

# Lattice Studies of Hadron Structure

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– QCDSF Collaboration –



Special mention:

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# Outline

## Lattice Simulation

Basics

Nucleon

Pion

Conclusions & Outlook

# Lattice Simulation

## Action

$$N_f = 2$$

$$S = S_G + S_F$$

$$S_G = \beta \sum_{x, \mu < \nu} \left( 1 - \frac{1}{3} \text{Re Tr } U_{\mu\nu}(x) \right)$$

$$S_F = \sum_x \left\{ \bar{\psi}(x)\psi(x) - \kappa \bar{\psi}(x)U_\mu^\dagger(x - \hat{\mu})[1 + \gamma_\mu]\psi(x - \hat{\mu}) \right. \\ \left. - \kappa \bar{\psi}(x)U_\mu(x)[1 - \gamma_\mu]\psi(x + \hat{\mu}) - \frac{1}{2}\kappa c_{SW} g \bar{\psi}(x)\sigma_{\mu\nu}F_{\mu\nu}(x)\psi(x) \right\}$$



$$\partial_\mu A_\mu^{\text{imp}} = 2m_q P$$

Clover Fermions

## Advantages

- Local
- Transfer matrix
- $O(a)$  improved
- Flavor symmetry
- Fast to simulate

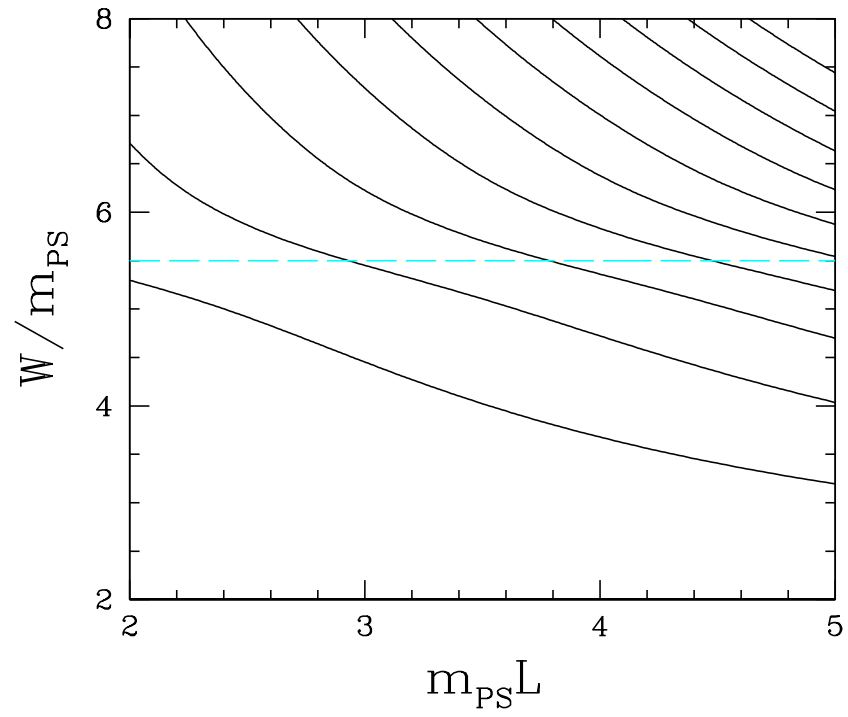
Prerequisite to making contact with  $SU(2)$  ChPT

- Finite size corrections
- Chiral extrapolation
- Determination of low-energy constants

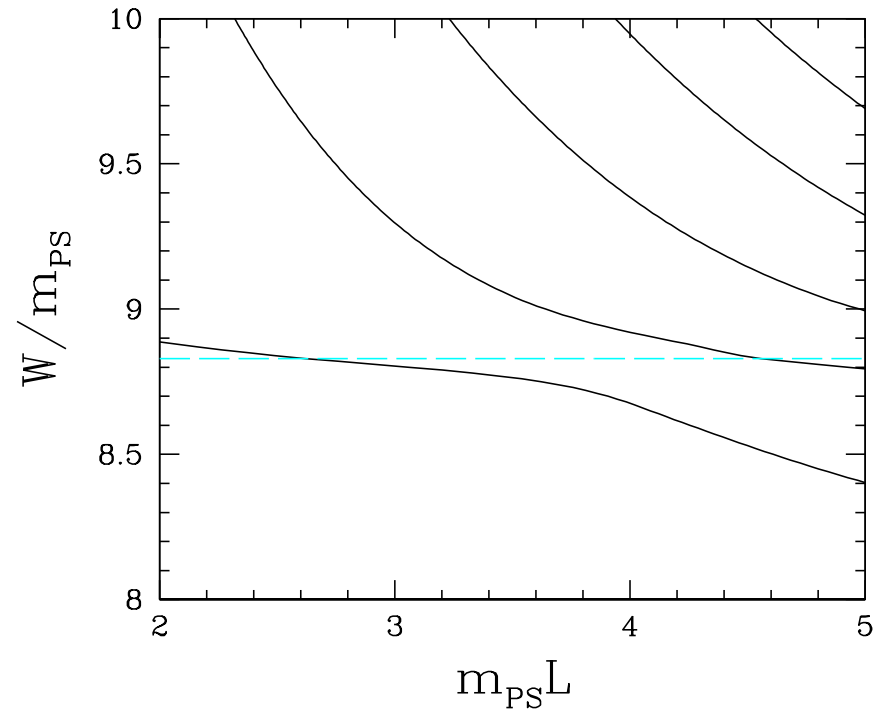
$\beta$	$\kappa_{\text{sea}}$	Volume	$a$ [fm]	$m_{PS}$ [MeV]
5.20	0.13420	$16^3 \times 32$	0.090	1010
5.20	0.13500	$16^3 \times 32$	0.090	830
5.20	0.13550	$16^3 \times 32$	0.090	620
5.25	0.13460	$16^3 \times 32$	0.084	990
5.25	0.13520	$16^3 \times 32$	0.084	830
5.25	0.13575	$24^3 \times 48$	0.084	600
5.25	0.13600	$24^3 \times 48$	0.084	430
5.26	0.13450	$16^3 \times 32$	0.083	1010
5.29	0.13400	$16^3 \times 32$	0.080	1170
5.29	0.13500	$16^3 \times 32$	0.080	930
5.29	0.13550	$24^3 \times 48$ $16^3 \times 32$ $12^3 \times 32$	0.080	810
5.29	0.13590	$24^3 \times 48$ $16^3 \times 32$ $12^3 \times 32$	0.080	590
5.29	0.13605	$24^3 \times 48$	0.080	480
5.29	0.13620	$24^3 \times 48$	0.080	390
5.29	0.13632	$40^3 \times 64$ $32^3 \times 64$ $24^3 \times 48$	0.080	250
5.29	0.13640	$64^3 \times 96$ $40^3 \times 64$	0.080	140
5.40	0.13500	$24^3 \times 48$	0.072	810
5.40	0.13560	$24^3 \times 48$	0.072	770
5.40	0.13610	$24^3 \times 48$	0.072	610
5.40	0.13625	$24^3 \times 48$	0.072	530
5.40	0.13640	$32^3 \times 64$	0.072	420
5.40	0.13660	$32^3 \times 64$	0.072	240

In progress →

$\rho \rightarrow \pi\pi$

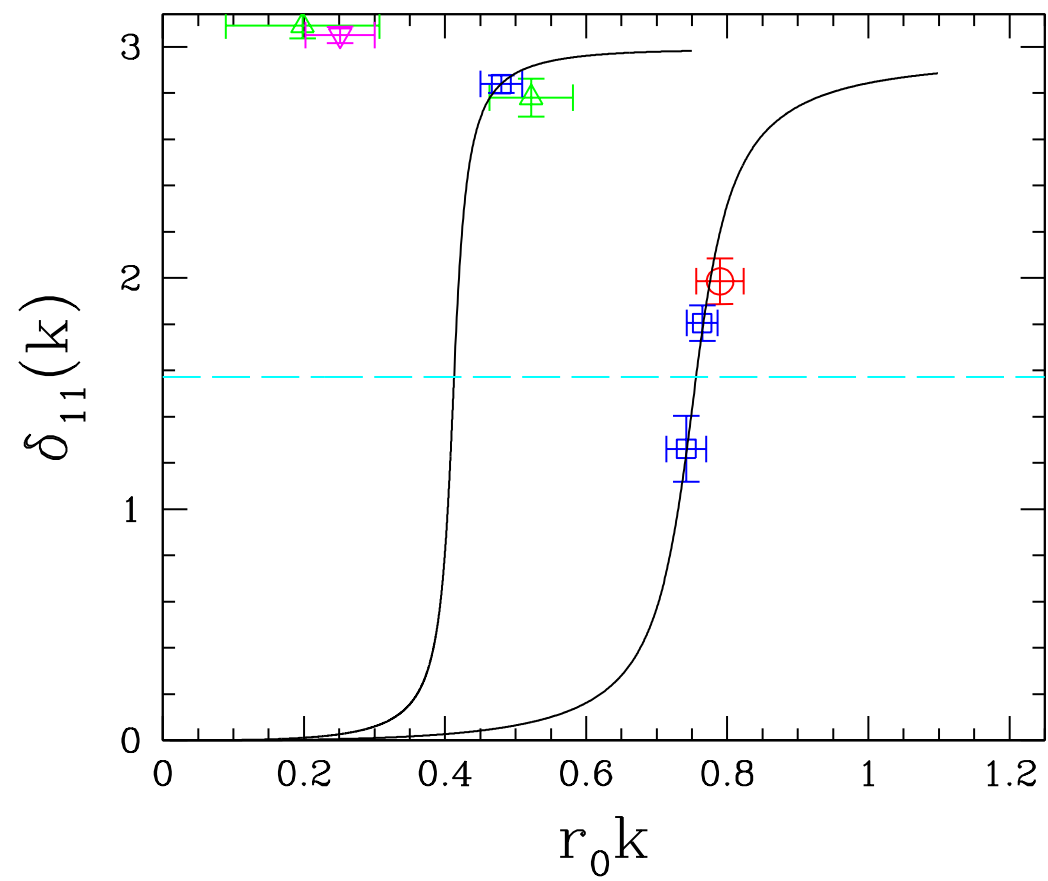


$\Delta \rightarrow N\pi$



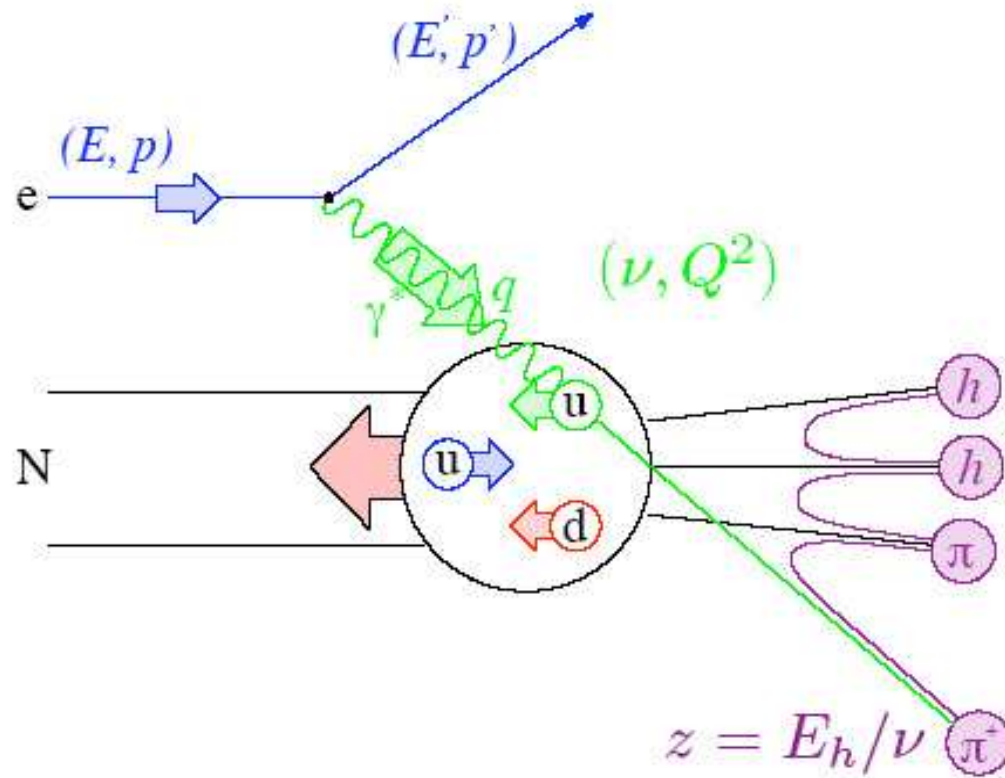


$\rho \rightarrow \pi\pi$

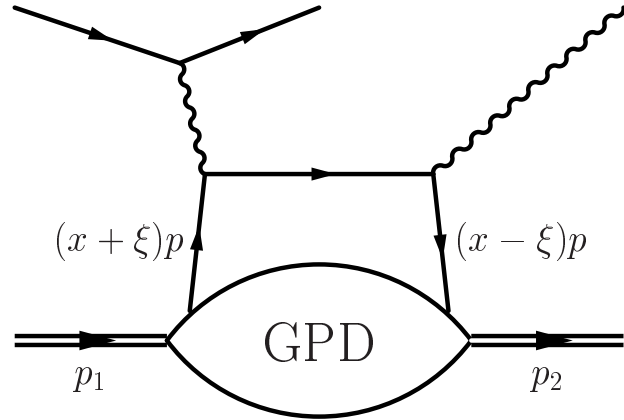


**Basics**

# Inclusive



# OPE



$$p = \frac{1}{2}(p_1 + p_2), \quad \Delta = p_2 - p_1, \quad q = \frac{1}{2}(q_1 + q_2)$$

$\xi = 0$ : Momentum transfer of the struck parton purely transverse, i.e.  $\Delta = \Delta_{\perp}$

$$J(q) J(-q) = \sum_n c_n \times \left\{ \begin{array}{l} \mathcal{O}_{\mu_1 \dots \mu_n}^q = \left(\frac{i}{2}\right)^{n-1} \bar{q} \gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2} \dots \overleftrightarrow{D}_{\mu_n} q \\ \mathcal{O}_{\sigma \mu_1 \dots \mu_n}^{5q} = \left(\frac{i}{2}\right)^n \bar{q} \gamma_{\sigma} \gamma_5 \overleftrightarrow{D}_{\mu_1} \dots \overleftrightarrow{D}_{\mu_n} q \\ \mathcal{O}_{\mu\nu \mu_1 \dots \mu_n}^{Tq} = \left(\frac{i}{2}\right)^n \bar{q} \sigma_{\mu\nu} \gamma_5 \overleftrightarrow{D}_{\mu_1} \dots \overleftrightarrow{D}_{\mu_n} q \end{array} \right.$$

## Nucleon

$$\langle p_1, s | \mathcal{O}_{\{\mu_1 \dots \mu_n\}}^q | p_2, s \rangle = \bar{u}(p_1, s) \left[ A_n^q(\Delta^2) \gamma_{\{\mu_1} \right. \\ \left. + B_n^q(\Delta^2) \frac{i\Delta^\alpha}{2m_N} \sigma_{\alpha\{\mu_1} \right] p_{\mu_2} \dots p_{\mu_n\}} u(p_2, s) + \dots$$

$$\langle p_1, s | \mathcal{O}_{\{\mu\mu_1 \dots \mu_n\}}^{5q} | p_2, s \rangle = \bar{u}(p_1, s) \left[ \tilde{A}_{n+1}^q(\Delta^2) \gamma_{\{\mu} \gamma_5 p_{\mu_1} \dots p_{\mu_n\}} \right] u(p_2, s) + \dots$$

$$\langle p_1, s | \mathcal{O}_{\mu\{\nu\mu_1 \dots \mu_n\}}^{Tq} | p_2, s \rangle = \bar{u}(p_1, s) \left[ A_{n+1}^{Tq}(\Delta^2) \sigma_{\mu\{\nu} \gamma_5 - \tilde{A}_{n+1}^{Tq}(\Delta^2) \left( \frac{\Delta^2}{2m_N^2} \sigma_{\mu\{\nu} - \frac{\Delta_\mu \Delta_\alpha}{2m_N^2} \sigma_{\alpha\{\nu} \right) \gamma_5 \right. \\ \left. + \bar{B}_{n+1}^{Tq}(\Delta^2) \epsilon_{\alpha\beta\mu\{\nu} \frac{\Delta_\alpha \gamma_\beta}{2m_N} \right] p_{\mu_1} \dots p_{\mu_n\}} u(p_2, s) + \dots$$

$$A_n^q(\Delta^2) = \int_0^1 dx x^{n-1} H^q(x, \Delta^2)$$

$$H^q(x, 0) = q(x)$$

$$B_n^q(\Delta^2) = \int_0^1 dx x^{n-1} E^q(x, \Delta^2)$$

$$\tilde{A}_n^q(\Delta^2) = \int_0^1 dx x^{n-1} \tilde{H}^q(x, \Delta^2)$$

$$\tilde{H}^q(x, 0) = \Delta q(x)$$

$$A_n^{Tq}(\Delta^2) = \int_0^1 dx x^{n-1} H^{Tq}(x, \Delta^2)$$

$$H^{Tq}(x, 0) = \delta q(x)$$

↑  
GFFs

↑  
GPDs

$$\boxed{\frac{1}{2}(A_2^q(0) + B_2^q(0)) = J^q}$$

Ji

$$A_1^q(\Delta^2) = F_1^q(\Delta^2)$$

$$B_1^q(\Delta^2) = F_2^q(\Delta^2)$$

$$\tilde{A}_1^q(\Delta^2) = g_A^q(\Delta^2)$$

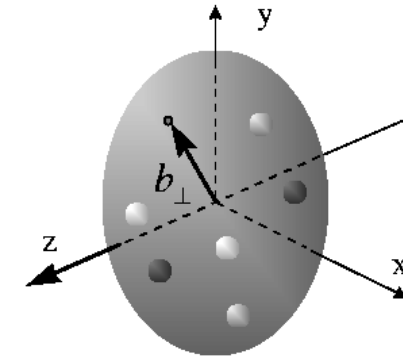
$$A_1^{Tq}(\Delta^2) = g_T^q(\Delta^2)$$

$$\Delta^2 = t = -Q^2$$

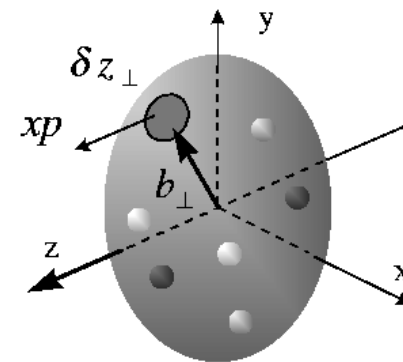
# Impact Parameter Space

Generically

$$A_n^q(\mathbf{b}_\perp^2) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{i \mathbf{b}_\perp \Delta_\perp} A_n^q(\Delta_\perp^2)$$



$$H^q(x, \mathbf{b}_\perp^2) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{i \mathbf{b}_\perp \Delta_\perp} H^q(x, \Delta_\perp^2)$$



Probability interpretation

$$H^q(x, \Delta^2) = \int_x^1 \frac{dy}{y} C\left(\frac{x}{y}, \Delta^2\right) q(y)$$

Similarly for  $\tilde{H}^q$  and  $H^{Tq}$

$$\int_0^1 dx x^n C(x, \Delta^2) = \frac{A_{n+1}(\Delta^2)}{A_{n+1}(0)}$$

*E.g.*  $\frac{1}{(1 - \Delta^2/M_n^2)^p}$



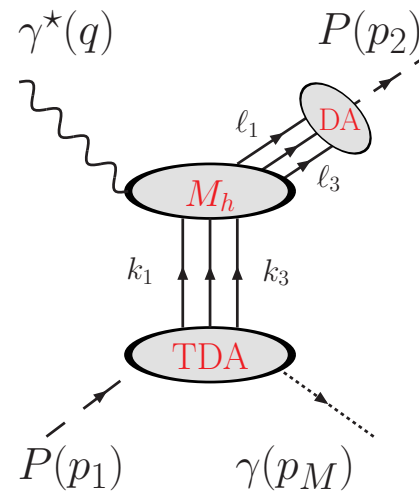
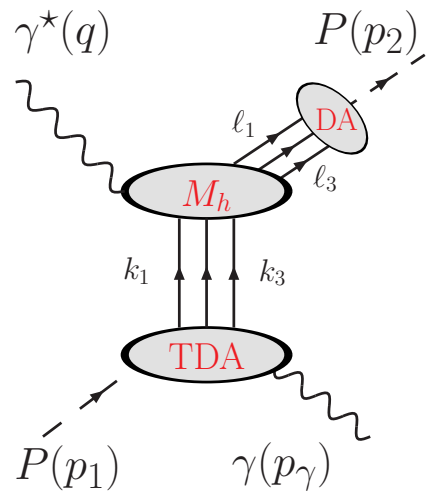
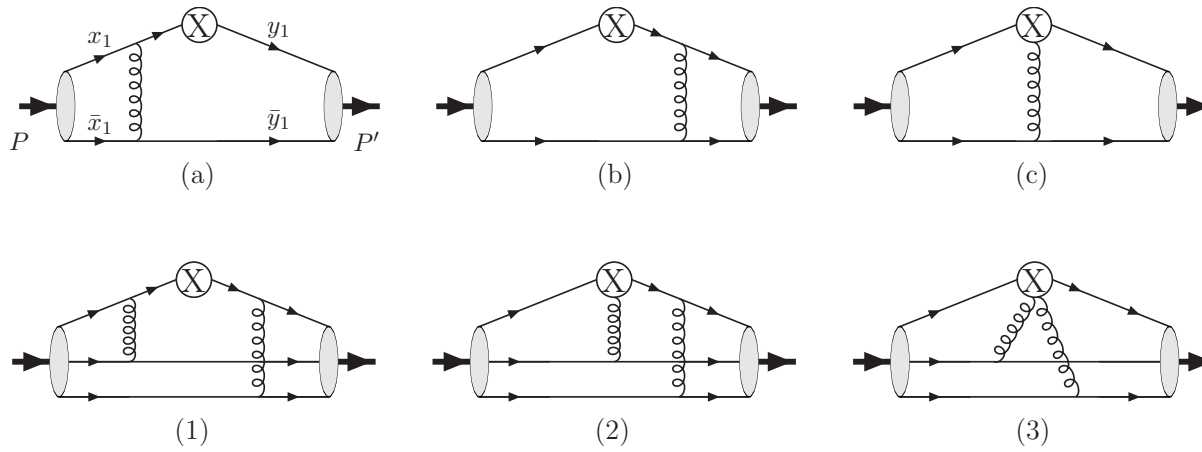
By inverse Mellin transform

$$H^q(x, \mathbf{b}_\perp^2) = \int_x^1 \frac{dy}{y} C\left(\frac{x}{y}, \mathbf{b}_\perp^2\right) q(y)$$

$$C(x, \mathbf{b}_\perp^2) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{i\mathbf{b}_\perp \Delta_\perp} C(x, \Delta_\perp^2)$$



# Exclusive

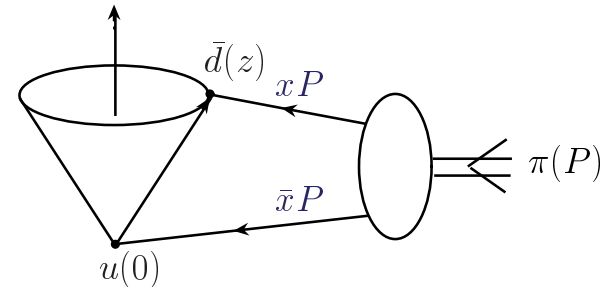


# Distribution Amplitudes Leading Twist

## Pion

$$\begin{aligned} & \langle 0 | \bar{d}(z) \gamma_\mu \gamma_5 U(z, 0) u(0) | \pi(p) \rangle \\ &= i f_\pi p_\mu \int_0^1 dx e^{ix(zp)} \phi_\pi(x, \mu^2) \end{aligned}$$

Projection onto LC coordinates



$$\phi_\pi(x, \mu^2) = \int_{|k_\perp| < \mu} d^2 k_\perp \phi_{BS}(x, k_\perp, \mu^2)$$

## Nucleon

$$\begin{aligned} & \langle 0 | \left( u_a^\uparrow(z_1) C \not{p} u_b^\downarrow(z_2) \right) \not{p} d_c^\uparrow(z_3) U_{aa'}(z_1, z_0) U_{bb'}(z_2, z_0) U_{cc'}(z_3, z_0) \epsilon_{a'b'c'} | p \rangle \\ &= \frac{f_N}{4} \not{p} N^\uparrow \int [dx] e^{i \sum x_i (z_i p)} \phi_N(x_1, x_2, x_3, \mu^2) \end{aligned}$$

Braun et al.

## Generic moment

$$\langle 0 | \mathcal{O}_{\rho \lambda_1 \dots \lambda_l \mu_1 \dots \mu_m \nu_1 \dots \nu_n \alpha}(0) | p \rangle = \phi_N^{lmn}(\mu^2) p_\rho p_{\lambda_1} \dots p_{\lambda_l} p_{\mu_1} \dots p_{\mu_m} p_{\nu_1} \dots p_{\nu_n} N_\alpha^\dagger(p)$$

$$\phi_N^{lmn}(\mu^2) = \int [dx] x_1^l x_2^m x_3^n \phi_N(x_1, x_2, x_3, \mu^2)$$

$$\begin{aligned} \mathcal{O}_{\rho \lambda_1 \dots \lambda_l \mu_1 \dots \mu_m \nu_1 \dots \nu_n \alpha}(0) &= \left( [i^l D_{\lambda_1} \dots D_{\lambda_l} u^\dagger(0)]_a C \gamma_\rho [i^m D_{\mu_1} \dots D_{\mu_m} u^\downarrow(0)]_b \right) \\ &\times [i^n D_{\nu_1} \dots D_{\nu_n} d^\dagger(0)]_{c \alpha} \epsilon_{abc} \end{aligned}$$

## Applications Two Examples

### Form factor

$$G_M(\Delta^2) = \int [dy][dz] \phi_N^*(y_1, y_2, y_3, \mu^2) T_M([y], [z], \Delta^2) \phi_N(z_1, z_2, z_3, \mu^2) + \text{HT}$$

$\Delta^2$  large

↑

Perturbative above scale  $\Delta^2$

### DVCS

$$H^q(x, \Delta^2, \xi) = \int [dy][dz] \phi_N^*(y_1, y_2, y_3, \mu^2) T_H^q([y], [z], x, \Delta^2, \xi) \phi_N(z_1, z_2, z_3, \mu^2) + \text{HT}$$

$\Delta^2$  large

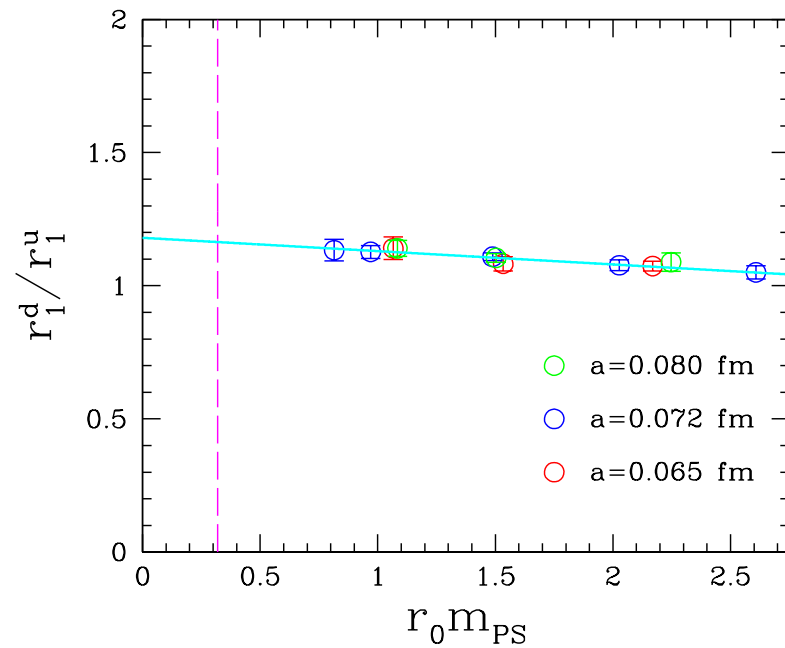
Hoodbhoy, Ji & Yuan

**Nucleon**

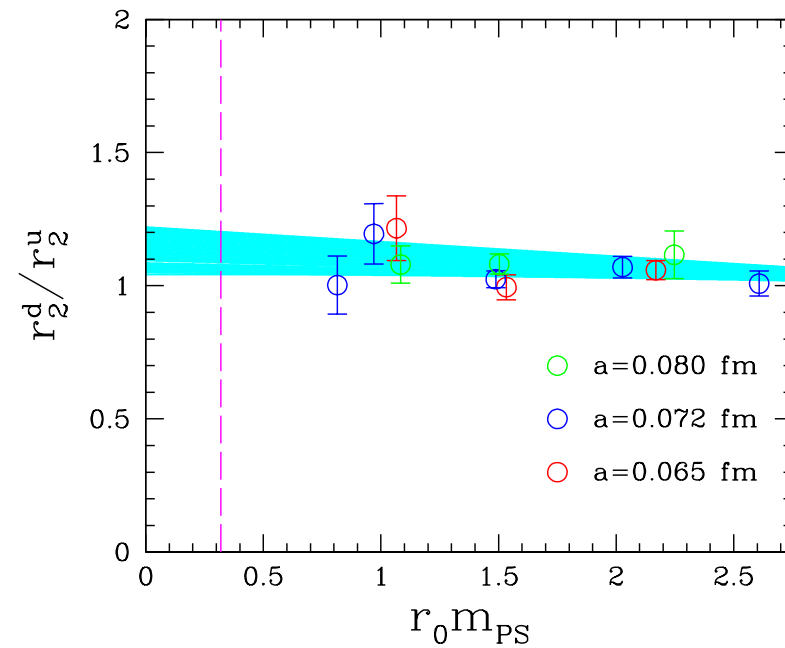
# Form Factors Revisited

$$F_i(Q^2) = F_i(0) \left( 1 - \frac{1}{6} r_i^2 Q^2 + O(Q^4) \right)$$

$$i = 1, 2$$



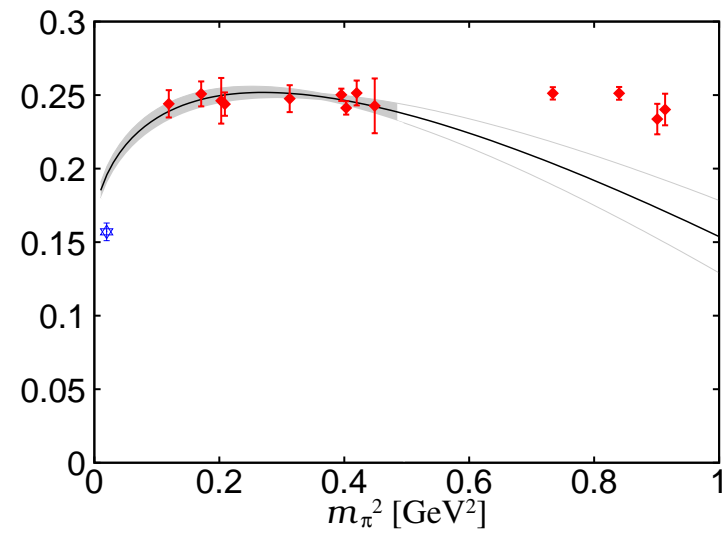
Proton



$$r_{1,2}^d > r_{1,2}^u$$

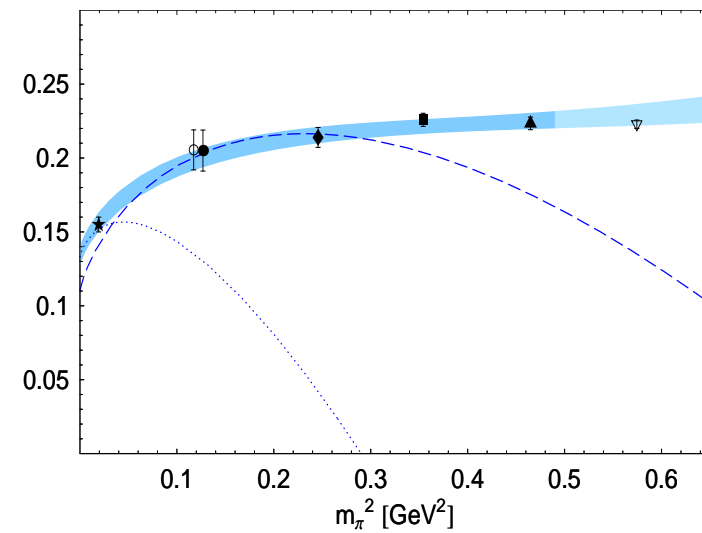
# Moments of Structure Functions

$$\langle x \rangle = \int dx x [u(x, Q^2) - d(x, Q^2)]$$



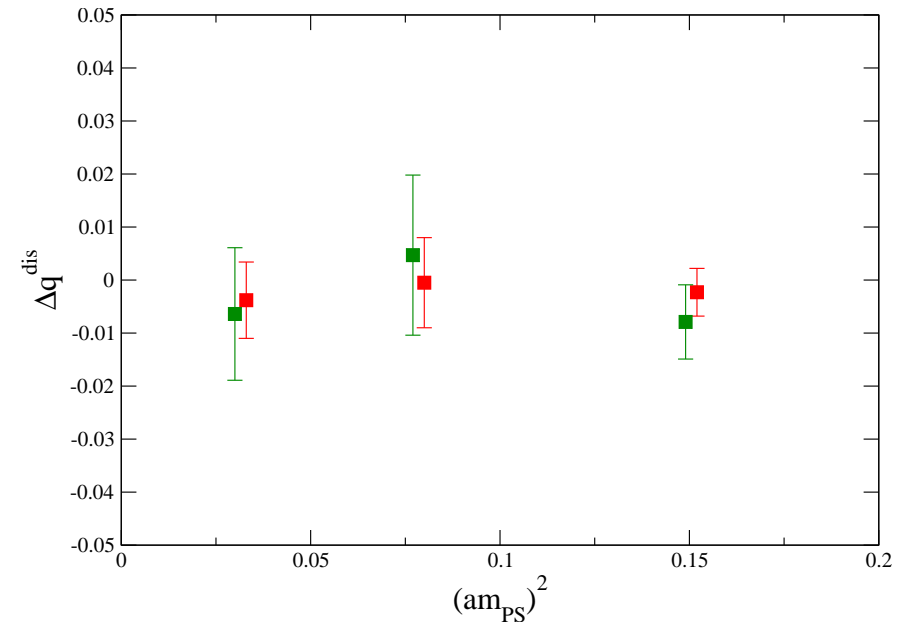
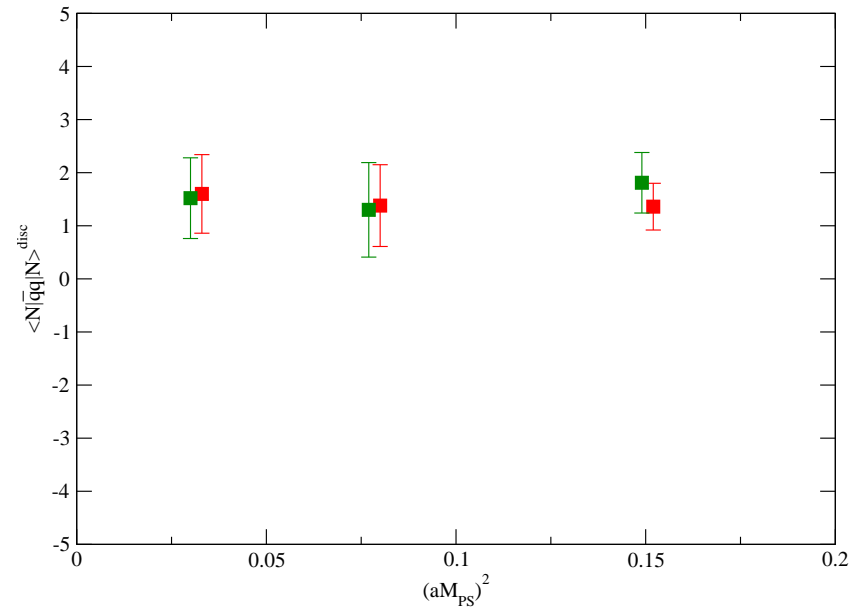
$$Q^2 = 4 \text{ GeV}^2$$

QCDSF



LHP

# Disconnected



Bali, Collins & Schäfer



# Spin Asymmetries

Transverse spin density

$\lambda_{\perp}$  quark spin  
 $s_{\perp}$  nucleon spin

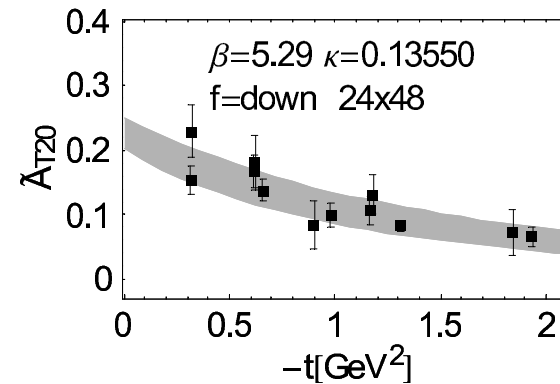
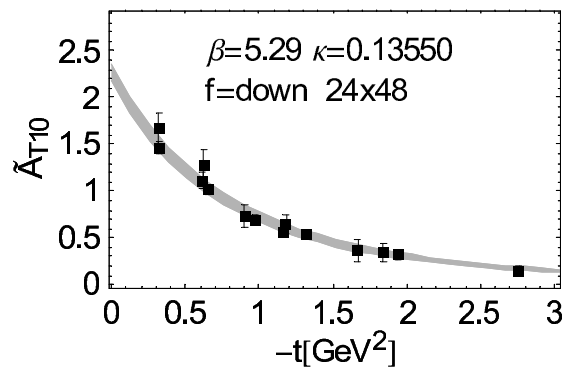
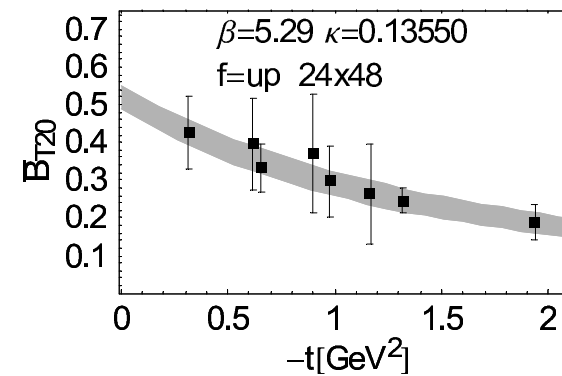
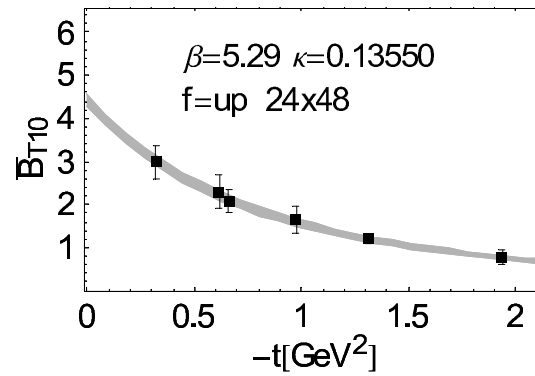
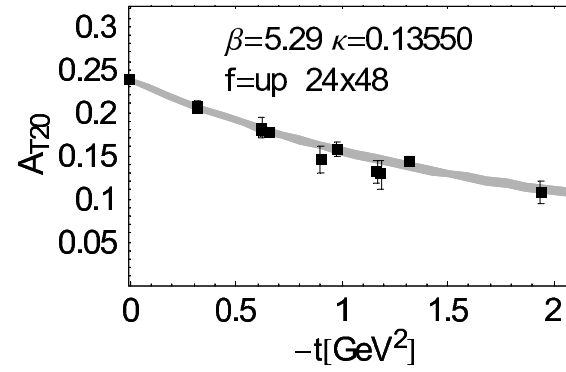
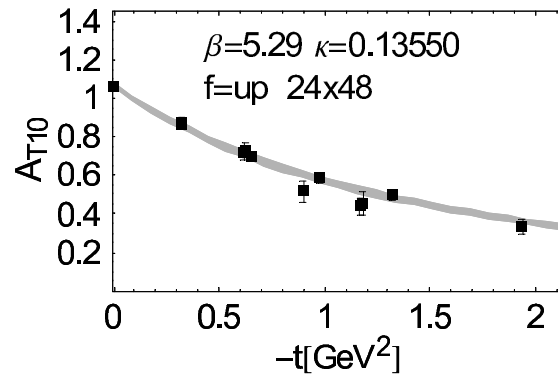


$$\begin{aligned} \langle p_+, s_{\perp} | \bar{q}(\mathbf{b}_{\perp}) [\gamma_+ - \lambda_{\perp i} \sigma_{+j} \gamma_5] q(\mathbf{b}_{\perp}) | p_+, s_{\perp} \rangle = & \left\{ A_1^q(\mathbf{b}_{\perp}^2) + \lambda_{\perp i} s_{\perp i} \left[ A_1^{Tq}(\mathbf{b}_{\perp}^2) \right. \right. \\ & - \frac{1}{4m_N^2} \Delta_{b_{\perp}} \tilde{A}_1^{Tq}(\mathbf{b}_{\perp}^2) \left. \right] - \frac{1}{m_N} \epsilon_{ij} b_{\perp j} \left[ s_{\perp i} B_1^q(\mathbf{b}_{\perp}^2)' + \lambda_{\perp i} \bar{B}_1^{Tq}(\mathbf{b}_{\perp}^2)' \right] \\ & \left. + \frac{1}{m_N^2} \lambda_{\perp i} (2b_{\perp i} b_{\perp j} - \mathbf{b}_{\perp}^2 \delta_{ij}) s_{\perp j} \tilde{A}_1^{Tq}(\mathbf{b}_{\perp}^2)'' \right\} \end{aligned}$$



Quadrupole

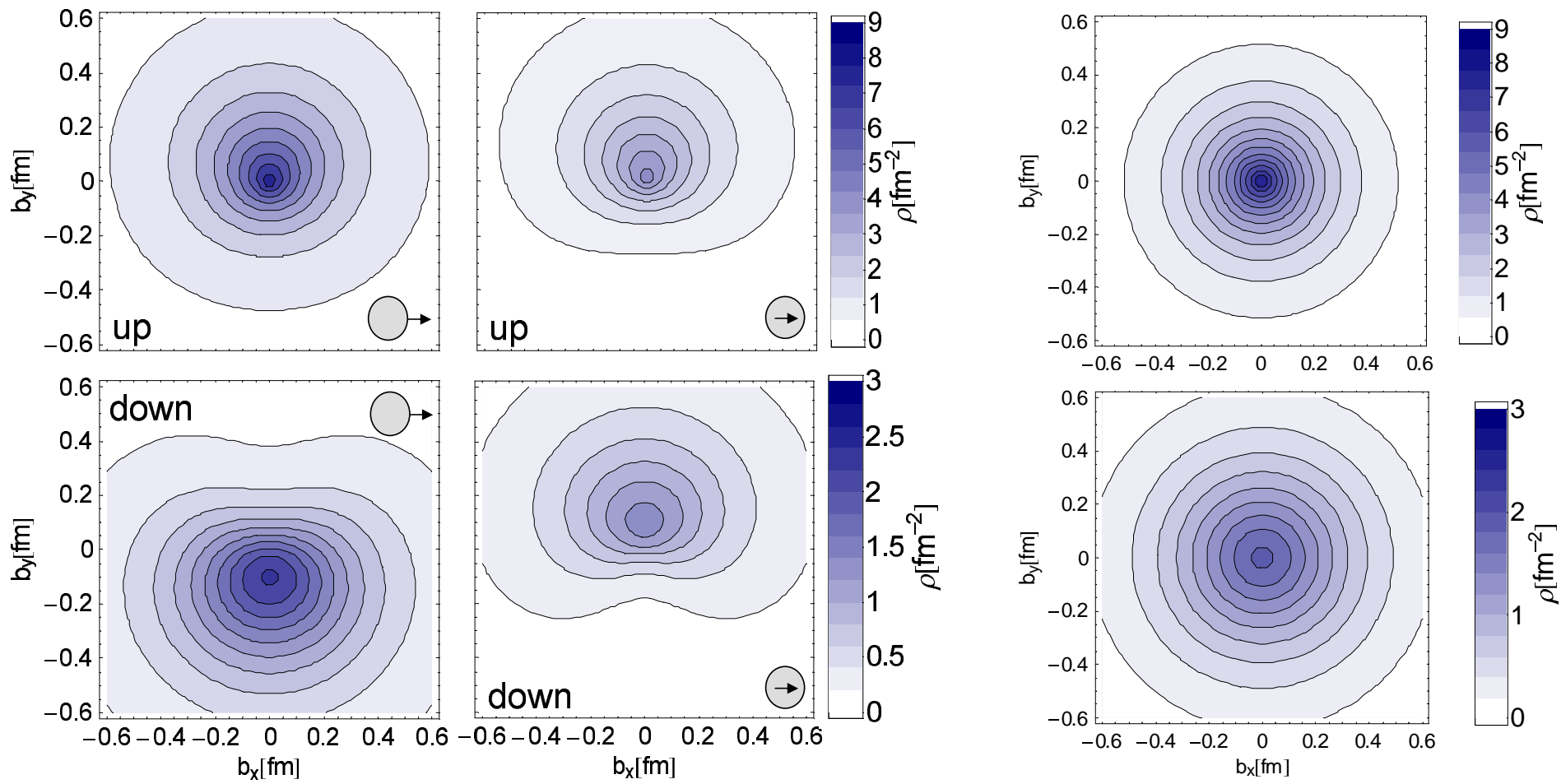
Diehl & Hägler



Dipole fit

1<sup>st</sup> moment

To be extrapolated to chiral limit

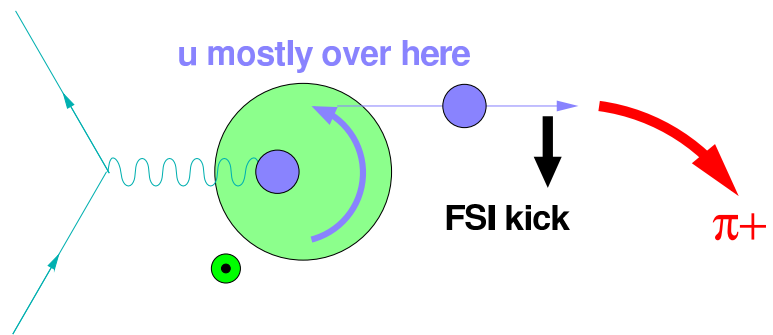


Sivers effect

Boer-Mulders effect

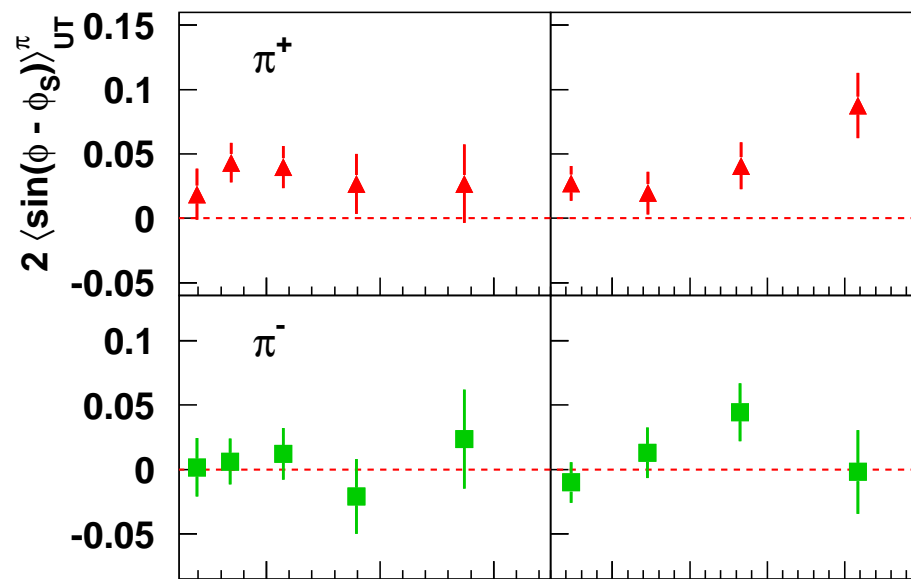
Unpolarized

$$r_T^d > r_T^u$$



Courtesy of G. Schnell

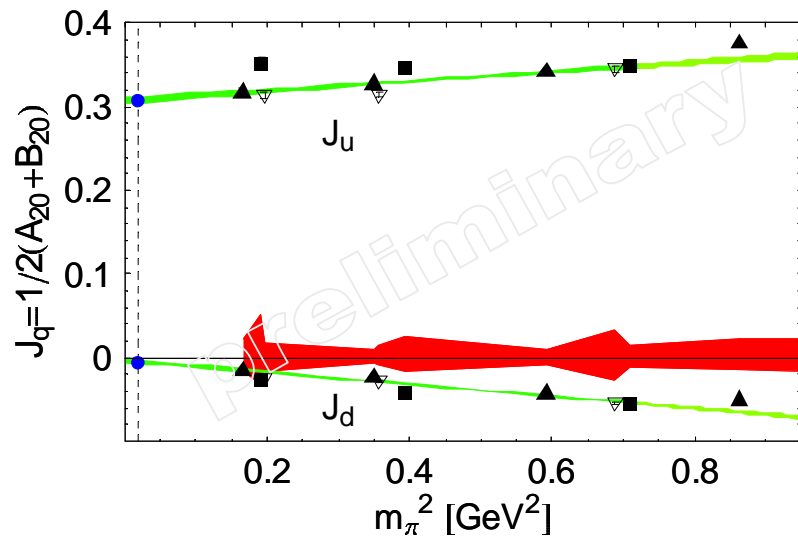
Sivers effect



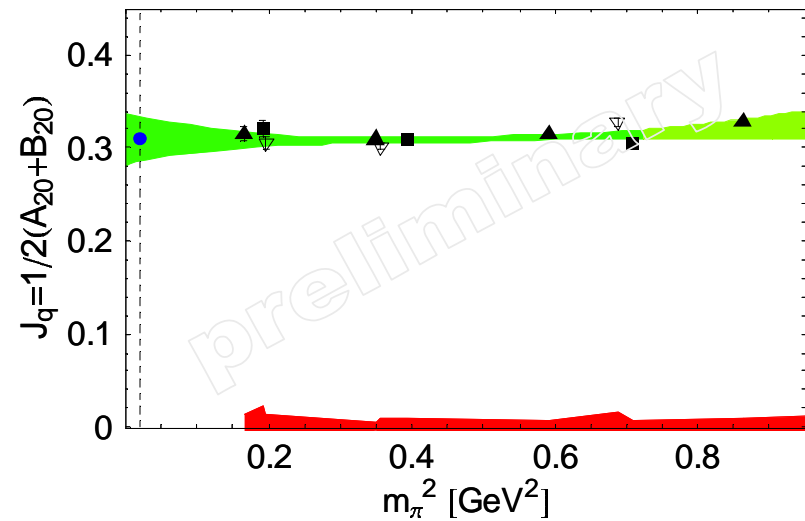
Hermes

Sivers effect  $\Leftarrow$  Orbital angular momentum

$$J^q = \frac{1}{2}(A_2^q(0) + B_2^q(0)) \equiv \frac{1}{2}\Delta\Sigma^q + L^q$$



$$J^u = 0.33(2) \quad J^d = -0.02(2)$$

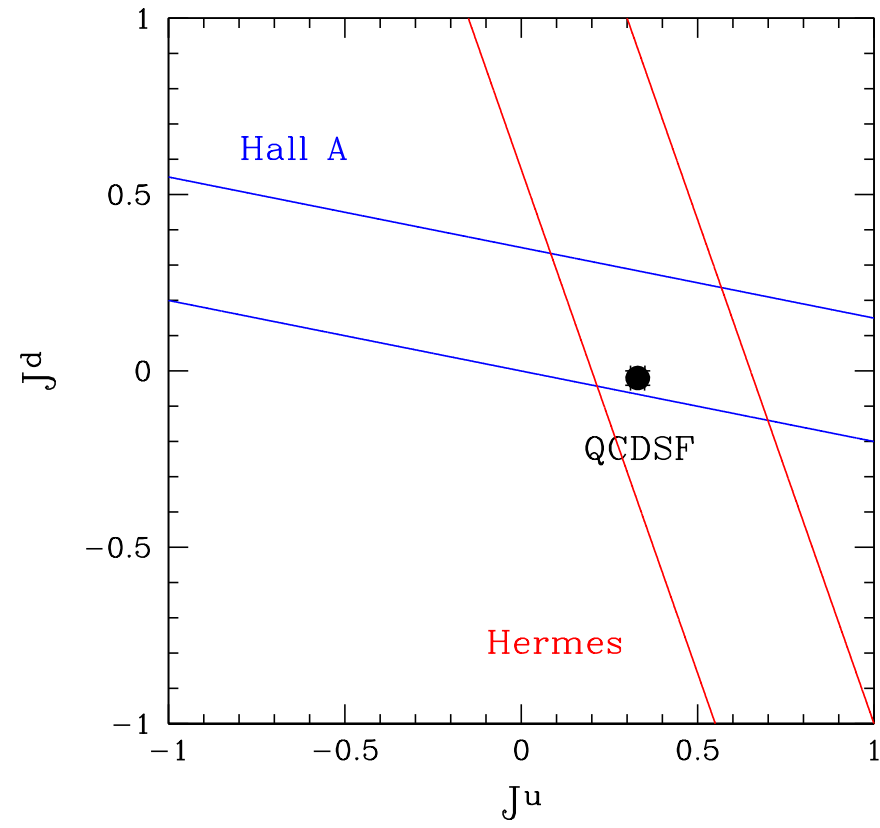


$$L^u = -0.13(4) \quad L^d = 0.15(4)$$

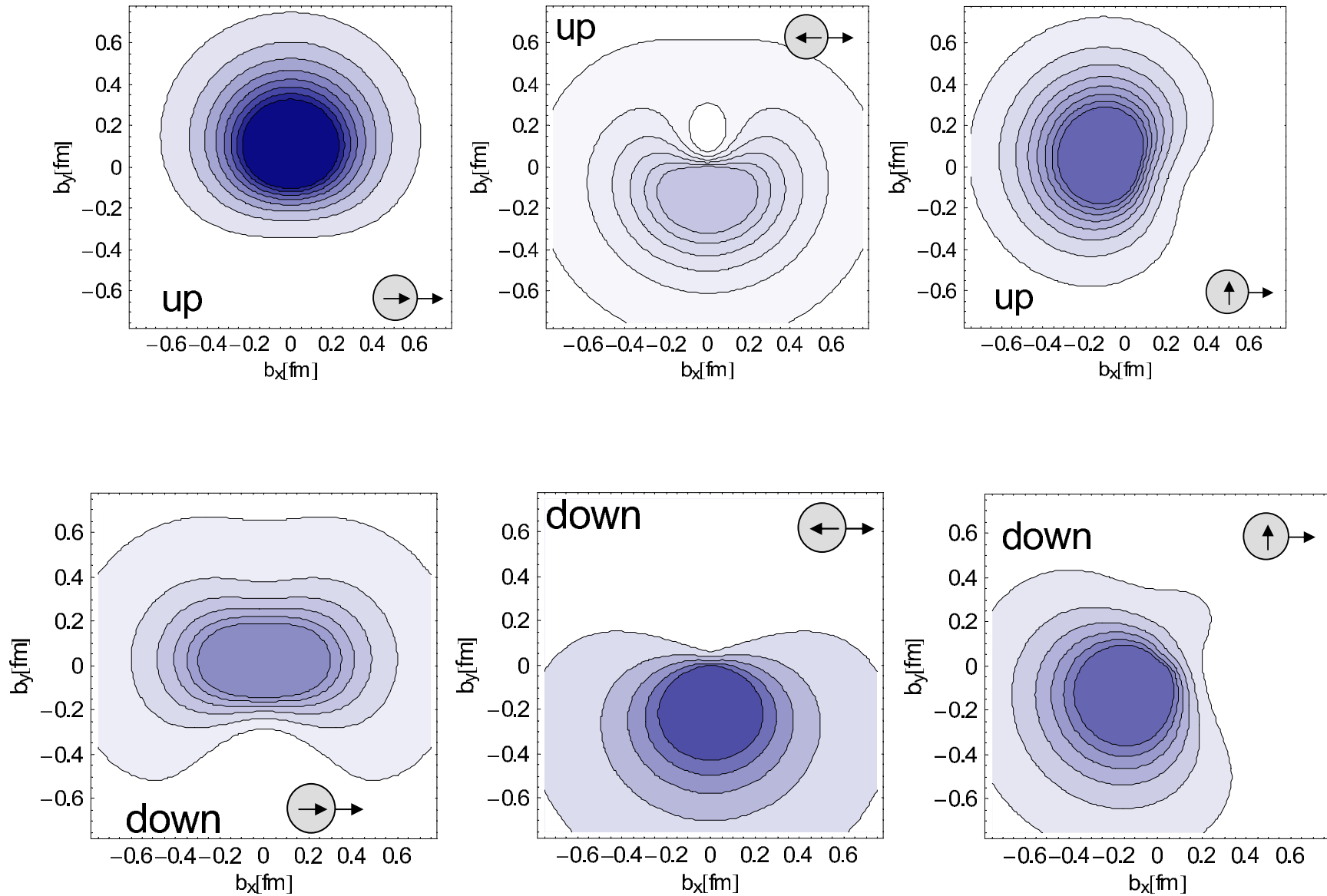
ChPT fit

Chen & Ji

# Experiment



# Nucleon and quarks both polarized Spin-orbit coupling



## Distribution Amplitudes

Expansion

$$\phi_N(x_1, x_2, x_3, \mu^2) = 120 x_1 x_2 x_3 \sum_{n=0}^{\infty} \sum_{l=0}^n c_{nl}(\mu_0) P_{nl}(x_1, x_2, x_3) L^{\gamma_{nl}/\beta_0} \quad \text{LLog}$$

$$L = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$$

$$c_{10} = \frac{7}{2} \left[ 3(\phi_N^{100}(\mu_0^2) + \phi_N^{001}(\mu_0^2)) - 2 \right]$$

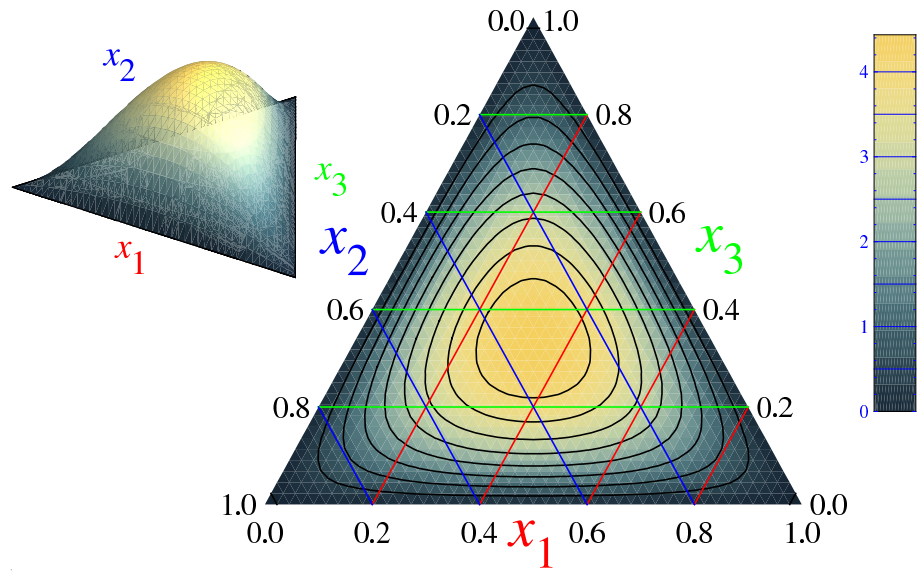
$$c_{11} = \frac{63}{2} (\phi_N^{100}(\mu_0^2) + \phi_N^{001}(\mu_0^2))$$

$$c_{21} = -\frac{126}{5} (\phi_N^{200}(\mu_0^2) + \phi_N^{002}(\mu_0^2) + 3\phi_N^{101}(\mu_0^2)) + \frac{18}{5}(4 + c_{10})$$

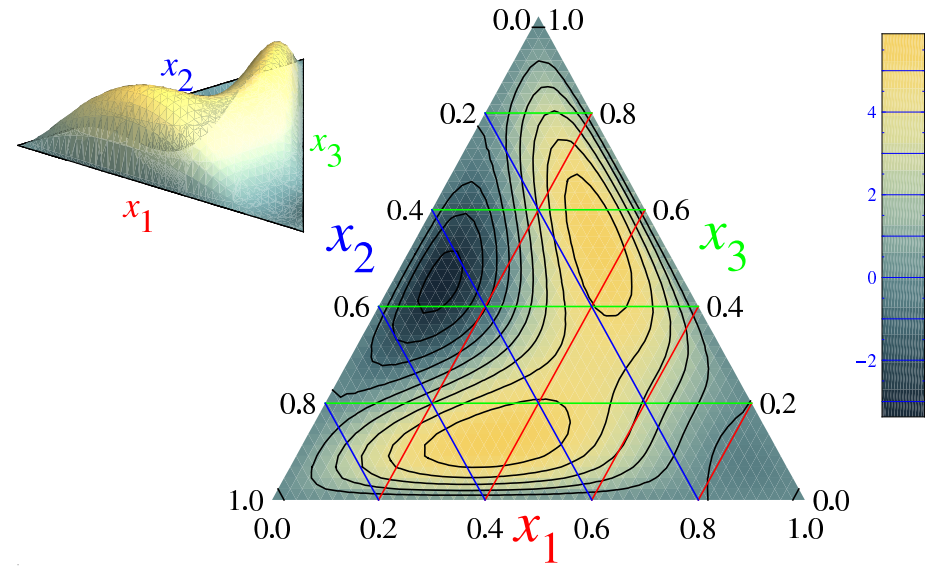
⋮



$\phi_N(x_1, x_2, x_3, \mu^2)$  Barycentric contour plot

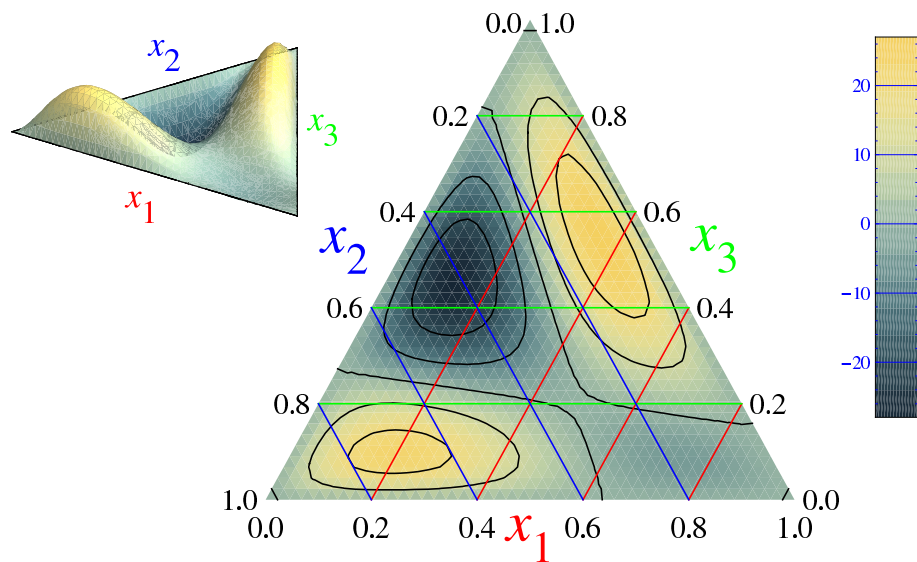


$\mu^2 \rightarrow \infty$

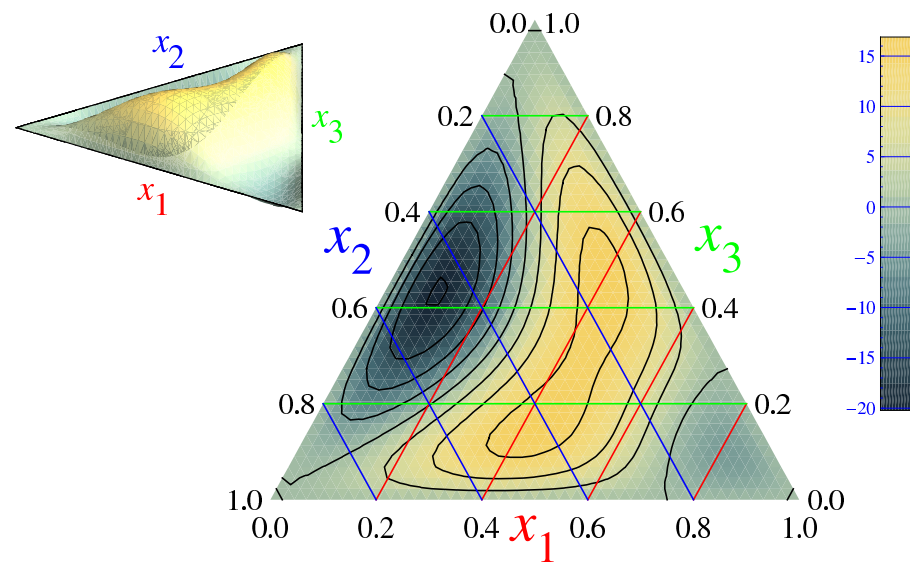


$\mu^2 = 4 \text{ GeV}^2$

## Nucleon



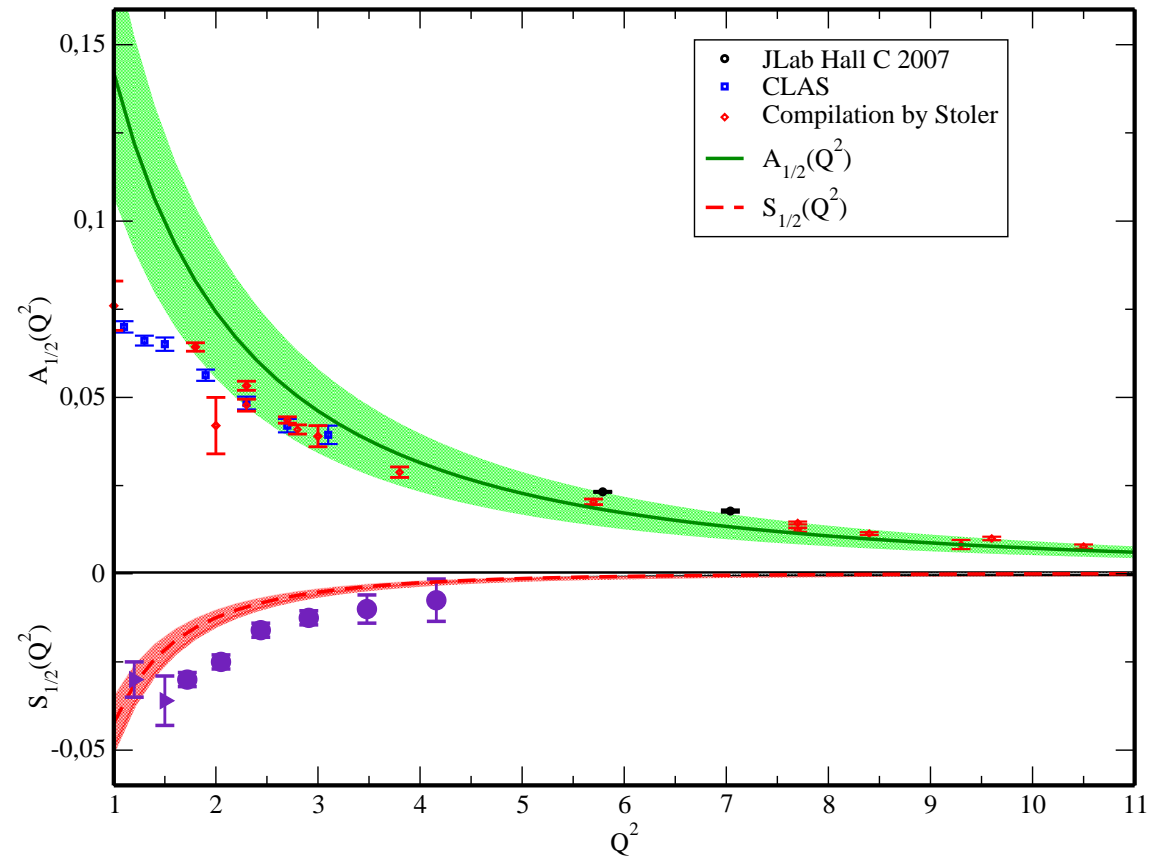
## $N^*(1535)$



Strong correlation of  $(u^\downarrow d^\uparrow)$  diquark

$$\mu^2 = 1 \text{ GeV}^2$$

$$\gamma^* N \rightarrow N^*(1535)$$



Braun et al.

**Pion**

# Spin Asymmetries

Transverse spin density

$\lambda_{\perp}$  quark spin



$$\langle p_+ | \bar{q}(\mathbf{b}_{\perp}) [\gamma_+ - \lambda_{\perp i} \sigma_{+j} \gamma_5] q(\mathbf{b}_{\perp}) | p_+ \rangle = \left\{ A_1^q(\mathbf{b}_{\perp}^2) - \frac{1}{m_{\pi}} \epsilon_{ij} b_{\perp j} \lambda_{\perp i} \bar{B}_1^{Tq}(\mathbf{b}_{\perp}^2)' \right\}$$

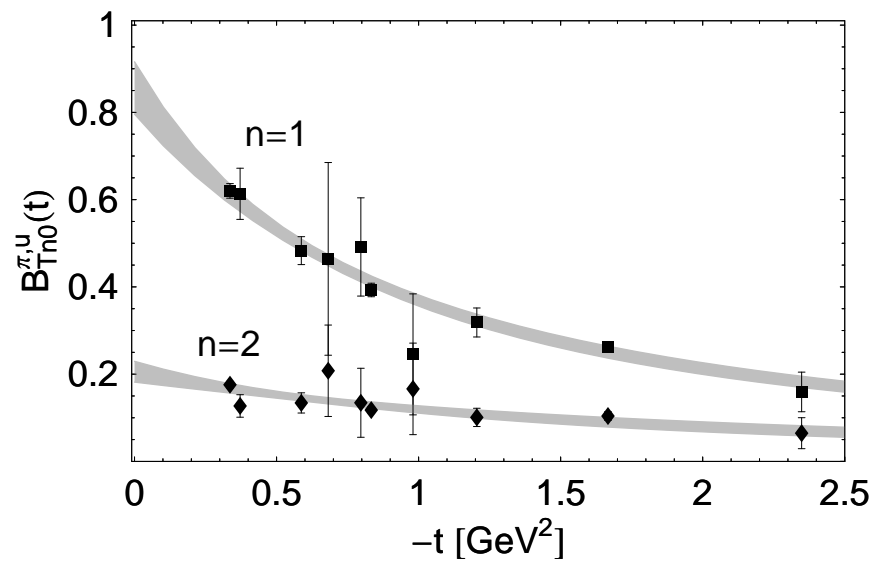


Monopole

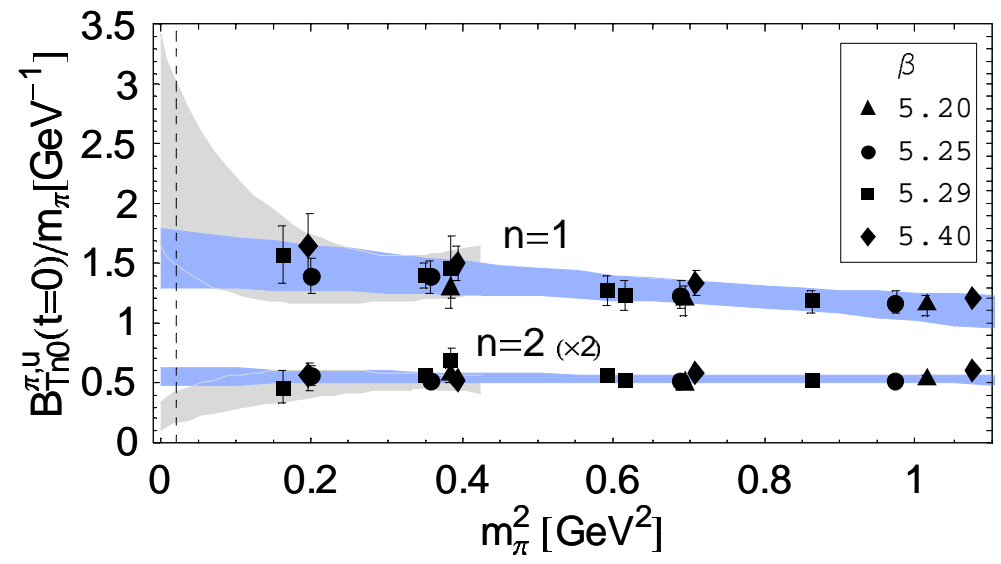


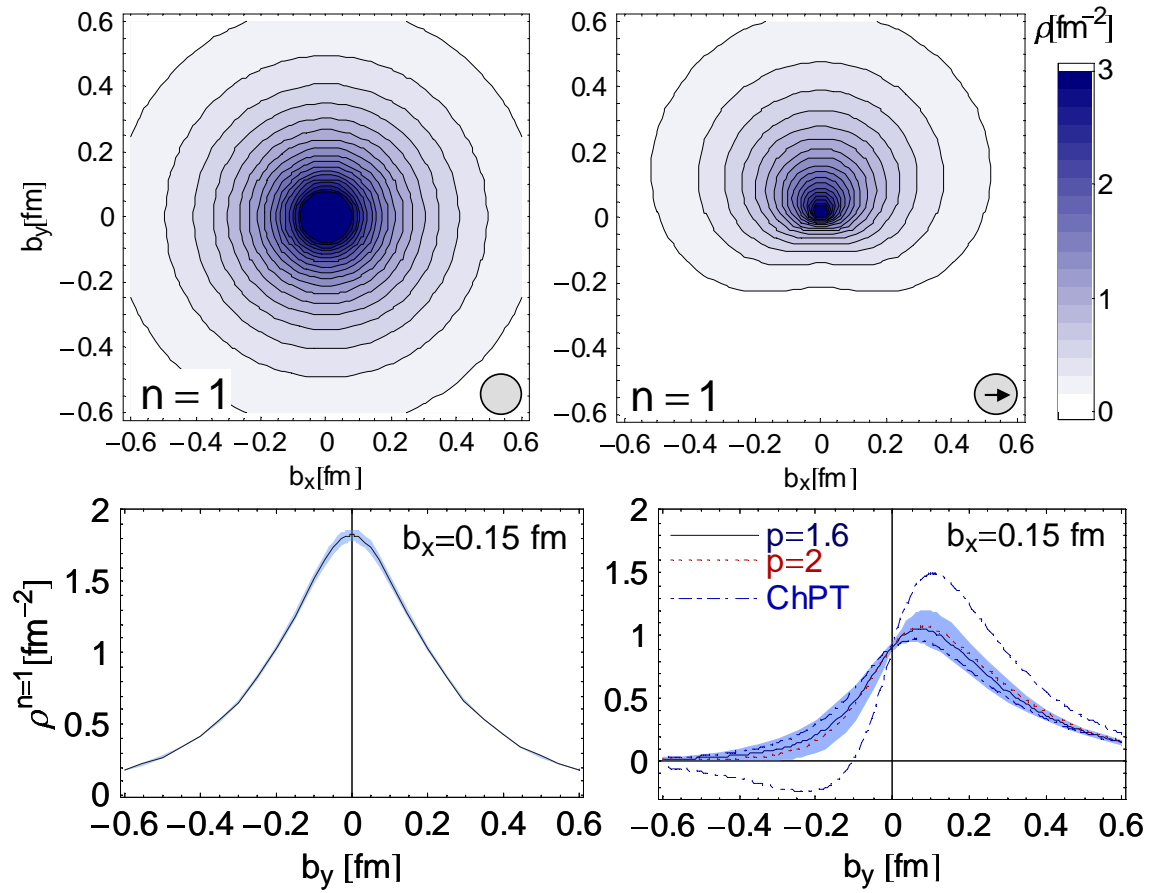
Dipole

Hägler et al.



*p*-pole fit





Boer-Mulders effect

ChPT: Diehl, Manashov & Schäfer

## Distribution Amplitude

Expansion

$$\phi_\pi(\xi, \mu^2) = \frac{3}{4}(1 - \xi^2) \left( 1 + \sum_{n=1}^{\infty} a_n(\mu_0) C_n^{3/2}(\xi) L^{\gamma_n/2\beta_0} \right)$$

$$L = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$$

$$\gamma_n = \frac{32}{3} \left( \sum_{k=1}^{n+1} \frac{1}{k} - \frac{3}{4} - \frac{1}{2(n+1)(n+2)} \right)$$

$$a_1 = \frac{5}{3} \langle \xi \rangle$$

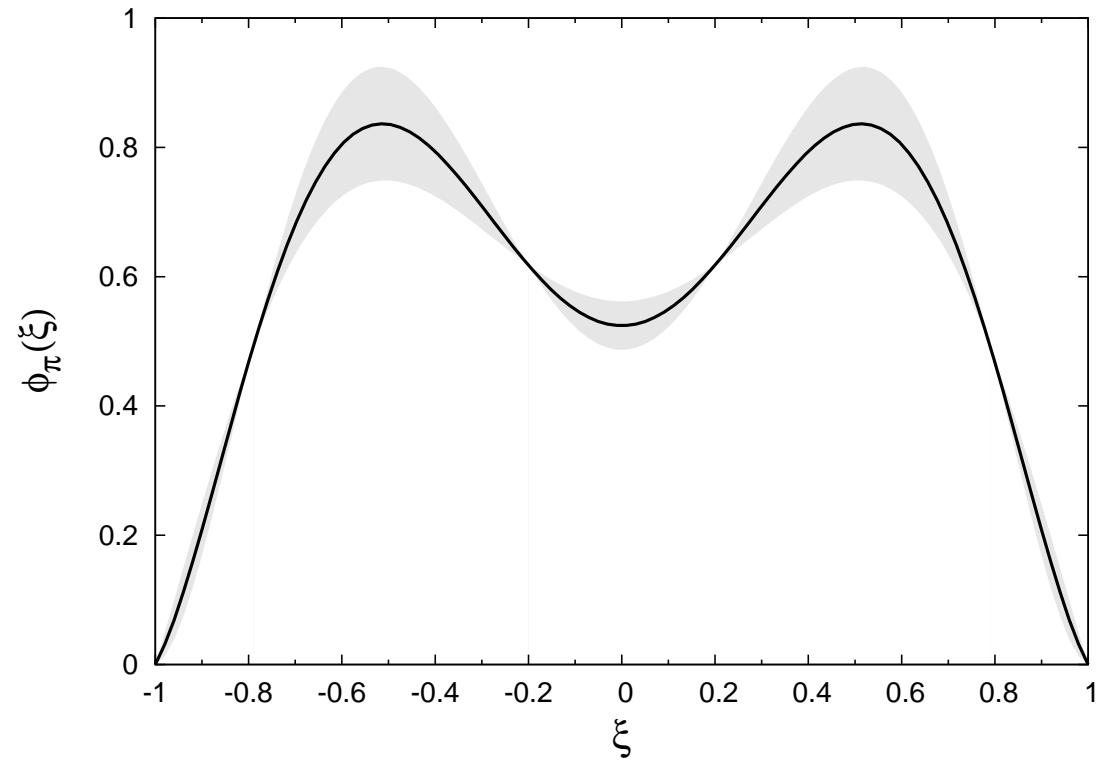
$$a_2 = \frac{7}{12} (5 \langle \xi^2 \rangle - 1)$$

⋮

$$\langle \xi^n \rangle = \int_{-1}^1 d\xi \xi^n \phi_\pi(\xi, \mu^2)$$



$$\phi_\pi(\xi, \mu^2)$$



$$\mu^2 = 4 \text{ GeV}^2$$

## Conclusions & Outlook

- Simulations at the physical pion mass with Wilson-type fermions are feasible now
- Extrapolations to infinite volume are largely under control
- Simulations with  $N_f = 2 + 1$  SLiNC fermions are under way

- Challenge: Evaluation of disconnected diagrams

- To exploit the full potential of lattice calculations, interactions with Experiment and Hadron Phenomenology are very important

- Improvement of algorithms
- Increase of computing power

FS corrections surprisingly well described by ChPT

