Pion Form Factor in Holographic QCD Backgrounds


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JLAB User Group Meeting 2009, June 9, 2009
Outline

Introduction
- Anti-de Sitter Space (AdS)
- Conformal Field Theory (CFT)
- Holographic Principle Original Conjecture

Holographic QCD
- Holographic QCD Action
- Chiral Symmetry Breaking
- Three Models with Different Backgrounds
- Model Calculation e.g.: Obtaining $g_5$

Pion Electromagnetic Form Factor

Results and Discussions
- Pion EM Form Factor
- Low-energy QCD observables
- Discussions

Conclusion
Anti-de Sitter Space

- Anti-de Sitter space: maximally symmetric space with negative curvature.
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- We used: 5-dimensional AdS (AdS$_5$) metric,

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where \( \eta_{\mu\nu} = \text{diag}(+,-,-,-) \).
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where \( R \): Ricci Scalar, \( n \): number of dimensions = 5.
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- Bulk/Louiville coordinate \(z \leftrightarrow \text{inverse energy scale } (Q \sim \frac{1}{z})\)
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- Bulk/Louiville coordinate $z \leftrightarrow$ inverse energy scale $(Q \sim \frac{1}{z})$
- one boundary at $z = 0$, our "brane".
- modified boundary $z = \epsilon$ with $\epsilon \equiv$ UV cutoff $\rightarrow$ "UV brane"
Conformal Field Theory

- Conformal symmetry: includes symmetry under scale transformation
  1. $x^\mu \rightarrow \lambda x^\mu$.
  2. scale invariance: physics looks the same at all scale.
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  4. no S-matrix: no asymptotic state.

- QCD is not conformal:
  1. quark mass
  2. QCD scale (\( \Lambda_{QCD} \))

- we address this issue in our model.
Holography

- in general, holographic principle:
  classical field theory coupled to gravity in (d+1) spatial dimensions
  corresponds to
d-dimensional quantum field theory without gravity on the surface.
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- original AdS/CFT conjecture: type IIB String theory in the \(\text{AdS}_5 \times S_5\) background is equivalent to Large \(\mathcal{N} = 4\) \(\text{SU}(N)\) Super YM in \(d=4\) (Maldacena)
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- practically:
  every CFT operator \(\mathcal{O}(x)\)
  is associated with
  \(\Psi(x)|_{z=\varepsilon}\) a bulk field at the “UV brane”
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- $\mathcal{O}$: operator with $\phi_0$ as source (the corresponding operator), constructed from the $\psi$’s fields.
- can work out two-point correlation function:

\[
\langle 0|\mathcal{O}(x)\mathcal{O}(y)|0 \rangle = \frac{\delta^2 Z_S[\phi_0]}{\delta \phi_0(x) \delta \phi_0(y)}.
\]
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Motivation: AdS/CFT $\rightarrow$ AdS/QCD approach

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deforming SYM $\rightarrow$ QCD.
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  3. holographic version QCD sum rule.
Holographic QCD continue

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  1. limit the ability of the fields to penetrate into the bulk
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Features of our holographic QCD:
1. Confinement.
2. Chiral Symmetry Breaking.
3. Vector Meson Dominance.
4. No running of QCD coupling constant.
The full 5-dimensional action:

$$S = \int d^5 x \ e^{-\Phi(z)} \ \sqrt{g} \ \text{Tr} \left\{ |DX|^2 + 3|X|^2 - \frac{1}{4g_5^2}(F_L^2 + F_R^2) \right\},$$

where $g \equiv |\det g_{MN}|$. 
Holographic QCD Action

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5. $\Sigma$: actually determined by the IR BC's, but in this model it is chosen to be an input parameter.
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Physical Mesons

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3. modified $D^M X$, $F^M_{VN}$ and $F^M_{AN}$:
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4. Pions:
   - $X = X_0 \exp(2i\pi^a t^a)$ (chiral perturbation style).
   - pion field $\pi^a$: dimensionless, related to $\tilde{\pi}^a$ of chiral Lagrangians via $\pi^a = \tilde{\pi}^a / f_\pi$, with $f_\pi = 93$ MeV.
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   - \( A_\mu \): decomposed into transverse piece \( A_{\mu\perp} \) and longitudinal piece \( \varphi \), \( A_\mu = A_{\mu\perp} + \partial_\mu \varphi \).
Recap of the Model

1. The Action in physical fields:

\[ S = \int d^5 x \, e^{-\Phi(z)} \sqrt{g} \, \text{Tr} \left\{ |D X|^2 + 3 |X|^2 - \frac{1}{2 g_5^2} (F_V^2 + F_A^2) \right\}. \]
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2. The model free parameters:
   - coupling constant: \( g_5 \)
   - confinement: \( Z_{IR} \)
   - chiral symmetry breaking: \( m_q \) and \( \sigma \)
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   - chiral symmetry breaking: \( m_q \) and \( \sigma \)

3. the low-energy QCD observables (to be calculated):
   \( m_\rho, F_\rho, \ m_{a_1}, \ F_{a_1}, \ m_\pi, \ f_\pi, \ g_{\rho\pi\pi} \) and \( \pi \) electromagnetic form factor \( (F_\pi) \)
Three different backgrounds

Table: Comparison of hard-wall, soft-wall soft-wall with modified scalar field $X(z)$ and Saxon-Woods hybrid model.

<table>
<thead>
<tr>
<th></th>
<th>hard-wall</th>
<th>soft-wall</th>
<th>Saxon-Woods hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>cut-off</td>
<td>cut-off at $z = z_0$</td>
<td>no cutoff</td>
<td>no cutoff</td>
</tr>
<tr>
<td>$\Phi(z)$</td>
<td>$\Phi(z) = 0$</td>
<td>$\Phi(z) = \kappa^2 z^2$</td>
<td>$e^{-\Phi(z)} = \frac{e^{\lambda^2 z_0^2} - 1}{e^{\lambda^2 z_0^2} + e^{\lambda^2 z^2} - 2}$</td>
</tr>
<tr>
<td>$\partial_z \psi(x, z)</td>
<td>_{z=z_0} = 0$</td>
<td>$\psi(x, z)</td>
<td>_{z \to \infty} = 0$</td>
</tr>
<tr>
<td>$X_0(z)$</td>
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</table>

with $\nu(z) = m_q z + \sigma z^3$
Equations of motion of the fields

1. Equation of motion of the Fourier transformed $\Psi(q, z)$:

$$\partial_z \left( \frac{1}{z} \partial_z V_\mu^a \right) + \frac{q^2}{z} V_\mu^a = 0$$

$$\left[ \partial_z \left( \frac{1}{z} \partial_z A_\mu^a \right) + \frac{q^2}{z} A_\mu^a - \frac{g_5^2 \nu(z)^2}{z^3} A_\mu^a \right] \perp = 0$$

$$\partial_z \left( \frac{1}{z} \partial_z \phi^a \right) + \frac{g_5^2 \nu(z)^2}{z^3} (\pi^a - \phi^a) = 0$$

$$- q^2 \partial_z \phi^a + \frac{g_5^2 \nu(z)^2}{z^2} \partial_z \pi^a = 0$$

where $\nu(z) = 2X_0(z) = m_q z + \sigma z^3$. 
Solving the Vector EOM and Obtaining $g_5$

1. Equation of motion of the Fourier transformed $V_\mu(q, z)$:

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2. Decompose $V^a_\mu(q, z) = \mathcal{V}(q, z) \tilde{V}^a_\mu(q)$,

$\tilde{V}^a_\mu(q)$: Fourier transform of the source of $J^a_\mu = \bar{q} \gamma_\mu t^a q$ at $z = \epsilon$

$\mathcal{V}(q, z)$: “bulk-to-boundary propagator”, normalized to $\mathcal{V}(q, \epsilon) = 1$. 

Solving the Vector EOM and Obtaining $g_5$

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   \[
   \partial_z \left( \frac{1}{z} \partial_z V^a_\mu \right) + \frac{q^2}{z} V^a_\mu = 0, 
   \]

2. decompose $V^a_\mu(q, z) = V(q, z) \tilde{V}^a_\mu(q)$,
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   \tilde{V}^a_\mu(q): \text{Fourier transform of the source of } J^a_\mu = \bar{q} \gamma_\mu t^a q \text{ at } z = \epsilon 
   \]
   $V(q, z)$: “bulk-to-boundary propagator”, normalized to $V(q, \epsilon) = 1$.

3. Boundary Conditions:
   - axial-like gauge, $V_z(q, z) = 0$.
   - $z_0$ boundary condition.

4. $V(q, z)$: Bessel functions.
Surface Term

1. Referring to the $5^{th}$-D surface ($z = \epsilon$) term, the 4D surface term $\rightarrow 0$.

2. Substituting EOM to the $F^2_V$ portion of the action:

$$S_{Surf} = -\frac{1}{2g_5^2} \int d^4x \; \tilde{V}_a^\mu(-q) \tilde{V}^{\mu a}(q) \frac{1}{z} \partial_z V(q, z) \Bigg|_{z=\epsilon}.$$
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$$S_{\text{Surf}} = -\frac{1}{2g_5^2} \int d^4x \, \tilde{V}^a_{\mu}(-q) \tilde{V}^{\mu a}(q) \frac{1}{z} \partial_z V(q, z) \bigg|_{z=\epsilon}.$$  

3. Isospin conservation ($q_{\mu} V^{\mu} = 0$):
   - $\tilde{V}^a_{\mu} \tilde{V}^{\mu a} \rightarrow \tilde{V}^a_{\mu} \tilde{V}^{b}_{\nu} \Pi^{\mu \nu} \delta^{ab}$.
   - $\Pi^{\mu \nu} \equiv \eta^{\mu \nu} - q^{\mu} q^{\nu} / q^2$. 

Surface Term
Two-point Correlation Function

Quadratic variation of the action with respect to the source $\tilde{V}$ produces the vector current two-point function:

$$\int d^4 x \, e^{i q x} \langle J^a_{\mu}(x) J^b_{\nu}(0) \rangle = \delta^{ab} \ \Pi_{\mu\nu} \ \Sigma_V(q^2)$$

$$\begin{align*}
\text{AdS/CFT} & = \left( \frac{\delta}{\delta \tilde{V}^a_{\mu}(-q)} \right) \left( \frac{\delta}{\delta \tilde{V}^b_{\nu}(q)} \right) S[\tilde{V}^c_{\alpha}] \\
\Rightarrow \Sigma_V(q^2) & = - \frac{1}{g_5^2} \frac{\partial_z V(q, z)}{z} \bigg|_{z=\epsilon}
\end{align*}$$

And in the large Euclidean $Q^2 = -q^2$ limit:

$$\Sigma_V(-Q^2) = \frac{Q^2}{2g_5^2} \ln Q^2.$$
Matching to the original QCD sum rule result for currents $J_\mu$, Shifman et al. (1979):

\[
\Sigma_V(-Q^2) = \frac{Q^2 N_c}{24\pi^2} \ln Q^2 ,
\]

\[
\Rightarrow g_5^2 = \frac{12\pi^2}{N_c} \rightarrow 4\pi^2 .
\]
1. Pion EM FF: obtained from $V_{\pi\pi}$, $V_{AA}$ and $V_{A\pi}$ terms in the action: longitudinal modes of $A$ contribute.

$|DX|^2 \supset (\partial_{\pi})(V_{\pi})$ and $(V_{\pi})(A)$.

$F^2_V \supset (\partial V)(AA)$.

$F^2_A$: does not contribute because the longitudinal modes cancel each other.
Pion Electromagnetic Form Factor

1. Pion EM FF: obtained from $V_\pi\pi$, $VAA$ and $VA\pi$ terms in the action: longitudinal modes of $A$ contribute. 

$|DX|^2 \supset (\partial_\pi)(V_\pi)$ and $(V_\pi)(A)$. 

$F^2_V \supset (\partial V)(AA)$. 

$F^2_A$: does not contribute because the longitudinal modes cancel each other.

2. The complete related action:

$$S^{V_\pi\pi}_{AdS} = \epsilon_{abc} \int d^4x \int dz \left[ \frac{1}{g_5^2 z} (\partial_z \partial_\mu \varphi^a) V^b_\mu (\partial_z \varphi^c) 
+ \frac{v(z)^2}{z^3} (\partial_\mu \pi^a - \partial_\mu \varphi^a) V^b_\mu (\pi^c - \varphi^c) \right].$$
1. The 3-point correlation function:

\[ \langle J^a_\pi(p_1) J^{\mu,b}_V(q) J^c_\pi(-p_2) \rangle = \epsilon^{abc} F(p_1^2, p_2^2, q^2) (p_1 + p_2)^\mu \times i(2\pi)^4 \delta^{(4)}(p_1 - p_2 + q). \]
Form Factor and 3-point Correlation Function

1. The 3-point correlation function:

\[
\langle J^a_\pi(p_1) J^{\mu,b}_V(q) J^c_\pi(-p_2) \rangle = \epsilon^{abc} F(p_1^2, p_2^2, q^2) (p_1 + p_2)^\mu \\
\times i(2\pi)^4 \delta^{(4)}(p_1 - p_2 + q).
\]

2. The dynamical form factor \( F(p_1^2, p_2^2, q^2) \) in terms of transition form factors:

\[
F(p_1^2, p_2^2, q^2) = \sum_{n,k=1}^{\infty} \frac{f_n f_k F_{nk}(q^2)}{(p_1^2 - M^2_n) (p_2^2 - M^2_k)},
\]

where \( F_{nk}(q^2) \) correspond to form factors for \( n \to k \) transitions.
Form Factor and 3-point Correlation Function

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\]

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\]
where \( F_{nk}(q^2) \) correspond to form factors for \( n \rightarrow k \) transitions.

3. the pion form factor \( F_\pi(q^2) \equiv F_{11}(q^2) \).
AdS QCD and 3-point Correlation Function

1. Trilinear variation of the action with respect to the source $\tilde{V}\tilde{\pi}\tilde{\pi}$ produces the 3-point function:

$$\langle J_{\pi}^{a}(p_{1})J_{V}^{\mu,b}(q)J_{\pi}^{c}(-p_{2}) \rangle^{AdS/CFT} = \left( \frac{\delta}{\delta \tilde{\pi}^{a}(p_{1})} \right) \left( \frac{\delta}{\delta \tilde{V}^{b,\mu}(q)} \right) \left( \frac{\delta}{\delta \tilde{\pi}^{c}(-p_{2})} \right) S_{AdS}^{V\pi\pi}.$$
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2. Pion EM FF:

$$F_\pi(q^2) = \int_0^{z_{IR}} dz \frac{V(q,z)}{f_\pi^2} \left\{ \frac{1}{g_5^2 z} [\partial_z \varphi(z)]^2 + \frac{v(z)^2}{z^3} [\pi(z) - \varphi(z)]^2 \right\}.$$
Figure: Spacelike scaling behavior for $F_\pi(Q^2)$ as a function of $Q^2 = -q^2$. The continuous line is the prediction of the hard-wall model with $1/z_0 = 323$ MeV. The dotted line is the prediction of the soft-wall model with $\kappa = m_\rho/2$. The crosses use modified $e^{-\Phi(z)}$ with $\lambda z_0 = 2.1$, and the pluses use $\lambda z_0 = 1$. The black stars are from a data compilation from CERN, the blue circles are from DESY, reanalyzed by Tadevosyan et al., the green triangle is data from DESY, and the red boxes and green diamonds are from Jefferson Lab.
Predictions for QCD observables

Table: Comparison of soft-wall model to modified $e^{-\Phi(z)}$ with $\lambda z_0 = 1$; values in MeV (except for $g_{\rho\pi\pi}$).

<table>
<thead>
<tr>
<th>Observable</th>
<th>Experiment</th>
<th>Soft-wall</th>
<th>$\lambda z_0 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_\pi$</td>
<td>139.6±0.0004</td>
<td>139.6</td>
<td>139.6</td>
</tr>
<tr>
<td>$m_\rho$</td>
<td>775.5±0.4</td>
<td>777.4</td>
<td>779.2</td>
</tr>
<tr>
<td>$m_{a_1}$</td>
<td>1230±40</td>
<td>1601</td>
<td>1596</td>
</tr>
<tr>
<td>$f_\pi$</td>
<td>92.4±0.35</td>
<td>87.0</td>
<td>92.0</td>
</tr>
<tr>
<td>$f_{\rho}^{1/2}$</td>
<td>346.2±1.4</td>
<td>261</td>
<td>283</td>
</tr>
<tr>
<td>$f_{a_1}^{1/2}$</td>
<td>433±13</td>
<td>558</td>
<td>576</td>
</tr>
<tr>
<td>$g_{\rho\pi\pi}$</td>
<td>6.03±0.07</td>
<td>3.33</td>
<td>3.49</td>
</tr>
</tbody>
</table>
### Predictions for QCD observables

**Table:** Comparison of hard-wall model to the modified $e^{-\Phi(z)}$ with $\lambda z_0 = 2.1$; values in MeV (except for $g_{\rho\pi\pi}$).

<table>
<thead>
<tr>
<th>Observable</th>
<th>Experiment</th>
<th>Hard wall</th>
<th>$\lambda z_0 = 2.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_\pi$</td>
<td>$139.6 \pm 0.0004$</td>
<td>139.6</td>
<td>139.6</td>
</tr>
<tr>
<td>$m_\rho$</td>
<td>$775.5 \pm 0.4$</td>
<td>775.3</td>
<td>777.5</td>
</tr>
<tr>
<td>$m_{a_1}$</td>
<td>$1230 \pm 40$</td>
<td>1358</td>
<td>1343</td>
</tr>
<tr>
<td>$f_\pi$</td>
<td>$92.4 \pm 0.35$</td>
<td>92.1</td>
<td>88.0</td>
</tr>
<tr>
<td>$f_{\rho}^{1/2}$</td>
<td>$346.2 \pm 1.4$</td>
<td>329</td>
<td>325</td>
</tr>
<tr>
<td>$f_{a_1}^{1/2}$</td>
<td>$433 \pm 13$</td>
<td>463</td>
<td>474</td>
</tr>
<tr>
<td>$g_{\rho\pi\pi}$</td>
<td>$6.03 \pm 0.07$</td>
<td>4.48</td>
<td>4.63</td>
</tr>
</tbody>
</table>
Discussions

1. the Saxon-Woods hybrid model interpolate between hard-wall and soft-wall model.
Discussions

1. the Saxon-Woods hybrid model interpolate between hard-wall and soft-wall model.

2. the SW hybrid model with parameter matching it to hard-wall model produces higher-order vector meson poles exhibiting Regge trajectory behavior $m_n^2 \sim n$

Table: Comparison of vector meson masses and decay constants (in MeV) for the hard-wall model and the model using modified $e^{-\Phi(z)}$ with $\lambda z_0 = 2.1$.

<table>
<thead>
<tr>
<th>Original hard wall</th>
<th>$m_\rho$</th>
<th>$F_\rho^{1/2}$</th>
<th>modified $e^{-\Phi(z)}$ with $\lambda z_0 = 2.1$</th>
<th>$m_\rho$</th>
<th>$F_\rho^{1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>775.6</td>
<td>392</td>
<td>777.5</td>
<td>325</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1780.2</td>
<td>734</td>
<td>1608.1</td>
<td>528</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2790.8</td>
<td>1029</td>
<td>2226.8</td>
<td>611</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3802.8</td>
<td>1298</td>
<td>2637.5</td>
<td>644</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4815.2</td>
<td>1549</td>
<td>2986.6</td>
<td>683</td>
<td></td>
</tr>
</tbody>
</table>
Conclusions

1. Motivated AdS/CFT correspondence, proposed by Maldacena in 1997, a five-dimensional framework for modeling low-energy properties of QCD is proposed by Erlich et al. in 2005.
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Conclusions

1. Motivated AdS/CFT correspondence, proposed by Maldacena in 1997, a five-dimensional framework for modeling low-energy properties of QCD is proposed by Erlich et al. in 2005.

2. This holographic QCD model naturally incorporates properties of QCD such as confinement and chiral symmetry breaking.

3. Prediction for low-energy QCD observables are off only by 10%.

4. Possible improvement on the model and future work:
   - including baryons in the model via Chern-Simons terms.
   - inclusion of strange quark (SU(3)×SU(3) chiral symmetry).
   - including running of the gauge coupling by considering logarithmic corrections to the AdS geometry.
AdS/CFT Dictionary

<table>
<thead>
<tr>
<th>AdS</th>
<th>CFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Field</td>
<td>Operator</td>
</tr>
<tr>
<td>Mass</td>
<td>Scaling Dimension</td>
</tr>
<tr>
<td>Non-Normalizable Mode</td>
<td>Source</td>
</tr>
<tr>
<td>Normalizable Mode</td>
<td>State ↔ VEV</td>
</tr>
<tr>
<td>Gauge Field</td>
<td>Global Symmetry</td>
</tr>
</tbody>
</table>
Holographic QCD Dictionary

1. fields ↔ operators:

<table>
<thead>
<tr>
<th>4D: $\mathcal{O}(x)$</th>
<th>5D: $\phi(x, z)$</th>
<th>$p$</th>
<th>$\Delta$</th>
<th>$(m_5)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{q}_L \gamma^{\mu} t^a q_L$</td>
<td>$A^{a}_{L\mu}$</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$\bar{q}_R \gamma^{\mu} t^a q_R$</td>
<td>$A^{a}_{R\mu}$</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$\bar{q}_R^{\alpha} q^\beta_L$</td>
<td>$(2/z) X^{\alpha\beta}$</td>
<td>0</td>
<td>3</td>
<td>$-3$</td>
</tr>
</tbody>
</table>

Table: Operators/fields of the model

2. masses ↔ scaling dimensions:

$$(\Delta - p)(\Delta + p - 4) = m^2_5$$ (Witten, Gubser et al.),

$m_5$: 5D masses of the fields $A^{a}_{L\mu}$, $A^{a}_{R\mu}$, and $X$

$\Delta$: the dimension of the corresponding $p$-form operator.
Table: Comparison of hard-wall, soft-wall soft-wall with modified scalar field $X(z)$ and hard-wall-soft-wall hybrid model.

<table>
<thead>
<tr>
<th>hard-wall</th>
<th>soft-wall</th>
<th>modified soft-wall</th>
<th>hard-wall-soft-wall hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>cut-off at $z = z_0$</td>
<td>no cutoff</td>
<td>no cutoff</td>
<td>no cutoff</td>
</tr>
<tr>
<td>$\phi(z) = 0$</td>
<td>$\phi(z) = \kappa^2 z^2$</td>
<td>$\phi(z) = \kappa^2 z^2$</td>
<td>$e^{-\phi(z)} = \frac{e^{\lambda^2 z^2_0} - 1}{e^{\lambda^2 z^2_0} + e^{\lambda^2 z^2} - 2}$</td>
</tr>
<tr>
<td>$\partial_z \Psi(x, z)</td>
<td>_{z=z_0} = 0$</td>
<td>$\Psi(x, z)</td>
<td>_{z \to \infty} = 0$</td>
</tr>
<tr>
<td>$X_0(z) = \frac{1}{2} \nu(z)$</td>
<td>$X_0(z) = \frac{1}{2} \nu(z)$</td>
<td>$X_0(z) = \frac{1}{2} \nu(z)(1 - e^{-A_c / \kappa^4 z^4})$</td>
<td>$X_0(z) = \frac{1}{2} \nu(z)$</td>
</tr>
</tbody>
</table>

with $\nu(z) = m_q z + \sigma z^3$