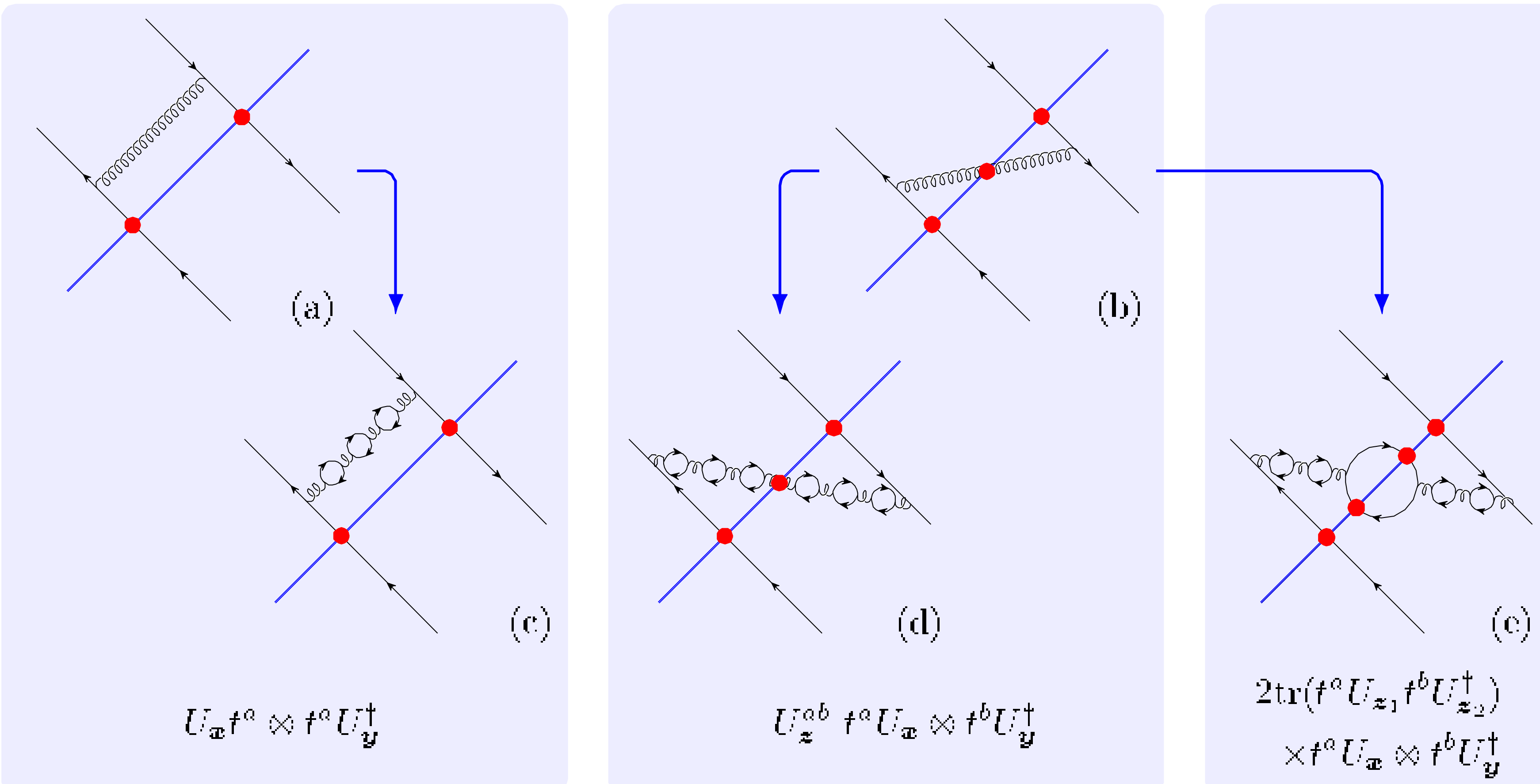
by replacing $N_f \to -6\pi\beta_2$ to obtain the scale of the running coupling:

$$\alpha_s(Q^2) = \frac{\alpha_\mu}{1 + \alpha_\mu \beta_2 \ln(Q^2/\mu^2)}$$

(Brodsky, Lepage, Mackenzie '83)

Running Coupling Corrections to All Orders

One has to insert fermion bubbles to all orders:



Running Coupling BK

Here's the BK equation with the running coupling corrections (H. Weigert, Yu. K. '06; I. Balitsky, '06):

$$\frac{\partial N(x_0, x_1, Y)}{\partial Y} = \frac{N_C}{2\pi^2} \int d^2 x_2$$

$$\times \left[\frac{\alpha_S(1/x_{02}^2)}{x_{02}^2} + \frac{\alpha_S(1/x_{12}^2)}{x_{12}^2} - 2\,\frac{\alpha_S(1/x_{02}^2)\,\alpha_S(1/x_{12}^2)}{\alpha_S(1/R^2)}\,\frac{\mathbf{x}_{20}\cdot\mathbf{x}_{21}}{x_{02}^2\, x_{12}^2} \right]$$

$$\times [N(x_0, x_2, Y) + N(x_2, x_1, Y) - N(x_0, x_1, Y) - N(x_0, x_2, Y)\, N(x_2, x_1, Y)]$$

where

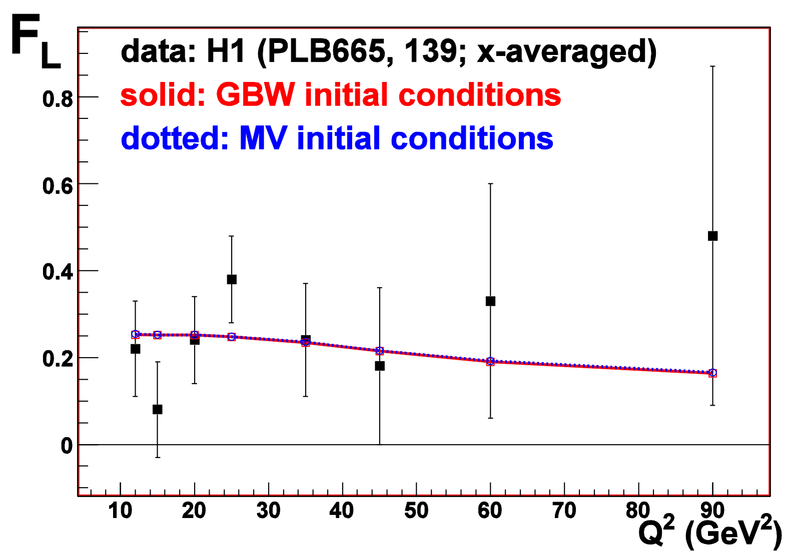
$$\ln R^2 \mu^2 = \frac{x_{20}^2 \ln(x_{21}^2\, \mu^2) - x_{21}^2 \ln(x_{20}^2\, \mu^2)}{x_{20}^2 - x_{21}^2} + \frac{x_{20}^2\, x_{21}^2}{\mathbf{x}_{20}\cdot\mathbf{x}_{21}}\,\frac{\ln(x_{20}^2 / x_{21}^2)}{x_{20}^2 - x_{21}^2}$$

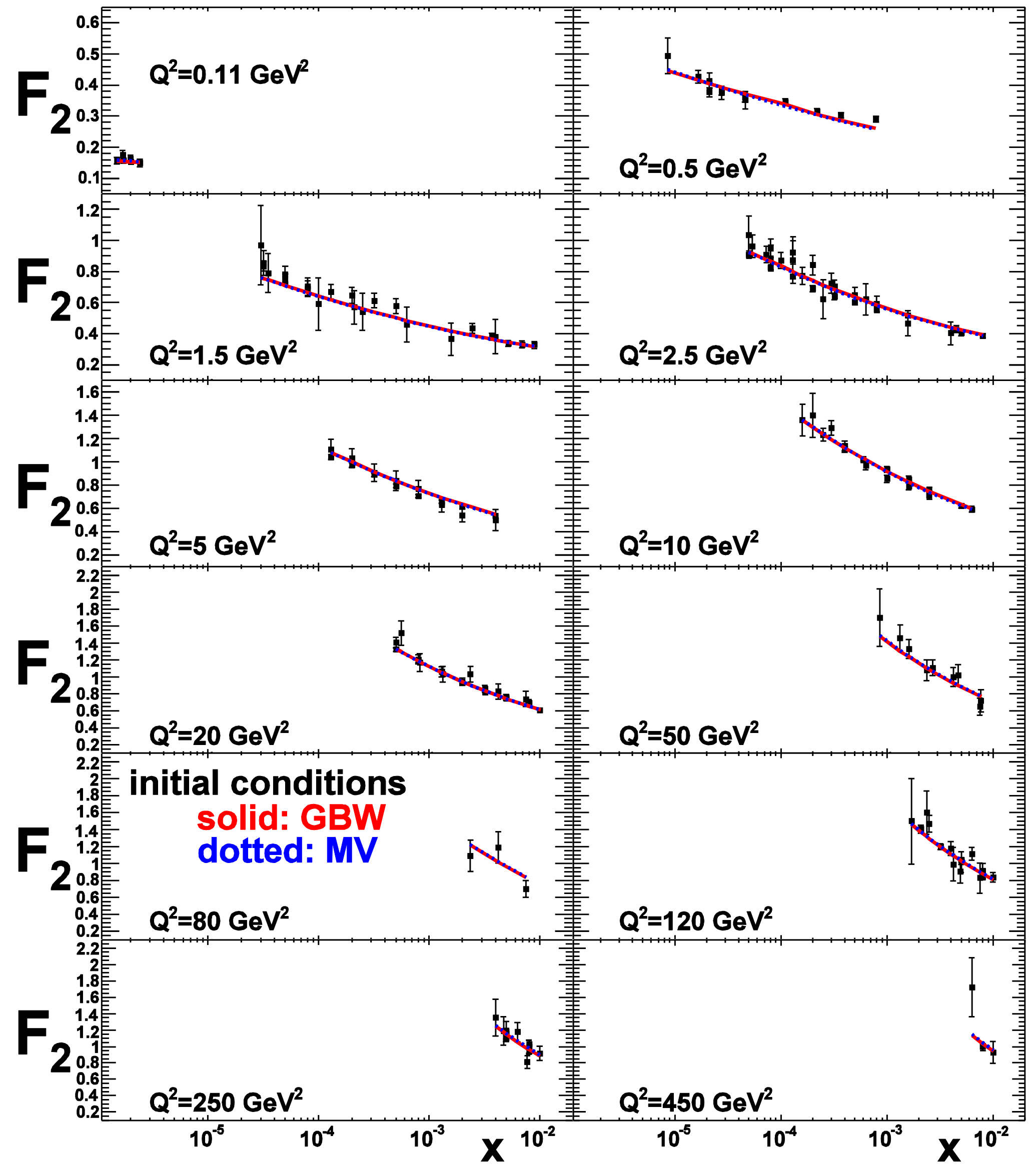
Comparison of rcBK with HERA F2 Data

DIS structure functions:

$$F_{2,\,L} = \frac{Q^2}{4\,\pi^2\,\alpha_{EM}}\,\sigma^{\gamma * p}_{tot,\,L}$$

from Albacete, Armesto, Milhano, Salgado '09



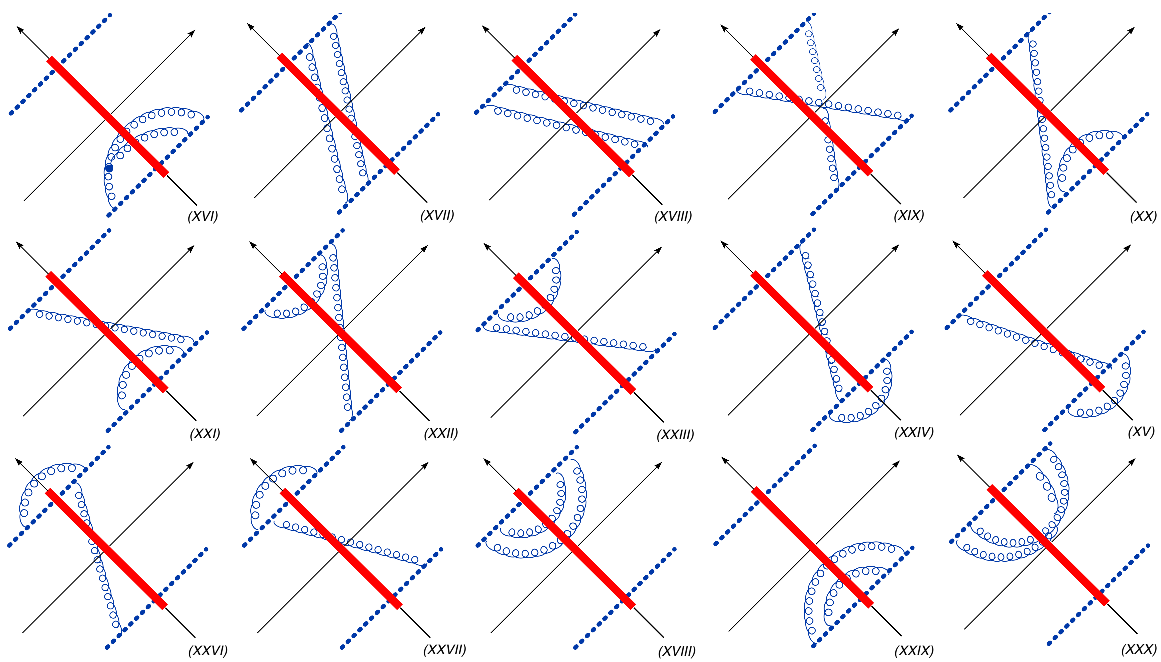


B. NLO BFKL/BK/JIMWLK

NLO BK

- NLO BK evolution was calculated by Ballitsky and Chirilli in 2007.
- It resums powers of $\alpha_s^2 Y$ (NLO) in addition to powers of $\alpha_s Y$ (LO).
- Here's a sampler of relevant diagrams (need kernel to order-α^2):

Diagrams with 2 gluons interaction



NLO BK

- The large-N_C limit:

$$
\begin{aligned}
\frac{d}{d\eta}N(x,y) = {} & \frac{\alpha_s N_c}{2\pi^2}\int d^2z \frac{(x-y)^2}{X^2Y^2}\left\{1+\frac{\alpha_s N_c}{4\pi}\left[\frac{11}{3}\ln(x-y)^2\mu^2 - \frac{11}{3}\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2}+\frac{67}{9}-\frac{\pi^2}{3}-2\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\right]\right\} \\
& \times [N(x,z)+N(z,y)-N(x,y)-N(x,z)N(z,y)] \\
& + \frac{\alpha_s^2 N_c^2}{8\pi^4}\int d^2z d^2z' \left\{-\frac{2}{(z-z')^4}+\left[\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}\right.\right. \\
& \left.\left. + \frac{(x-y)^2}{X^2Y'^2(z-z')^2}\right]\ln\frac{X^2Y'^2}{X'^2Y^2}\right\}[N(z,z')-N(x,z)N(z,z')-N(z,z')N(z',y)-N(x,z)N(z',y)+N(x,z)N(z,y) \\
& + N(x,z)N(z,z')N(z',y)]. \qquad (136)
\end{aligned}
$$

(yet to be solved numerically)

NLO JIMWLK

- Very recently NLO evolution has been calculated for other Wilson line operators (not just dipoles), most notably the 3-Wilson line operator (Grabovsky '13, Balitsky & Chirilli '13, Kovner, Lublinsky, Mulian '13, Balitsky and Grabovsky '14). See talk by I. Balitsky later today!

- The NLO JIMWLK Hamiltonian was constructed as well (Kovner, Lublinsky, Mulian '13, '14).

- However, the equations do not close, that is, the operators on the right hand side can not be expressed in terms of the operator on the left. Hence can't solve.

- To find the expectation values of the corresponding operators, one has to perform a lattice calculation with the NLO JIMWLK Hamiltonian, generating field configurations to be used for averaging the operators.

NLO Dipole Evolution at any N_C

- NLO BK equation is the large-N_C limit of (Balitsky and Chrilli ' 07)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat U_x \hat U_y^\dagger\} \qquad (5)\\
&= \frac{\alpha_s}{2\pi^2}\int d^2z\, \frac{(x-y)^2}{X^2Y^2}\Big\{1+\frac{\alpha_s}{4\pi}\Big[b\ln(x-y)^2\mu^2 - b\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2} + (\frac{67}{9}-\frac{\pi^2}{3})N_c - \frac{10}{9}n_f \\
&\qquad - 2N_c\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big]\Big\}\,[\mathrm{Tr}\{\hat U_x\hat U_z^\dagger\}\mathrm{Tr}\{\hat U_z\hat U_y^\dagger\} - N_c\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}] \\
&+ \frac{\alpha_s^2}{16\pi^4}\int d^2z d^2z' \Bigg[\Big(-\frac{4}{(z-z')^4} + \Big\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]} \\
&+ \frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big] + \frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big) \\
&\qquad \times [\mathrm{Tr}\{\hat U_x\hat U_z^\dagger\}\mathrm{Tr}\{\hat U_z\hat U_{z'}^\dagger\}\mathrm{Tr}\{\hat U_{z'}\hat U_y^\dagger\} - \mathrm{Tr}\{\hat U_x\hat U_z^\dagger\hat U_{z'}U_y^\dagger\hat U_z\hat U_{z'}^\dagger\} - (z'\to z)] \\
&+ \Big\{\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big] - \frac{(x-y)^4}{X^2Y'^2X'^2Y^2}\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\ \mathrm{Tr}\{\hat U_x\hat U_z^\dagger\}\mathrm{Tr}\{\hat U_z\hat U_{z'}^\dagger\}\mathrm{Tr}\{\hat U_{z'}\hat U_y^\dagger\} \\
&+ 4n_f\Big\{\frac{4}{(z-z')^4} - 2\frac{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}{(z-z')^4(X^2Y'^2-X'^2Y^2-)}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\mathrm{Tr}\{t^a\hat U_x t^b\hat U_y^\dagger\}[\mathrm{Tr}\{t^a\hat U_z t^b\hat U_{z'}^\dagger\} - (z'\to z)]\Bigg]
\end{aligned}
$$

NLO Corrections

- Note also that two iterations of NLO evolution kernel is parametrically of the same order as a combination of one LO and one NNLO kernels:

$$(\alpha_s^2 \, Y)^2 \sim (\alpha_s \, Y) \, (\alpha_s^3 \, Y)$$

- Does this mean that NLO kernel can only be inserted once into the LO evolution?
- Things simplify if you know the solution of the equation. For instance, in DGLAP case, perturbative expansion in the kernel naturally translates into the perturbative expansion in the anomalous dimensions.
- Nonlinear equations are hard. Let's consider the linear BFKL evolution.

The Problem

- We want to find the BFKL Green function. It satisfies the BFKL equation

$$\partial_Y G(k,k',Y) = \int d^2q\, K(k,q)\, G(q,k',Y)$$

with the initial condition

$$G(k,k',Y=0) = \frac{1}{2\pi k}\delta(k-k')$$

- K(k,q) represents a BFKL kernel at an unspecified order in α_s.
- We need to find the eigenfunctions and eigenvalues for the kernel.

BFKL Equation in *N*=4 SYM Theory

- The form of the BFKL equation's solution is straightforward to determine in *N*=4 SYM theory: there the eigenfunctions are fixed by conformal symmetry and are simply $E^{n,\nu}$ (eigenfunctions of the Casimir operators of the Mobius group).

- In the angle-independent case at hand $E^{n,\nu}$'s reduce to simple powers of momentum k and we write the BFKL Green function in *N*=4 SYM theory as

$$G(k, k', Y) = \int\limits_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\alpha\, \chi_0(\nu) + \alpha^2\, \chi_1(\nu) + \ldots\right] Y} \left(\frac{k^2}{k'^2}\right)^{i\,\nu}$$

- Perturbative expansion takes place in the exponent (the eigenvalue).

Solving BFKL Equation in QCD

- QCD is not a conformal theory: we can not fix the all-order BFKL eigenfunctions by a symmetry argument.
- While simple powers are eigenfunctions for the LO kernel, they are not eigenfunctions for the NLO kernel due to the running coupling effects:

$$\int d^2q \, K^{\mathrm{LO+NLO}}(k,q) \; q^{2\gamma-2} = \left[\bar{\alpha}_\mu \, \chi_0(\gamma) - \bar{\alpha}_\mu^2 \, \beta_2 \, \chi_0(\gamma) \, \ln \frac{k^2}{\mu^2} - \frac{1}{2} \, \bar{\alpha}_\mu^2 \, \beta_2 \, \chi_0'(\gamma) + \bar{\alpha}_\mu^2 \, \chi_1(\gamma) \right] k^{2\gamma-2}.$$

LO BFKL eigenvalue

1-loop running coupling

Conformal NLO terms

NLO terms

$$\bar{\alpha}_\mu \equiv \frac{\alpha_\mu \, N_c}{\pi}$$

$$\chi_0(\gamma) = 2\,\psi(1) - \psi(\gamma) - \psi(1-\gamma) \qquad \beta_2 = \frac{11\,N_c - 2\,N_f}{12\,N_c}$$

The Strategy

- Since the BFKL kernel is known perturbatively up to NLO

$$K(k,q) = \bar{\alpha}_\mu \, K^{\mathrm{LO}}(k,q) + \bar{\alpha}_\mu^2 \, K^{\mathrm{NLO}}(k,q) + \mathcal{O}(\bar{\alpha}_\mu^3)$$

  it appears logical to construct the eigenfunctions order-by-order in the coupling as well. (Solving NLO BFKL equation exactly would exceed the precision of the approximation as NLO^2 = LO x NNLO.)

G. Chirilli, Yu.K. '13

- To find the eigenfunctions we thus write

$$H_\gamma(k) = k^{2\gamma-2} \, \left[1 + \bar{\alpha}_\mu \, f_\gamma(k) + \ldots\right]$$

  and (perturbatively) impose the eigenfunction condition

$$\int d^2q \, K^{\mathrm{LO+NLO}}(k,q) \, H_\gamma(q) = \Delta(\gamma) \, H_\gamma(k)$$

  where the eigenvalue $\Delta(\gamma)$ is also an unknown.

NLO BFKL Solution

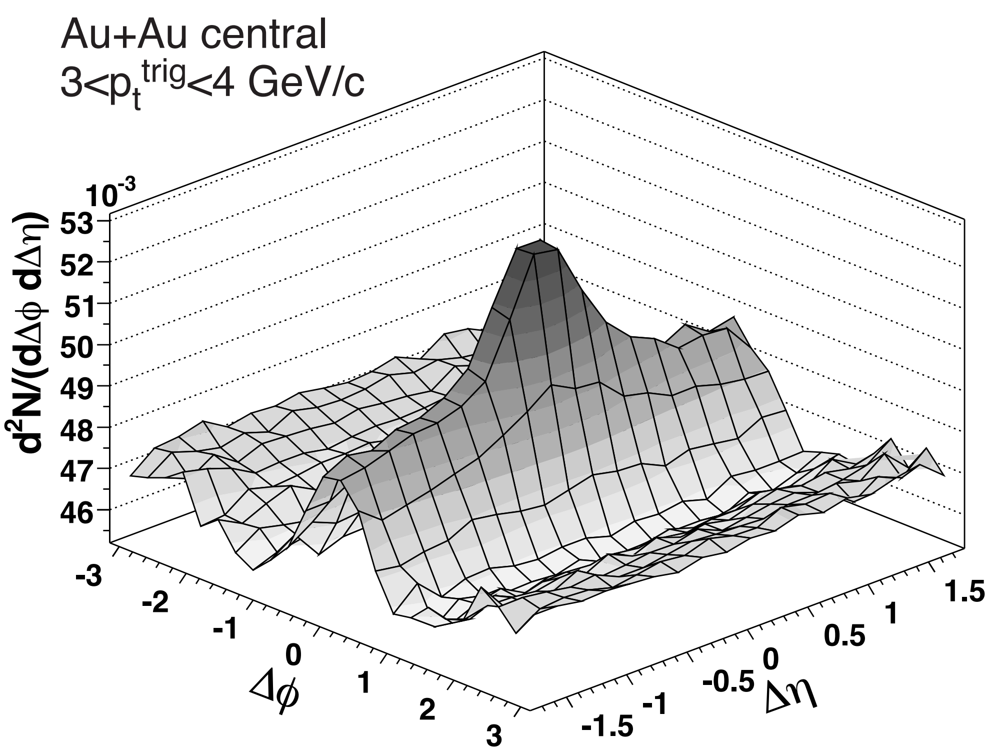
$$\boxed{G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2}\, e^{\left[\bar{\alpha}_\mu\, \chi_0(\nu)+\bar{\alpha}_\mu^2\, \chi_1(\nu)\right] Y}\, H_{\frac{1}{2}+i\,\nu}(k)\, \left[H_{\frac{1}{2}+i\,\nu}(k')\right]^*}$$

- Note that the perturbative expansion is present both in the exponent and in the eigenfunctions (G. Chirilli, Yu.K. '13).
- The procedure can be repeated at higher orders in α_s and was implemented at NNLO already (G. Chirilli, Yu.K. '14).
- See talk by G. Chirillli next!

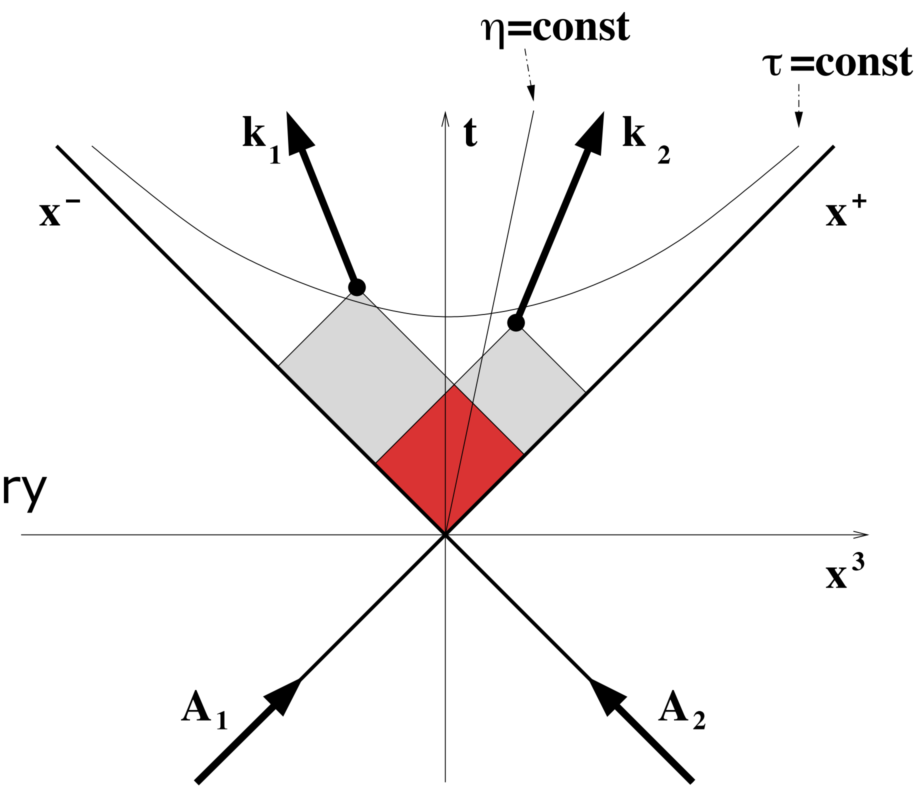
Long-Range Rapidity Correlations

Ridge in heavy ion collisions

- Heavy ion collisions, along with high-multiplicity p+p and p+A collisions, are known to have long-range rapidity correlations, known as 'the ridge':



Origin of rapidity correlations



Causality demands that long-range rapidity correlations originate at very early times (cf. explanation of the CMB homogeneity in the Universe)

Gavin, McLerran, Moschelli '08;
Dumitru, Gelis, McLerran, Venugopalan '08.

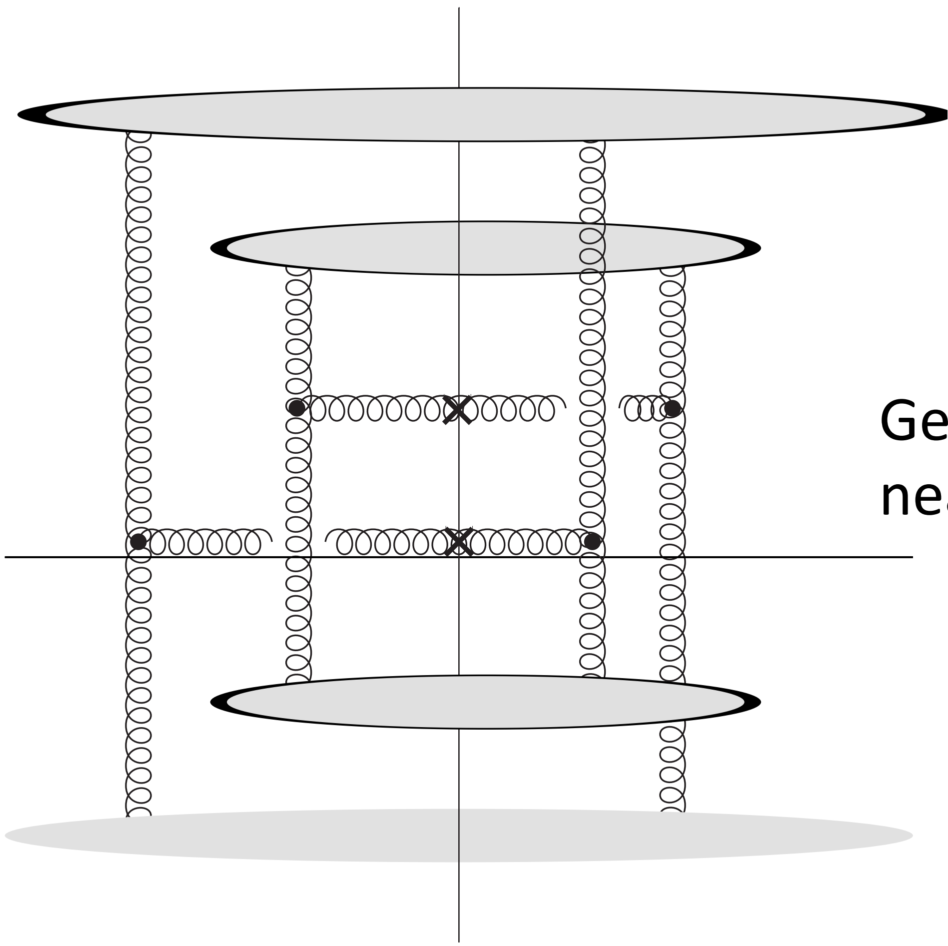
Ridge in CGC

- There are two explanations of the ridge in CGC:

  – Long-range rapidity-independent fields are created at early times, with correlations generated soon after and with azimuthal collimation produced by radial hydro flow. (Gavin, McLerran, Moschelli '08)

  – Both long-range rapidity correlations and the azimuthal correlations are created in the collision due to a particular class of diagrams referred to as the "Glasma graphs".

“Glasma” graphs



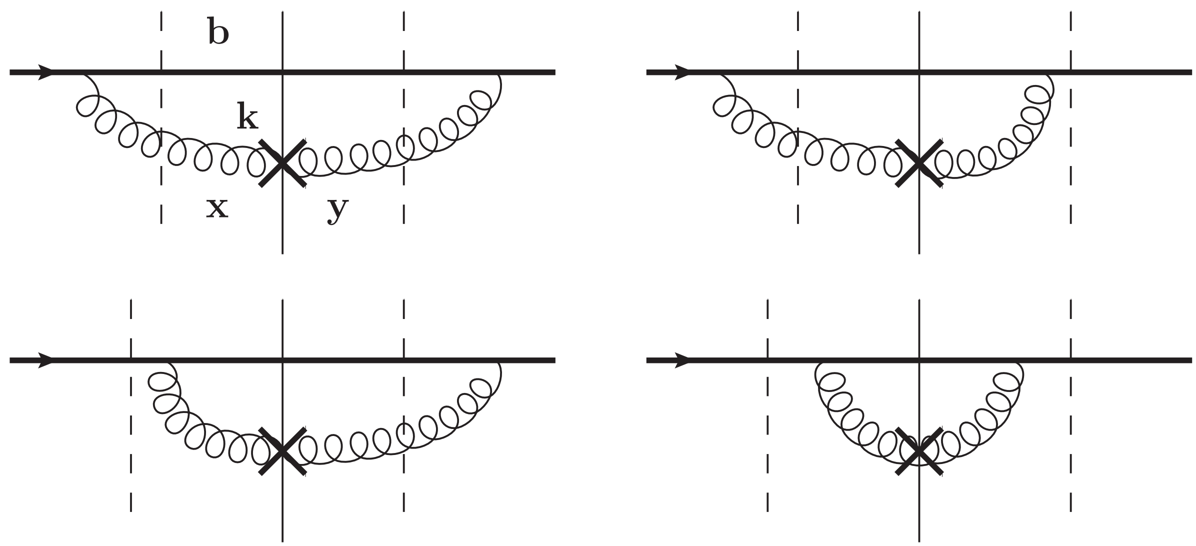
Generate back-to-back and near-side azimuthal correlations.

Dumitru, Gelis, McLerran, Venugopalan ‘08.

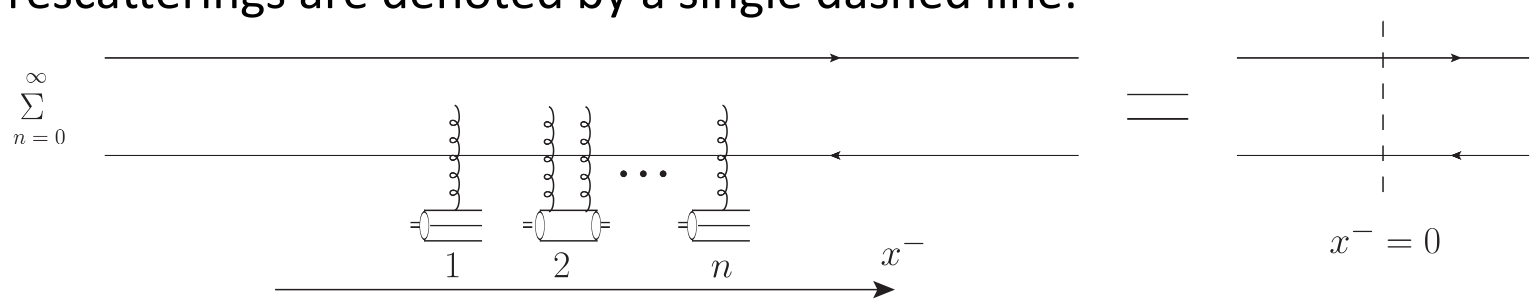
(i) Single gluon production in pA

Single gluon production in pA

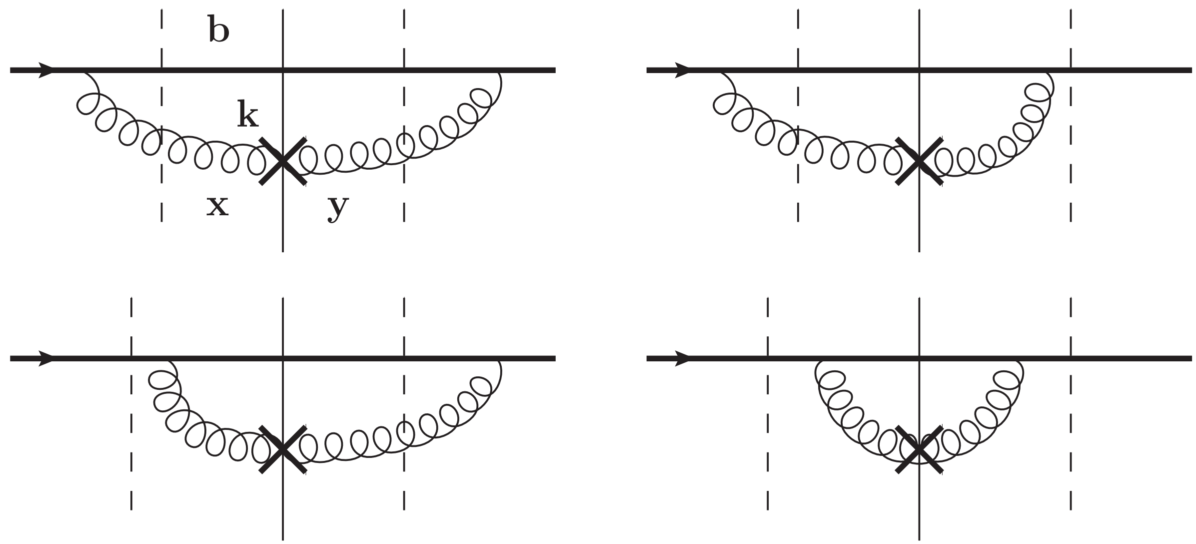
Model the proton by a single quark (can be easily improved upon).
The diagrams are shown below (Yu.K., A. Mueller '97):



Multiple rescatterings are denoted by a single dashed line:



Single gluon production in pA



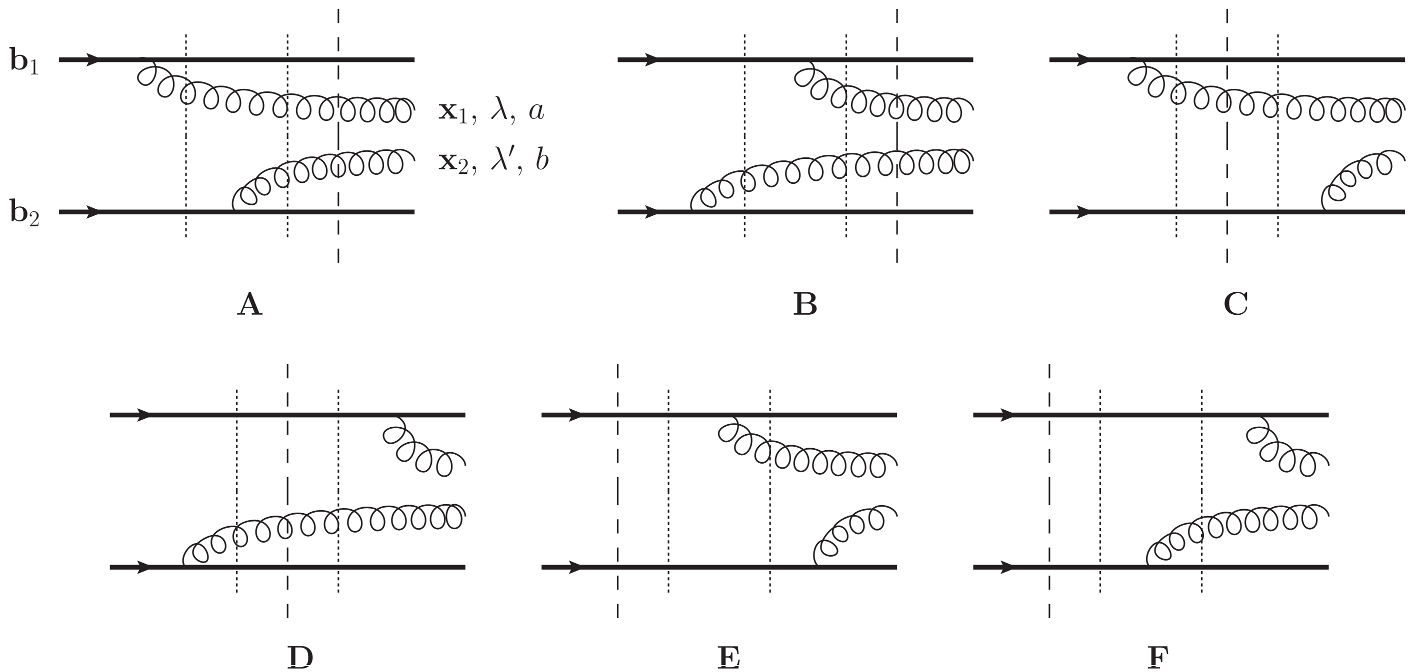
The gluon production cross section can be readily written as (U = Wilson line in **adjoint** representation, represents gluon interactions)

$$\left\langle \frac{d\sigma^{pA_2}}{d^2k\, dy\, d^2b} \right\rangle = \frac{\alpha_s\, C_F}{4\,\pi^4} \int d^2x\, d^2y\; e^{-i\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, \frac{\mathbf{x}-\mathbf{b}}{|\mathbf{x}-\mathbf{b}|^2} \cdot \frac{\mathbf{y}-\mathbf{b}}{|\mathbf{y}-\mathbf{b}|^2}$$

$$\times\; \left\langle \frac{1}{N_c^2-1}\, Tr[U_{\mathbf{x}} U_{\mathbf{y}}^\dagger] \;-\; \frac{1}{N_c^2-1}\, Tr[U_{\mathbf{x}} U_{\mathbf{b}}^\dagger] \;-\; \frac{1}{N_c^2-1}\, Tr[U_{\mathbf{b}} U_{\mathbf{y}}^\dagger] \;+\; 1 \right\rangle$$

(ii) Two-gluon production in heavy-light ion collisions

The process

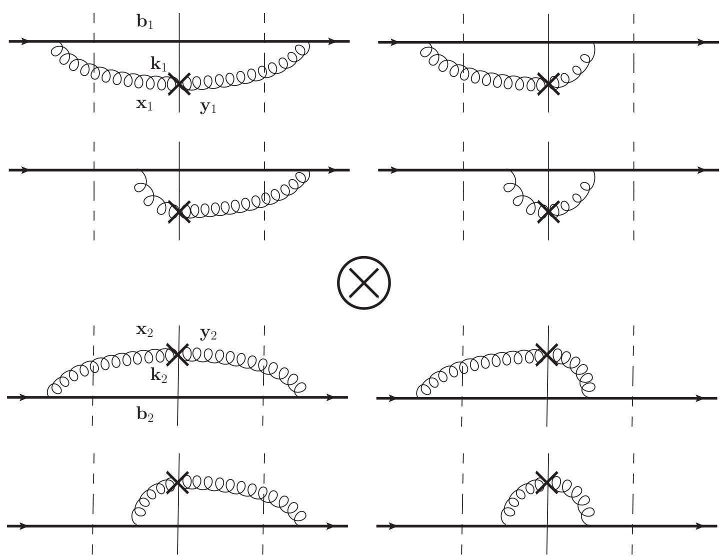


Solid horizontal lines = quarks in the incoming nucleons.
Dashed vertical line = interaction with the target.
Dotted vertical lines = energy denominators.

Two-gluon production cross section

- "Squaring" the single gluon production cross section yields

$$\frac{d\sigma}{d^2k_1 dy_1 d^2k_2 dy_2} = \frac{\alpha_s^2\, C_F^2}{16\,\pi^8} \int d^2B\, d^2b_1\, d^2b_2\, T_1(\mathbf{B}-\mathbf{b}_1)\, T_1(\mathbf{B}-\mathbf{b}_2)\, d^2x_1\, d^2y_1\, d^2x_2\, d^2y_2\, e^{-i\,\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{y}_1)-i\,\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{y}_2)}$$
$$\times\ \frac{\mathbf{x}_1-\mathbf{b}_1}{|\mathbf{x}_1-\mathbf{b}_1|^2}\cdot\frac{\mathbf{y}_1-\mathbf{b}_1}{|\mathbf{y}_1-\mathbf{b}_1|^2}\ \frac{\mathbf{x}_2-\mathbf{b}_2}{|\mathbf{x}_2-\mathbf{b}_2|^2}\cdot\frac{\mathbf{y}_2-\mathbf{b}_2}{|\mathbf{y}_2-\mathbf{b}_2|^2}$$
$$\times\ \Bigg\langle\left(\frac{1}{N_c^2-1}\,Tr[U_{\mathbf{x}_1}U^\dagger_{\mathbf{y}_1}] - \frac{1}{N_c^2-1}\,Tr[U_{\mathbf{x}_1}U^\dagger_{\mathbf{b}_1}] - \frac{1}{N_c^2-1}\,Tr[U_{\mathbf{b}_1}U^\dagger_{\mathbf{y}_1}] + 1\right)$$
$$\times\ \left(\frac{1}{N_c^2-1}\,Tr[U_{\mathbf{x}_2}U^\dagger_{\mathbf{y}_2}] - \frac{1}{N_c^2-1}\,Tr[U_{\mathbf{x}_2}U^\dagger_{\mathbf{b}_2}] - \frac{1}{N_c^2-1}\,Tr[U_{\mathbf{b}_2}U^\dagger_{\mathbf{y}_2}] + 1\right)\Bigg\rangle$$



(cf. Kovner & Lublinsky, '12)

Two-gluon production cross section

- The "crossed" diagrams give

$$\frac{d\sigma_{crossed}}{d^2k_1 dy_1 d^2k_2 dy_2} = \frac{1}{[2(2\pi)^3]^2} \int d^2B\, d^2b_1\, d^2b_2\, T_1(\mathbf{B}-\mathbf{b}_1)\, T_1(\mathbf{B}-\mathbf{b}_2)\, d^2x_1\, d^2y_1\, d^2x_2\, d^2y_2$$

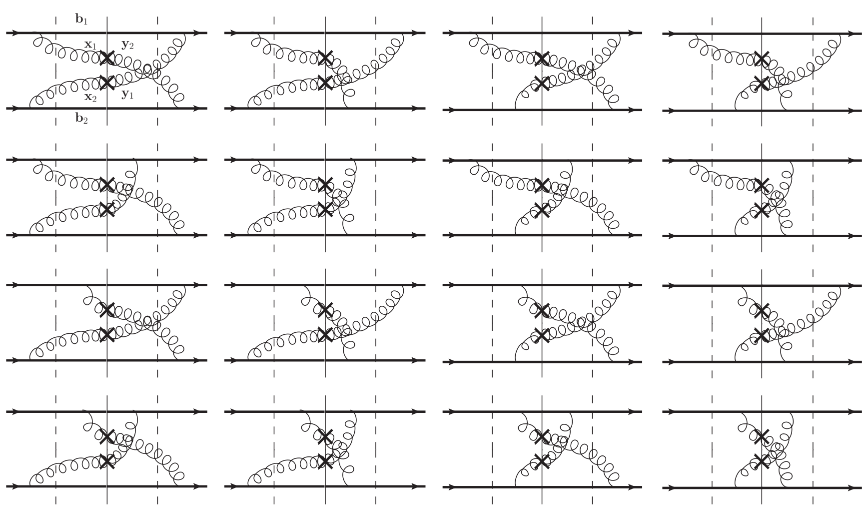
$$\times \left[e^{-i\,\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{y}_2)-i\,\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{y}_1)} + e^{-i\,\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{y}_2)+i\,\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{y}_1)} \right]$$

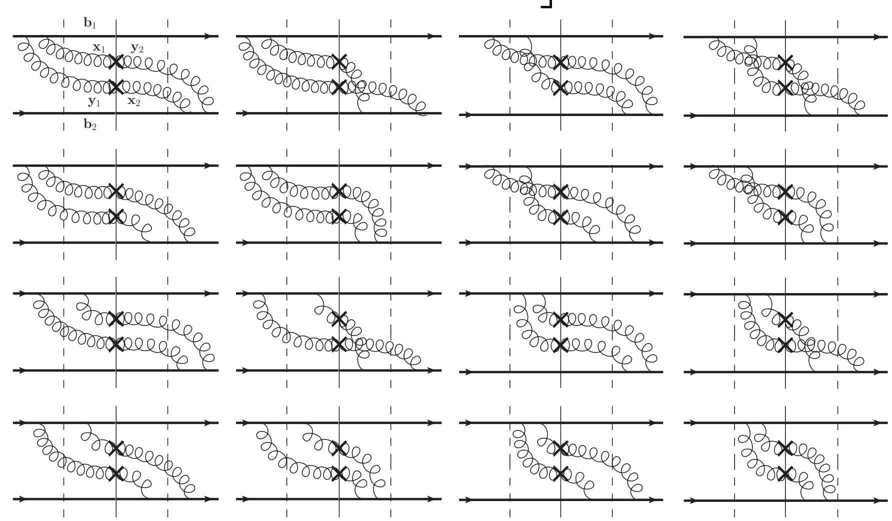
$$\times\ \frac{16\,\alpha_s^2}{\pi^2}\,\frac{C_F}{2N_c}\,\frac{\mathbf{x}_1-\mathbf{b}_1}{|\mathbf{x}_1-\mathbf{b}_1|^2}\cdot\frac{\mathbf{y}_2-\mathbf{b}_2}{|\mathbf{y}_2-\mathbf{b}_2|^2}\,\frac{\mathbf{x}_2-\mathbf{b}_2}{|\mathbf{x}_2-\mathbf{b}_2|^2}\cdot\frac{\mathbf{y}_1-\mathbf{b}_1}{|\mathbf{y}_1-\mathbf{b}_1|^2}$$

$$\times \Big[Q(\mathbf{x}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{y}_2) - Q(\mathbf{x}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{b}_2) - Q(\mathbf{x}_1,\mathbf{y}_1,\mathbf{b}_2,\mathbf{y}_2) + S_G(\mathbf{x}_1,\mathbf{y}_1) - Q(\mathbf{x}_1,\mathbf{b}_1,\mathbf{x}_2,\mathbf{y}_2)$$

$$+ Q(\mathbf{x}_1,\mathbf{b}_1,\mathbf{x}_2,\mathbf{b}_2) + Q(\mathbf{x}_1,\mathbf{b}_1,\mathbf{b}_2,\mathbf{y}_2) - S_G(\mathbf{x}_1,\mathbf{b}_1) - Q(\mathbf{b}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{y}_2) + Q(\mathbf{b}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{b}_2)$$

$$+ Q(\mathbf{b}_1,\mathbf{y}_1,\mathbf{b}_2,\mathbf{y}_2) - S_G(\mathbf{b}_1,\mathbf{y}_1) + S_G(\mathbf{x}_2,\mathbf{y}_2) - S_G(\mathbf{x}_2,\mathbf{b}_2) - S_G(\mathbf{b}_2,\mathbf{y}_2) + 1 \Big]$$





Two-gluon production cross section

- The “crossed” diagrams give

$$\frac{d\sigma_{crossed}}{d^2k_1 dy_1 d^2k_2 dy_2} = \frac{1}{[2(2\pi)^3]^2} \int d^2B\, d^2b_1\, d^2b_2\, T_1(\mathbf{B}-\mathbf{b}_1)\, T_1(\mathbf{B}-\mathbf{b}_2)\, d^2x_1\, d^2y_1\, d^2x_2\, d^2y_2$$

$$\times \left[e^{-i\,\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{y}_2)-i\,\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{y}_1)} + e^{-i\,\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{y}_2)+i\,\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{y}_1)} \right]$$

$$\times \frac{16\,\alpha_s^2}{\pi^2}\, \frac{C_F}{2N_c}\, \frac{\mathbf{x}_1-\mathbf{b}_1}{|\mathbf{x}_1-\mathbf{b}_1|^2} \cdot \frac{\mathbf{y}_2-\mathbf{b}_2}{|\mathbf{y}_2-\mathbf{b}_2|^2}\, \frac{\mathbf{x}_2-\mathbf{b}_2}{|\mathbf{x}_2-\mathbf{b}_2|^2} \cdot \frac{\mathbf{y}_1-\mathbf{b}_1}{|\mathbf{y}_1-\mathbf{b}_1|^2}$$

$$\times \Big[Q(\mathbf{x}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{y}_2) - Q(\mathbf{x}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{b}_2) - Q(\mathbf{x}_1,\mathbf{y}_1,\mathbf{b}_2,\mathbf{y}_2) + S_G(\mathbf{x}_1,\mathbf{y}_1) - Q(\mathbf{x}_1,\mathbf{b}_1,\mathbf{x}_2,\mathbf{y}_2)$$

$$+ Q(\mathbf{x}_1,\mathbf{b}_1,\mathbf{x}_2,\mathbf{b}_2) + Q(\mathbf{x}_1,\mathbf{b}_1,\mathbf{b}_2,\mathbf{y}_2) - S_G(\mathbf{x}_1,\mathbf{b}_1) - Q(\mathbf{b}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{y}_2) + Q(\mathbf{b}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{b}_2)$$

$$+ Q(\mathbf{b}_1,\mathbf{y}_1,\mathbf{b}_2,\mathbf{y}_2) - S_G(\mathbf{b}_1,\mathbf{y}_1) + S_G(\mathbf{x}_2,\mathbf{y}_2) - S_G(\mathbf{x}_2,\mathbf{b}_2) - S_G(\mathbf{b}_2,\mathbf{y}_2) + 1 \Big]$$

- We introduced the adjoint color-dipole and color quadrupole amplitudes:

$$S_G(\mathbf{x}_1,\mathbf{x}_2,y) \equiv \frac{1}{N_c^2-1}\, \left\langle Tr[U_{\mathbf{x}_1} U^\dagger_{\mathbf{x}_2}] \right\rangle$$

$$Q(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) \equiv \frac{1}{N_c^2-1}\, \left\langle Tr[U_{\mathbf{x}_1} U^\dagger_{\mathbf{x}_2} U_{\mathbf{x}_3} U^\dagger_{\mathbf{x}_4}] \right\rangle$$

Two-gluon production: properties

$$\frac{d\sigma}{d^2k_1 dy_1 d^2k_2 dy_2} = \frac{\alpha_s^2\, C_F^2}{16\,\pi^8} \int d^2B\, d^2b_1\, d^2b_2\, T_1(\mathbf{B}-\mathbf{b}_1)\, T_1(\mathbf{B}-\mathbf{b}_2)\, d^2x_1\, d^2y_1\, d^2x_2\, d^2y_2\, e^{-i\,\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{y}_1)-i\,\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{y}_2)}$$
$$\times\ \frac{\mathbf{x}_1-\mathbf{b}_1}{|\mathbf{x}_1-\mathbf{b}_1|^2}\cdot\frac{\mathbf{y}_1-\mathbf{b}_1}{|\mathbf{y}_1-\mathbf{b}_1|^2}\ \frac{\mathbf{x}_2-\mathbf{b}_2}{|\mathbf{x}_2-\mathbf{b}_2|^2}\cdot\frac{\mathbf{y}_2-\mathbf{b}_2}{|\mathbf{y}_2-\mathbf{b}_2|^2}$$
$$\times\ \left\langle\left(\frac{1}{N_c^2-1}\,Tr[U_{\mathbf{x}_1}U_{\mathbf{y}_1}^\dagger]\ -\ \frac{1}{N_c^2-1}\,Tr[U_{\mathbf{x}_1}U_{\mathbf{b}_1}^\dagger]\ -\ \frac{1}{N_c^2-1}\,Tr[U_{\mathbf{b}_1}U_{\mathbf{y}_1}^\dagger]\ +\ 1\right)\right.$$
$$\left.\times\ \left(\frac{1}{N_c^2-1}\,Tr[U_{\mathbf{x}_2}U_{\mathbf{y}_2}^\dagger]\ -\ \frac{1}{N_c^2-1}\,Tr[U_{\mathbf{x}_2}U_{\mathbf{b}_2}^\dagger]\ -\ \frac{1}{N_c^2-1}\,Tr[U_{\mathbf{b}_2}U_{\mathbf{y}_2}^\dagger]\ +\ 1\right)\right\rangle$$

- The cross section is symmetric under (ditto for the “crossed” term)

$\mathbf{k}_1 \leftrightarrow \mathbf{k}_2$ (just coordinate relabeling)

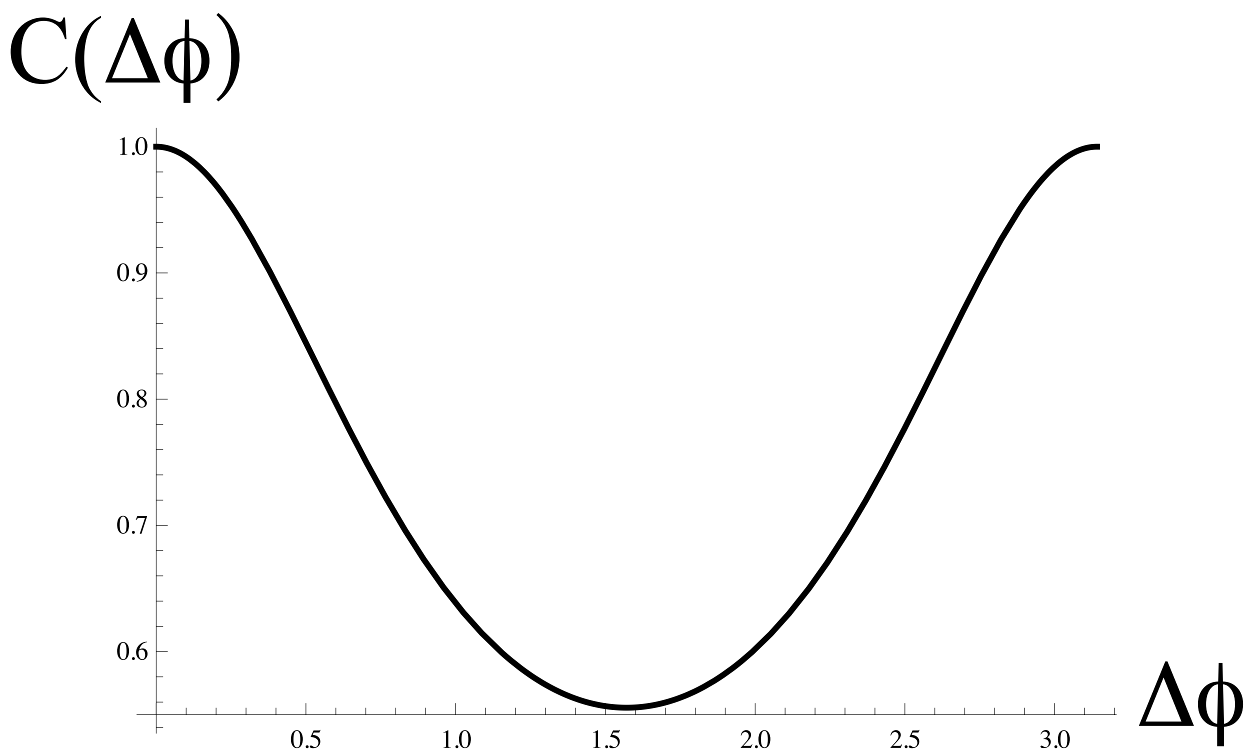
$\mathbf{k}_2 \rightarrow -\mathbf{k}_2$ as $Tr\left[U_{\mathbf{x}}\,U_{\mathbf{y}}^\dagger\right] = Tr\left[U_{\mathbf{y}}\,U_{\mathbf{x}}^\dagger\right]$

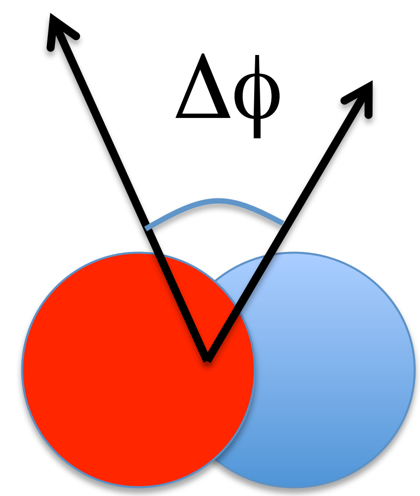
- Hence the correlations generate only even azimuthal harmonics

$\sim \cos 2\,n\,(\phi_1-\phi_2)$ and there should be two ridges!

Correlation function

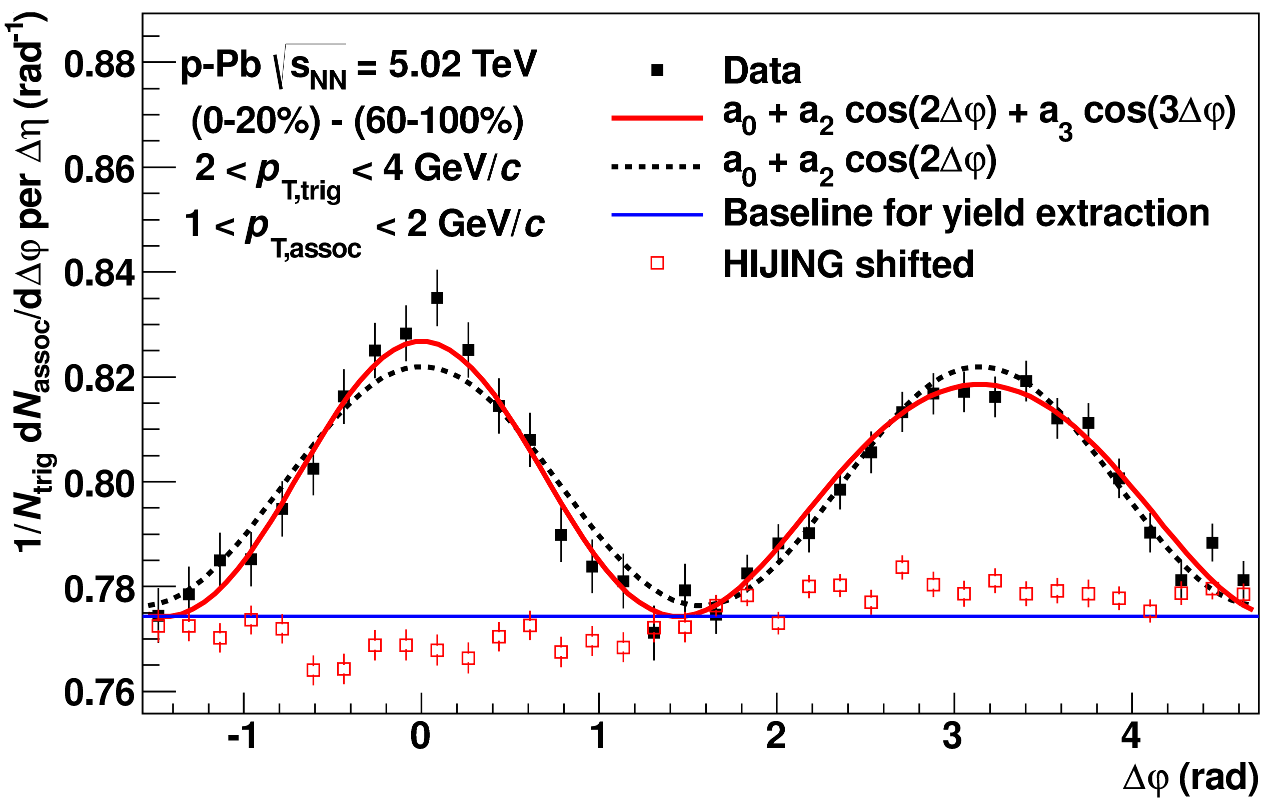
May look like this (a toy model; two particles far separated in rapidity, jets subtracted, pA and AA):





Dumitru, Gelis, McLerran, Venugopalan ’08; Kovner, Lublinsky ‘10;
Yu.K., D. Wertepny ‘12; Lappi, Srednyak, and Venugopalan ‘09

LHC p+Pb data from ALICE



- These are high-multiplicity collisions: it is possible that quark-gluon plasma is created in those, with the hydrodynamics contributing to these correlations.

- Saturation approach in this framework is lacking the odd harmonics, like cos (3 $\Delta\phi$) , etc. May they be generated at subleading orders?

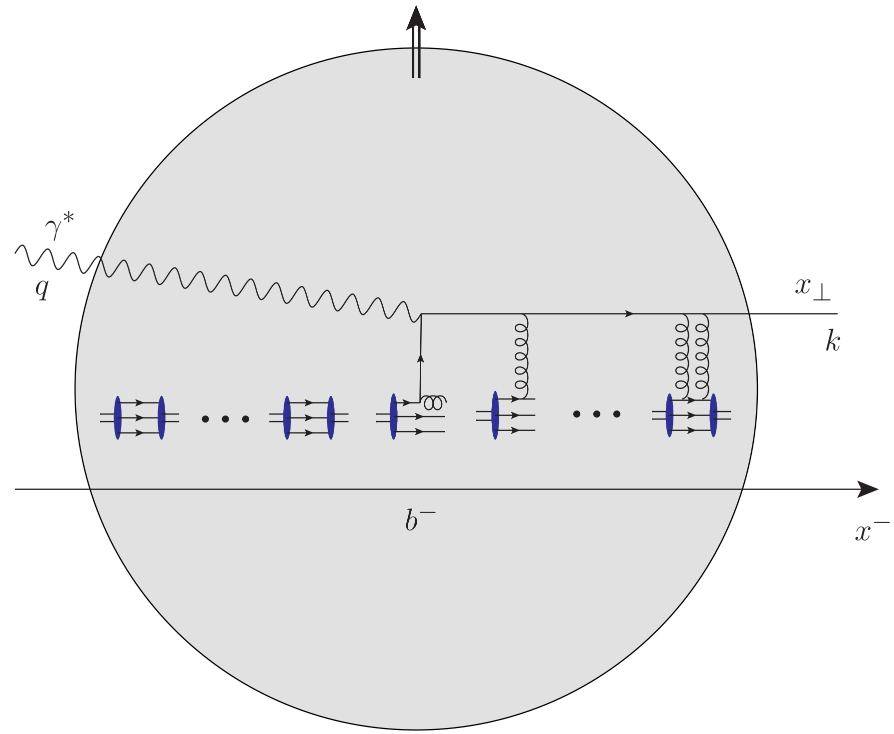
Connections to TMDs and Spin

Quark Production

- Start with inclusive classical quark production cross section in SIDIS.

- The kinematics is standard:

$$s \sim Q^2 \gg \perp^2$$



- The result is

Wigner distribution

$$\frac{d\sigma^{\gamma^*+A\to q+X}}{d^2k\,dy} = A\int \frac{dp^+\,d^2p\,db^-}{2(2\pi)^3}\int d^2x\,d^2y\;W\left(p, b^-, \frac{\underline{x}+\underline{y}}{2}\right)$$
$$\times \int \frac{d^2k'}{(2\pi)^2}\, e^{-i\,(\underline{k}-\underline{k}')\cdot(\underline{x}-\underline{y})}\,\frac{d\hat{\sigma}^{\gamma^*+N\to q+X}}{d^2k'\,dy}(p,q)\, D_{\underline{x}\,\underline{y}}[+\infty, b^-]$$

LO cross sect

Wilson lines

Wilson lines

- Here

$$D_{\underline{x}\,\underline{y}}[+\infty, b^-] = \left\langle \frac{1}{N_c} \, \mathrm{Tr} \left[V_{\underline{x}}[+\infty, b^-] \, V_{\underline{y}}^\dagger[+\infty, b^-] \right] \right\rangle$$

is the quark dipole scattering S-matrix with

$$V_{\underline{x}}[b^- \, , \, a^-] \equiv \mathcal{P} \, \exp \left[\frac{ig}{2} \int\limits_{a^-}^{b^-} dx^- A^+(x^+ = 0, x^-, \underline{x}) \right]$$

denoting Wilson lines.

- In the quasi-classical approximation D_{xy} is real!

Compare with TMD definitions:

- TMDs are defined through the correlator

$$\Phi_{ij}(x, \underline{k}; P, S) \equiv \int \frac{dx^- \, d^2 x_\perp}{2(2\,\pi)^3} \, e^{i\left(\frac{1}{2}\,x\,P^+\,x^- - \underline{x}\cdot\underline{k}\right)} \, \langle P, S | \bar{\psi}_j(0)\, \mathcal{U}\, \psi_i(x^+ = 0, x^-, \underline{x}) | P, S \rangle$$

with the gauge links

$$\mathcal{U}^{SIDIS} = V_{\underline{0}}^\dagger[+\infty\,,\,0]\, V_{\underline{x}}[+\infty\,,\,x^-]$$
$$\mathcal{U}^{DY} = V_{\underline{0}}[0\,,\,-\infty]\, V_{\underline{x}}^\dagger[x^-\,,\,-\infty]$$

- The correlator is decomposed in terms of quark TMDs as

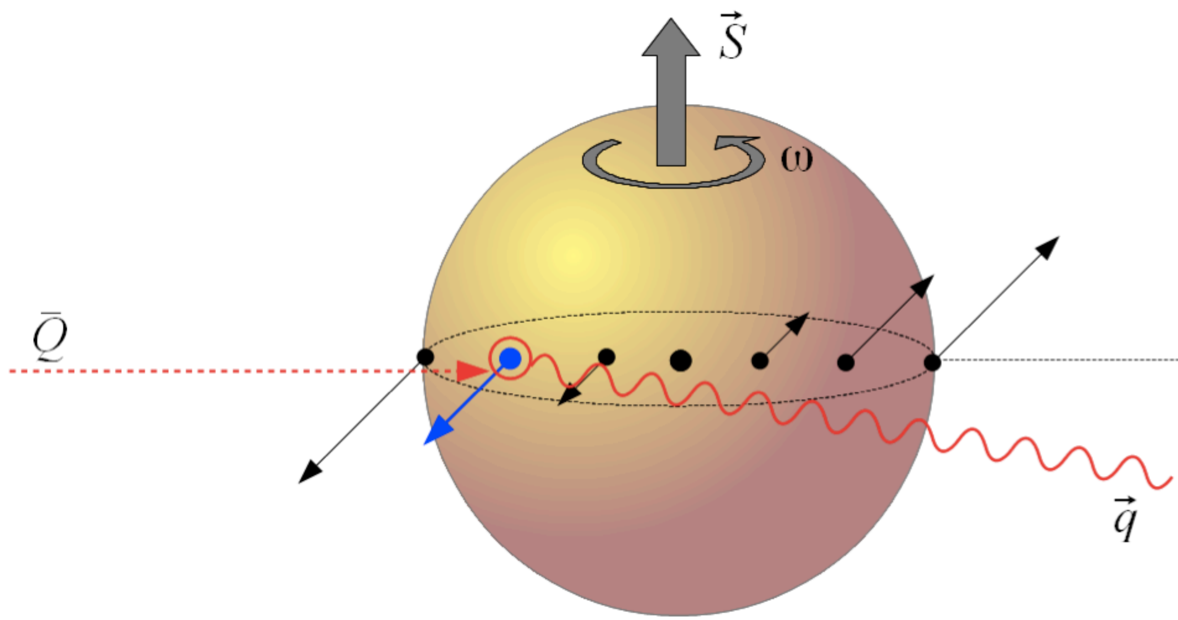
$$\Phi_{ij}(x, \underline{k}; P, S) = \frac{M}{2\,P^+} \left[f_1(x, k_T)\, \frac{P\cdot\gamma}{M} + \frac{1}{M^2}\, f_{1\,T}^\perp(x, k_T)\, \epsilon_{\mu\nu\rho\sigma}\, \gamma^\mu\, P^\nu\, k_\perp^\rho\, S_\perp^\sigma - \frac{1}{M}\, g_{1s}(x, \underline{k})\, P\cdot\gamma\,\gamma^5 \right.$$
$$\left. - \frac{1}{M}\, h_{1T}(x, k_T)\, i\,\sigma_{\mu\nu}\,\gamma^5\, S_\perp^\mu\, P^\nu - \frac{1}{M^2}\, h_{1s}^\perp(x, \underline{k})\, i\,\sigma_{\mu\nu}\,\gamma^5\, k_\perp^\mu\, P^\nu + h_1^\perp(x, k_T)\, \sigma_{\mu\nu}\, \frac{k_\perp^\mu\, P^\nu}{M^2} \right]_{ij}$$

TMDs and Small-x Physics

- SIDIS TMDs and quark production are related in a simple way
- Can use our small-x expertise to calculate the Wilson line correlator, either in the classical approximation or including evolution
- Partly done for Sivers function, resulting in the following insight (see the talk by M. Sievert on Thursday)

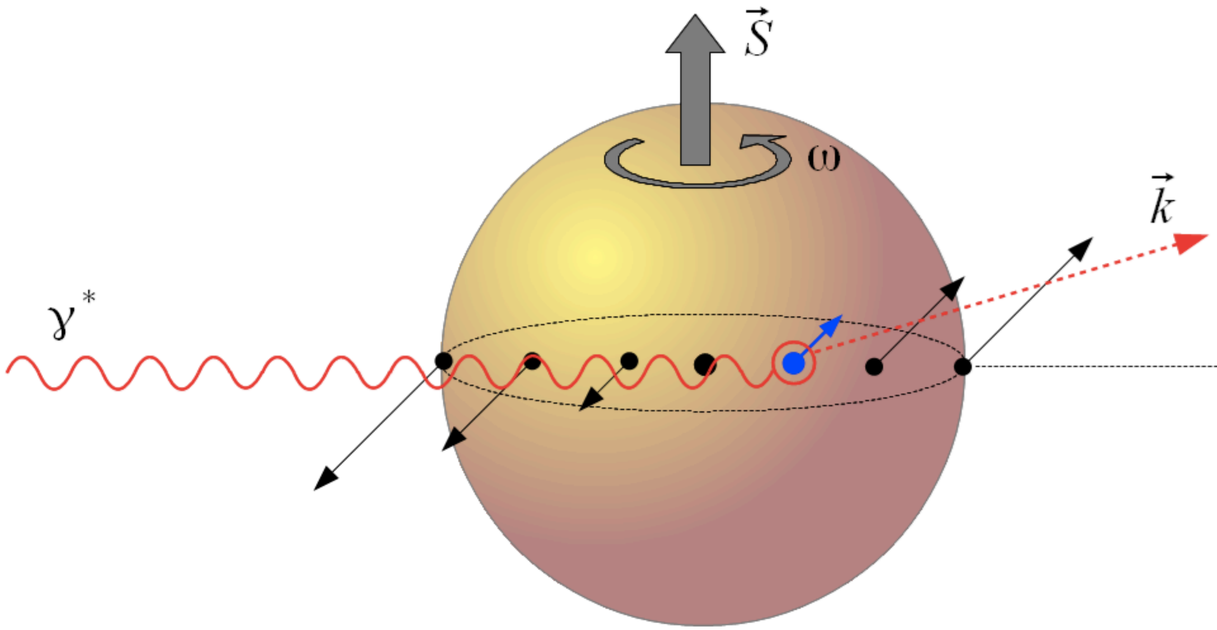
Classical picture of STSA in Drell-Yan

- Think of a transversely polarized proton as a rotating disk with the axis perpendicular to the collision axis
- The proton is not transparent: it has some amount of screening/ shadowing (e.g. gray disk, black disk, etc.)
- Incoming anti-quark (in DY) is more likely to interact near the “front” of the proton: hence, due to the rotation, the outgoing virtual photon is more likely to be produced **left-of-beam**, thus generating STSA.



Classical picture of STSA in SIDIS

- Ditto for SIDIS: except now the incoming virtual photon is more likely to interact near the “back” of the proton, in order for the produced quark to be able to escape out of the proton remnants.
- Owing to the same rotation, the outgoing quark is more likely to be produced **right-of-beam**, thus generating STSA in SIDIS with the **opposite sign** compared to STSA in DY!
- See talk by M. Sievert on Thursday!



Conclusions

- I have way too many slides
- All small-x evolution equations are known up to NLO, with rc corrections to LO kernel known as well.
- We know how to systematically construct the solution of the BFKL evolution at any order in the coupling.
- Long-range rapidity correlations received a lot of attention recently, due to their observation in pp, pA and AA.
- Connections between small-x and TMD physics are being explored recently, with the hope of achieving mutually beneficial results.

Backup Slides

The main principle

- Saturation physics is based on the existence of a large internal momentum scale Q_S which grows with both energy s and nuclear atomic number A

$$\boxed{Q_S^2 \sim A^{1/3}\ s^{\lambda}}$$

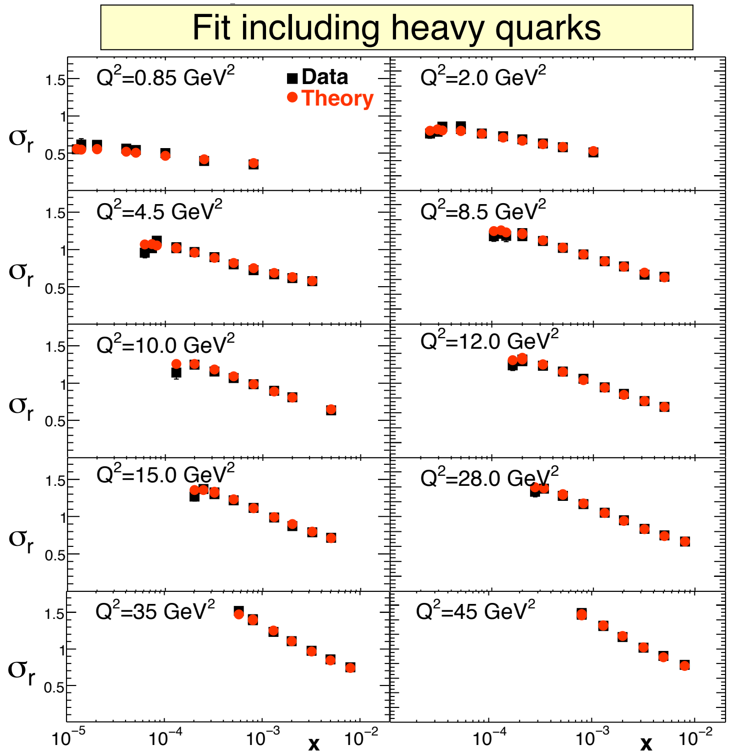
such that

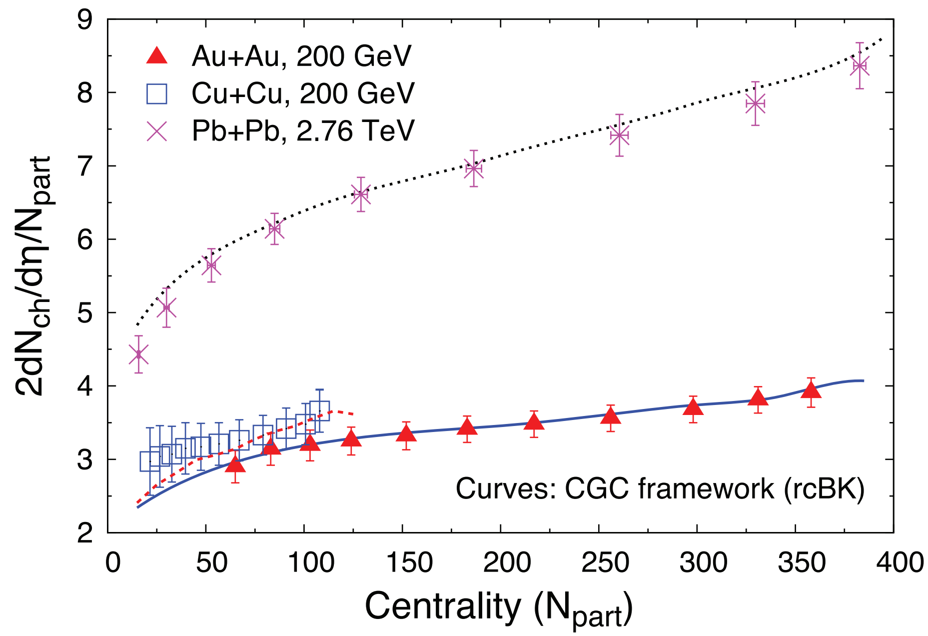
$$\boxed{\alpha_S = \alpha_S\left(Q_S\right) << 1}$$

- and we can calculate total cross sections, particle spectra and multiplicities, etc, from <u>first principles</u>.

More running-coupling phenomenology

- rcBK has been very successful in describing the DIS HERA data (Albacete et al, 2011) and heavy ion collisions (Albacete and Dumitru, ’10):





LHC line is a prediction!

- Seems like to do serious phenomenology one needs running coupling corrections for diffractive evolution.

Solutions of Evolution Equations

- DGLAP equation
  - derived: 1972 (QED), 1977 (QCD)
  - solved: 1972, 1974
- LO BFKL equation
  - derived: 1977, 1978
  - solved: 1978
- NLO BFKL equation
  - derived: 1998 (Fadin&Lipatov, Camici&Ciafaloni)
  - solved: 2013 (see talk by G. Chirilli)