Systematic Error Analysis of the Mo/Tsai Inclusive Radiative Corrections Scheme

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TALK OUTLINE

• Overview of the Small Angle GDH experiment
• Overview of the Mo and Tsai radiative corrections scheme
  • Elastic tail subtraction systematic error
  • Inelastic radiative corrections systematic error
• Results for the Small Angle GDH experiment nitrogen data set
SMALL ANGLE GDH

- Jefferson Lab Hall A experiment
  - Polarized He3 gas target
  - Nitrogen data set as well
  - Inclusive measurement
- Low $Q^2$ data (0.02 to 0.35 GeV$^2$)
- Systematic error analysis focused on low $Q^2$ nitrogen data set
- Data set split among 6°/9° scattering and also partial/full $W$ coverage

$$Q^2 = -q^2 = 4E_s E_p \sin^2 \frac{\theta}{2}$$
$$W^2 = M_p^2 + 2M_p \nu - Q^2$$
$$\nu = E_s - E_p$$

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‘Classic’ radiative corrections scheme
• L.W. Mo and Y.S. Tsai, Rev. Mod. Phys 41, 205 (1969)
• Y.S. Tsai, SLAC-PUB-848 (1971)

• Target structure accounted via structure functions
  • Scheme is model independent
• Mathematic formulation is non-covariant
• Diff between hard/soft photons done with $\Delta$ parameter
  • Results should be independent of parameter
• Systematic error analysis applies to set of FORTRAN codes: ROSETAIL and RADCOR
  • Your mileage may vary
Elastic tail is accounting of all possible ways electron can lose energy and then scatter elastically into the detectors. Calculation includes:

- Elastic Form Factors
  - FF’s are calculated in first Born approximation (only single photon exchange)
  - Correction factor applied to take into account higher order virtual photon diagrams
  - Ignore anything happening at the hadron vertex
- Bremmstrahlung (emission of real photons)
  - Internal bremm. – occurs within the Coulombic field of the target nucleus
  - External bremm. – occurs within the Coulombic field of anything but the target
    - Technically not bremm. but external tail also includes collisional/ionizational energy loss (colliding with atomic electrons)
- Multiple photon corrections
  - Mo/Tsai assume single bremm. photon is emitted but in reality an infinite number of photons share the energy of that one photon
  - Mo/Tsai apply correction to both internal/external corrections to account for this
- Additional data found to check the fit and fill in low $Q^2$ portion

Systematic uncertainty to first Born approximation assumption?
- Should be small because nitrogen is a light nucleus and saGDH is at low $Q^2$
- Source papers:
- They estimate the contribution as:
  \[
  \delta = \pi \alpha Z \sin \frac{\theta}{2} \left/ \left( 1 + \sin \frac{\theta}{2} \right) \right.
  \]

Results in systematic error for $Z = 7$ and $\theta = 6.0$ of $\delta \sim 0.8\%$
Mo/Tsai give their virtual photon corrections as sum of (call it $\tilde{F}$):

- **Vacuum:**
  \[ \delta_{\text{vac}} = \frac{2\alpha}{\pi} \left( -\frac{5}{9} + \frac{1}{3} \ln\left(\frac{Q^2}{m_e^2}\right) \right) \]

- **Vertex (non-infrared divergent):**
  \[ \delta_{\text{vertex}} = \frac{2\alpha}{\pi} \left( -1 + \frac{3}{4} \ln\left(\frac{Q^2}{m_e^2}\right) \right) \]

- **Schwinger (soft-photon/non-infrared divergent):**
  \[ \delta_{S} = \frac{\alpha}{\pi} \left( \frac{\pi^2}{6} - \Phi(\cos^2\theta) \right) \]

- There is also a normalization term for the external bremsstrahlung:
  - Given as \[ 1/\Gamma\left(1 + bt\right) \] where b=4/3 and t is the sum of the radiation lengths
  - Obviously could have vacuum loops according to muon and tau leptons
  - Could also have quark loops in the vacuum diagram
  - And also hadron vertex photon emission!
AN UPDATED ‘FBAR’

A full formula (not in limit $Q^2 >> m_e^2$) for vacuum contribution is given in a variety of references:


\[
\delta_{\text{vac},\mu,\tau}^e = \frac{2\alpha}{\pi} \left[ \frac{-5}{9} + \frac{4m_i^2}{3Q^2} + \frac{1}{3} \sqrt{1 + \frac{4m_i^2}{Q^2}} \left( 1 - \frac{2m_i^2}{Q^2} \right) \ln \left( \frac{\sqrt{1 + 4m_i^2/Q^2} + 1}{\sqrt{1 + 4m_i^2/Q^2}^2 + 1} \right) \right]
\]

• Ignoring quark loops because they’re sensitive to the quark mass and all parameterizations I found of this effect are at $Q^2 > 1 \text{ GeV}^2$
• Ignoring $\gamma – Z$ interference terms as well
• Comparison between the M/T and updated vacuum diagram ‘FBar’ $(1 + \delta)$:
  • MT: $Q^2 = 0.05 \text{ GeV}^2$ -> ‘FBar’ = 1.0577 & $Q^2 = 0.02 \text{ GeV}^2$ -> ‘FBar’ = 1.053
  • Updated: $Q^2 = 0.05 \text{ GeV}^2$ -> ‘FBar’ = 1.0625 & $Q^2 = 0.02 \text{ GeV}^2$ -> ‘FBar’ = 1.056

Systematic for ‘Fbar’ term is $\delta \approx 0.4\%$. 

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MT neglect any kind of target radiation for the internal elastic tail

- Sources for including this effect:

\[
R_t = 1 + \frac{\alpha}{\pi b t_r} \left[ \left( \frac{1 + 2\tau}{2\tau \sqrt{1 + 1/\tau}} \right) \ln(1 + 2\tau + 2\tau \sqrt{1 + 1/\tau}) - 1 + 2\ln(\eta_s) \right] 
\]

\[
b t_r = \frac{2\alpha}{\pi} (\ln(Q^2/m_e^2) - 1)
\]

\[
\tau = \frac{Q^2}{4M_T^2}
\]

\[
\eta_s = 1 + 2 \frac{E}{M_T} \sin^2 \frac{\theta}{2}
\]

- At \(\nu = 1200\) MeV (\(E = 2135\) Mev/\(\theta = 6°\)):
  - \(R_t = 1.00019\) so the systematic is negligible for nitrogen scattering at saGDH kinematics
Multiple photon correction applied to both internal and external tail

- Correction given by MT and Stein in the following papers
  - Y.S. Tsai, “Radiative Corrections to Electron Scattering.” SLAC-PUB-848 1971
- MP is sizable correction (at saGDH kinematics) to tail ranging in value from ~0.60 to ~0.90 as you go from low-ν to high-ν
  - Especially important where tail is large!
- Quoting Tsai: “There is some uncertainty in the validity of the [MP] factor. We know that this factor is correct when \( \omega_s/E_s \) and \( \omega_p/(E_p+w_p) \) are small.”

\[
\delta_{MP} = \left( \frac{\omega_s}{E_s} \right)^{(b(t_a+t_b)+bt_r/2)} \left( \frac{\omega_p}{E_p+w_p} \right)^{(b(t_a+t_b)+bt_r/2)}
\]

\[
\omega_s = E_s - \frac{E_p}{1 - 2E_p/M_T \sin^2 \frac{\theta}{2}}
\]

\[
\omega_p = \frac{E_s}{1 - 2E_s/M_T \sin^2 \frac{\theta}{2}} - E_p
\]

\[
bt_r = 2\alpha/\pi [\ln(Q^2/m^2) - 1]
\]

Energy of incoming (outgoing) photon
Guthrie Miller’s thesis offers an alternative MP correction term

- Assuming only external effects Miller gives exact external bremm. tail as (this includes multiple photon processes)

\[
\frac{d\sigma}{d\Omega dE'} = \int_{E-\omega}^{E} \pi(E, E_1, t_b) \frac{d\sigma}{d\Omega}(E_1, \theta) \pi(E_1', E', t_a) dE_1
\]

- New MP factor:

\[
\left( \frac{k}{\sqrt{EE'}} \right)^{bt_a+bt_b} \frac{1}{\Gamma(1 + bt_b + bt_a)} 
\times [t_b w(E, E - \omega) \eta^2 \frac{d\sigma}{d\Omega}(E - \omega, \theta) + t_a w(E' + \omega, E') \frac{d\sigma}{d\Omega}(E, \theta)]
\]

- \( k \) is soft-photon limiting energy
- Can compare exact equation to approximate equation to determine \( k/\text{MP} \) contribution
MULTIPLE SOFT-PHOTON CORRECTION COMPARISON

saGDH kinematics:

- Big difference at large $\nu$ for nitrogen
- Agreement at lower $\nu$ (all photons are soft)
  - Tsai (SLAC PUB) stated that MT factor is correct here so this makes sense
- How does it affect the tail subtraction?

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Tsai MP Correction

• Assume MP correction also holds/applies to internal elastic tail
• Same effect anywhere the tail is large
• Use Miller factor for final analysis

Miller MP Correction

SUMMARY OF ELASTIC TAIL RESULTS

Total Elastic Tail Systematic Error for saGDH:

• **2%** for loop diagrams/first Born approximation/internal tail integration/soft-photons/energy-peaking approximation
  - Estimate soft-photon systematic error by looking at factor’s sensitivity to form factor fit and value of equivalent radiator for internal contribution
  - Energy-peaking approximation systematic is estimated in Tsai’s SLAC PUB
• **4-15%** for form factor error (comparing world data to fit)
  - Break it up into internal and external contributions
    - External limited to world data within $Q^2$ range of data
    - Internal $1/Q^4$ weighted over all of world data
• **1% and less** for choice of external straggling function
  - Compare MT and Miller external straggling functions
  - Apply bin-by-bin
• Elastic tails calculated using monte-carlo
  - Takes into account acceptance/extended target effects
Some errors carry over from the elastic tail so don’t redo them:

- ‘Fbar’ (higher order virtual photon corrections)
- Energy-peaking approximation used in evaluating external corrections
- Born approximation

New potential sources of error

- Angle-peaking approximation in internal bremsstrahlung
- Soft photon correction factor
- Interpolation/extrapolation error on unfolding procedure
  - Including using a model as source for lowest energy setting to RADCOR input

Inelastic continuum can be regarded as a summation of many discrete levels…
**INELASTIC INTERNAL BREMSSTRAHLUNG**

**Full expression of the inelastic internal bremsstrahlung**

- Inelastic isn’t limited to $W = M_T$ so the inelastic internal contribution is the elastic internal tail integrated over all possible $M_f$:

$$
\frac{d\sigma_r}{d\Omega dE}(\omega > \Delta) = \frac{\alpha^3}{2\pi} \frac{E_p}{E_s M_T} \int_{-1}^{1} d(cos\theta_k) \int_{\Delta}^{\omega_{max}(cos\theta_k)} \frac{\omega d\omega}{q^4} B_{\mu\nu} T^{\mu\nu}
$$

- **Soft photons**

$$
\frac{d\sigma_r}{d\Omega dE}(E_s, E_p) = e^{-\delta_r(\Delta)} (q^2) \frac{d\sigma}{d\Omega dE} + \frac{d\sigma_r}{d\Omega dE}(\omega > \Delta)
$$

- $\Delta$ parameter avoids a divergence at $\omega = 0$

  - Slightly tweaked version of (B.6) of MT, but it keeps choice of fbar consistent across calculations (See B. Badalek et al. arxiv:hep-ph/940328v1 (1994).)

- **Structure functions evaluated at most probable energy loss kinematics**

  - Takes into account photons radiating away energy
The equivalent correction in the angle-peaking approximation is

- Dropping the soft-photon terms from the integrals. SLAC-PUB-848 (1971) (C.23)

\[
\frac{d\sigma_r}{d\Omega dE}(E_s, E_p) = \tilde{F}(q^2) \left[ \left( \frac{R\Delta}{E_s} \right)^{bt_r} \left( \frac{\Delta}{E_p} \right)^{bt_r} \frac{d\sigma}{d\Omega dE}(E_s, E_p) \right.
\]

\[
+ \int_{E_p + \Delta}^{E_{p\max}} dE_p' \frac{d\sigma}{d\Omega dE}(E_s, E_p') \frac{bt_r}{2(E_p' - E_p)} \phi \left( \frac{E_p' - E_p}{E_p'} \right)
\]

Only internal radiation

\[
+ \int_{E_s - R\Delta}^{E_s - \Delta} dE_s' \frac{d\sigma}{d\Omega dE}(E_s', E_p) \frac{bt_r}{2(E_s' - E_s)} \phi \left( \frac{E_s' - E_s}{E_s} \right)
\]

\]

Angle peaking approximation is used because:

- Significantly faster ( <1 min to run compared to multiple hours for full integral)
- Majority of photon’s are emitted in same direction as incoming/outgoing electrons
Use P.E. Bosted and V. Mamyan model for comparison

• Full integral computation time ~one week on my desktop
  • Python calculation
  • Proton calculation quicker

• Systematic is contribution to total radiated cross section
  • Comparison between:
    • (Low e + Ext + Int + Coll)
    • (Low e + Ext + Full Int + Coll)

• Assume difference is a function of $E_p$ and fit
INELASTIC MULTIPLE SOFT PHOTONS

Compare soft-photon term from Mo/Tsai to Guthrie Miller

- Separate but similar soft-photon terms for each integral
- Difference in soft-photon terms is smaller for inelastic
  - Sufficient to use an approx value for Miller $k$
- Systematic error is variance between MT term and Miller terms for a range of $k$'s

MT

$$dE_s: \left(\frac{E_s - E'_s}{E_p R}\right)^{b(t_a + t_r)} \left(\frac{E_s - E'_s}{E_s}\right)^{b(t_b + t_r)}$$

$$dE_p: \left(\frac{E'_p - E_p}{E'_p}\right)^{b(t_a + t_r)} \left(\frac{R(E'_p - E_p)}{E_s}\right)^{b(t_b + t_r)}$$

Use P.E. Bosted and V. Mamyan model for comparison
UNFOLDING SYSTEMATIC ERROR

- Vary lowest energy model input
- Values for scale guided by comparison of radiated Bosted model vs. data

n2: Bosted 2135 MeV/ 6 degrees

Interpolated/smoothed data

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Summary of Inelastic Results

Total Inelastic Systematic Error for saGDH:

- 1.5% for loop diagrams/first Born approximation/energy-peaking approximation
  - Energy-peaking approximation systematic is estimated in Tsai’s SLAC PUB
- <1 – 3% differences in soft-photon terms
- <1 – 4% for angle peaking approximation
- <1 – 2% error in the unfolding procedure (driven by extrapolation)
- <1 – 8% error from the use of an input model for the lowest extrapolation energy
- Note: variance on unfolding procedure with different scattering angles within the angular acceptance is negligible
CONCLUSION

• Biggest potential systematic error in the Mo and Tsai inclusive radiation scheme comes from handling of soft-photon corrections
  • Error coming from form factor parameterization isn’t a limitation of MT
• With requisite hardware possible to replace all peaking approximations with full integrations
  • Did not consider removing energy-peaking approximation in this analysis
• Full results written up and can be found in my tech-note at

THANK YOU
For the elastic tail, the full expression for the internal bremsstrahlung is

\[
\sigma_{\text{exact}} = \left( \frac{d^2\sigma}{d\Omega dE_p} \right)_{\text{ex}} = \frac{\alpha^3}{2\pi} \left( \frac{E_p}{E_s} \right) \int_{-1}^{1} \frac{2M_T\omega d(\cos\theta_k)}{q^4(u_0 - |\vec{u}|\cos\theta_k)} \\
\times \left( \tilde{W}_2(q^2) \left\{ \frac{-am^2}{x^3} \left[ 2E_s(E_p + \omega) + \frac{q^2}{2} \right] - \frac{a'm^2}{y^3} \left[ 2E_p(E_s + \omega) + \frac{q^2}{2} \right] \right\} \\
- 2 + 2\nu(x^{-1} - y^{-1})\{m^2(s \cdot p - \omega^2) + (s \cdot p)[2E_sE_p - (s \cdot p) + \omega(E_s - E_p)]\} \\
+ x^{-1} \left[ 2(E_sE_p + E_s\omega + E_p^2) + \frac{q^2}{2} - (s \cdot p) - m^2 \right] \right) \right) \\
- y^{-1} \left[ 2(E_pE_s + E_p\omega + E_s^2) + \frac{q^2}{2} - (s \cdot p) - m^2 \right) \right) \right) \\
+ \tilde{W}_1(q^2) \left[ \left( \frac{a}{x^3} + \frac{a'}{y^3} \right) m^2(2m^2 + q^2) + 4 + 4\nu(x^{-1} - y^{-1})(s \cdot p)(s \cdot p - 2m^2) \right. \\
+ (x^{-1} - y^{-1})(2s \cdot p + 2m^2 - q^2) \right) \right),
\]
MT give an exact form of the internal bremm. in eq. B5

- Exact means 1\textsuperscript{st} Born approximation and no target radiation
- The equation is an integral with potential for a divide by zero error. MT say to ignore this small point in numerical integration.
- ROSETAIL has custom integration routine to account for this
- What about using a more modern integration method?
- Potentially big enough deal that Maximon /Williamson wrote a paper on how to avoid divide by 0. (Paper also helped speed up the calculation)
- Comparison between ROSETAIL/python integration of B5 (nb/MeV sr):
  - RT: $v = 10$ MeV $\rightarrow$ XS = 2282 \\
  - PY: $v = 10$ MeV $\rightarrow$ XS = 2290

- Systematic for numerical integration is $\sim$0.4%.
  - Systematic contribution to total tail (internal + external) is $\sim$0.2%
• Use difference method to correct quasi-elastic peak and ratio method for rest of spectrum
• Helps control systematic error
  • Systematic errors are applied to the RC correction factor and then propagated in the standard fashion
• Use the Bosted model method for bin-centering
  • Small correction
• Kept the absolute value of the statistical uncertainty constant
Determine systematic by applying Bosted ratio method to data I could unfold.

Comparison of the two gives the systematic:

- Take weighted average (RC_Bosted/RC_Unfold) for each spectrum:
  - Weight is 1 over the propagated systematic error on the above ratio.
- Then average each spectrum to get systematic.
- Limit comparison to lowest $W$ of spectrum I ultimately want to apply this method to:
  - 4209 / 6 degrees:
    - $W > 1575$ -> Sys = 4.7%
  - 3775 / 9 degrees:
    - $W > 1281$ -> Sys = 6.5%
  - 4404 / 9 degrees:
    - $W > 1641$ -> Sys = 4.5%