The Thomas Jefferson National Accelerator Facility
Theory Group Preprint Series

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The Southeastern Universities Research Association (SURA) operates the Thomas Jefferson National Accelerator Facility for the United States Department of Energy under contract DE-AC05-84ER40150.

JLAB-THY-97-10
April 1, 1997
hep-ph/9704207

Nonforward Parton Distributions

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Applications of perturbative QCD to deeply virtual Compton scattering and hard exclusive electroproduction processes require a generalization of usual parton distributions for the case when long-distance information is accumulated in nonforward matrix elements $\langle p'|O(0,z)|p\rangle$ of quark and gluon light-cone operators. We describe two types of nonperturbative functions parametrizing such matrix elements: double distributions $F(x,y;t)$ and nonforward distribution functions $\mathcal{F}_\zeta(X;t)$, discuss their spectral properties, evolution equations which they satisfy, basic uses and general aspects of factorization for hard exclusive processes.

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I. INTRODUCTION

The standard feature of applications of perturbative QCD to hard processes is the introduction of phenomenological functions accumulating information about nonperturbative long-distance dynamics. The well-known examples are the parton distribution functions $f_{p/H}(x)$ [1] used in perturbative QCD approaches to hard inclusive processes and distribution amplitudes $\varphi_\pi(x)$, $\varphi_N(x_1, x_2, x_3)$, which naturally emerge in the asymptotic QCD analyses of hard exclusive processes [2-7]. Recently, it was argued [8,9] that the amplitudes of hard exclusive ρ-meson electroproduction processes at small x are determined by the same gluon distribution function $f_g(x)$ used for description of hard inclusive processes (see also [10]). Furthermore, it was proposed [11] to use another exclusive process of deeply virtual Compton scattering (DVCS) for measuring quark distribution functions inaccessible in inclusive measurements (nonforward Compton amplitudes were discussed also in [12,9]). In fact, the kinematics of hard elastic electroproduction processes (DVCS can be also treated as one of them) requires the presence of the longitudinal component in the momentum transfer $r \equiv p - p'$ from the initial hadron to the final: $r_\parallel = \zeta p$, where ζ for $Q^2 >> |t|, m_H^2$ reduces to the Bjorken variable $x_{Bj} = Q^2/2(pq)$ associated with the virtual photon momentum q. This means that the nonperturbative matrix element $\langle p'| \ldots |p\rangle$ is essentially asymmetric and, strictly speaking, the distributions which appear in the hard elastic electroproduction amplitudes differ from those studied in inclusive processes. In the latter case, one always deals with a symmetric situation when the same momentum p appears in both brackets of the hadronic matrix element $\langle p| \ldots |p\rangle$.

For diffractive processes, one deals with a kinematic situation when both the variable ζ specifying the longitudinal momentum asymmetry of the nonperturbative matrix element $\langle p'| \ldots |p\rangle$ and the absolute value of the momentum transfer $t \equiv (p'-p)^2$ are small. Studying the DVCS process, one should be able to consider the whole region $0 \leq \zeta \leq 1$ and $t \sim 1\,GeV^2$ [13]. In this situation, one deals with essentially *nonforward* (or *off-forward* in terminology of ref. [11]) kinematics for the matrix element $\langle p'| \ldots |p\rangle$. The basics of the pQCD approach incorporating asymmetric/off-forward parton distributions were formulated in refs. [11,14,15,13]. A detailed analysis of pQCD factorization for hard meson electroproduction processes was given in ref. [16]. Applications of asymmetric gluon distributions to elastic diffractive J/ψ electroproduction were discussed in [17-19]. In a recent paper [20], the off-forward quark distributions were studied within the MIT bag model†. Thus, there is an increasing interest in the studies of these new types of hadron distributions, their general properties and applications.

Our goal in the present paper is to give a detailed description of the approach outlined in our earlier papers [14,15]. The basic idea of refs. [14,15] is that constructing a consistent pQCD approach for amplitudes of hard exclusive electroproduction processes one should treat the initial momentum p and the longitudinal part of the momentum transfer r on equal footing by introducing double distributions $F(x,y)$, which specify the fractions of p and $r_\parallel$ carried by the constituents of the nucleon. These distributions have hybrid properties: they look like distribution functions with respect to x and like distribution amplitudes with respect to y. Writing matrix elements of composite operators in terms of double distributions is the starting point of constructing the pQCD parton picture. Another important step is taking into account the logarithmic scaling violation. The evolution kernels $R(x,y;\xi,\eta)$ for double distributions have a remarkable property: they produce the GLAPD evolution kernels $P(x/\xi)$ [22-24] when integrated over y, while integrating $R(x,y;\xi,\eta)$ over x one obtains the BL-type evolution kernels $V(y,\eta)$ [6,7] for the relevant distribution amplitudes‡. One can use these properties of the kernels to construct formal solutions of the one-loop evolution equations for the double distributions. The longitudinal momentum transfer $r_\parallel$ is proportional to p: $r_\parallel = \zeta p$ and, for this reason, it is convenient to parametrize matrix elements $\langle p - r| \ldots |p\rangle$ by *asymmetric distribution functions* $\mathcal{F}_\zeta(X)$ specifying the total light-cone fractions Xp, $(X-\zeta)p$ of the initial hadron momentum p carried by the "outgoing" and "returning" partons§. It should be emphasized that double distributions $F(x,y)$

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† Just before completing the present paper, I was informed by M. Strikman about a numerical study [21] of the evolution of the asymmetric gluon distribution.

‡ Originally, the evolution equation for the pion distribution amplitude in QCD was derived and solved in ref. [5], where the anomalous dimension matrix Z_{nk} was used instead of $V(x,y)$ (see also [25]).

§ The asymmetric distribution functions defined in ref. [15] are similar to, but not coinciding with the $t \to 0$ limit of the

are universal functions in the sense that they do not depend on the momentum asymmetry parameter ζ while the asymmetric distribution functions $\mathcal{F}_\zeta(X)$ form a family of X-dependent functions changing their shape when ζ is changed. The functions $\mathcal{F}_\zeta(X)$ also have hybrid properties. In the region $X \geq \zeta$ the returning parton carries a positive fraction $(X-\zeta)p$ of the initial momentum p, and hence $\mathcal{F}_\zeta(X)$ is similar to the usual parton distribution $f(X)$. On the other hand, in the region $0 \leq X \leq \zeta$ the difference $X - \zeta$ is negative, *i.e.*, the second parton should be treated as propagating together with the first one. The partons in this case share the longitudinal momentum transfer $r_\parallel = \zeta p$ in fractions $Y \equiv X/\zeta$ and $1 - Y$. This means that in the region $X \leq \zeta$ the function $\mathcal{F}_\zeta(X)$ looks like a distribution amplitude. It is possible to formulate equations governing the evolution of the asymmetric distribution functions $\mathcal{F}_\zeta^g(X)$ and establish relations between these functions, double distributions $F(x,y)$ and usual distribution functions $f(x)$ [14,15].

Constructing a QCD parton picture for hard electroproduction processes, it is very important to know spectral properties of the relevant parton distributions $F(x,y)$ and $\mathcal{F}_\zeta(X)$. Using the approach [26] based on the α-representation analysis [27-30], it is possible to prove that double distributions $F(x,y)$ have a natural property that both x and y satisfy the "parton" constraints $0 \leq x \leq 1$, $0 \leq y \leq 1$ for any Feynman diagram contributing to $F(x,y)$. A less obvious restriction $0 \leq x + y \leq 1$ guarantees that the argument $X = x + y\zeta$ of the asymmetric distribution $\mathcal{F}_\zeta(X)$ also changes between the limits 0 and 1. An important observation here is that $X = 0$ can be obtained only if both $x = 0$ and $y = 0$. Because of vanishing phase space for such a configuration, one may expect that asymmetric distributions $\mathcal{F}_\zeta(X)$ vanish for $X = 0$. This property is very essential, because the hard subprocess amplitudes usually contain $1/X$ factors. When $\mathcal{F}_\zeta(0) \neq 0$, one faces a singularity $\mathcal{F}_\zeta(X)/X$ at the end-point of the integration region $0 \leq X \leq 1$. Since such a singularity is not integrable, factorization of short- and long-distance contributions does not work in that case.

The paper is organized in the following way. In Section II, we consider parton distributions in a toy scalar model. Despite its simplicity, it shares many common features with the realistic QCD case. In particular, the spectral properties of distribution functions are not affected by the numerators of quark and gluon propagators, derivatives in triple-gluon vertices, *etc.* Hence, studying a scalar model we just concentrate on the denominator structure of the relevant momentum integrals, which is the same in both theories. We start with the simplest example of the usual (forward) distribution $f(x)$ and then consider more and more complicated functions: the double distribution $F(x,y)$, asymmetric distribution function $\mathcal{F}_\zeta(X)$ and nonforward distribution $\mathcal{F}_\zeta(X;t)$. Explicit expressions for these functions at one-loop level are obtained with the help of the α-representation. Using the latter one can easily establish the spectral properties of the distribution functions. The α-representation also provides a very effective starting point for a general analysis of factorization and large-Q^2 behavior of elastic amplitudes. In Section III, we outline the all-order extension of the one-loop analysis. We give an all-order definition of the double distribution function $F(x,y)$ and demonstrate that it has the spectral properties $0 \leq \{x, y, x+y\} \leq 1$. We show how one can use the α-representation analysis for finding integration regions responsible for the leading large-Q^2 contributions. We also discuss modifications of twist counting rules in QCD due to cancellations between different gluonic contributions in Feynman gauge and other complications which appear in gauge theories. In Section IV, we give definitions of nonforward distributions $\mathcal{F}_\zeta(X;t)$ in QCD. Just like the usual distribution functions $f(x)$ and distribution amplitudes $\varphi(y)$, the new distributions depend on the factorization scale μ, *i.e.*, it is more appropriate to use the notation $\mathcal{F}_\zeta(X;t;\mu)$ for the nonforward distributions rather than simply $\mathcal{F}_\zeta(X;t)$. Evolution equations governing the μ-dependence of the nonforward distributions are discussed in Sections V and VI. We show how one can obtain evolution kernels for nonforward distributions using already known kernels $B(u,v)$ of the evolution equation for the light-cone operators [31]. Since this equation has an operator form, substituting it into a specific matrix element one can convert $B(u,v)$ into desired evolution kernels. In particular, taking $\langle p| \ldots |p\rangle$ one obtains the GLAPD kernels, choosing $\langle 0| \ldots |p\rangle$ one gets BL-type kernels while resorting to $\langle p'| \ldots |p\rangle$ and parametrizing the matrix elements through $F(x,y)$ or $\mathcal{F}_\zeta(X)$ one ends up with the kernels $R(x,y;\xi,\eta)$ and $W_\zeta(X,Z)$ governing the evolution of double and asymmetric/nonforward distributions, respectively. In Section V, we discuss the derivation

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off-forward parton distributions introduced by X.Ji [11], see Section IX.

of the evolution kernels $W_\zeta(X,Z)$ for the nonforward distributions. We show, in particular, that in the region $0 \le \{X,Z\} \le \zeta$, the kernels $W_\zeta(X,Z)$ reduce to the BL-type kernels $V(X,Z)$ calculated for rescaled variables X/ζ, Z/ζ. This result is very natural, since $\mathcal{F}_\zeta(X)$ can be treated as a distribution amplitude when $X \le \zeta$. In the opposite limit $\zeta \le \{X,Z\} \le 1$, the evolution is similar to that of the GLAP equation, the basic distinction being the difference between the outgoing X,Z and returning $X' \equiv X-\zeta$, $Z' \equiv Z-\zeta$ momentum fractions. We show that writing the kernels $W_\zeta(X,Z)$ in terms of the fractions X,X',Z,Z' in the region $\zeta \le \{X,Z\} \le 1$ gives the functions $W(X,X';Z,Z')$ which have the symmetry property with respect to the interchange of initial and final partons: $W(X,X';Z,Z') = W(X',X;Z',Z)$. For $\zeta = 0$ one has $X = X', Z = Z'$ and the kernels $W_{\zeta=0}(X,Z)$ acquire the GLAPD form. In Section VI, we discuss the QCD evolution of the nonforward distributions. Qualitatively, the evolution can be described in the following way. Due to the GLAP-type evolution in the $X > \zeta$ region, the momenta of partons decrease, and distributions shift into the regions of smaller X. However, when the parton momentum degrades to values smaller than the momentum transfer $r = \zeta p$, the further evolution is like that for a distribution amplitude: it tends to make the distribution symmetric (or antisymmetric) with respect to the central point $X = \zeta/2$ of the $(0,\zeta)$ segment. In section VII, we briefly discuss two basic uses of nonforward distributions: deeply virtual Compton scattering and hard elastic meson electroproduction. In particular, we show how to combine the definition of the gluon distribution through the matrix element of the gauge-invariant gluonic operator $G^a_{\mu\alpha}(0)E_{ab}(0,z;A)G^a_{\alpha\nu}(z)$ with the usual Feynman rules formulated for the vector potential A^a_μ. In Section VIII, we discuss possible sources of pQCD factorization breaking for hard elastic electroproduction processes, due to singularities at the end-points of the integration region. In particular, we emphasize the importance of establishing the $\mathcal{F}_\zeta(0) = 0$ property for the nonforward distributions. In Section IX, we compare our notations, definitions and terminology with those used by other authors (off-forward parton distributions $H(x,\xi;t)$ introduced by X. Ji [11] and non-diagonal distributions $f(x_1,x_2)$ defined by Collins, Frankfurt and Strikman [16]). Section X contains concluding remarks.

II. FORWARD AND NONFORWARD DISTRIBUTIONS IN SCALAR TOY MODEL

A. Introductory remarks

The nonforward parton distributions $\mathcal{F}_\zeta(X;t;\mu)$ parametrizing matrix elements $\langle p'|\mathcal{O}(0,z)|p\rangle$ of composite two-body operators $\mathcal{O}(0,z)$ on the light cone $z^2=0$ depend on four parameters. In addition to the "usual" parton variable X specifying the fraction Xp of the initial hadron momentum p carried by the active parton (more formally, X may be treated as the Fourier-conjugate parameter to (pz)), the functions $\mathcal{F}_\zeta(X;t;\mu)$ also depend on the invariant momentum transfer $t=(p'-p)^2$, the longitudinal momentum asymmetry parameter $\zeta = (rz)/(pz)$ (where $r \equiv p - p'$) and the evolution/factorization scale μ. The latter characterizes the subtraction procedure for singularities that appear on the light cone $z^2=0$ (in general, μ may be different from the scale μ_R introduced by the R-operation for ordinary UV divergences, but the usual convention is to take $\mu = \mu_R$). Furthermore, depending on the type of the composite operator $\mathcal{O}(0,z)$, one would get quark, antiquark, flavor-singlet, flavor-nonsinglet, gluonic, spin-dependent, spin-independent, *etc.* distributions. In this situation, we propose to follow a step by step approach. We will start with simplest examples and then gradually proceed to more complicated ones. For this reason, we consider first a toy scalar model. The lowest nontrivial level corresponds to one loop Feynman diagrams. The relevant integrals are easily calculable, and their study provides useful information about the structure of the nonforward distributions, especially about their spectral properties, because the latter are insensitive to numerators of quark and gluon propagators and other complications brought in by the spin structure of the realistic QCD case.

B. Forward distribution functions

Our starting point is the scalar analog of the usual "forward" parton distribution functions $f(x)$. Consider a one-loop box diagram for a scalar version of the virtual forward Compton amplitude (Figs.1a,b). Both incoming and outgoing virtual "photons" have momentum $q \equiv q' - \zeta p$, where q' and p are lightlike momenta $(q')^2 = 0, p^2 = 0$.

The "photons" couple with the constant e to a massive scalar "quark" field ϕ. The initial and final hadrons are imitated by massless scalar particles with the momentum p. Their coupling to the quarks is specified by a constant g. In these notations, $q^2 \equiv -Q^2 = -2\zeta(pq')$. Since $(pq) = (pq')$, the parameter ζ coincides with the Bjorken variable $\zeta = x_{Bj} \equiv Q^2/2(pq)$. Using the α-representation for the scalar propagators

$$\frac{1}{m^2-k^2-i\epsilon} = i\int_0^\infty e^{i\alpha(k^2-m^2+i\epsilon)}\,d\alpha \tag{2.1}$$

and calculating the resulting Gaussian integral over the loop momentum k we obtain for the diagram 1a:

$$T_a(p,q) = -\frac{e^2g^2}{16\pi^2}\int_0^\infty \exp\left\{i\left[2(pq')\,\alpha_1\frac{\alpha_3-\zeta(\alpha_2+\alpha_3+\alpha_4)}{\alpha_1+\alpha_2+\alpha_3+\alpha_4} - \lambda\,(m^2-i\epsilon)\right]\right\}\frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4}{\lambda^2}. \tag{2.2}$$

We use the shorthand notation $\lambda \equiv \alpha_1+\alpha_2+\alpha_3+\alpha_4$. The large-$Q^2$ asymptotics is determined by integration over the region where the coefficient accompanying $2(pq')$ vanishes. Otherwise, the integrand rapidly oscillates and the result of integration is exponentially suppressed. Integration over $\alpha_1 \sim 0$ region is evidently the simplest possibility. Other variants are $\alpha_1+\alpha_2+\alpha_3+\alpha_4 \sim 0$ or $\alpha_3 - \zeta(\alpha_2+\alpha_3+\alpha_4) \sim 0$. It is easy to check that the leading power behavior is generated by the $\alpha_1 \sim 0$ integration, which gives

$$T_a(p,q) = -\frac{ie^2g^2}{16\pi^2}\int_0^\infty \frac{1}{2(pq')(\alpha_3/\tilde\lambda - \zeta + i\epsilon)}\,\frac{e^{-i\tilde\lambda(m^2-i\epsilon)}}{\tilde\lambda^2}\,d\alpha_2 d\alpha_3 d\alpha_4 + O(1/Q^4)\,, \tag{2.3}$$

where $\tilde\lambda \equiv \alpha_2+\alpha_3+\alpha_4$. Introducing the distribution function

$$f(x) = \frac{ig^2}{16\pi^2}\int_0^\infty \delta\left(x - \frac{\alpha_3}{\alpha_2+\alpha_3+\alpha_4}\right)\frac{e^{-i\tilde\lambda(m^2-i\epsilon)}}{\tilde\lambda^2}\,d\alpha_2 d\alpha_3 d\alpha_4\,, \tag{2.4}$$

we can write the leading-power contribution in the parton form:

$$T_a^{as}(p,q) = -\int_0^1 \frac{e^2}{2(pq')(x-\zeta+i\epsilon)}f(x)\,dx = -\int_0^1\frac{e^2}{(xp+q)^2+i\epsilon}f(x)\,dx \equiv \int_0^1 t_a(xp,q)\,f(x)\,dx\,. \tag{2.5}$$

At the last step, we introduced the parton subprocess amplitude

$$t_a(xp,q) = -\frac{e^2}{(xp+q)^2+i\epsilon}\,. \tag{2.6}$$

Hence, the parameter x can be treated as the fraction of the initial momentum p carried by the quark interacting with the virtual photon. Note that the limits $0 \le x \le 1$ necessary for this interpretation of x are automatically imposed by the α-representation of $f(x)$. A similar result holds for the u-channel diagram 1b:

$$T_b^{as}(p,q) = \int_0^1\frac{e^2}{2(pq')(x+\zeta-i\epsilon)}f(x)\,dx = -\int_0^1\frac{e^2}{(xp-q)^2+i\epsilon}f(x)\,dx \equiv \int_0^1 t_b(xp,q)\,f(x)\,dx\,. \tag{2.7}$$

The distribution function $f(x)$ is defined here by the same α-parameter integral (2.4). The latter can be easily calculated to give

$$f(x) = \frac{g^2}{16\pi^2m^2}(1-x)\,\theta(0\le x\le 1)\,. \tag{2.8}$$

Note, that $f(x)$ is purely real. Due to singularity at $x=\zeta$ in Eq.(2.4), the total amplitude $T \equiv T_a + T_b$ has both real and imaginary parts. Since $x \ge 0$ and $\zeta \ge 0$, its imaginary part is given by the s-channel contribution $T_a(p,q)$ only:

$$\frac{1}{\pi e^2}\,\mathrm{Im}\,T^{as}(p,q) = \int_0^1 \mathrm{Im}\;t_a(xp,q)\,f(x)\,dx = \int_0^1\frac{1}{2(pq')}\delta(x-\zeta)\,f(x)\,dx = \frac{f(\zeta)}{2(pq')} = \frac{1}{2(pq')}\,\frac{g^2}{16\pi^2m^2}(1-\zeta)\,. \tag{2.9}$$

The real part of T is given by T_b and by the real part of T_a:

$$\mathrm{Re}\, T_a^{as}(p,q) = \int_0^1 \mathrm{Re}\; t_a(xp,q)\, f(x)\, dx = -\frac{e^2}{2(pq')}\, \mathrm{P} \int_0^1 \frac{f(x)}{x-\zeta}\, dx\,, \tag{2.10}$$

where P stands for the principal value prescription.

To translate these results into the OPE language, we write the contribution of the diagrams $1a, b$ in the coordinate representation:

$$T(p,q) = \int \langle p|\, \phi(0)\phi(z)\, |p\rangle \left(e^{-i(qz)} + e^{i(qz)}\right) D_m(z^2)\, d^4z\,. \tag{2.11}$$

The large-Q^2 asymptotics of $T(p,q)$ is given by the leading light-cone behavior of both the quark propagator $D_m(z^2) = 1/4i\pi^2(z^2 - i\epsilon) + \ldots$ and the matrix element $\langle p|\, \phi(0)\phi(z)\, |p\rangle$

$$T(p,q) = \int \frac{e^{-i(qz)} + e^{i(qz)}}{4i\pi^2(z^2 - i\epsilon)} \langle p|\, \phi(0)\phi(z)\, |p\rangle|_{z^2=0}\, d^4z \; + O(1/Q^4). \tag{2.12}$$

Defining the parton distribution function $f(x)$ by

$$\langle p\,|\, \phi(0)\phi(z)\,|\, p\rangle|_{z^2=0} = \int_0^1 \frac{1}{2}\left(e^{-ix(pz)} + e^{ix(pz)}\right) f(x)\, dx\,, \tag{2.13}$$

we rederive the parton formulas (2.5), (2.7). Basically, the integral (2.13) can be treated as a Fourier representation for the light-cone matrix element $\langle p\,|\, \phi(0)\phi(z)\,|\, p\rangle|_{z^2=0} \equiv \tilde{f}(pz)$ which is a function of the only variable (pz). However, to derive the spectral constraint $-1 \le x \le 1$ for the Fourier partner of (pz) and establish the property $f(x) = f(-x)$, one should incorporate the fact that $\tilde{f}(pz)$ is given by Feynman integrals with specific analytic properties and that we have the same scalar field ϕ at both points 0 and z. The α-representation which we used above is one of the most effective (though perturbative) ways to take these properties into account. In ref. [26] (see also Section III below), the α-representation was used to prove that the constraint $0 \le x \le 1$ in Eq.(2.13) and similar (but more complicated) constraints for multiparton distributions and distribution amplitudes hold for any Feynman diagram. Two other approaches to studying spectral properties of parton distributions are described in refs. [32,33].

Anticipating comparison with the nonforward distributions discussed below, it is worth emphasizing here that the Bjorken ζ-parameter is not present in Eq.(2.13) defining the parton distribution function $f(x)$. It appears only after one calculates the Compton amplitude $T(p,q)$.

C. Double distributions

Now, consider a one-loop box diagram for the scalar analog of the deeply virtual Compton scattering amplitude (Figs.1c,d). Using the same basic light-cone momenta p, q' as in the forward case, we write the momentum of the incoming virtual photon as $q \equiv q' - \zeta p$. The outgoing real photon carries the lightlike momentum q'. The momentum conservation requires that the final hadron has the momentum $(1-\zeta)p$, *i.e.*, in this kinematics we have a lightlike momentum transfer $r \equiv \zeta\, p$. Since the initial momenta q, p are identical to those of the forward amplitude, the parameter ζ coincides with the Bjorken variable $x_{Bj} \equiv Q^2/2(pq)$. In the α-representation, the contribution of the diagram 1c is

$$T_c(p,q,q') = -\frac{e^2g^2}{16\pi^2}\int_0^\infty \exp\left\{i\left[2(pq')\,\alpha_1\frac{\alpha_3 - \zeta(\alpha_3+\alpha_4)}{\alpha_1+\alpha_2+\alpha_3+\alpha_4} - \lambda\,(m^2 - i\epsilon)\right]\right\}\frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4}{\lambda^2}\,. \tag{2.14}$$

The large-Q^2 limit is again governed by the small-α_1 integration which gives

$$T_c(p,q,q') = -\frac{ie^2g^2}{16\pi^2}\int_0^\infty \frac{1}{2(pq')(\alpha_3/\tilde{\lambda} - \zeta(1-\alpha_2/\tilde{\lambda}) + i\epsilon)}\frac{e^{-i\tilde{\lambda}(m^2-i\epsilon)}}{\tilde{\lambda}^2}\, d\alpha_2 d\alpha_3 d\alpha_4 + O(1/Q^4)\,. \tag{2.15}$$

In the forward case, the ratio $\alpha_3/\tilde{\lambda}$ was substituted by the variable x which was interpreted then as the fraction of the initial hadron momentum carried by the active quark. The result expressed by Eq.(2.15) contains also another ratio $\alpha_2/\tilde{\lambda}$. So, let us introduce the *double distribution*

$$F(x,y) = \frac{ig^2}{16\pi^2}\int_0^\infty \delta\left(x - \frac{\alpha_3}{\alpha_2+\alpha_3+\alpha_4}\right)\delta\left(y - \frac{\alpha_2}{\alpha_2+\alpha_3+\alpha_4}\right)\frac{e^{-i\tilde{\lambda}(m^2-i\epsilon)}}{\tilde{\lambda}^2}\, d\alpha_2 d\alpha_3 d\alpha_4\,. \tag{2.16}$$

It is easy to see that both variables x, y vary between 0 and 1. Furthermore, their sum is also confined within these limits: $0 \le x+y \le 1$. Hence, $F(x,y) = \theta(x+y\le 1)\, F(x,y)$. Using Eq.(2.16), we write the leading-power contribution of $T_c(p,q,q')$ in terms of the double distribution:

$$\begin{aligned} T_c^{as}(p,q,q') &= -\int_0^1\int_0^1 \frac{e^2}{2(pq')(x+y\zeta-\zeta+i\epsilon)}\, F(x,y)\, dx\, dy \\ &= -\int_0^1\int_0^1 \frac{e^2}{(xp+yr+q)^2+i\epsilon}\, F(x,y)\, dx\, dy \equiv \int_0^1\int_0^1 t_c(xp+yr,q,q')\, F(x,y)\, \theta(x+y\le 1)\, dx\, dy\,. \end{aligned} \tag{2.17}$$

The parton subprocess amplitude t_c is given by

$$t_c(xp+yr,q,q') = -\frac{e^2}{(xp+yr+q)^2+i\epsilon}\,. \tag{2.18}$$

Hence, the momentum $xp+yr$ of the quark interacting with the virtual photon originates both from the initial hadron momentum p (term xp) and the momentum transfer (term yr). In a similar way, for the u-channel diagram 1d, we get

$$\begin{aligned} T_d^{as}(p,q',\zeta) &= \int_0^1\int_0^1 \frac{e^2}{2(pq')(x+y\zeta-i\epsilon)}\, F(x,y)\, dx\, dy \\ &= -\int_0^1\int_0^1 \frac{e^2}{(xp+yr-q')^2+i\epsilon}\, F(x,y)\, dx\, dy \equiv \int_0^1\int_0^1 t_d(xp+yr,q,q')\, F(x,y)\, \theta(x+y\le 1)\, dx\, dy\,, \end{aligned} \tag{2.19}$$

with the same double distribution $F(x,y)$ given by Eq. (2.16). In the explicit form,

$$F(x,y) = \frac{g^2}{16\pi^2m^2}\,\theta\,(0\le x+y\le 1)\,. \tag{2.20}$$

Again, $F(x,y)$ is purely real. Comparing the α-representations for $f(x)$ and $F(x,y)$, we obtain the reduction formula for the double distribution $F(x,y)$:

$$\int_0^{1-x} F(x,y)\, dy = f(x)\,. \tag{2.21}$$

Due to the restrictions $x \ge 0, y \ge 0$, the imaginary part of the total amplitude $T \equiv T_c + T_d$ is given by the s-channel contribution alone:

$$\begin{aligned} \frac{1}{\pi e^2}\,\mathrm{Im}\, T_c(\zeta,Q^2) &= \frac{1}{2(pq)}\int_0^1\int_0^1 \delta(x+y\zeta-\zeta)F(x,y)\,\theta(x+y\le 1)\, dx\, dy \\ &= \frac{1}{2\zeta(pq)}\int_0^\zeta F(x,1-x/\zeta)\, dx = \frac{1}{2(pq)}\int_0^1 F(\bar{y}\zeta,y)\, dy \equiv \frac{1}{2(pq)}\,\Phi(\zeta). \end{aligned} \tag{2.22}$$

The last form is similar to the expression for $\mathrm{Im}\, T$ in the forward case: one should just use the function $\Phi(\zeta)$ instead of $f(\zeta)$. Moreover, the integral defining $\Phi(\zeta)$ looks similar to that appearing in the reduction formula (2.21). Still, the two integrals are not identical and, in general, $\Phi(\zeta) \ne f(\zeta)$. Using the explicit form of $F(x,y)$ for our toy model, we obtain

$$\Phi(\zeta) = \frac{g^2}{16\pi^2m^2}\,\theta(0\le\zeta\le 1). \tag{2.23}$$

The factor $(1-\zeta)$ present in $f(\zeta)$ (see Eq.(2.8)), does not appear here. Note, however, that the difference is small for small ζ.

In the OPE language, the basic change compared to the forward case is that we should deal now with the asymmetric matrix element $\langle p-r\,|\,\phi(0)\phi(z)\,|\,p\rangle$. Our definition of the double distribution $F(x,y)$ corresponds to the following parametrization

$$\langle p-r\,|\,\phi(0)\phi(z)\,|\,p\rangle|_{z^2=0} = \int_0^1 \int_0^1 \frac{1}{2}\left(e^{-ix(pz)-iy(rz)} + e^{ix(pz)-i\bar{y}(rz)}\right) F(x,y)\,\theta(x+y\le 1)\,dx\,dy\,. \qquad (2.24)$$

Taking the limit $r=0$ in Eq.(2.24) gives the matrix element defining the usual parton distribution function $f(x)$, and we reobtain the reduction formula (2.21). Again, this definition of $F(x,y)$ can be treated as a Fourier representation for a function of two independent variables (pz) and (rz), with the spectral constraints $x\ge 0$, $y\ge 0$, $x+y\le 1$ dictated by the analytic structure of the relevant Feynman integrals. An important feature implied by the representation (2.24) is the absence of the ζ-dependence in the double distribution $F(x,y)$. The asymmetric matrix element (2.24), of course, has ζ-dependence. But it appears only through the ratio $(rz)/(pz)$ of variables in the exponential factor. In this treatment, ζ characterizes the "longitudinal momentum asymmetry" of the matrix elements. The fact that for the deeply virtual Compton amplitude T the parameter ζ coincides with the Bjorken variable $x_{Bj}=Q^2/2(pq)$ is a specific feature of a particular process. The matrix element itself accumulates a process-independent information and, hence, has quite a general nature.

Thus, despite the fact that the momenta p and r are proportional to each other $r=\zeta p$, there is a clear distinction between them, since p and r specify the momentum flow in two different channels. For $r=0$, the momentum flows only in the s-channel and the total momentum entering into the composite operator vertex is zero. In this case, the matrix element coincides with the usual distribution function. The partons entering the composite vertex then carry the fractions x_i $(i=1,2)$ of the initial proton momentum. In general, $-1<x_i<1$, but when x_i is negative, we should interpret the parton as going out of the composite vertex and returning to the final hadron. In other words, x_i can be redefined to secure that the integral always runs over the segment $0\le x\le 1$. In this parton picture, the spectators take the remaining momentum $(1-x)p$. On the other hand, if the total momentum flowing through the composite vertex is r, the matrix element has the structure of the distribution amplitude in which the momentum r splits into the fractions yr and $(1-y)r\equiv\bar{y}r$ carried by the quark fields attached to that vertex. In a combined situation, when both p and r are nonzero, the initial quark has momentum $xp+yr$, while the final one carries the momentum $xp-\bar{y}r$. Both the initial active quark and the spectator carry positive fractions of the lightlike momentum p: $x+\zeta y$ for the active quark and $\bar{x}-\zeta y=(1-x-y)+(1-\zeta)y$ for the spectator. However, the total fraction of the initial momentum p carried by the quark returning the fraction xp into the hadron matrix element is given by $x-\bar{y}\zeta$ and it may take both positive and negative values.

D. Asymmetric distribution functions

Since $(rz)=\zeta(pz)$, the variable y appears in eq.(2.24) only in the $x+y\zeta\equiv X$ combination, where X can be treated as the *total* fraction of the initial hadron momentum p carried by the active quark. Since $\zeta\le 1$ and $x+y\le 1$, the variable X satisfies a natural constraint $0\le X\le 1$. Integrating the double distribution $F(X-y\zeta,y)$ over y gives the *asymmetric distribution function*

$$\mathcal{F}_\zeta(X)=\theta(X\ge\zeta)\int_0^{\bar{X}/\bar{\zeta}} F(X-y\zeta,y)\,dy+\theta(X\le\zeta)\int_0^{X/\zeta} F(X-y\zeta,y)\,dy\,, \qquad (2.25)$$

where $\bar{\zeta}\equiv 1-\zeta$. The basic distinction between the double distribution $F(x,y)$ and the asymmetric distribution function $\mathcal{F}_\zeta(X)$ is that the former is a universal function in the sense that it does not depend on the momentum asymmetry parameter ζ while the latter is explicitly labelled by it. Hence, we deal now with a family of asymmetric distribution functions $\mathcal{F}_\zeta(X)$ whose shape changes when ζ is changed. In our toy model,

$$\mathcal{F}_\zeta(X)=\frac{g^2}{16\pi^2m^2}\left\{\frac{X}{\zeta}\,\theta(0\le X\le\zeta)+\frac{1-X}{1-\zeta}\,\theta(\zeta\le X\le 1)\right\}. \qquad (2.26)$$

One can see that when $\zeta\to 0$, the limiting curve for $\mathcal{F}_\zeta(X)$ reproduces the usual distribution function:

$$\mathcal{F}_{\zeta=0}(X)=f(X)\,. \qquad (2.27)$$

In general, this formula also follows directly from the definition of $\mathcal{F}_\zeta(X)$ and the reduction formula (2.21) for the double distribution $F(x,y)$.

The fraction $(X-\zeta)\equiv X'$ of the original hadron momentum p carried by the "returning" parton differs from X by ζ: $X-X'=\zeta$ [9]. Since X changes from 0 to 1 and $\zeta\ne 0,1$, the fraction X' can be either positive or negative, *i.e.*, the asymmetric distribution function has two components corresponding to the regions $1\ge X\ge\zeta$ and $0\le X\le\zeta$. In the region $X>\zeta$ (Fig.2*a*), where the initial parton momentum Xp is larger than the momentum transfer $r=\zeta p$, the function $\mathcal{F}_\zeta(X)$ can be treated as a generalization of the usual distribution function $f(x)$ for the asymmetric case when the final hadron momentum p' differs by ζp from the initial momentum p. In this case, $\mathcal{F}_\zeta(X)$ describes a parton going out of the hadron with a positive fraction Xp of the original hadron momentum and then coming back into the hadron with a changed (but still positive) fraction $(X-\zeta)p$. The parameter ζ specifies the momentum asymmetry of the matrix element.

In the region $X<\zeta$ (Fig.2*b*), the "returning" parton has a negative fraction $(X-\zeta)$ of the light-cone momentum p. Hence, it is more appropriate to treat it as a parton going out of the hadron and propagating along with the original parton. Writing X as $X=Y\zeta$, we see that both partons carry now positive fractions $Y\zeta p\equiv Yr$ and $\bar{Y}r\equiv(1-Y)r$ of the momentum transfer r. The asymmetric distribution function in the region $X=Y\zeta<\zeta$ looks like a distribution amplitude $\Psi_\zeta(Y)$ for a $\phi\phi$-state with the total momentum $r=\zeta p$:

$$\Psi_\zeta(Y)=\int_0^Y F((Y-y)\zeta,y)\,dy. \qquad (2.28)$$

In our model,

$$\Psi_\zeta(Y)=\frac{g^2}{16\pi^2m^2}\,Y\,\theta(0\le Y\le 1). \qquad (2.29)$$

Both $F(x,y)$ and $\mathcal{F}_\zeta(X)$ in our model are purely real. This result is determined, in fact, by general properties of the definition of the double distribution, Eq.(2.16). Indeed, let us introduce the Feynman parameters β_i by $\alpha_i=\tilde{\lambda}\beta_i$. After integrating over $\tilde{\lambda}$, the only possible source of imaginary contributions is the denominator factor $1/(m^2-i\epsilon)$. However, since $m^2>0$, this factor is always positive and the integral is purely real. This would not happen if the initial "hadron" has a sufficiently large mass $M>2m$. In this case, instead of $1/(m^2-i\epsilon)$ we would get $1/(-M^2\beta_3(1-\beta_3)+m^2-i\epsilon)$ if the final hadron has the same mass M. Then, the denominator is not positive-definite if $M>2m$, and the integral has both real and imaginary part. Clearly, the imaginary part appears because the initial hadron can decay into its constituents. If such a possibility is excluded, the double distributions $F(x,y)$ and, hence, the asymmetric distribution functions $\mathcal{F}_\zeta(X)$ are purely real.

The usual parton distributions $f(x)$ are often related to imaginary parts, or more precisely, s- and u-channel discontinuities of parton-hadron amplitudes**. Note, that in our approach, the parton distributions are defined by form-factor-type matrix elements which depend only on momentum invariants p^2, p'^2, r^2 irrelevant to such discontinuities (so far we even were setting them to zero). The variable X in our definition just reflects a more complicated structure of the operator vertex. To illustrate this point, we write $\mathcal{F}_\zeta(X)$ in the momentum representation (see Fig.3*a*)

$$\mathcal{F}_\zeta(X)=\frac{i}{(2\pi)^4}\int\frac{\delta(X-(kq')/(pq'))\,d^4k}{(k^2-m^2+i\epsilon)((p-k)^2-m^2+i\epsilon)((r-k)^2-m^2+i\epsilon)}\,. \qquad (2.30)$$

**In a recent paper, L. Frankfurt *et al.* [21] discuss also discontinuities in the context of the non-diagonal distribution functions.

The $\delta(X - (kq')/(pq'))$-function here corresponds to composite operator (denoted by a blob on Fig.3*a*). Using the α-representation, one can take the Gaussian k-integral and obtain a representation similar to Eq.(2.16), which finally gives our purely real result (2.26).

It is worth emphasizing that the parton representations (2.5), (2.17) and (2.33) below are valid for the total Compton amplitude: there is no need to split the latter into its real and imaginary parts in order to define the parton distribution. To make a parallel with the traditional approach in which the parton distributions are defined through the discontinuities of parton-hadron amplitudes, let us calculate the k-integral above using the Sudakov decomposition

$$k = \xi p + \eta q' + k_\perp \ , \ 2(pq') \equiv s \ , \tag{2.31}$$

which gives

$$\mathcal{F}_\zeta(X) = \frac{is}{2(2\pi)^4} \int d^2k_\perp \int_{-\infty}^{\infty} \frac{d\eta}{[X\eta s - k_\perp^2 - m^2 + i\epsilon][(X-1)\eta s - k_\perp^2 - m^2 + i\epsilon][(X-\zeta)\eta s - k_\perp^2 - m^2 + i\epsilon]} \, . \tag{2.32}$$

Looking at the location of singularities for the η-integral, we immediately see that a nonzero result is obtained only when $0 \le X \le 1$. Furthermore, in the region $\zeta \le X \le 1$, the integral over η is given by residue at $\eta = -(k_\perp^2 - i\epsilon)/(1-X)s$, which corresponds to substituting the ordinary propagator $-1/[(p-k)^2 + i\epsilon]$ by the $\delta((p-k)^2)$-function for the line with momentum $(p-k)$. In other words, for $\zeta \le X \le 1$, our one-loop model for the function $\mathcal{F}_\zeta(X)$ is totally given by the residue corresponding to the s-channel cut through the parton-hadron scattering amplitude (see Fig.3*b*). On the other hand, in the region $0 \le X \le \zeta$, the integral over η is given by residue at $\eta = (k_\perp^2 + m^2 - i\epsilon)/Xs$, which corresponds to cutting the line with momentum k (see Fig.3c). Such a cut cannot be related to s- or u-channel discontinuities ††. In both cases, one can say that $\mathcal{F}_\zeta(X)$ originates from a parton-hadron scattering amplitude $\mathcal{T} = i\mathcal{F}$ whose imaginary part is given by one or another type of discontinuities. In our treatment, the only important fact is that the amplitude $\mathcal{T}$ is purely imaginary so that the distribution function $\mathcal{F}_\zeta(X)$ is real. As we have seen above, the function $\mathcal{F}_\zeta(X)$ can be written in several different ways, *e.g.*, in the α-representation which can be integrated without taking any residues.

In terms of $\mathcal{F}_\zeta(X)$, the virtual Compton amplitude $T_{c+d}(p,q,q')$ can be written as

$$T_{c+d}^{as}(p,q,q') = -\frac{e^2}{2(pq)} \int_0^1 \left[\frac{1}{X - \zeta + i\epsilon} - \frac{1}{X - i\epsilon} \right] \mathcal{F}_\zeta(X)\, dX. \tag{2.33}$$

For a real function $\mathcal{F}_\zeta(X)$, the imaginary part of $T_{c+d}^{as}(p,q,q')$ is determined by that of the short-distance amplitude (terms in square brackets). Since $\mathcal{F}_\zeta(X)$ linearly vanishes as $X \to 0$, the singularity $1/(X - i\epsilon)$ of the u-channel diagram 1*d* gives a vanishing imaginary part. As a result, the imaginary part of the whole amplitude is generated by the $1/(X - \zeta + i\epsilon)$ singularity coming from the s-channel diagram 1*c*:

$$\frac{1}{\pi e^2} \operatorname{Im} T_c(\zeta, Q^2) = \frac{1}{2(pq)} \int_0^1 \delta(X - \zeta) \mathcal{F}_\zeta(X)\, dX = \frac{1}{2(pq)} \mathcal{F}_\zeta(\zeta). \tag{2.34}$$

Hence, the integral $\Phi(\zeta)$ in Eq.(2.22) is equal to $\mathcal{F}_\zeta(\zeta)$, *i.e.*, to the asymmetric distribution function $\mathcal{F}_\zeta(X)$ taken at the point $X = \zeta$. The parameter ζ appears in $\mathcal{F}_\zeta(\zeta)$ twice: first as the parameter specifying the longitudinal momentum asymmetry of the matrix element and then as the momentum fraction at which the imaginary part appears. As one may expect, it appears for $X = x_{Bj} = \zeta$, just like in the forward case. Note, however, that the momentum $(X - \zeta)p$ of the "returning" parton vanishes when $X = \zeta$. In other words, the imaginary part appears in a highly asymmetric configuration in which the fraction of the original hadron momentum carried by the second parton vanishes. Hence, $\mathcal{F}_\zeta(\zeta)$ in general differs from the function $f(\zeta)$. The latter corresponds to a symmetric

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††I am grateful to L. Frankfurt for attracting my attention to this point and correspondence.

configuration in which the final parton has momentum equal to that of the initial one. As discussed earlier, in our toy model $f(\zeta)/\mathcal{F}_\zeta(\zeta) = f(\zeta)/\Phi(\zeta) = 1 - \zeta$, *i.e.*, $\mathcal{F}_\zeta(\zeta)$ is larger than $f(\zeta)$, though the difference is small for small values of ζ.

The fact that $\mathcal{F}_\zeta(X)$ vanishes for $X = 0$ has a rather general nature. Note, that for small X the function $\mathcal{F}_\zeta(X)$ is given by its $X \le \zeta$ component

$$\mathcal{F}_\zeta(X)|_{X \le \zeta} = \int_0^{X/\zeta} F(X - y\zeta, y)\, dy \, . \tag{2.35}$$

The size of the integration region is proportional to X and, as a result, $\mathcal{F}_\zeta(X)$ vanishes like const$\times X$ or faster for any double distribution $F(x,y)$ which is finite for small x and y.

In the coordinate representation, the asymmetric distribution function can be defined through the matrix element

$$\langle p' \,|\, \phi(0)\phi(z) \,|\, p \rangle|_{z^2=0} = \int_0^1 \frac{1}{2} \left(e^{-iX(pz)} + e^{i(X-\zeta)(pz)} \right) \mathcal{F}_\zeta(X)\, dX, \tag{2.36}$$

with $\zeta = 1 - (p'z)/(pz)$. To re-obtain the relation between $\mathcal{F}_\zeta(X)$ and the double distribution function $F(x,y)$, one should combine this definition with Eq.(2.24). The $\zeta \to 0$ reduction formula (2.27) trivially follows from Eq.(2.36).

Using translation invariance, we can write representation for a more general light-cone operator:

$$\langle p' \,|\, \phi(uz)\phi(vz) \,|\, p \rangle|_{z^2=0} = \int_0^1 \frac{1}{2} \left(e^{-iXv(pz) + i(X-\zeta)u(pz)} + e^{-iXu(pz) + i(X-\zeta)v(pz)} \right) \mathcal{F}_\zeta(X)\, dX. \tag{2.37}$$

This formula explicitly shows that if the parton corresponding to $\phi(vz)$ has momentum Xp, then the momentum of the parton related to $\phi(uz)$ is $(X - \zeta)p$ and *vice versa*.

E. Nonforward distributions

Writing the momentum of the virtual photon as $q = q' - \zeta p$ is equivalent to using the Sudakov decomposition in the light-cone "plus" (p) and "minus" (q') components in a situation when there is no transverse momentum. An essential advantage of expressing the amplitudes in the α-representation is that it explicitly shows the dependence of the diagram on the relevant momentum invariants. This means that we can derive the parton picture both for zero and non-zero invariant momentum transfers $t = (p' - p)^2$ without bothering about an optimal choice of the basic vectors for the external momenta. Maintaining for simplicity $p^2 = p'^2 = 0$, we get for the diagram 1*c*

$$T_c(p,q,q') = -\frac{e^2 g^2}{16\pi^2} \int_0^\infty \exp\left\{ i \left[\frac{\alpha_1(2(pq)\alpha_3 - Q^2(\alpha_3 + \alpha_4)) + t\alpha_2\alpha_4}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} - \lambda\,(m^2 - i\epsilon) \right] \right\} \frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4}{\lambda^2} \, . \tag{2.38}$$

The small-α_1 integration then gives

$$T_c^{as}(p,q,q') = -\frac{ie^2 g^2}{16\pi^2} \int_0^\infty \frac{e^{it\alpha_2\alpha_4/\bar\lambda - i\bar\lambda(m^2 - i\epsilon)}}{2(pq)(\alpha_3/\bar\lambda - \zeta(1 - \alpha_2/\bar\lambda) + i\epsilon)} \frac{d\alpha_2 d\alpha_3 d\alpha_4}{\bar\lambda^2} + O(1/Q^4) \, , \tag{2.39}$$

where $\zeta = Q^2/2(pq) \equiv x_{Bj}$. Hence, introducing the t-dependent double distribution

$$F(x,y;t) = \frac{ig^2}{16\pi^2} \int_0^\infty \delta\left(x - \alpha_3/\bar\lambda\right) \delta\left(y - \alpha_2/\bar\lambda\right) e^{it\alpha_2\alpha_4/\bar\lambda - i\bar\lambda(m^2 - i\epsilon)} \frac{d\alpha_2 d\alpha_3 d\alpha_4}{\bar\lambda^2} \tag{2.40}$$

we obtain the same parton formula, but with a modified parton distribution $F(x,y;t)$

$$T_c^{as}(p,q,q') = -\int_0^1 \int_0^1 \frac{e^2}{2(pq)(x + y\zeta - \zeta + i\epsilon)} F(x,y;t)\, dx\, dy \, . \tag{2.41}$$

Moreover, the dependence on t appears *only* through the t-dependence of $F(x,y;t)$. Similarly, we can write down the α-representation for the u-channel diagram 1*d*:

$$T_d(p,q,q') = -\frac{e^2 g^2}{16\pi^2} \int_0^\infty \exp\left\{ i\left[\frac{\alpha_1(-2(pq')\alpha_3 - Q^2(\alpha_2+\alpha_4)) + t\alpha_2\alpha_4}{\alpha_1+\alpha_2+\alpha_3+\alpha_4} - \lambda\,(m^2 - i\epsilon)\right]\right\} \frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4}{\lambda^2}. \quad (2.42)$$

Using $2(pq') = 2(pq) + t$ and integrating over small α_1 gives the parton formula

$$T_d^{as}(p,q,q') = \int_0^1 \int_0^1 \frac{e^2}{2(pq)(x + y\zeta + xt/2(pq) - i\epsilon)}\, F(x,y;t)\, dx\, dy \quad (2.43)$$

with the same t-dependent function $F(x,y;t)$. In our model,

$$F(x,y;t) = \frac{g^2}{16\pi^2}\, \frac{\theta\,(0 \le x + y \le 1)}{m^2 - ty(1-x-y)}. \quad (2.44)$$

The parton subprocess amplitude in this case has the $O(t/2(pq)) = O(\zeta t/Q^2)$ correction term which can be neglected in the large-Q^2, fixed-t limit. Then the parton amplitude again depends only on the combination $x + y\zeta$, and it makes sense to introduce the *nonforward distribution*

$$\mathcal{F}_\zeta(X;t) = \int_0^{\min\{X/\zeta,\, \bar{X}/\bar{\zeta}\}} F(X - y\zeta, y; t)\, dy\,, \quad (2.45)$$

which can be treated as the finite-t generalization of the asymmetric distribution function $\mathcal{F}_\zeta(X)$ (or more precisely, $\mathcal{F}_\zeta(X)$ is the $t=0$ idealization of $\mathcal{F}_\zeta(X;t)$). In our simple model, it can be calculated analytically:

$$\mathcal{F}_\zeta(X;t) = \frac{g^2}{16\pi^2}\left\{ \frac{4\tau\,\theta\,(X \ge \zeta)}{-t\bar{X}\sqrt{1+\tau^2}} \ln\left(\tau + \sqrt{1+\tau^2}\right) + \theta\,(X \le \zeta) \int_0^{X/\zeta} \frac{dy}{m^2 - ty\,(\bar{X} - y\bar{\zeta})} \right\} \quad (2.46)$$

where $\tau = \sqrt{(-t/4m^2)(1-X)/(1-\zeta)}$. The function $\mathcal{F}_\zeta(X;t)$ falls off with increasing $|t|$ like a form factor.

The t-dependent distributions $F(x,y;t)$ and $\mathcal{F}_\zeta(X;t)$ in our model are purely real. Indeed, introducing again the Feynman parameters β_i by $\alpha_i = \tilde{\lambda}\beta_i$ and integrating over $\tilde{\lambda}$ gives the denominator factor $1/(-t\beta_2\beta_4 + m^2 - i\epsilon)$. However, since $t \le 0$, this factor is always positive and the integral is purely real. Imaginary part for $F(x,y;t)$ and $\mathcal{F}_\zeta(X;t)$ would appear only if the initial hadron mass satisfies $M^2 > 4m^2$.

For real distributions, the imaginary part of the total Compton amplitude can be calculated by taking the imaginary part of the short-distance amplitude which picks out the function $\mathcal{F}_\zeta(\zeta;t)$

$$\mathcal{F}_\zeta(\zeta;t) = \frac{g^2}{16\pi^2 m^2 T\sqrt{1+T^2}} \ln\left(T + \sqrt{1+T^2}\right), \quad (2.47)$$

where $T = \sqrt{(-t/4m^2)(1-\zeta)}$.

In the OPE approach, the nonforward distribution is given by the matrix element

$$\langle p' \,|\, \phi(0)\phi(z) \,|\, p\rangle|_{z^2=0} = \int_0^1 \frac{1}{2}\left(e^{-iX(pz)} + e^{i(X-\zeta)(pz)}\right) \mathcal{F}_\zeta(X;t)\, dX. \quad (2.48)$$

Taking the local limit $z = 0$, we obtain the following sum rule for $\mathcal{F}_\zeta(X;t)$

$$\int_0^1 \mathcal{F}_\zeta(X;t)\, dX = \langle p' \,|\, \phi(0)\phi(0) \,|\, p\rangle = F(t)\,, \quad (2.49)$$

where $F(t)$ is the toy model analog of a hadronic form factor.

III. ALL-ORDER ANALYSIS

A. Handbag diagram to all orders

Using the α-representation, one can write down the contribution of any diagram in terms of functions of the α-parameters specified by the structure of the diagram. Since the object of our interest is the matrix element of

a two-body operator, we can extract it from the simplest handbag diagrams, *i.e.*, those in which the q vertex is connected to the q' vertex by a single propagator. The contribution of any diagram of this type can be written as (see, *e.g.*, [27,30])

$$T^{(i)}(p,q,q') = i^l \frac{P(\text{c.c.})}{(4\pi i)^{zd/2}} \int_0^\infty \prod_{\sigma=1} d\alpha_\sigma D^{-d/2}(\alpha)$$
$$\exp\left\{ iq^2 \frac{\alpha_1 A_L(\alpha)}{D(\alpha)} + is\frac{\alpha_1 A_s(\alpha)}{D(\alpha)} + iu\frac{\alpha_1 A_u(\alpha)}{D(\alpha)} + it\frac{A_t(\alpha)}{D(\alpha)} - i\sum_\sigma \alpha_\sigma(m_\sigma^2 - i\epsilon)\right\}, \quad (3.1)$$

where $s = (p+q)^2, u = (p-q')^2$ and $t = (p-p')^2$ are the Mandelstam variables, d is the space-time dimension, P(c.c.) is the relevant product of the coupling constants, z is the number of loops of the diagram and l is the number of its internal lines. Finally, D, A_s, A_u, A_t, A_L are functions of the α-parameters uniquely determined for each diagram.

To describe them, we need definitions of a tree and a 2-tree of a graph. A tree (2-tree) of a graph G is a subgraph of G which consists of one (two) connected components each of which has no loops. Any tree G_1^k (2-tree G_2^l) of G is determined by the set of lines σ which should be removed from the initial graph G to produce G_1^k (G_2^l). The product of the α_σ-parameters associated with these lines will be referred to as α-tree (α-2-tree). The function $D(\alpha)$ is called the determinant of the graph. It is given by the sum of all α-trees of the graph G. By $B(i_1,\ldots,i_m|j_1,\ldots,j_n)$ we denote the sum of all α-2-trees possessing the property that the vertices $i_1,\ldots,i_m$ belong to one component, $j_1,\ldots,j_n$ to the other, while the vertices not indicated explicitly may belong to either component. In these notations,

$$\alpha_1 A_L(\alpha) = B(q|p,q',p'); \quad \alpha_1 A_s(\alpha) = B(q,p|q',p'); \quad \alpha_1 A_u(\alpha) = B(q,p'|p,q'); \quad A_t(\alpha) = B(q,q'|p,p'). \quad (3.2)$$

The mnemonics is straightforward: the square of the total momentum entering into one of the components (due to momentum conservation, it does not matter which one) just gives the relevant momentum invariant (see Fig.4). To get all the 2-trees corresponding to this invariant, one should make all possible cuts resulting in such a separation of external momenta. Note, that α_1 must be present in all terms of $B(q|p,q',p')$, $B(q,p|q',p')$ and $B(q,p'|p,q')$ because the vertices q, q' in these cases belong to different components. On the other hand, for $B(q,q'|p,p')$ these vertices are in the same component. As a result, there are terms in $A_t(\alpha)$ which do not contain α_1 as a factor, *i.e.*, $A_t(\alpha) = \alpha_1 A_t^{(1)}(\alpha) + A_t^{(0)}(\alpha)$ with $A_t^0(\alpha) \ne 0$ and $A_t^{(1)}(\alpha) \ne 0$ for $\alpha_1 = 0$. Similarly, the function $D(\alpha)$ can be written as $D(\alpha) = \alpha_1 D_1(\alpha) + D_0(\alpha)$, where $D_1(\alpha)$ is the determinant for the graph G_1 obtained from G by deleting the line σ_1, while $D_0(\alpha)$ is that for the graph G_0 resulting from G by contracting the line σ_1 into a point (and gluing the vertices q, q' into a single point). One can see that the function $D_0(\alpha)$ can also be written in terms of the same α-2-trees:

$$D_0(\alpha) = \{B(q|p,q',p') + B(q,p|q',p') + B(q,p'|p,q') + B(q,p,p'|q')\}/\alpha_1$$
$$= A_L(\alpha) + A_s(\alpha) + A_u(\alpha) + A_R(\alpha)\,, \quad (3.3)$$

where $A_R(\alpha)$ is the function corresponding to the cut separating out the momentum invariant q'^2. To get the leading large-Q^2 asymptotics, we integrate over the region $\alpha_1 \sim 0$. This gives

$$T^{as}(p,q,q') = \sum_{\text{diagr.}} i^{l+1} \frac{P(\text{c.c.})}{(4\pi i)^{zd/2}} \int_0^\infty \prod_{\sigma=2} d\alpha_\sigma D_0^{-d/2}(\alpha) \left[q^2 \frac{A_L(\alpha)}{D_0(\alpha)} + s\frac{A_s(\alpha)}{D_0(\alpha)} + u\frac{A_u(\alpha)}{D_0(\alpha)} + t\frac{A_t^{(1)}(\alpha)}{D_0(\alpha)} - m^2 + i\epsilon \right]^{-1}$$
$$\times \exp\left\{ it\frac{A_t^{(0)}(\alpha)}{D_0(\alpha)} - i\sum_{\sigma=2} \alpha_\sigma(m_\sigma^2 - i\epsilon)\right\}. \quad (3.4)$$

Using $s = -Q^2 + 2(pq), u = -2(pq) + t$ and neglecting t and m^2 compared to $O(Q^2)$ terms in the denominator factor, we transform it into

$$-Q^2 \frac{A_L(\alpha) + A_s(\alpha)}{D_0(\alpha)} + 2(pq)\frac{A_s(\alpha) - A_u(\alpha)}{D_0(\alpha)} + i\epsilon\,.$$

This expression has the structure similar to that of the one-loop contributions (2.39),(2.42). In particular, it can be converted into the form of the s-channel term (2.39) if we denote $[A_s(\alpha) - A_u(\alpha)]/D_0(\alpha)$ by x and $[A_L(\alpha) + A_s(\alpha)]/D_0(\alpha)$ by $1-y$. Analogously, to make it look like the u-channel term (2.42), we should take $[A_s(\alpha) - A_u(\alpha)]/D_0(\alpha) = -x$ and $[A_L(\alpha) + A_s(\alpha)]/D_0(\alpha) = y$. If we want to have *positive* x, we should perform the first identification in the region where $A_s(\alpha) > A_u(\alpha)$ and use the second one in the region where $A_s(\alpha) < A_u(\alpha)$. In other words, we define the t-dependent double distribution by

$$F(x,y;t) = \sum_{\text{diagr.}} i^{l-1} \frac{P(\text{c.c.})}{(4\pi i)^{ld/2}} \int_0^\infty \prod_{\sigma=2} d\alpha_\sigma D_0^{-d/2}(\alpha) \exp\left\{ it \frac{A_t^{(0)}(\alpha)}{D_0(\alpha)} - i\sum_{\sigma=2} \alpha_\sigma (m_\sigma^2 - i\epsilon) \right\}$$
$$\left[\delta\left(1 - y - \frac{A_L(\alpha) + A_s(\alpha)}{D_0(\alpha)}\right) \delta\left(x - \frac{A_s(\alpha) - A_u(\alpha)}{D_0(\alpha)}\right) \theta(A_s(\alpha) > A_u(\alpha)) \right.$$
$$\left. + \delta\left(y - \frac{A_L(\alpha) + A_s(\alpha)}{D_0(\alpha)}\right) \delta\left(x - \frac{A_u(\alpha) - A_s(\alpha)}{D_0(\alpha)}\right) \theta(A_s(\alpha) < A_u(\alpha)) \right] . \tag{3.5}$$

An intuitive interpretation is that when $A_s(\alpha) > A_u(\alpha)$, the quark *takes* the momentum xp from the initial hadron. Its total momentum is $xp + yr$. Alternatively, when $A_s(\alpha) < A_u(\alpha)$, the quark *returns* the momentum xp to the final state, and its total returning momentum is $xp - (1-y)r$. Due to Eq.(3.3), we automatically have $0 \le x \le 1$, $0 \le y \le 1$. Furthermore, since $x + y = [A_L(\alpha) + A_u(\alpha)]/D_0(\alpha) \le 1$ in the first region and $x + y = [A_R(\alpha) + A_s(\alpha)]/D_0(\alpha) \le 1$ in the second one, we always have the restriction $x + y \le 1$.

Again, introducing the Feynman parameters $\beta_i = \alpha_i/\lambda$ and the common scale λ given by the sum of all α_i-parameters, we can integrate over λ to see that the resulting denominator factor $1/(-tA_t^{(0)}(\alpha)/D_0(\alpha) + m^2)$ is positive for $t \le 0$, and the double distribution is purely real.

The same definition of the parameters x, y based on the α-representation can be used in the realistic case of spin-$\frac{1}{2}$ quarks. However, one should take into account that the quark lines in that case are oriented. Depending on their direction, we should interpret the parton with momentum $xp + yr$ either as a quark or as an antiquark.

The nonforward distributions $\mathcal{F}_\zeta(X;t)$ can be obtained from the double distributions using Eq.(2.45). The restrictions $x, y \ge 0$, $x + y \le 1$ guarantee that the total fraction X satisfies the basic constraint $0 \le X \le 1$. Furthermore, if the double distribution $F(x,y;t)$ is finite for all relevant x, y, the nonforward distribution $\mathcal{F}_\zeta(X;t)$ vanishes (at least linearly) as $X \to 0$.

B. Alpha-representation and factorization

Using the α-representation, we can write each perturbative diagram contributing to the virtual Compton scattering amplitude $T(p,q,q')$ in any field theory model, including QCD (see Fig.5)

$$T^{(i)}(p,q,q') = i^l \frac{P(\text{c.c.})}{(4\pi i)^{ld/2}} \int_0^\infty \prod_\sigma d\alpha_\sigma D^{-d/2}(\alpha)\, G(\alpha,p,q,q';m_\sigma)$$
$$\exp\left\{ -iQ^2 \frac{B_L(\alpha) + B_s(\alpha)}{D(\alpha)} + 2i(pq)\frac{B_s(\alpha) - B_u(\alpha)}{D(\alpha)} \right\}$$
$$\exp\left\{ it \frac{B_t(\alpha) + B_u(\alpha)}{D(\alpha)} + iM^2 \frac{B_1(\alpha) + B_2(\alpha)}{D(\alpha)} - i\sum_\sigma \alpha_\sigma (m_\sigma^2 - i\epsilon) \right\} . \tag{3.6}$$

The only difference is the presence of the preexponential factor $G(\alpha,p,q,q';m_\sigma)$ due to the numerator structure of the QCD propagators and vertices. It has a polynomial dependence on the momentum invariants. The functions $B(\alpha)$ are defined by the relevant 2-trees, *e.g.*, $B_L(\alpha) = B(q|p,q',p')$, *etc.*

In the region where Q^2 and $2(pq) = \zeta Q^2$ are large, all the contributions having a power-like behavior on Q^2 can only come from the integration region inside which all the ratios $A_L/D, A_s/D, A_u/D$ vanish: if any of them is larger than some constant ρ, the integrand rapidly oscillates and the resulting contribution from such an integration region is exponentially suppressed.

Since A_L, A_s, A_u and D are given by sums of products of non-negative α-parameters, there are two basic possibilities to arrange $A_i/D = 0$. In the first case, called the "short-distance regime", A_i vanishes faster than D when some of the α-parameters tend to zero (small α correspond to large virtualities k^2, *i.e.*, to "short" distances). The second possibility, called the "infrared regime", occurs if D goes to infinity faster than A_i when some of the α-parameters tend to infinity (large α correspond to small momenta k, *i.e.*, to the infrared limit). One can also imagine a combined regime, when $A_i/D = 0$ because some α-parameters vanish and some are infinite.

There exists a simple rule using which one can easily find the lines σ whose α-parameters must be set to zero and those whose α-parameters must be taken infinite to assure that $A_i/D = 0$. First, one should realize that $A_i/D = 0$ means that the corresponding diagram of a scalar theory (in which $G = 1$) has no dependence on the relevant momentum invariant (Q^2, s or u in our case). As the second step, one should incorporate the well-known analogy between the Feynman diagrams and electric circuits [34]: the α_σ-parameters may be interpreted as the resistances of the corresponding lines σ. In other words, $\alpha_\sigma = 0$ corresponds to short-circuiting the line σ while $\alpha_\sigma = \infty$ corresponds to its removal from the diagram. Hence, the problem is to find the sets of lines $\{\sigma\}_{SD}$, $\{\sigma\}_{IR}$ whose contraction into point (for $\{\sigma\}_{SD}$) or removal from the diagram (for $\{\sigma\}_{IR}$) produces the diagram which in a scalar theory does not depend on p_i^2. Thus, the rule determining possible types of the powerlike contributions is the following: if the part of the diagram corresponding to a short-distance subprocess is contracted into a point and the part corresponding to soft exchange is removed from the diagram, the resulting diagram ("reduced diagram", cf. [35,16]) should have no dependence on large momentum invariants.

Some examples are shown in Fig.6. The simplest possibility is to contract into point some subraph H containing the photon vertices q, q' (Fig.6a). The reduced diagram depends only on small invariants t, M^2 and masses m. The long-distance part corresponds to a nonforward distribution. This is the standard OPE configuration. However, since q'^2 is not a large momentum invariant: $q'^2 = 0$, there is a less trivial possibility shown in Fig.6b. In this case, there are two long-distance parts: one is given by a nonforward distribution again and the other can be interpreted as the distribution amplitude (hadronic component) of a real photon. Exchange of soft quanta between the two long-distance parts of Fig.6b corresponds to a combined SD-IR regime (Fig.6c): the α-parameters of lines inside H vanish while those belonging to the soft subgraph S tend to infinity.

One can easily invent other, more complicated configurations. Fortunately, not all of them are equally important: different configurations have different Q^2-behavior. The power counting is based on the observation that in the essential region of integration $\alpha_\sigma \sim 1/Q^2$ for lines in the short-distance subgraph H and $\alpha_\sigma \sim Q^2/p^4$ for lines in the soft subgraph S (p^2 is some generic small scale, say, M^2 or m^2). In the momentum representation, this corresponds to $k \sim Q$ for the H-lines and $k \sim p^2/Q$ for the S-lines. As a result, in a theory with dimensionless coupling constants, we can use the dimensional analysis to derive that the contribution due to H behaves like Q^{4-d_H}, where d_H is the sum of dimensions (in mass units) of the fields associated with the external lines of H. We should also take into account extra numerator factors brought by these external lines. For instance, each external quark line adds a Dirac spinor $u(p)$, two of them give $u(p)\bar{u}(p) \sim \hat{p}$, and $\hat{p}$ can combine with $\hat{q}$ from H to give $(pq) \sim Q^2$. This means that each external quark line can bring an extra $Q^{1/2}$ factor. Note, that $1/2$ is the spin of the quark. Similarly, an external gluon line can add a p^μ factor. Combined with q_μ from H it gives $(pq) \sim Q^2$, i.e., the gluon line can bring an extra $Q = Q^1$ factor for the whole amplitude. Again, "1" is the spin of the gluon. Hence, each external quark or gluon line can give the factor $Q^{s_i - d_i} = Q^{-t_i}$ where $t_i = d_i - s_i$ is its twist. Note also that calculating the virtual Compton ampitude we do not convolute the vector indices μ, ν of the initial and final photon lines with momentum-dependent vectors. Hence, each external photon line gives only the factor Q^{-1} due to its dimension. Thus, the counting rule for the contribution of the hard subgraph H is

$$t_H(Q) \lesssim Q^{4-N-\Sigma_i t_i} , \tag{3.7}$$

where N is the number of external photon lines of the hard subgraph and summation is over quark and gluon external lines of H. For the simplest hard subgraph with two external quark lines this gives $t_H(Q) \lesssim Q^0$, a scaling behavior as expected. For the configuration 6b, the estimate is $t_H(Q) \lesssim Q^{-1}$. Hence, the contribution of Fig.6b is power-suppressed compared to that of Fig.6a. Note that since the gluons have zero twist, the hard subgraph can have an arbitrary number of extra gluon lines without changing its power behavior. A similar power counting estimate [38] based on $k \sim p^2/Q$ can be obtained for the soft subgraph S:

$$t_S(Q) \lesssim Q^{-\Sigma_j t_j}, \tag{3.8}$$

where the summation is over the external lines of S. Hence, exchanging a soft quark (Fig.6d) produces the $1/Q^2$ suppression (S has then two external quark lines each having $t = 1$), while the exchange of any number of soft gluons is not necessarily accompanied by a suppression factor, at least on diagram by diagram level (for more details, see discussion in the next subsection). For the combined SD-IR configuration, the power counting estimate is

$$t_{HS}(Q) \lesssim Q^{4-N-\Sigma_H t_i} Q^{-\Sigma_S t_j}. \tag{3.9}$$

It is convenient to describe the power-low behavior of $T(Q^2)$ in terms of the Mellin transformation

$$T(Q^2) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(\frac{Q^2}{M^2}\right)^J \Phi(J)\, dJ. \tag{3.10}$$

Then the statement that $T(Q^2) \sim (1/Q^2)^n$ is equivalent to saying that the Mellin transform $\Phi(J)$ has a pole at $J = -n$. Take as an example the Mellin transform of the scalar diagram shown in Fig.7a (it is essentially identical to the diagram 1c):

$$\Phi_c(J) = -\frac{e^2 g^2}{16\pi^2} \Gamma(-J) \int_0^\infty \left[i\alpha_1 \frac{(\alpha_3+\alpha_4) - \alpha_3/\zeta}{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \right]^J \exp\left\{ i\,t\,\alpha_2\alpha_4/\lambda - i\lambda\,(m^2 - i\epsilon) \right\} \frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4}{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{d/2}}. \tag{3.11}$$

Small-α_1 integration corresponds to the simplest SD-regime 6a and generates the pole $1/(J+1)$ corresponding to the $1/Q^2$ asymptotic behavior. The relevant reduced graph is shown in Fig.7b.

Another possibility to kill the dependence on large variables is to take $\alpha_3 = \alpha_4 = 0$ which corresponds to the reduced graph shown in Fig.7c. To describe a simultaneous vanishing of two α-parameters, we use the common scale $\rho = \alpha_3 + \alpha_4$ and the Feynman parameters $\gamma_i = \alpha_i/\rho$. The resulting ρ-integral $\rho^J \rho d\rho$ gives the pole $1/(J+2)$ corresponding to a non-leading behavior $1/Q^4$.

Furthermore, contracting the whole diagram into point (*i.e.* taking $\alpha_i = 0$ for all α-parameters) we also obtain a reduced graph which does not depend on large variables. In this case, we introduce the common scale $\lambda = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and the relative parameters $\beta_i = \alpha_i/\lambda$. In $d = 4$ dimensions, the integrand behaves like $\lambda^J \lambda^3 d\lambda/\lambda^2$ which produces the pole $1/(J+2)$ generating a non-leading behavior $1/Q^4$. However, if we take a scalar model in $d = 6$ space-time dimensions, then the integrand behaves like $\lambda^J \lambda^3 d\lambda/\lambda^3$ and small-λ integration generates the leading pole $1/(J+1)$. Note that in this case after the λ-integration we still have the factor β_1^J capable of producing another $1/(J+1)$ pole due to small-β_1 integration. Hence, the total singularity of this diagram in 6 dimensions is $1/(J+1)^2$, which gives $T(Q^2) \sim (\ln Q^2)/Q^2$. This corresponds, of course, to the scaling violation *i.e.*, to evolution of the nonforward distribution. One can even extract the relevant evolution kernel from the remaining integral over $\beta_2, \beta_3, \beta_4 = 1 - \beta_2 - \beta_3$ (the result, in fact, can be read off Eq.(2.26)). Another observation is that if we simply integrate over small-α_1 region, the remaining integral $d\alpha_2 d\alpha_3 d\alpha_4/\tilde\lambda^3$ logarithmically diverges in the region of small $\tilde\lambda \equiv \alpha_2 + \alpha_3 + \alpha_4$. This is the standard UV divergence of a matrix element of a light-cone operator in a theory with dimensionless coupling constants.

Taking $\alpha_2 \to \infty$, we incorporate the IR regime corresponding to the reduced graph 7d. If the quark corresponding to the σ_2 line is massless, the α_2 integral in this limit is $\alpha_2^{-J} d\alpha_2/\alpha_2^2$. It produces the $1/(J+1)$ pole corresponding to the leading $1/Q^2$ behavior. In the previous section, we did not see this contribution because the quark masses were assumed to be nonzero for all the lines. For nonzero mass, the factor $\exp[-i\alpha_2 m^2]$ suppresses the large-α_2 integration and no poles in the J-plane are produced. In other words, the IR regime should be taken into account only for massless (or nearly massless) fields. Note, that in QCD the IR regime for the virtual Compton amplitude also gives $1/Q^2$ behavior for massless quarks (see Eq.(3.8)). However, in QCD this is a non-leading contribution compared to the scaling behavior produced by the purely SD regime 7b.

C. QCD and gauge invariance

After the SD-dominance is established, the next step is to write the contribution of the SD configuration in the coordinate representation (Fig.8a)

$$T(p,q,q') = \int e^{-i(qz)} d^4z \int \langle p' \,|\, \phi(z_2) C(z, z_1, z_2) \phi(z_1) \,|\, p\rangle\, d^4z_1\, d^4z_2 \tag{3.12}$$

(where ϕ is a generic notation for the quark fields $\psi, \bar\psi$ and the gluon field A) and expand the bilocal matrix element $\langle p' \,|\, \phi(z_2) \ldots \phi(z_1) \,|\, p\rangle$ in powers of $(z_2 - z_1)^2$. Since we already know from the α-representation analysis that the virtualities inside the SD-subgraph are $O(Q^2)$, extra powers of $(z_2 - z_1)^2$ for simply dimensional reasons result in extra powers of $1/Q^2$, and the leading large-Q^2 behavior will be given by the lowest term of this expansion corresponding to the lowest-twist composite operator. Parametrizing the nonforward matrix elements of the light-cone operators by formulas analogous to Eq.(2.37) gives the parton formulas similar to Eq.(2.33). Of course, this is just a general idea how to obtain the QCD parton picture for the SD-dominated amplitudes. Its practical implementation depends on specific properties of a particular process under consideration.

The most important complication in QCD is due to the gauge nature of the gluonic field. In Feynman gauge, the gluon vector potential A_μ has zero twist, and we should perform an infinite summation over the external gluonic lines both for the SD-subgraphs H and infrared subgraphs S. Consider the sum of gluon insertions into the quark propagator. It is well-known (see, *e.g.*, [36–39]) that after summation

$$S^c(\xi - \eta) + \int S^c(\xi - z)\gamma^\mu g A_\mu(z) S^c(z - \eta)\, d^4z + \ldots = E(\xi,\eta;A)\, S^c(\xi - \eta)\left[1 + O(G)\right] \tag{3.13}$$

all the A-fields can be accumulated in the path-ordered exponential

$$E(\xi,\eta;A) \equiv P\exp\left(ig\int_\eta^\xi A_\mu(z) dz^\mu \right) \tag{3.14}$$

while $O(G)$ term depends on the gluonic fields only through the tensor $G_{\mu\nu}$ and its covariant derivatives. Since $G_{\mu\nu}$ is asymmetric with respect to the interchange of the indices μ, ν, it should be treated as a twist-1 field. For the simplest SD configuration possessing a single long-distance part, combining the E-factors of all internal lines of the SD-subgraph, one gets gauge-invariant operators, *e.g.*, $\bar q(z_1)\gamma_\nu E(z_1, z_2; A) q(z_2)$.

If the lowest-order SD-configuration contains two long-distance parts (like in Fig.6b), the gluonic corrections include insertions into the external lines of the SD subgraph 8b. The resulting path-ordered exponentials $E_n(\xi,\infty;A)$ then go to infinity along the relevant light-cone directions, *e.g.*, q' or p in case of hard electroproduction processes. However, for color-singlet channels there are at least two such exponentials and their long-distance tails cancel each other so that only the factors $E(\xi,\eta;A)$ related to SD-subgraph vertices ξ,η remain. The basic effect of the exponential factor $E(\xi,\eta;A)$ is that expanding operators $\mathcal{O}(\xi,\eta)$ into the Taylor series, *e.g.*,

$$\bar q(\xi)\gamma_\nu E(\xi,\eta;A) q(\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} \Delta^{\nu_1}\Delta^{\nu_2}\ldots\Delta^{\nu_n} \bar q(\xi)\gamma_\nu D_{\nu_1} D_{\nu_2}\ldots D_{\nu_n} q(\xi) \quad ; \quad \Delta = \eta - \xi \tag{3.15}$$

one gets local operators $\bar q\gamma_\nu D_{\nu_1} D_{\nu_2}\ldots D_{\nu_n} q$ containing covariant derivatives $D^\nu = \partial^\nu - igA^\nu$ rather than ordinary ones.

The cancellation of $E_n(\xi,\infty;A)$ factors is very important for the success of the standard factorization program. Only after such a cancellation, the long-distance information is accumulated in universal matrix elements of gauge invariant light-cone operators. To illustrate the difference between a color-singlet and a non-color singlet channel, consider matrix element $J(p,q') = \langle 0|E_{q'}(0,\infty;A)\psi(0)|p\rangle$ of the quark field $\psi(0)$ coming out of a state with momentum p and taken together with the accompanying gluonic field A which is then absorbed by a q' channel quark collecting the gluonic A-fields into the $E_{q'}(0,\infty;A)$ factor (see Fig.8c). Note that the latter can be written as

$$E_{q'}(0,\infty) = P\exp\left(\int_0^\infty \mathcal{A}(t)\,dt\right) = 1 + \int_0^\infty \mathcal{A}(t)\,dt + \int_0^\infty \mathcal{A}(t)\,dt \int_0^t \mathcal{A}(t_1)\,dt_1 + \ldots$$
$$= 1 + \int_0^\infty \mathcal{A}(t)\,E_{q'}(0,t)\,dt \tag{3.16}$$

where $\mathcal{A}(t) = igq'_\mu A^\mu(tq')$. Substituting this result into the matrix element

$$J(p,q') = \langle 0|\psi(0)|p\rangle + \int_0^\infty \langle 0|\mathcal{A}(t)E_{q'}(0,t)\psi(0)|p\rangle\,dt\,, \tag{3.17}$$

shifting the arguments of all fields in the second term by tq' and performing the Taylor expansion

$$E_{q'}(-t,0)\,\psi(-tq') = \sum_0^\infty \frac{(-t)^n}{n!}(q'D)^n\psi(0) \tag{3.18}$$

we can take the integral over t to get

$$J(p,q') = \langle 0|\psi|p\rangle - \sum_{n=0}^\infty \frac{i^{n+1}}{(pq')^{n+1}}\langle 0|\mathcal{A}\,(q'D)^n\psi|p\rangle\,, \tag{3.19}$$

where all the fields are taken at the origin. In fact, since

$$\langle 0|A_\nu D_{\nu_1}\ldots D_{\nu_n}\psi|p\rangle = p_\nu p_{\nu_1}\ldots p_{\nu_n}a_n(\mu^2)\,, \tag{3.20}$$

the rhs of Eq.(3.19) does not depend on (pq') (cf. [40]). Note, that the new representation for $J(p,q')$, unlike the original one, is not explicitly gauge invariant. However, the ψ-term can be represented as

$$\langle 0|\psi|p\rangle = \langle 0|i\frac{(q'\partial)}{(pq')}\psi|p\rangle \tag{3.21}$$

and we can combine it with the first term from the sum to get a term containing a covariant derivative $D = \partial - igA$:

$$J(p,q') = \langle 0|i\frac{(q'D)}{(pq')}\psi|p\rangle - \sum_{n=1}^\infty \frac{i^{n+1}}{(pq')^{n+1}}\langle 0|\mathcal{A}\,(q'D)^n\psi|p\rangle \tag{3.22}$$

Repeating this trick, *i.e.*, representing the term outside the sum as

$$\langle 0|i\frac{(q'D)}{(pq')}\psi|q\rangle = \langle 0|i^2\frac{(q'\partial)}{(pq')}\frac{(q'D)}{(pq')}\psi|p\rangle \tag{3.23}$$

and combining this term with the $n=1$ term from the sum one obtains

$$J(p,q') = \langle 0|i^2\frac{(q'D)^2}{(pq')^2}\psi|p\rangle - \sum_{n=2}^\infty \frac{i^{n+1}}{(pq')^{n+1}}\langle 0|\mathcal{A}\,(q'D)^n\psi|p\rangle\,. \tag{3.24}$$

It is clear now that we can write $J(p,q')$ in a manifestly gauge-invariant form (cf. [41]):

$$J(p,q') = \lim_{n\to\infty}\frac{i^n}{(pq')^n}\langle 0|(q'D)^n\psi|p\rangle \equiv \langle 0|\left(\frac{i(q'D)}{(pq')}\right)^\infty\psi|p\rangle\,. \tag{3.25}$$

In perturbation theory, matrix elements $\langle 0|(q'D)^n\psi|p\rangle$ for finite n have ultraviolet divergences which can be regulated in a standard way, *e.g.*, by the dimensional regularization. After renormalization, we get one-loop terms like $g^2\gamma_n\log\mu^2$. However, the anomalous dimension γ_n contains the usual $(\sum_j^n 1/j)$ term [22] which behaves like $\log n$ for large n. Hence, taking the formal limit $n\to\infty$ one encounters a logarithmic singularity, which requires an additional regularization on top of dimensional regularization (cf. [42]). The parameter characterizing the extra regularization can be taken proportional to μ, *i.e.*, matrix element $\langle 0|E_{q'}(0,\infty;A)\psi(0)|p\rangle$ is the simplest example

of a long-distance object with a double-logarithmic dependence on the UV cut-off [43]. Such objects ("collinear" or "jet" factors [44,40,45]) play an important role in pQCD studies of Sudakov effects. However, within the standard factorization approach, presence of non-cancelling double logarithms of Q^2 (reflected by double logarithms $\log^2\mu^2$ in long-distance matrix elements) is treated as a failure of the factorization program, since the amplitudes in that case are not given by a convolution of parton distributions defined through matrix elements of light-cone operators.

Another signature of Sudakov effects is the presence of the IR contributions (see Fig.6c). Again, since all the hadrons participating in a hard exclusive scattering process are color singlets, summing over all soft gluon insertions one would get a path-ordered exponential over a closed contour, and by Stokes theorem

$$\langle 0|P\exp\left\{ig\oint A_\mu(z)\,dz^\mu\right\}|0\rangle = 1 + \langle 0|O(G)|0\rangle\,, \tag{3.26}$$

where $O(G)$ depends on the gluon field only through the field strength tensor $G_{\mu\nu}$ which has non-zero twist generating a power suppression of the net IR regime contribution.

IV. NONFORWARD DISTRIBUTIONS IN QCD

A. Quark distributions.

Let us discuss now the nonforward parton distributions in the realistic QCD case. For quarks, we should take into account that the field $\psi_a(z)$ contains both the a-quark annihilation operator and the $\bar a$-antiquark creation operator, *i.e.*, the matrix element of the same light-cone operator $\bar\psi_a(z)\ldots\psi_a(0)$ determines distribution functions both for the quark and antiquark. Another complication is related to spin. There are two leading-twist operators $\bar\psi_a(0)\gamma_\mu E(0,z;A)\psi_a(z)$ and $\bar\psi_a(0)\gamma_5\gamma_\mu E(0,z;A)\psi_a(z)$, where, as discussed above, $E(0,z;A)$ is the path-ordered exponential (3.14) which makes the operators gauge-invariant. In the forward case, the first one gives the spin-averaged distribution functions $f(x)$ while the second one is related to the spin-dependent structure functions $g_1(x)$. In this paper, we will concentrate on the $\bar\psi_a\gamma_\mu E(0,z;A)\psi_a$ operators and gluonic operators with which it mixes under renormalization. The relevant nonforward matrix element can be written as[‡‡]

$$\langle p',s'\,|\,\bar\psi_a(0)\hat z E(0,z;A)\psi_a(z)\,|\,p,s\rangle|_{z^2=0} \tag{4.1}$$
$$= \bar u(p',s')\hat z u(p,s)\int_0^1\left(e^{-iX(pz)}\mathcal{F}^a_\zeta(X;t) - e^{i(X-\zeta)(pz)}\mathcal{F}^{\bar a}_\zeta(X;t)\right)dX$$
$$+\,\bar u(p',s')\frac{\hat z\hat r - \hat r\hat z}{2M}u(p,s)\int_0^1\left(e^{-iX(pz)}\mathcal{K}^a_\zeta(X;t) - e^{i(X-\zeta)(pz)}\mathcal{K}^{\bar a}_\zeta(X;t)\right)dX,$$

where M is the nucleon mass and s,s' specify the nucleon polarization. Throughout the paper, we use the "hat" (rather than "slash") convention $\hat z \equiv z^\mu\gamma_\mu$. In Eq.(4.1), the quark and antiquark contributions are explicitly separated. The exponential $e^{-iX(pz)}$ associated with the functions $\mathcal{F}^a_\zeta(X;t)$ and $\mathcal{K}^a_\zeta(X;t)$ indicates that the field $\psi_a(z)$ corresponds to the a-quark taking the momentum Xp from the nucleon. When the momentum Xp is taken from the nucleon by an a-antiquark, the corresponding annihilation operator is in $\bar\psi_a(0)$, and the functions $\mathcal{F}^{\bar a}(X;t)$ and $\mathcal{K}^{\bar a}(X;t)$ are accompanied by the exponential $e^{i(X-\zeta)(pz)}$ corresponding to the momentum at the $\psi_a(z)$-vertex. The antiquark terms come with the minus sign because the creation and annihilation operators for them appear in the reversed order.

As emphasized by X. Ji [11], the parametrization of this nonforward matrix element must include both the non-flip term described by the functions $\mathcal{F}_\zeta(X;t)$ and the spin-flip term[§§] characterized by the functions which we denote by $\mathcal{K}_\zeta(X;t)$. Taking the $O(z)$ term of the Taylor expansion gives the sum rules [11]

[‡‡] Two other definitions of the nonforward parton distributions in terms of matrix elements of composite operators proposed by X.Ji [11] and Collins, Frankfurt and Strikman [16] are discussed in Section IX.

[§§] The possibility of a spin-flip in nonforward matrix elements was discussed earlier in refs. [46,47].

$$\int_0^1 \left[\mathcal{F}_\zeta^a(X;t) - \mathcal{F}_\zeta^{\bar a}(X;t)\right] dX = F_1^a(t), \tag{4.2}$$

$$\int_0^1 \left[\mathcal{K}_\zeta^a(X;t) - \mathcal{K}_\zeta^{\bar a}(X;t)\right] dX = F_2^a(t) \tag{4.3}$$

relating the nonforward distributions $\mathcal{F}_\zeta^a(X;t)$, $\mathcal{K}_\zeta^a(X;t)$ to the a-flavor components of the Dirac and Pauli form factors, respectively (see also [47] and [48]). The spin-flip terms disappear only if $r = 0$. In the weaker $r^2 \equiv t = 0$ limit, they survive, *e.g.*, $F_2^a(0) = \kappa^a$ is the a-flavor contribution to the anomalous magnetic moment. In the formal $t = 0$ limit, the nonforward distributions $\mathcal{F}_\zeta^a(X;t)$, $\mathcal{K}_\zeta^a(X;t)$ reduce to the asymmetric distribution functions $\mathcal{F}_\zeta^a(X)$, $\mathcal{K}_\zeta^a(X)$. It is worth mentioning here that for a massive target (nucleons in our case) there is a kinematic restriction [9]

$$-t \gtrsim \zeta^2 M^2/\bar\zeta. \tag{4.4}$$

Hence, for fixed ζ, the formal limit $t \to 0$ is not physically reachable. However, many results (evolution equations being the most important example) obtained in the formal $t = 0$, $M = 0$ limit are still applicable.

In the region $X \geq \zeta$, the initial quark momentum Xp is larger than the momentum transfer $r = \zeta p$, and we can treat $\mathcal{F}_\zeta^a(X)$ as a generalization of the usual distribution function $f_a(x)$. When $\zeta \to 0$, the limiting curve for $\mathcal{F}_\zeta(X)$ reproduces $f_a(X)$:

$$\mathcal{F}_{\zeta=0}^a(X) = f_a(X) \ \ ; \ \ \mathcal{F}_{\zeta=0}^{\bar a}(X) = f_{\bar a}(X). \tag{4.5}$$

The spin-flip asymmetric distribution functions $\mathcal{K}_\zeta(X)$ do not necessarily vanish in the $\zeta \to 0$ limit. However, the relevant nucleon matrix element $\bar u(p')(\hat z\hat r - \hat r\hat z)u(p)$ is proportional to ζ and the spin-flip term is invisible in the forward case.

In the region $X < \zeta$, one can define $Y = X/\zeta$ and treat the function $\mathcal{F}_\zeta^a(X)$ as the distribution amplitude $\Psi_\zeta^a(Y)$. In particular, the non-flip part in this region can be written as

$$\zeta\, \bar u(p')\hat z u(p) \int_0^1 \left[e^{-iY(rz)}\mathcal{F}_\zeta^a(\zeta Y) - e^{-i(1-Y)(rz)}\mathcal{F}_\zeta^{\bar a}(\zeta Y)\right] dY = \zeta\, \bar u(p')\hat z u(p) \int_0^1 e^{-iY(rz)}\Psi_\zeta^a(Y)\, dY\,, \tag{4.6}$$

where the distribution amplitude $\Psi_\zeta^a(Y)$ is defined by

$$\Psi_\zeta^a(Y) = \mathcal{F}_\zeta^a(Y\zeta) - \mathcal{F}_\zeta^{\bar a}(\bar Y\zeta). \tag{4.7}$$

The function $\Psi_\zeta^a(Y)$ gives the probability amplitude that the initial nucleon with momentum p is composed of the final nucleon with momentum $(1-\zeta)p \equiv p - r$ and a $\bar q q$-pair in which the total pair momentum r is shared in fractions Y and $1 - Y \equiv \bar Y$.

B. Gluon Distribution

For gluons, the nonforward distribution can be defined through the matrix element

$$\begin{aligned} &\langle p' \,|\, z_\mu z_\nu G^a_{\mu\alpha}(0) E^{ab}(0,z;A) G^b_{\alpha\nu}(z) \,|\, p\rangle|_{z^2=0} \\ &= \bar u(p')\hat z u(p)(z\cdot p) \int_0^1 \frac{1}{2}\left[e^{-iX(pz)} + e^{i(X-\zeta)(pz)}\right] \mathcal{F}_\zeta^g(X;t)\, dX \\ &+ \bar u(p') \frac{\hat z\hat r - \hat r\hat z}{2M} u(p)(z\cdot p) \int_0^1 \frac{1}{2}\left[e^{-iX(pz)} + e^{i(X-\zeta)(pz)}\right] \mathcal{K}_\zeta^g(X;t)\, dX\,. \end{aligned} \tag{4.8}$$

The exponentials $e^{-iX(pz)}$ and $e^{i(X-\zeta)(pz)}$ are accompanied here by the same function $\mathcal{F}_\zeta^g(X;t)$ reflecting the fact that gluon and "antigluon" is the same thing. Again, the contribution from the region $0 < X < \zeta$ can be written as

$$\bar u(p')\hat z u(p)\,(z\cdot r) \int_0^1 e^{-iY(rz)}\,\Psi_\zeta^g(Y;t)\, dY + \text{“}\mathcal{K}\text{” term}, \tag{4.9}$$

with the generalized $Y \leftrightarrow \bar Y$ symmetric distribution amplitude $\Psi_\zeta^g(Y;t)$ given by

$$\Psi_\zeta^g(Y;t) = \frac{1}{2}\left(\mathcal{F}_\zeta^g(Y\zeta;t) + \mathcal{F}_\zeta^g(\bar Y\zeta;t)\right). \tag{4.10}$$

In the formal $t = 0$ limit, the nonforward distributions $\mathcal{F}_\zeta^g(X;t)$, $\mathcal{K}_\zeta^g(X;t)$ convert into the asymmetric distribution functions $\mathcal{F}_\zeta^g(X)$, $\mathcal{K}_\zeta^g(X)$. Finally, in the $\zeta = 0$ limit, $\mathcal{F}_\zeta^g(X)$ reduces to the usual gluon density

$$\mathcal{F}_{\zeta=0}^g(X) = X f_g(X). \tag{4.11}$$

C. Flavor-singlet and valence quark distributions

In our original definition (4.1) of the quark distributions, the exponentials $\exp[-iX(pz)]$ and $\exp[i(X-\zeta)(pz)]$ are accompanied by different functions $\mathcal{F}_\zeta^a(X;t)$ and $\mathcal{F}_\zeta^{\bar a}(X;t)$), respectively. In many cases, it is convenient to introduce the flavor-singlet quark operator

$$\mathcal{O}_Q(uz,vz) = \sum_a \mathcal{O}_a^{(+)}(uz,vz) \tag{4.12}$$

where

$$\mathcal{O}_a^{(+)}(uz,vz) = \frac{i}{2}\left[\bar\psi_a(uz)\hat z E(uz,vz;A)\psi_a(vz) - \bar\psi_a(vz)\hat z E(vz,uz;A)\psi_a(uz)\right]. \tag{4.13}$$

The nonforward distribution function $\mathcal{F}_\zeta^Q(X;t)$ for the flavor-singlet quark combination (4.12)

$$\langle p',s' |\, \mathcal{O}_Q(uz,vz) \,|\, p,s\rangle|_{z^2=0} = \bar u(p',s')\hat z u(p,s) \int_0^1 \frac{i}{2}\left[e^{-ivX(pz)+iuX'(pz)} - e^{ivX'(pz)-iuX(pz)}\right]\mathcal{F}_\zeta^Q(X;t)\, dX + \text{“}\mathcal{K}\text{”} \tag{4.14}$$

(where $X' \equiv X - \zeta$) can be expressed as the sum of "$a + \bar a$" distributions:

$$\mathcal{F}_\zeta^Q(X;t) = \sum_a (\mathcal{F}_\zeta^a(X;t) + \mathcal{F}_\zeta^{\bar a}(X;t)). \tag{4.15}$$

Writing the contribution from the $0 < X < \zeta$ region as

$$\zeta \bar u(p')\hat z u(p)\,(z\cdot r) \int_0^1 e^{-iY(rz)}\Psi_\zeta^Q(Y;t)\, dY + \text{“}\mathcal{K}\text{” term}, \tag{4.16}$$

we introduce the flavor-singlet quark distribution amplitude $\Psi_\zeta^Q(Y;t)$ which has the antisymmetry property $\Psi_\zeta^Q(Y;t) = -\Psi_\zeta^Q(\bar Y;t)$ with respect to the $Y \leftrightarrow \bar Y$ transformation.

Another combination of quark operators

$$\mathcal{O}_a^{(-)}(uz,vz) = \frac{1}{2}\sum_a \left[\bar\psi_a(uz)\hat z E(uz,vz;A)\psi_a(vz) + \bar\psi_a(vz)\hat z E(vz,uz;A)\psi_a(uz)\right] \tag{4.17}$$

corresponds to the valence combinations $\mathcal{F}_\zeta^{V_a}(X;t) \equiv \mathcal{F}_\zeta^a(X;t) - \mathcal{F}_\zeta^{\bar a}(X;t)$:

$$\langle p',s' |\, \mathcal{O}_a^{(-)}(uz,vz) \,|\, p,s\rangle|_{z^2=0} = \bar u(p',s')\hat z u(p,s) \int_0^1 \frac{1}{2}\left[e^{-ivX(pz)+iuX'(pz)} + e^{ivX'(pz)-iuX(pz)}\right]\mathcal{F}_\zeta^{V_a}(X,t)\, dX + \text{“}\mathcal{K}\text{”}. \tag{4.18}$$

In both cases (see Eqs.(4.14),(4.18)), two possible exponential factors are accompanied by the same distribution function, just like for the gluon distribution. In the region $0 < X < \zeta$, the function $\mathcal{F}_\zeta^{V_a}(X;t)$ can be written in terms of the flavor-nonsinglet distribution amplitide $\Psi_\zeta^{V_a}(Y;t)$ which is symmetric $\Psi_\zeta^{V_a}(Y;t) = \Psi_\zeta^{V_a}(\bar Y;t)$ with respect to the $Y \leftrightarrow \bar Y$ interchange.

V. EVOLUTION EQUATIONS FOR NONFORWARD DISTRIBUTIONS

A. General formalism

Near the light cone $z^2 \sim 0$, the bilocal operators $\phi(0)\phi(z)$ develop logarithmic singularities $\ln z^2$, so that the formal limit $z^2 \to 0$ is singular. Calculationally, these singularities manifest themselves as ultraviolet divergences for the light-cone operators. The divergences are removed by a subtraction prescription characterized by some scale μ: $\mathcal{F}_\zeta(X;t) \to \mathcal{F}_\zeta(X;t;\mu)$. In QCD, the gluonic operator

$$\mathcal{O}_g(uz, vz) = z_\mu z_\nu G^a_{\mu\alpha}(uz) E^{ab}(uz, vz; A) G^b_{\nu\alpha}(vz) \tag{5.1}$$

mixes under renormalization with the flavor-singlet quark operator. At one loop (*i.e.*, in the leading logarithm approximation), the easiest way to get the evolution equations for nonforward distributions is to use the Balitsky-Braun evolution equation [31] for the light-cone operators***. For the flavor-singlet case, it reads

$$\mu \frac{d}{d\mu} \mathcal{O}_a(0, z) = \int_0^1 \int_0^1 \sum_b B_{ab}(u, v) \mathcal{O}_b(uz, \bar{v}z)\, \theta(u + v \leq 1)\, du\, dv\,, \tag{5.2}$$

where $\bar{v} \equiv 1 - u$ and $a, b = g, Q$. For valence distributions, there is no mixing, and their evolution is generated by the QQ-kernel alone. Inserting the BB-equation (5.2) between chosen hadronic states and parametrizing the matrix elements by appropriate distributions, one can get the well-known evolution kernels such as GLAPD and BL-type kernels and also to calculate the new kernels $R^{ab}(x, y; \xi, \eta)$ and $W^{ab}_\zeta(X, Z)$. The kernels $R^{ab}(x, y; \xi, \eta)$ govern the evolution of the double distributions:

$$\mu \frac{d}{d\mu} F^a(x, y; t; \mu) = \int_0^1 \int_0^1 \sum_b R^{ab}(x, y; \xi, \eta)\, F^b(\xi, \eta; t; \mu)\, \theta(\xi + \eta \leq 1)\, d\xi\, d\eta\,, \tag{5.3}$$

where a and b denote g or Q. Another set of kernels $W^{ab}_\zeta(X, Z)$ dictates the evolution of the nonforward distributions and asymmetric distribution functions:

$$\mu \frac{d}{d\mu} \mathcal{F}^a_\zeta(X; t; \mu) = \int_0^1 \sum_b W^{ab}_\zeta(X, Z)\, \mathcal{F}^b_\zeta(Z; t; \mu)\, dZ\,. \tag{5.4}$$

The evolution of the double distributions will be briefly discussed later in Section VI. Here we will discuss the structure of the $W^{ab}_\zeta(X, Z)$ kernels. Since the form of the equation is not affected by the t-dependence, "t" will not be explicitly indicated in what follows.

Before starting the actual calculations, one should take into account that the gluon distribution $\mathcal{F}^g_\zeta(X)$ is accompanied by the sum of two exponentials while the flavor singlet quark distribution $\mathcal{F}^Q_\zeta(X)$ with which it mixes is accompanied by the difference. This sign change is, in fact, compensated by the extra (pz) factor in the rhs of the gluon distribution definition. The set of evolution equations for $\mathcal{F}^Q_\zeta(X)$, $\mathcal{F}^g_\zeta(X)$ can be obtained by substituting the definitions of the gluon (4.8) and quark (4.14) distributions into the BB-equation and performing the Fourier transformation with respect to the (pz)-variable. For this procedure, the (pz)-factor is equivalent to differentiation d/dX while $1/(pz)$ results in an integration over X. Note, that both operations change the relative sign of the exponentials. Hence, it is convenient to introduce first the auxiliary kernels $M^{ab}_\zeta(X, Z)$ which would appear in the absence of the (pz) mismatch. They are directly related by

$$M^{ab}_\zeta(X, Z) = \int_0^1 \int_0^1 B_{ab}(u, v)\, \delta(X - \bar{u}Z + v(Z - \zeta))\, \theta(u + v \leq 1)\, du\, dv \tag{5.5}$$

***Instead of the original kernels $K^{ab}(u, v)$ from ref. [31], we prefer to use the kernels $B_{ab}(u, v) = -K^{ab}(\bar{u}, v)$ which have the symmetry property $B_{ab}(u, v) = B_{ab}(v, u)$.

to the Balitsky-Braun kernels $B(u, v)$, which we write here in the form given in ref. [15]:

$$\begin{aligned}
B_{QQ}(u, v) &= \frac{\alpha_s}{\pi} C_F \left(1 + \delta(u)[\bar{v}/v]_+ + \delta(v)[\bar{u}/u]_+ - \frac{1}{2}\delta(u)\delta(v)\right)\,, \\
B_{gQ}(u, v) &= \frac{\alpha_s}{\pi} C_F \left(2 + \delta(u)\delta(v)\right)\,, \\
B_{Qg}(u, v) &= \frac{\alpha_s}{\pi} N_f \,(1 + 4uv - u - v)\,, \\
B_{gg}(u, v) &= \frac{\alpha_s}{\pi} N_c \left(4(1 + 3uv - u - v) + \frac{\beta_0}{2N_c} \delta(u)\delta(v) + \left\{\delta(u)\left[\frac{\bar{v}^2}{v} - \delta(v)\int_0^1 \frac{dz}{z}\right] + \{u \leftrightarrow v\}\right\}\right).
\end{aligned} \tag{5.6}$$

The W-kernels are related to the M-kernels by

$$W^{gg}_\zeta(X, Z) = M^{gg}_\zeta(X, Z) \quad , \quad W^{QQ}_\zeta(X, Z) = M^{QQ}_\zeta(X, Z), \tag{5.7}$$

$$\frac{\partial}{\partial X} W^{gQ}_\zeta(X, Z) = -M^{gQ}_\zeta(X, Z) \quad , \quad W^{Qg}_\zeta(X, Z) = -\frac{\partial}{\partial X} M^{Qg}_\zeta(X, Z)\,. \tag{5.8}$$

Hence, to get $W^{gQ}_\zeta(X, Z)$ we should integrate $M^{gQ}_\zeta(X, Z)$ with respect to X. The integration constant can be fixed from the requirement that $W^{gQ}_\zeta(X, Z)$ vanishes for $X > 1$. Then

$$W^{gQ}_\zeta(X, Z) = \int_X^1 M^{gQ}_\zeta(\tilde{X}, Z)\, d\tilde{X}\,. \tag{5.9}$$

Integrating the delta-function in eq.(5.5) produces four different types of the θ-functions, each of which corresponds to a specific evolution regime for the nonforward distributions. In two extreme cases, when $\zeta = 0$ or $\zeta = 1$, the evolution equation reduces to known GLAP and BL-type equations, respectively.

B. BL-type evolution kernels

When $\zeta = 1$, the initial momentum coincides with the momentum transfer and $\mathcal{F}_\zeta(X)$ reduces to a distribution amplitude whose evolution is governed by the BL-type kernels:

$$W^{ab}_{\zeta=1}(X, Z) = V^{ab}(X, Z). \tag{5.10}$$

Taking $\zeta = 1$ in Eq.(5.5) we obtain

$$M^{ab}_{\zeta=1}(X, Z) \equiv U^{ab}(X, Z) = \int_0^1 \int_0^1 B_{ab}(u, v)\, \delta(X - \bar{u}Z - v(1 - Z))\, \theta(u + v \leq 1)\, du\, dv\,. \tag{5.11}$$

Eliminating the δ-function, one would observe that in the regions $X \leq Z$ and $X \geq Z$ the $U^{ab}(X, Z)$ kernels are given by different analytic expressions. However, from the representation (5.11) and the symmetry property $B_{ab}(u, v) = B_{ab}(v, u)$ it follows that $U^{ab}(\bar{X}, \bar{Z}) = U^{ab}(X, Z)$. Hence, it is sufficient to know the U-kernels in the $X \leq Z$ region only. The basic function $U^{ab}_0(X, Z) \equiv \theta(X \leq Z)\, U^{ab}(X, Z)$ can be calculated from

$$U^{ab}_0(X, Z) = \frac{1}{Z} \int_0^X B_{ab}\left(\bar{v} - (X - v)/Z, v\right) dv\,. \tag{5.12}$$

The total kernel $U^{ab}(X, Z)$ then can be written as

$$U^{ab}(X, Z) = \theta(X \leq Z)\, U^{ab}_0(X, Z) + \theta(Z \leq X)\, U^{ab}_0(\bar{X}, \bar{Z})\,.$$

One can easily derive a table of $B \to U_0$ conversion formulas for all the structures present in the BB-kernels:

$$\delta(u)\,\delta(v) \to \delta(Z-X) \ , \ 1 \to \frac{X}{Z} \ , \ \delta(u)\frac{\bar v}{v} \to 0 \ , \ \delta(u)\left(\frac{\bar v}{v}\right)^2 \to 0,$$
$$\delta(v)\frac{\bar u}{u} \to \left(\frac{X}{Z}\right)\frac{1}{Z-X} \ , \ \delta(v)\frac{\bar u^2}{u} \to \left(\frac{X}{Z}\right)^2\frac{1}{Z-X},$$
$$u+v \to \frac{X}{Z}\left(1-\frac{X}{2Z}\right) \ , \ uv \to \frac{X^2}{Z}\left(\frac{1}{2}-\frac{X}{6Z}-\frac{X}{3}\right). \tag{5.13}$$

Using Eqs.(5.6) and this table, we can get the BL-type kernels $V^{ab}(X,Z)$. Before doing this, we note that the BL-type kernels appear as a part of the asymmetric kernel $W_\zeta^{ab}(X,Z)$ even in the general $\zeta \neq 1,0$ case. As explained earlier, if X is in the region $X \le \zeta$, then the function $\mathcal{F}_\zeta(X)$ can be treated as a distribution amplitude $\Psi_\zeta(Y)$ with $Y = X/\zeta$. For this reason, when both X and Z are smaller than ζ, we would expect that the kernels $W_\zeta^{ab}(X,Z)$ must simply reduce to the BL-type evolution kernels $V^{ab}(X/\zeta, Z/\zeta)$. Indeed, the relation (5.5) can be written as

$$M_\zeta^{ab}(X,Z) = \frac{1}{\zeta}\int_0^1\int_0^1 B_{ab}(u,v)\,\delta(X/\zeta - \bar u Z/\zeta - v(1-Z/\zeta))\,\theta(u+v\le 1)\,du\,dv. \tag{5.14}$$

Comparing this expression with the representation for the $U_0^{ab}(X,Z)$ kernels, we conclude that, in the region where $X/\zeta \le 1$ and $Z/\zeta \le 1$, the kernels $M_\zeta^{ab}(X,Z)$ are given by

$$M_\zeta^{ab}(X,Z)|_{0\le\{X,Z\}\le\zeta} = \frac{1}{\zeta}U^{ab}(X/\zeta, Z/\zeta). \tag{5.15}$$

From the expressions connecting the W- and M-kernels, we obtain the following relations between the asymmetric evolution kernels $W_\zeta^{ab}(X,Z)$ in the region $0 \le \{X,Z\} \le \zeta$ (let us denote them by $L_\zeta^{ab}(X,Z) \equiv W_\zeta^{ab}(X,Z)|_{0\le\{X,Z\}\le\zeta}$) and the BL-type kernels $V^{ab}(X,Z)$:

$$L_\zeta^{QQ}(X,Z) = \frac{1}{\zeta}V^{ab}(X/\zeta, Z/\zeta) \ ; \ L_\zeta^{gQ}(X,Z) = V^{gQ}(X/\zeta, Z/\zeta) \ ;$$
$$L_\zeta^{Qg}(X,Z) = \frac{1}{\zeta^2}V^{ab}(X/\zeta, Z/\zeta) \ ; \ L_\zeta^{gg}(X,Z) = \frac{1}{\zeta}V^{QQ}(X/\zeta, Z/\zeta). \tag{5.16}$$

Explicit calculations based on Eqs.(5.5)-(5.9), (5.10), (5.16) give

$$V^{QQ}(X,Z) = \frac{\alpha_s}{\pi}C_F\left\{\left[\frac{X}{Z}\left(1+\frac{1}{Z-X}\right)\theta(X<Z)\right]_+ + \{X\to\bar X, Z\to\bar Z\}\right\} \tag{5.17}$$
$$V^{Qg}(X,Z) = \frac{\alpha_s}{\pi}N_f\left\{\frac{X}{Z}\left[2(2-X)+\frac{1-X}{Z}\right]\theta(X<Z) - \{X\to\bar X, Z\to\bar Z\}\right\}, \tag{5.18}$$
$$V^{gQ}(X,Z) = \frac{\alpha_s}{\pi}C_F\left\{\left(2-\frac{X^2}{Z}\right)\theta(X<Z) + \frac{(1-X)^2}{1-Z}\,\theta(X>Z)\right\}, \tag{5.19}$$
$$V^{gg}(X,Z) = \frac{\alpha_s}{\pi}N_c\left\{2\frac{X^2}{Z}\left(3-2X+\frac{1-X}{Z}\right) + \frac{1}{Z-X}\left(\frac{X}{Z}\right)^2\right.$$
$$\left. + \delta(X-Z)\left[\frac{\beta_0}{2N_c} - \int_0^1\frac{dz}{1-z}\right]\right\}\theta(X<Z) + \{X\to\bar X, Z\to\bar Z\}. \tag{5.20}$$

Note, that the $V^{gQ}(X,Z)$ kernel can be represented as the sum

$$V^{gQ}(X,Z) = \frac{\alpha_s}{\pi}C_F + \frac{\alpha_s}{\pi}C_F\left\{\left(1-\frac{X^2}{Z}\right)\theta(X<Z) - \{X\to\bar X, Z\to\bar Z\}\right\} \tag{5.21}$$

of a constant term and a kernel which is explicitly antisymmetric with respect to the $\{X\to\bar X, Z\to\bar Z\}$ transformation. In fact, the constant term does not contribute to evolution since the flavor-singlet distribution amplitude $\Psi^Q(Z)$ with which it is convoluted is antisymmetric $\Psi^Q(Z) = -\Psi^Q(\bar Z)$. For the same reason, the convolution of $V^{gQ}(X,Z)$ with $\Psi^Q(Z)$ determining the evolution correction to $\mathcal{F}_\zeta^g(X)$ behaves like X^2 for small X.

Furthermore, the BL-type kernels also govern the evolution in the region corresponding to transitions from a fraction Z which is larger than ζ to a fraction X which is smaller than ζ. Indeed, using the δ-function to calculate the integral over u, we get

$$M_\zeta^{ab}(X,Z)|_{X\le\zeta\le Z} = \frac{1}{Z}\int_0^{X/\zeta} B_{ab}\Big([1-X/Z-v(1-\zeta/Z)], v\Big)\,dv, \tag{5.22}$$

which has the same analytic form (5.12) as the expression for $M_\zeta^{ab}(X,Z)$ in the region $X\le Z\le\zeta$. For QQ, gg and Qg kernels, this automatically means that $W_\zeta^{ab}(X,Z)|_{X\le\zeta\le Z}$ is given by the same analytic expression as $L_\zeta^{ab}(X,Z)$ for $X<Z$. Because of integration, to get $W_\zeta^{gQ}(X,Z)$ one should also know $M_\zeta^{gQ}(X,Z)$ in the region $\zeta\le X\le Z$. However, our explicit calculation confirms that $W_\zeta^{gQ}(X,Z)$ in the transition region $X\le\zeta\le Z$ is given by the same expression as $L_\zeta^{gQ}(X,Z)$ for $X<Z$.

Note, that the evolution jump through the critical fraction ζ is irreversible: the δ-function in Eq.(5.14) requires that $X/\zeta = v + (1-u-v)Z/\zeta$ or $X\le\zeta$ if $Z\le\zeta$. To put it in words, when the parton momentum degrades in the evolution process to values smaller than the momentum transfer $\zeta p \equiv r$, further evolution is like that for a distribution amplitude: the momentum can decrease or increase up to the r-value but cannot exceed this value.

C. Region $Z\ge\zeta$, $X\ge\zeta$

Recall, that when $X>\zeta$, the initial quark momentum Xp is larger than the momentum transfer $r=\zeta p$, and we can treat the asymmetric distribution function $\mathcal{F}_\zeta^a(X)$ as a generalization of the usual distribution function $f_a(X)$ for a somewhat skewed kinematics. Hence, we can expect that evolution in the region $\zeta<X\le 1$, $\zeta<Z\le 1$ is similar to that generated by the GLAP equation. In particular, it has the basic property that the evolved fraction X is always smaller than the original fraction Z. The relevant kernels are given by

$$M_\zeta^{ab}(X,Z)|_{\zeta\le X\le Z\le 1} = \frac{1}{Z}\int_0^{\frac{1-X/Z}{1-\zeta/Z}} B_{ab}\Big([1-X/Z-v(1-\zeta/Z)], v\Big)\,dv. \tag{5.23}$$

Introducing the integration variable $w \equiv v(1-\zeta/Z)/(1-X/Z)$, we obtain the expression in which the arguments of the BB-kernels are treated in a more symmetric way

$$M_\zeta^{ab}(X,Z)|_{\zeta\le X\le Z\le 1} = \frac{Z-X}{ZZ'}\int_0^1 B_{ab}(\bar w(1-X/Z), w(1-X'/Z'))\,dw, \tag{5.24}$$

where $X'\equiv X-\zeta$ and $Z'\equiv Z-\zeta = v/(1-X'/Z')$ are the "returning" partners of the original fractions X,Z. Moreover, since $Z-X = Z'-X'$, the kernels $M_\zeta^{ab}(X,Z)$ are given by functions symmetric with respect to the interchange of X,Z with X',Z'. This observation can be used to check the results of calculations. However, since we are dealing with the asymmetric situation $X>X'$, $Z>Z'$, other practical applications of this symmetry are not evident at the moment. Again, we can easily obtain a table for transitions from the B_{ab}-kernels to the M^{ab}-kernels for the region $\zeta\le X\le Z\le 1$:

$$\delta(u)\,\delta(v) \to \delta(Z-X) \ ; \ 1 \to \frac{Z-X}{ZZ'} \ ; \ (u+v) \to \frac{Z-X}{2ZZ'}\left[2-\frac{X}{Z}-\frac{X'}{Z'}\right];$$
$$uv \to \frac{Z-X}{6ZZ'}\left(1-\frac{X}{Z}\right)\left(1-\frac{X'}{Z'}\right) \ ; \ \left(\delta(u)\frac{\bar v}{v}+\delta(v)\frac{\bar u}{u}\right) \to \frac{1}{Z-X}\left[\frac{X}{Z}+\frac{X'}{Z'}\right];$$
$$\left(\delta(u)\frac{\bar v^2}{v}+\delta(v)\frac{\bar u^2}{u}\right) \to \frac{1}{Z-X}\left[\left(\frac{X}{Z}\right)^2+\left(\frac{X'}{Z'}\right)^2\right]. \tag{5.25}$$

Introducing the notation $P_\zeta^{ab}(X,Z) \equiv W_\zeta^{ab}(X,Z)|_{\zeta \le X \le Z \le 1}$ and using the formulas given above, we calculate the P-kernels:

$$P_\zeta^{QQ}(X,Z) = \frac{\alpha_s}{\pi} C_F \left\{ \frac{1}{Z-X}\left[1+\frac{XX'}{ZZ'}\right] - \delta(X-Z)\int_0^1 \frac{1+z^2}{1-z}\,dz \right\} \to \frac{1}{Z} P_{QQ}(X/Z), \quad (5.26)$$

$$P_\zeta^{Qg}(X,Z) = \frac{\alpha_s}{\pi} N_f \frac{1}{ZZ'}\left\{\left(1-\frac{X}{Z}\right)\left(1-\frac{X'}{Z'}\right)+\frac{XX'}{ZZ'}\right\} \to \frac{1}{Z^2} P_{Qg}(X/Z), \quad (5.27)$$

$$P_\zeta^{gQ}(X,Z) = \frac{\alpha_s}{\pi} C_F \left\{\left(1-\frac{X}{Z}\right)\left(1-\frac{X'}{Z'}\right)+1\right\} \to \frac{X}{Z} P_{gQ}(X/Z), \quad (5.28)$$

$$P_\zeta^{gg}(X,Z) = \frac{\alpha_s}{\pi} N_c \left\{ 2\left[1+\frac{XX'}{ZZ'}\right]\frac{Z-X}{ZZ'} + \frac{1}{Z-X}\left[\left(\frac{X}{Z}\right)^2+\left(\frac{X'}{Z'}\right)^2\right] \right.$$
$$\left. + \delta(X-Z)\left[\frac{\beta_0}{2N_c} - 2\int_0^1 \frac{dz}{1-z}\right]\right\} \to \frac{X}{Z^2} P_{gg}(X/Z). \quad (5.29)$$

They also have a symmetric form. The arrows indicate how the asymmetric kernels $P_\zeta^{ab}(X,Z)$ are related to the GLAPD kernels in the $\zeta = 0$ limit when $Z = Z'$ and $X = X'$. Deriving these relations, one should take into account that the asymmetric gluon distribution function $\mathcal{F}_\zeta^g(X)$ reduces in the limit $\zeta = 0$ to $X f_g(X)$ rather than to $f_g(X)$.

In the region $Z > \zeta$, the evolution is one-sided: the evolved fraction X is smaller than Z. Furthermore, if $Z \le \zeta$ then also $X \le Z$, *i.e.*, distributions in the $X > \zeta$ regions are not affected by the distributions in the $X < \zeta$ regions. Hence, just like in the GLAP case, information about the initial distribution in the $Z > \zeta$ region is sufficient for calculating its evolution in this region. This situation may be contrasted with the evolution of distributions in the $Z < \zeta$ regions: in that case one should know the asymmetric distribution functions in the whole domain $0 < Z < 1$.

Qualitatively, the evolution in the $X, Z > \zeta$ region proceeds just like in the GLAP evolution: the distributions shift to smaller and smaller values of X. In the GLAP case, the distributions approach the $\delta(x)$ form condensing at a single point $x = 0$. In the asymmetric case, the whole region $Z < \zeta$ works like a "black hole" for the partons: after they end up there, they will never come back to the $X > \zeta$ region. Inside the $Z < \zeta$ region, the evolution is governed by the BL-equation transforming the $\Psi_\zeta(Y)$ distribution amplitudes into their asymptotic forms like $Y\bar{Y}, Y\bar{Y}(Y-\bar{Y})$ for the quarks and $(Y\bar{Y})^2, (Y\bar{Y})^2(Y-\bar{Y})$ for the gluons; a particular form is dictated by the symmetry properties of the relevant operators.

VI. ASYMPTOTIC SOLUTIONS OF EVOLUTION EQUATIONS

A. Evolution of asymmetric distribution function

To describe the qualitative features of the QCD evolution of the nonforward distributions, we will consider the simplest case, *i.e.*, the evolution equation for the flavor-nonsinglet (valence) functions. Then Qg, gQ and gg kernels do not contribute, and the evolution is completely determined by the QQ-kernel. The multiplicatively renormalizable operators in this case were originally found in ref. [5]

$$\mathcal{O}_n = (z\partial_+)^n \, \bar\psi \lambda^a \hat{z} C_n^{3/2}(z \overleftrightarrow{\partial} / z\partial_+)\psi . \quad (6.1)$$

Here we use the symbolic notation $(z \overleftrightarrow{\partial} / z\partial_+)$ of ref. [5], with $\overleftrightarrow{\partial} = \overrightarrow{\partial} - \overleftarrow{\partial}$, $\partial_+ = \overrightarrow{\partial} + \overleftarrow{\partial}$ and $C_n^{3/2}(y)$ being the Gegenbauer polynomials. This means that the Gegenbauer moments

$$\mathcal{C}_\zeta(n,\mu) = \int_0^1 C_n^{3/2}(2Z/\zeta - 1)\mathcal{F}_\zeta(Z;\mu)\,dZ \quad (6.2)$$

of the asymmetric distribution function $\mathcal{F}_\zeta(X;\mu)$ have a simple evolution:

$$\mathcal{C}_\zeta(n,\mu) = \mathcal{C}_\zeta(n,\mu_0)\left[\frac{\ln \mu_0/\Lambda}{\ln \mu/\Lambda}\right]^{\gamma_n/\beta_0}, \quad (6.3)$$

where $\beta_0 = 11 - \frac{2}{3}N_f$ is the lowest coefficient of the QCD β-function and γ_n are the non-singlet anomalous dimensions [50,51]

$$\gamma_n = C_F\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} + 2\sum_{j=2}^{n+1}\frac{1}{j}\right]. \quad (6.4)$$

For $n = 0$, the Gegenbauer moment coincides with the ordinary one and, since $\gamma_0 = 0$, the area under the curve remains constant. Other Gegenbauer moments decrease as μ increases. For the ordinary moments of the nonforward distribution

$$\mathcal{M}_N(\zeta,\mu) \equiv \int_0^1 \mathcal{F}_\zeta(X;\mu)X^N\,dX, \quad (6.5)$$

using explicit expression for the Gegenbauer polynomials we can derive the following expansion over the multiplicatively renormalizable combinations $\mathcal{C}_\zeta(n,\mu)$:

$$\mathcal{M}_N(\zeta,\mu) = \zeta^N N!(N+1)!\sum_{n=0}^{N}(-1)^n\frac{2(2n+3)}{(N+n+3)!(N-n)!}\,\mathcal{C}_\zeta(n,\mu). \quad (6.6)$$

We can also write the expression which gives the evolved moments $\mathcal{M}_N(\zeta,\mu)$ in terms of the original ones:

$$\mathcal{M}_N(\zeta,\mu) = \zeta^N N!(N+1)!\sum_{n=0}^{N}\frac{(-1)^n 2(2n+3)}{(N+n+3)!(N-n)!}\left[\frac{\ln\mu_0/\Lambda}{\ln\mu/\Lambda}\right]^{\gamma_n/\beta_0}\sum_{k=0}^{n}\frac{(-1)^k(k+n+2)!}{2\,k!\,(k+1)!\,(n-k)!}\mathcal{M}_k(\zeta,\mu_0). \quad (6.7)$$

With increasing N, the number of contributing Gegenbauer moments $\mathcal{C}_\zeta(n,\mu)$ in Eq.(6.6) increases. An important observation is that the non-evolving (and ζ-independent, but t-dependent) term $\mathcal{C}(0)$ contributes to each moment. As a result, in the $\mu \to \infty$ limit, all the moments tend to constant values determined by the $n = 0$ term in the sum (6.6):

$$\mathcal{M}_N(\zeta,\mu\to\infty) = \zeta^N\frac{6}{(N+2)(N+3)}\,\mathcal{C}(0) = \int_0^\zeta \frac{\mathcal{C}(0)}{\zeta}\,6\,(X/\zeta)(1-X/\zeta)\,X^N\,dX. \quad (6.8)$$

This means that in the limit $\mu \to \infty$, the function $\mathcal{F}_\zeta(X;\mu\to\infty)$ completely disappears from the region $X \ge \zeta$, *i.e.*, it reduces to the distribution amplitude $\Psi_\zeta(Y)$ which ultimately tends to the usual asymptotic shape $6Y(1-Y)$ in the $Y = X/\zeta$ variable:

$$\mathcal{F}_\zeta(X;\mu\to\infty) = 6\,\mathcal{C}(0)X(1-X/\zeta)/\zeta^2. \quad (6.9)$$

One may also be interested in finding expressions showing how the function $\mathcal{F}_\zeta(X;\mu)$ changes its shape from an arbitrary original curve $\mathcal{F}_\zeta(X;\mu_0)$ to the asymptotic one. Note, that the Gegenbauer moments for $\zeta < 1$ involve integration regions in which the argument $C_n^{3/2}(2Z/\zeta - 1)$ of the polynomials extends beyond the segment $(-1,1)$ where they form an orthogonal set of functions. Hence, a formal inversion of the Gegenbauer moments is only possible for $\zeta = 1$. In this case, the inversion produces the standard solution of the evolution equation for a distribution amplitude [5,6]

$$\mathcal{F}_{\zeta=1}(X;\mu) = \sum_{n=0}^{\infty}\frac{4(2n+3)}{(n+1)(n+2)}X\bar{X}\,C_n^{3/2}(2X-1)\left[\frac{\ln\mu_0/\Lambda}{\ln\mu/\Lambda}\right]^{\gamma_n/\beta_0}\int_0^1 C_n^{3/2}(2Z-1)\mathcal{F}_{\zeta=1}(Z;\mu_0)\,dZ. \quad (6.10)$$

Thus, if the initial distribution coincides with one of the eigenfunctions $X\bar{X}\,C_n^{3/2}(2X-1)$, the evolution is very simple: the function just decreases in magnitude without changing its form. An attractive feature of such a situation is that approximating the initial distribution amplitude by a few lowest Gegenbauer polynomials one obtains a simple model of its evolution. Inspired by this observation, one may be tempted to construct a similar representation for the evolution of the asymmetric distribution function. Using the expansion of the light-cone operator $\bar{\psi}(0)\lambda^a \hat{z}\psi(z)$ over the multiplicatively renormalizable operators $\mathcal{O}_n$ (see [31])

$$\bar{\psi}(0)\lambda^a \hat{z}\psi(z) = \sum_{n=0}^{\infty}(-1)^n \frac{2(2n+3)}{n!}\int_0^1 (u\bar{u})^{n+1}\mathcal{O}_n(uz)\,du \tag{6.11}$$

and inserting it into the nonforward matrix element, we obtain

$$\mathcal{F}_\zeta(X;\mu) = \sum_{n=0}^{\infty}(-1)^n \frac{2(2n+3)}{n!}\left(\frac{\ln\mu_0/\Lambda}{\ln\mu/\Lambda}\right)^{\gamma_n/\beta_0} \zeta^n C_\zeta(n,\mu_0)\int_0^1 (u\bar{u})^{n+1}\delta^{(n)}(X-u\zeta)\,du\,. \tag{6.12}$$

Integrating $(u\bar{u})^{n+1}\delta^{(n)}(X-u\zeta)$ over u, we get the Gegenbauer polynomials $C_n^{3/2}(2X/\zeta-1)$ accompanied by the spectral condition $X \leq \zeta$. This means that the formal integration does not give a correct result for functions which do not vanish outside the region $X \leq \zeta$. For such functions, one should first perform the summation over n (which is, of course, practically impossible) and only then take the u-integral.

Another limit in which the integral over u can be taken safely is $\zeta = 0$. For small ζ, the Gegenbauer polynomials are dominated by the senior power Z^n and in the $\zeta \to 0$ limit we obtain

$$\mathcal{F}_\zeta(X;\mu) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\delta^{(n)}(X)\left(\frac{\ln\mu_0/\Lambda}{\ln\mu/\Lambda}\right)^{\gamma_n/\beta_0}\int_0^1 \mathcal{F}_\zeta(Z;\mu_0)\,Z^n\,dZ\,, \tag{6.13}$$

i.e., the usual result that the moments of parton densities have a simple GLAP evolution. Note, that in this case, the functions which evolve without changing their shape are $\delta^{(n)}(x)$. From a practical point of view, this observation is of little use. Modeling the solutions of the GLAP equations is known to be a rather complicated excercise usually involving a numerical integration of the evolution equations.

Hence, the representation (6.12) should be understood only in the sense of (mathematical) distributions in X rather than functions. To get meaningful results, one should integrate them over X with some smooth function. In particular, integrating it with X^N, one obtains the formula (6.6) for the evolution of the X^N moments of nonforward distributions.

B. Evolution of double distribution

Solving the evolution equation for the valence double distribution $F(x,y;\mu)$ defined by

$$\begin{aligned}&\langle p-r,s'|\,\mathcal{O}^{(-)}(uz,vz)|_\mu\,|p,s\rangle|_{z^2=0}\\ &= \bar{u}(p',s')\hat{z}u(p,s)\int_0^1 \frac{1}{2}\left[e^{-ix(pz)-iy(rz)}+e^{ix(pz)-i\bar{y}(rz)}\right]F(x,y;\mu)\,\theta(x+y\leq 1)\,dx\,dy\,,\end{aligned} \tag{6.14}$$

we can give an alternative derivation of the asymptotic form of the valence nonforward distribution $\mathcal{F}_\zeta(X;\mu)$. The μ-dependence of $F(x,y;\mu)$ is governed by the evolution equation

$$\mu\frac{d}{d\mu}F(x,y;\mu) = \int_0^1 d\xi\int_0^1 R_{QQ}(x,y;\xi,\eta)F(\xi,\eta;\mu)d\eta. \tag{6.15}$$

Since the integration over y converts $F(x,y)$ into the parton distribution function $f(x)$, whose evolution is governed by the GLAP equation

$$\mu\frac{d}{d\mu}f(x;\mu) = \int_x^1 \frac{d\xi}{\xi}P_{QQ}(x/\xi)f(\xi;\mu)d\xi, \tag{6.16}$$

the kernel $R_{QQ}(x,y;\xi,\eta)$ must have the property

$$\int_0^{1-x} R_{QQ}(x,y;\xi,\eta)dy = \frac{1}{\xi}P_{QQ}(x/\xi). \tag{6.17}$$

For a similar reason, integrating $R_{QQ}(x,y;\xi,\eta)$ over x one should get the BL-type kernel:

$$\int_0^{1-y} R_{QQ}(x,y;\xi,\eta)dx = V_{QQ}(y,\eta). \tag{6.18}$$

Explicit calculation gives for $R_{QQ}(x,y;\xi,\eta)$ the following result

$$\begin{aligned}R_{QQ}(x,y;\xi,\eta) &= \frac{\alpha_s}{\pi}C_F\frac{1}{\xi}\Big\{\theta(0\leq x/\xi\leq \min\{y/\eta,\bar{y}/\bar{\eta}\}) - \frac{1}{2}\delta(1-x/\xi)\delta(y-\eta)\\ &+ \frac{\theta(0\leq x/\xi\leq 1)x/\xi}{(1-x/\xi)}\left[\frac{1}{\eta}\delta(x/\xi-y/\eta)+\frac{1}{\bar{\eta}}\delta(x/\xi-\bar{y}/\bar{\eta})\right] - 2\delta(1-x/\xi)\delta(y-\eta)\int_0^1\frac{z}{1-z}dz\Big\}.\end{aligned} \tag{6.19}$$

It can also be obtained from the BB-kernel $B_{QQ}(u,v)$ using the relation

$$R_{QQ}(x,y;\xi,\eta) = \frac{1}{\xi}B_{QQ}(y-\eta x/\xi,\bar{y}-\bar{\eta}x/\xi). \tag{6.20}$$

It is easy to verify that the spectral constraint $x+y\leq 1$ is not violated by the evolution: the kernel $R_{QQ}(x,y;\xi,\eta)$ has the property that $x+y\leq 1$ if $\xi+\eta\leq 1$. Using our expression for $R_{QQ}(x,y;\xi,\eta)$ and explicit forms of the $P_{QQ}(x/\xi)$ and $V_{QQ}(y,\eta)$ kernels (see Eqs. (5.26), (5.17)) one can check that $R_{QQ}(x,y;\xi,\eta)$ satisfies the reduction formulas (6.17) and (6.18). To solve the evolution equation, we combine the standard methods used to find solutions of the underlying GLAP and BL evolution equations. To solve the GLAP equation, one should consider the moments with respect to x. Multiplying Eq.(6.15) by x^n, integrating over x and utilizing the property $R_{QQ}(x,y;\xi,\eta) = R_{QQ}(x/\xi,y;1,\eta)/\xi$, we get

$$\mu\frac{d}{d\mu}F_n(y;\mu) = \int_0^1 R_n(y,\eta)F_n(\eta;\mu)d\eta\,, \tag{6.21}$$

where $F_n(y;\mu)$ is the nth x-moment of $F(x,y;\mu)$

$$F_n(y;\mu) = \int_0^{1-y} x^n F(x,y;\mu)dx \tag{6.22}$$

and the kernel $R_n(y,\eta)$ is given by

$$R_n(y,\eta) = \frac{\alpha_s}{\pi}C_F\left\{\left[\left(\frac{y}{\eta}\right)^{n+1}\left[\frac{1}{n+1}+\frac{1}{\eta-y}\right]\theta(y\leq\eta)+\{y\to\bar{y},\eta\to\bar{\eta}\}\right]+\delta(y-\eta)\left[\frac{1}{2}-\int_0^1\frac{dz}{z}\right]\right\}. \tag{6.23}$$

It is straightforward to check that $R_n(y,\eta)$ has the property

$$R_n(y,\eta)w_n(\eta) = R_n(\eta,y)w_n(y),$$

where $w_n(y) = (y\bar{y})^{n+1}$. Hence, the eigenfunctions of $R_n(y,\eta)$ are orthogonal with the weight $w_n(y) = (y\bar{y})^{n+1}$, *i.e.*, they are proportional to the Gegenbauer polynomials $C_k^{n+3/2}(y-\bar{y})$ (cf. [6,49]). Now, we can write the general solution of the evolution equation

$$F_n(y;\mu) = 2\,(y\bar{y})^{n+1}\frac{(2n+1)!\,(2n+2)!}{n!\,(n+1)!}\sum_{k=0}^{\infty}\frac{(2k+2n+3)\,k!}{(2n+k+2)!}C_k^{n+3/2}(y-\bar{y})\left[\frac{\log(\mu_0/\Lambda)}{\log(\mu/\Lambda)}\right]^{\gamma_k^{(n)}/\beta_0}A_{nk}(\mu_0)\,, \tag{6.24}$$

where

$$A_{nk}(\mu_0) = \int_0^1 F_n(y;\mu_0)\, C_k^{n+3/2}(y-\bar{y})\, dy \tag{6.25}$$

and the anomalous dimensions $\gamma_k^{(n)}$ are related to the eigenvalues of the kernel $R_n(y,\eta)$. They coincide with the standard non-singlet anomalous dimensions γ_N (6.4): $\gamma_k^{(n)} = \gamma_{n+k}$. Since $\gamma_0^{(0)} = 0$, while all other anomalous dimensions are positive, in the formal $\mu \to \infty$ limit we have $F_0(y, \mu \to \infty) \sim y\bar{y}$ and $F_n(y, \mu \to \infty) = 0$ for all $n \geq 1$. This means that

$$F(x, y; \mu \to \infty) \sim \delta(x)\, y\bar{y}, \tag{6.26}$$

i.e., in each of its variables, the limiting function $F(x, y; \mu \to \infty)$ acquires the characteristic asymptotic form dictated by the nature of the variable: $\delta(x)$ is specific for the distribution functions [50,51], while the $y\bar{y}$-form is the asymptotic shape for the lowest-twist two-body distribution amplitudes [5,6]. For the asymmetric distribution function this gives $\mathcal{F}_\zeta(X, \mu \to \infty) \sim (X/\zeta^2)(1 - X/\zeta)$. This result was already obtained in the previous subsection.

VII. BASIC USES OF NONFORWARD DISTRIBUTIONS

A. Deeply virtual Compton scattering

Using the parametrization for the matrix elements of the quark operator, we can easily write a parton-type representation for the handbag contribution to the DVCS amplitude:

$$T^{\mu\nu}(p,q,q') = \frac{1}{2\,(pq')} \sum_a e_a^2 \left[\left(-g^{\mu\nu} + \frac{1}{p\cdot q'}(p^\mu q'^\nu + p^\nu q'^\mu) \right) \left\{ \bar{u}(p')\hat{q}' u(p) T_F^a(\zeta) + \frac{1}{2M}\bar{u}(p')(\hat{q}'\hat{r} - \hat{r}\hat{q}')u(p) T_K^a(\zeta) \right\} \right.$$
$$\left. + i\epsilon^{\mu\nu\alpha\beta} \frac{p_\alpha q'_\beta}{(pq')} \left\{ \bar{u}(p')\gamma_5 \hat{q}' u(p)\, T_G^a(\zeta) + \frac{(q'r)}{2M}\bar{u}(p')\gamma_5 u(p) T_P^a(\zeta) \right\} \right] \tag{7.1}$$

where $\hat{q}' \equiv \gamma_\mu q'^\mu$, and $T^a(\zeta)$ are the invariant amplitudes depending on the scaling variable ζ. In particular,

$$T_F^a(\zeta) = -\int_0^1 \left[\frac{1}{X - \zeta + i\epsilon} + \frac{1}{X - i\epsilon} \right] (\mathcal{F}_\zeta^a(X;t) + \mathcal{F}_\zeta^{\bar{a}}(X;t))\ dX\,. \tag{7.2}$$

Since the nucleon is the lowest bound state in the 3-quark system, the nonforward distribution function for $t < 0$ is real. Hence, the imaginary part of $T_F^a(\zeta)$ can be produced only by singularities of the terms in the square brackets. Taking into account that the nonforward distributions vanish for $X = 0$, we conclude that only the term containing $1/(X - \zeta + i\epsilon)$ generates the imaginary part:

$$\frac{1}{\pi}\,\mathrm{Im}\, T_V^a(\zeta) = \mathcal{F}_\zeta^a(\zeta;t) + \mathcal{F}_\zeta^{\bar{a}}(\zeta;t) \tag{7.3}$$

with a similar expression for $\mathrm{Im}\, T_A^a(\zeta)$. As discussed in Section 1, the function $\mathcal{F}_\zeta^a(\zeta;t)$ does not coincide with the usual parton distribution $f_a(\zeta)$, even in the formal $t \to 0$ limit. To get the real part of the $1/(X - \zeta + i\epsilon)$ terms, one should use the principal value prescription

$$\mathrm{Re}\, T_V^a(\zeta) = -\mathrm{P}\int_0^1 (\mathcal{F}_\zeta^a(X;t) + \mathcal{F}_\zeta^{\bar{a}}(X;t)) \frac{dX}{X - \zeta}\,. \tag{7.4}$$

Since the principal value prescription is based on cancellation of $X < \zeta$ and $X > \zeta$ parts of the integral, it makes sense to preserve $\mathcal{F}_\zeta^a(X;t)$ as a single function. Splitting it into $X < \zeta$ and $X > \zeta$ components, one would simply get two divergent expressions for the real part of the amplitude.

Let us study how these formulas are modified by the evolution. At one loop, the $\ln Q^2$ term can be easily calculated using the coordinate representation:

$$T_1(p,q,q') = \frac{\alpha_s}{2\pi} \int d^4z\, e^{-i(qz)} \int_0^1 \int_0^1 \langle p' \,|\, \bar{\psi}(uz) \frac{\hat{z}}{2i\pi(z^2)^2} \psi(\bar{v}z) \,|\, p \rangle \ln z^2\, B_{QQ}(u,v)\, du\, dv\,. \tag{7.5}$$

Parametrizing the matrix element by the nonforward distribution (4.1), we obtain for the s-channel short-distance amplitude

$$t_1^{(s)}(p,q,q') = \ln Q^2 \int_0^1 \int_0^1 (qp) \frac{B_{QQ}(u,v)\, du\, dv}{(q + (\bar{v}X - uX')p)^2 + i\epsilon} = \frac{1}{2} \ln Q^2 \int_0^1 \int_0^1 \frac{B_{QQ}(u,v)\, du\, dv}{(1-u)X' - vX + i\epsilon}\,, \tag{7.6}$$

where $X' = X - \zeta$. Using explicit expression for the $B_{QQ}(u,v)$ kernel, we obtain

$$-\frac{1}{X' + i\epsilon} \to t_1^s(X) = -\frac{1}{X' + i\epsilon} \left\{ 1 + \frac{\alpha_s}{2\pi} C_F \left[\frac{3}{2} + \ln\left(\frac{X' + i\epsilon}{-\zeta + i\epsilon} \right) \right] \ln Q^2 \right\}\,. \tag{7.7}$$

A similar expression can be derived for the evolution of the u-channel-type term:

$$-\frac{1}{X - i\epsilon} \to t_1^u(X) = -\frac{1}{X - i\epsilon} \left\{ 1 + \frac{\alpha_s}{2\pi} C_F \left[\frac{3}{2} + \ln(X/\zeta) \right] \ln Q^2 \right\}\,. \tag{7.8}$$

Clearly, the u-channel term can be obtained from the s-channel one by the change $X' \to X$, $\zeta \to -\zeta$. In the region $X < \zeta$, both t_1^u and t_1^s are real. Furthermore, it is easy to establish that the correction terms in both cases vanish when integrated with the asymptotic distribution $6X(1 - X/\zeta)/\zeta$, explicitly showing that the latter does not evolve with Q^2. Note that $t_1^u(X)$ is purely real in the whole range $0 \leq X \leq 1$, while $t_1^s(X)$ is purely real only in the region $X < \zeta$. For $X \geq \zeta$, it has an imaginary part:

$$t_1^s(X) = -\mathrm{P}\frac{1}{X - \zeta} + i\pi\delta(X - \zeta) + \frac{\alpha_s}{2\pi} C_F \left\{ \frac{3}{2}\left(-\mathrm{P}\frac{1}{X - \zeta} + i\pi\delta(X - \zeta) \right) \right.$$
$$\left. + i\pi \left[\frac{\theta(X \geq \zeta)}{(X - \zeta)} \right]_+ - \left[\frac{\ln|X/\zeta - 1|}{X - \zeta} \right]_+ \right\} \ln Q^2\,. \tag{7.9}$$

This information can be used to write down the expression showing the leading logarithm evolution of the function $\mathcal{F}_\zeta(\zeta;Q^2)$ determining the imaginary part of the amplitude:

$$\mathcal{F}_\zeta(\zeta;Q^2) = \mathcal{F}_\zeta(\zeta;Q_0^2) + \frac{\alpha_s}{2\pi} C_F \ln Q^2/Q_0^2 \int_\zeta^1 \left\{ \delta(X - \zeta)\left(\frac{3}{2} - \int_0^1 \frac{dz}{1-z} \right) + \frac{1}{X - \zeta} \right\} \mathcal{F}_\zeta(X;Q_0^2)\, dX\,. \tag{7.10}$$

Evidently, the expression in the braces is given by the asymmetric evolution kernel $P_\zeta^{QQ}(\zeta, X)$ (5.26). For the usual distribution function the analogous equation contains the GLAPD kernel $P(\zeta/X)$:

$$f(\zeta;Q^2) = f(\zeta;Q_0^2) + \frac{\alpha_s}{2\pi} C_F \ln Q^2/Q_0^2 \int_\zeta^1 \left\{ \delta(X - \zeta)\left(\frac{3}{2} - 2\int_0^1 \frac{dz}{1-z} \right) + \frac{1 + (\zeta/X)^2}{X - \zeta} \right\} f(X;Q_0^2)\, dX\,. \tag{7.11}$$

The comparison of the two expressions shows that evolution of the function $\mathcal{F}_\zeta(\zeta;Q^2)$ is not identical to that of $f(\zeta;Q^2)$. Recall also that in the forward case the lowest-order amplitude is proportional to $1/(X - \zeta + i\epsilon) + 1/(X + \zeta - i\epsilon)$.

B. Gluonic contribution to hard exclusive meson electroproduction

The kinematics of hard exclusive meson electroproduction processes $\gamma^* p \to M p'$ is very close to that of the virtual Compton scattering, especially in a situation when one can neglect the mass of the final meson. Again, one can use the α-representation rules to determine possible regimes capable of producing a powerlike contribution for large Q^2. The basic difference is the absence of the regime analogous to short-circuiting a subgraph containing the photon vertices, since instead of the final photon described by an elementary field we have now a bound state. Hence, the leading short-distance regime corresponds to contraction into point of a subgraph which contains the virtual photon vertex and located in the middle between the two long-distance-sensitive pp'- and q'-components of the diagram. The pp'-component is described by the nonforward distribution function while the q'-part is parametrized by the meson distibution amplitude.

Depending on the type of lines connecting the short-distance subgraph with the $\langle p'|\ldots|p\rangle$ matrix element, one deals either with quark (Fig.9a) or gluonic (Fig.10) contributions to the lowest-order amplitude. The structure of the quark contribution is similar to that of the hard-gluon-exchange contribution to the meson electromagnetic form factor, with the distribution amplitude of the initial state substituted by the quark nonforward distribution. There is also an analog of the soft contribution to the meson form factor (see Fig.9b). It corresponds to the infrared regime $\alpha_{\sigma_2} \to \infty$.

Let us concentrate here on the gluonic contribution which requires a proper handling of restrictions imposed by gauge invariance. Using the coordinate representation for the hard propagators, we can write the contribution of Fig.10a as

$$T_g^a(p,p',q') = \int \langle q', M|\bar\psi(0)\,\gamma^\alpha S^c(-z_1)\tau^a\gamma^\mu S^c(z_1-z_2)\tau^b\gamma^\nu\psi(z_2)|0\rangle\,\langle p'|A_\mu^a(z_1)A_\nu^b(z_2)|p\rangle\, d^4z_1 d^4z_2\,, \tag{7.12}$$

where τ^a, τ^b are the $SU(3)$ color matrices. The first matrix element here can be expressed through the meson distribution amplitude $\varphi(\tau)$ while the second one is related to the asymmetric gluon distribution. Other 3 lowest-order diagrams can be written in a similar way. Applying formally the power counting (see Eq.(3.7) and discussion preceding it), we may conclude that each gluonic contribution has an extra Q^2 factor compared to the quark term, since the quarks have twist 1 while the twist of the gluon vector potential A_μ is zero. Technically, the enhancement appears when the $p_\mu p_\nu$ factor from the matrix element $\langle p'|A_\mu^a(z_1)A_\nu^b(z_2)|p\rangle$ combines with the q'_μ, q'_ν factors from hard propagators and polarization vectors, thus producing the estimate $\langle p'|AA|p\rangle \sim Q^2$. However, the power counting formulas like (3.7) only give an upper estimate for the relevant contribution. The actual behavior is determined by the twist $t_\mathcal{O}$ of the composite operator $\mathcal{O}$ constructed from the elementary fields corresponding to the external lines of the SD-subgraph. It is well-known that the simplest *gauge-invariant* composite operator containing two gluonic fields is $G_{\mu\rho}G_\nu^\rho$, and its twist equals 2 rather than 0, just like for the lowest-twist $\bar\psi\ldots\psi$ operator. Diagrammatically, this means that, in Feynman gauge, the leading-power terms of 4 lowest-order diagrams completely cancel each other and the total result is suppressed by $1/Q^2$ compared to leading contributions of separate diagrams. In general, picking out non-leading power terms (higher twist contributions) is a notoriously difficult problem of perturbative QCD. However, in our case, the cancellation of leading terms is guaranteed by gauge invariance of the total result. Hence, choosing a gauge in which the combination $q'_\mu q'_\nu\langle p'|A_\mu^a(z_1)A_\nu^b(z_2)|p\rangle$ is prevented from producing the $(q'p)^2$ factor, we would eliminate the artificially enhanced terms on diagram by diagram basis. This goal is achieved if one uses the gauge $q'^\mu A_\mu(z;q')=0$. Then A_μ can be expressed in terms of the field-strength tensor $G_{\mu\rho}$ (see, *e.g.*, [52])

$$A_\mu(z;q') = q'^\rho \int_0^\infty G_{\mu\rho}(z+\sigma q')\,e^{-\epsilon\sigma}\,d\sigma. \tag{7.13}$$

This representation also makes it easy to parametrize the matrix element $\langle p'|A_\mu^a(z_1)A_\nu^b(z_2)|p\rangle$ in terms of the gauge-invariant gluon distribution:

$$\langle p'\,|\,A_\mu^a(z_1;q')A_\nu^b(z_2;q')\,|\,p\rangle|_{(z_1-z_2)^2=0} = \frac{\delta^{ab}}{N_c^2-1}\,\frac{\bar u(p')\hat q' u(p)}{2(q'\cdot p)}\left(-g_{\mu\nu}+\frac{p_\mu q'_\nu + p_\nu q'_\mu}{(p\cdot q')}\right) \tag{7.14}$$
$$\times\int_0^1\left(e^{-iX(pz_1)+iX'(pz_2)}+e^{iX'(pz_1)-iX(pz_2)}\right)\frac{\mathcal{F}_\zeta^g(X)}{(X-i\epsilon)(X'+i\epsilon)}\,dX \;+\; \text{“}\mathcal{K}\text{”}.$$

In ref. [8], the amplitude of hard diffractive electroproduction was calculated for the longitudinal polarization of both the virtual photon ($\epsilon_{\gamma^*}^\mu = (q'^\mu + \zeta p^\mu)/Q$) and produced vector meson ($\epsilon_V^\mu = q'^\mu/m_V$). In this case, the contribution of Fig.10a in the $(q'A)=0$ gauge can be written as

$$T_{LL}^a(p,q',r) \sim \bar u(p')\hat q' u(p)\int_0^1 d\tau\,\varphi_V(\tau)\int_0^1 \mathrm{Sp}\left\{\hat\epsilon_V\hat\epsilon_{\gamma^*}\frac{\zeta\hat p-\tau\hat q'}{(\zeta p-\tau q')^2}\gamma_\mu\frac{(X-\zeta)\hat p+\tau\hat q'}{((X-\zeta)p+\tau q')^2}\gamma_\nu\right\}$$
$$\times\left(-g_{\mu\nu}+\frac{p_\mu q'_\nu+p_\nu q'_\mu}{(p\cdot q')}\right)\frac{\mathcal{F}_\zeta^g(X)}{(X-i\epsilon)(X-\zeta+i\epsilon)}\,dX\,, \tag{7.15}$$

where $\varphi_V(\tau)$ is the distribution amplitude of the longitudinal vector meson. This gives

$$T_{LL}^a(p,q,r) \sim \frac{\bar u(p')\hat q' u(p)}{Qm_V}\int_0^1 \varphi_V(\tau)\,\frac{d\tau}{\tau}\int_0^1 \frac{\mathcal{F}_\zeta^g(X)}{X(X-\zeta+i\epsilon)}\,dX\,. \tag{7.16}$$

Other diagrams give similar contributions, differing only in the τ-dependent factor. For Fig.10b, one should substitute $1/\tau$ by $1/\bar\tau$, while Figs.10c,d both have $1/\tau\bar\tau$ factor. Since $1/\tau+1/\bar\tau = 1/\tau\bar\tau$, the total contribution also has the $1/\tau\bar\tau$ structure

$$T_{LL}(p,q,r) \sim \frac{\sqrt{1-\zeta}}{Qm_V}\int_0^1 \varphi_V(\tau)\,\frac{d\tau}{\tau\bar\tau}\int_0^1 \frac{\mathcal{F}_\zeta^g(X)}{X(X-\zeta+i\epsilon)}\,dX\,, \tag{7.17}$$

where $\sqrt{1-\zeta}$ comes from $\bar u(p') = \sqrt{1-\zeta}\,\bar u(p)$. The ampitude $T_{LL}(p,q,r)$ has imaginary part due to the factor $1/(X-\zeta+i\epsilon)$:

$$\frac{1}{\pi}\,\mathrm{Im}\,T_{LL}(\zeta) \sim \frac{\sqrt{1-\zeta}}{\zeta Q m_V}\,\mathcal{F}_\zeta^g(\zeta)\int_0^1 \frac{\varphi_V(\tau)}{\tau\bar\tau}\,d\tau\,. \tag{7.18}$$

In ref. [8], the gluonic matrix element was approximated by the gluon distribution function $f_g(\zeta)$. To get our result from that of ref. [8], one should substitute there $f_g(\zeta)$ by $\sqrt{1-\zeta}\,\mathcal{F}_\zeta^g(\zeta)/\zeta$.

Though the asymmetric distribution function $\mathcal{F}_\zeta^g(X)$ coincides with $Xf_g(X)$ in the limit $\zeta=0$, in general these two functions differ when $\zeta\neq 0$. As discussed earlier, the imaginary part appears for $X=\zeta$, *i.e.*, in an asymmetric configuration in which the second gluon carries a vanishing fraction of the original hadron momentum, while $\zeta f_g(\zeta)$ corresponds to a symmetric configuration in which the final gluon has the momentum equal to that of the initial one.

VIII. FACTORIZATION AND END-POINT EFFECTS

A. General remarks

The standard question about pQCD applications for hard processes is whether factorization of short- and long-distance contributions is maintained in higher orders. Since Feynman integrals can be written in different representations, one can approach the factorization problem in various ways. In particular, the classic studies of deep inelastic scattering in QCD [53,50,51] relied on the operator product expansion in which the coordinate representation plays a crucial role. The claims that factorization also holds for a more complicated Drell-Yan process [54,55] were supported by studies [56,35,57] based on the analysis in the momentum representation (see, however, [43]). The early studies of exclusive processes in QCD which started with the analysis of the large-Q^2 behavior of the pion EM form factor also incorporated both the OPE-like coordinate representation methods [2,4] and momentum-representation oriented approaches [3,6]. Factorization was intensively studied in the following years (see [45,59] and references therein). Referring an interested reader to ref. [16] for a recent momentum-representation analysis of factorization for hard exclusive electroproduction processes, here we briefly discuss possible sources of factorization breaking analysing them within our approach [5] based on the combined use of α-representation and the OPE-type methods.

B. Structure of the lowest-order term

Exclusive processes are rather vulnerable to factorization breaking. In contrast to inclusive cross sections, factorization for exclusive ampitudes may fail even at the tree level. Hence it is a good idea just to write down the lowest-order contribution and carefully look at it. Take the DVCS amplitude (7.2). It has terms $1/(X - i\epsilon)$ and $1/(X - \zeta + i\epsilon)$ which are singular for $X = 0$ and $X = \zeta$, respectively. An immediate question is whether these singularities appear within the region of integration and if yes, whether they are inside that region or at its end-points. To be prepared to address this question, we performed a detailed study of spectral properties of nonforward distributions. Our α-representation analysis shows that $0 \leq X \leq 1$. Since the singularity $1/(X-\zeta+i\epsilon)$ is inside the integration region, we can write it as $\mathrm{P}\{1/(X-\zeta)\} - i\pi\delta(X-\zeta)$: it generates both real and imaginary part of the amplitude. On the other hand, the $1/(X - i\epsilon)$ singularity is at the end-point, and the relevant real part is given by a divergent integral unless the nonforward distribution $\mathcal{F}_\zeta(X)$ vanishes at $X = 0$. Hence, to claim factorization for the real part, it is absolutely necessary to give the arguments that $\mathcal{F}_\zeta(0) = 0$. In our analysis, we derived $\mathcal{F}_\zeta(X)$ from the double distribution $F(x, y)$. The basic expression for $\mathcal{F}_\zeta(X)$ shows that $\mathcal{F}_\zeta(X) \sim X$ for any $F(x, y)$ which is finite as $x, y \to 0$. One can get $\mathcal{F}_\zeta(0) \neq 0$ only if $F(x, y)$ is singular for $x = 0$, *e.g.*, if it behaves like $\delta(x)$ and does not vanish when $y = 0$. If $F(x, y)$ has such a behavior, there should be a special reason for it.

Similarly, for the meson electroproduction, the integral over τ contains the factor $1/\tau(1-\tau)$ singular at the endpoints $\tau = 0$, $\tau = 1$. Again, the factorization formula makes sense only if the distribution amplitude $\varphi(\tau)$ vanishes for $\tau = 0, 1$. Since $\varphi(\tau)$ is analogous to the $\zeta = 1$ limit of a nonforward distribution, we may expect that it also vanishes at $\tau = 0$ because of small phase space for the $\tau \to 0$ configuration. Furthermore, since for massless quarks $\varphi(1-\tau) = \pm\varphi(\tau)$, if $\varphi(\tau)$ vanishes at $\tau = 0$, it also vanishes for $\tau = 1$.

Of course, even if the vanishing at end-points holds for any diagram of perturbation theory, this still does not mean that the nonperturbative functions have the same property. So, a cautious statement might be that if in perturbation theory some function does not vanish at a particular end-point, it is unlikely that it will vanish non-perturbatively. If it vanishes perturbatively, there is some hope that this property is preserved for the nonperturbative function.

The standard procedure to get an educated guess concerning the end-point behavior of hadron distribution amplitude $\varphi(\tau, \mu)$ is to study the asymptotic $\mu \to \infty$ limit of their evolution. This idea is equivalent to saying that $\varphi(\tau, \mu)$ has the same behavior at the end-points as the relevant BL evolution kernel $V(\tau, \tau')$. In particular, in refs. [58] it was shown that $\varphi_L^{as}(\tau) \sim \tau(1-\tau)$ for the longitudinally polarized ρ-meson while $\varphi_T^{as}(\tau) \sim 1 - 2\tau\bar{\tau}$ for the transversely polarized one. This result excludes the transverse case from straightforward pQCD applications. This fact was repeatedly emphasized in refs. [8,9,16,19].

Similar estimates of the end-point behavior of the distribution amplitudes follow from QCD sum rule considerations. In particular, if perturbative term $\Pi^{pert}(\tau, M^2)$ of the QCD sum rule (M^2 is the SVZ-Borel parameter)

$$f_\rho \varphi(\tau) e^{-m_\rho^2/M^2} + \text{higher states} = \Pi^{pert}(\tau, M^2) + \text{condensates} \tag{8.1}$$

vanishes for $\tau = 0$ and $\tau = 1$, one can argue that because of quark-hadron duality, $\varphi(\tau)$ should also vanish at the end-points. For the longitudinally polarized ρ-meson, we have indeed $\Pi_L^{pert}(\tau, M^2) \sim \tau(1-\tau)$. However, for transverse polarization, $\Pi_T^{pert}(\tau, M^2) \sim const$, and integrating $\varphi_T(\tau)/\tau$ over τ one faces a logarithmic divergence.

Note, furthermore, that both quark and gluon propagators of the simplest hard subraph have denominators proportional to τ. However, for a longitudinally polarized virtual photon, only the $O(\tau)$ term in the numerator of the quark propagator survives which converts the $1/\tau^2$ singularity of the hard amplitude into $1/\tau$. This will not happen if the photon is transversely polarized. Hence, for transverse polarization one would face the integral with $\varphi_T(\tau)/\tau^2$ which linearly diverges if $\varphi_T(\tau) \sim const$. Such a strong divergence indicates that the amplitude may be dominated by the IR-regime (see Fig.9*b*) with soft quark exchange.

C. Double-flow regime

One of the lessons from the discussion above is that taking into account only the denominators of the "hard" quark and gluon propagators one is guaranteed to get a $1/\tau^2$ factor capable of destroying factorization from the very start. It is the cancellation of one power of τ by a numerator factor in case of a longitudinally polarized virtual photon which makes the factorization possible. In the absence of this cancellation, even if we take $\varphi_L(\tau) \sim \tau(1-\tau)$, the integral would logarithmically diverge. One may object that in such a situation factorization still works if $\varphi_L(\tau)$ vanishes faster than τ as $\tau \to 0$. Note, however, that evolution generates terms proportional to τ:

$$\varphi_L(\tau, \mu^2) = \varphi_L(\tau, Q^2) + \ln Q^2/\mu^2 \int_0^1 V(\tau, \tau')\varphi_L(\tau', \mu^2)\, d\tau' ,$$

since $V(\tau, \tau') \sim \tau$ for small τ. In the presence of nonzero masses m or other infrared cut-offs, one should change $1/\tau$ by $1/(\tau + m^2/Q^2)$. As a result, the logarithmic divergence converts the τ integral into an extra $\ln Q^2/m^2$. Together with the evolution logarithm $\ln Q^2/\mu^2$, they would amount to a double logarithm in a one loop diagram of Fig.11 type. It should be emphasized that this is not a Sudakov double logarithm. In particular, in two loops one would only get $\ln^3 Q^2$ ($\ln^2 Q^2$ from evolution and $\ln Q^2$ from the τ-integral) rather than $\ln^4 Q^2$. The possibility to get an extra logarithm in the form-factor-type amplitudes was discovered a long time ago in a scalar model (see *e.g.*, ref. [60]). In a scalar model, there are no numerator factors to moderate the $1/\tau^2$ singularity, hence such a possibility is always realized. In ref. [5], the diagram for a scalar analog of the pion form factor was studied with the help of the α-representation and the Mellin transformation. It was shown that, in the superrenormalizable $\phi^3_{(4)}$ model, this diagram has the $\ln[Q^2/m^2]/Q^4$ behavior despite the fact that there is no logarithmic evolution in this model. The logarithm appears because the leading SD-pole $1/(J+2)$ for the Mellin transform of this diagram can be obtained in two ways: from the small-ρ_L integration ($\rho_L = \alpha_1 + \alpha_2$) and from the small-$\rho_R$ integration ($\rho_R = \alpha_4 + \alpha_5$). There are no other possibilities. In particular, small-λ integration ($\lambda = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$) gives a non-leading pole $1/(J+3)$. Hence, the leading term comes from a configuration in which the large momentum Q flows simultaneously through two subgraphs $V_L = \{\sigma_1, \sigma_2\}$ and $V_R = \{\sigma_4, \sigma_5\}$ while the momentum through the intermediate line σ_3 is small. Such a configuration was called in ref. [5] the double-flow regime.

In a renormalizable $\phi^3_{(6)}$ model the diagram shown in Fig.11 has the $\ln^2[Q^2/m^2]/Q^4$ behavior because the leading SD-pole $1/(J+2)$ can be obtained in three ways: from small-λ integration, from small-ρ_L integration and from small-ρ_R integration. The factorization for a scalar analog of the pion form factor in the $\phi^3_{(6)}$ model was studied in more detail in ref. [61]. It was shown there, in particular, that the $\ln^2 Q^2/m^2$ behavior of the one-loop diagram results from the overlap of the evolution and the double-flow regime. In ref. [5], it was emphasized that the presence of the double-flow regime is a natural feature of exclusive ampitudes. Hence, to establish factorization, one should first check whether it is present or not. For the pion form factor in QCD (and other renormalizable models with spin-$\frac{1}{2}$ quarks) its absence to all orders was demonstrated in ref. [5].

A rather peculiar double-flow contribution appears in a two-loop pQCD diagram for the nucleon form factors [62]. Its specifics is that it works for a term in which one takes only quark masses in the numerators of the propagators of the intermediate lines. Proceeding by a routine calculation, it is rather difficult to detect such a contribution among a wide variety of two loop terms. However, it is rather easy to find it if one has a guiding principle, such as the requirement that both the left and right components of a double-flow configuration should simultaneously give the leading power behavior.

IX. COMPARISON WITH OTHER APPROACHES AND NOTATIONS

In our definitions of various distribution functions, we took the relevant matrix element and expressed it through an integral representation over the momentum fractions, incorporating the spectral condition $0 \leq X \leq 1$. Another approach is to introduce distribution functions by making a Fourier transform of the matrix element with respect to (pz) (cf. [63,32,33]). One can easily derive the result of such a procedure by rewriting our representations in a form with a universal exponential in the r.h.s. Consider, *e.g.*, the matrix element for the quark operator:

$$\begin{aligned}&\langle p', s' \,|\, \bar{\psi}_a(0)\hat{z}E(0,z;A)\psi_a(z)\,|\,p,s\rangle|_{z^2=0} \\ &= \bar{u}(p',s')\hat{z}u(p,s)\int_{-1+\zeta}^{1} e^{-i\widetilde{X}(pz)}\left[\mathcal{F}_\zeta^a(\widetilde{X};t)\,\theta(0\le\widetilde{X}\le 1) - \mathcal{F}_\zeta^{\bar a}(\zeta-\widetilde{X};t)\,\theta(-1+\zeta\le X\le\zeta)\right]d\widetilde{X} + \text{"}\mathcal{K}\text{"}\,.\end{aligned} \tag{9.1}$$

The Fourier transformation would project out the function

$$\widetilde{\mathcal{F}}_\zeta^a(\widetilde{X};t) = \mathcal{F}_\zeta^a(\widetilde{X};t)\,\theta(0\le\widetilde{X}\le 1) - \mathcal{F}_\zeta^{\bar a}(\zeta-\widetilde{X};t)\,\theta(-1+\zeta\le\widetilde{X}\le\zeta) \tag{9.2}$$

which coincides with the quark distribution for $\zeta \le \widetilde{X} \le 1$, reduces to the (minus) antiquark distribution for $-1+\zeta \le \widetilde{X} \le 0$ and is given by their difference for $0 \le \widetilde{X} \le \zeta$. The $\widetilde{X}$-variable changes within the segment $(-1+\zeta, 1)$ centered at $X = \zeta/2$, with the total range length equal to $2-\zeta$. To avoid the non-symmetric and ζ-dependent limits, one can introduce the variable (cf. [11])

$$\tilde{x} \equiv \frac{\widetilde{X}-\zeta/2}{1-\zeta/2} \tag{9.3}$$

which changes from -1 to 1. The ratio

$$\xi \equiv \frac{\zeta}{1-\zeta/2} \tag{9.4}$$

is an alternative parameter characterizing the longitudinal momentum asymmetry of the non-forward matrix element. For $t=0$ and a massless hadron, it varies between 0 and 2. The reversed relations are

$$\zeta = \frac{\xi}{1+\xi/2}\ ,\quad \widetilde{X} = \frac{\tilde{x}+\xi/2}{1+\xi/2}\ ,\quad \widetilde{X}-\zeta = \frac{\tilde{x}-\xi/2}{1+\xi/2}\,. \tag{9.5}$$

Using translation invariance (cf. Eq.(4.14)), one can easily derive that the operator with the quark fields taken at symmetric points $-z/2, z/2$ has a rather compact representation in terms of the $\tilde{x}$-variable:

$$\langle p', s' \,|\, \bar{\psi}_a(-z/2)\hat{z}E(-z/2,z/2;A)\psi_a(z/2)\,|\,p,s\rangle|_{z^2=0} = \bar{u}(p',s')\hat{z}u(p,s)\int_{-1}^{1} e^{-i\tilde{x}(Pz)}H_a(\tilde{x},\xi;t)\,d\tilde{x} + \text{"}E_a\text{"}\,, \tag{9.6}$$

where $P=(p+p')/2$ is the average momentum of the initial and final hadron (note, that $(Pz) = (1-\zeta/2)(pz) = (pz)/(1+\xi/2)$). This representation is equivalent to the definition of the *off-forward parton distributions* $H_a(\tilde{x},\xi;t)$, $E_a(\tilde{x},\xi;t)$ introduced by X. Ji [11] (see also [12]). Basically, the latter are related to our non-forward distributions by

$$\widetilde{\mathcal{F}}_\zeta^a(\widetilde{X};t) = (1+\xi/2)H_a(\tilde{x},\xi;t)\,, \tag{9.7}$$

and similarly for other functions. The off-forward distributions $H_a(\tilde{x},\xi;t)$, *etc.* are defined both for positive and negative $\tilde{x}$. Depending on the value of $\tilde{x}$, one can distinguish three different components: quark ($\xi/2 \le \tilde{x} \le 1$), antiquark ($-1 \le \tilde{x} \le -\xi/2$) and mixed "quark minus antiquark" ($-\xi/2 \le \tilde{x} \le \xi/2$) components of H. The mixed component corresponds evidently to the region $0 \le \widetilde{X} \le \zeta$ of the $\widetilde{X}$-variable in which the nonforward distributions can be treated as distribution amplitudes. Since $\widetilde{X}(pz) = (\tilde{x}+\xi/2)(Pz)$ and $(\widetilde{X}-\zeta)(pz) = (\tilde{x}-\xi/2)(Pz)$, the partons in this picture carry momenta $(\tilde{x}+\xi/2)P$ and $(\tilde{x}-\xi/2)P$. Using Eqs.(9.5),(9.7), one can relate our QQ evolution kernels with those given in ref. [13]. The gluonic matrix element can be also represented in the form of Eq.(9.6):

$$\langle p' \,|\, z_\mu z_\nu G^a_{\mu\alpha}(-z/2)E^{ab}(-z/2,z/2;A)G^b_{\alpha\nu}(z/2)\,|\,p\rangle|_{z^2=0} \tag{9.8}$$

$$= \frac{1}{2}\bar{u}(p')\hat{z}u(p)\,(Pz)\int_{-1}^{1} e^{-i\tilde{x}(Pz)}H_g(\tilde{x},\xi;t)\,d\tilde{x} + \text{"}E_g\text{"}\,. \tag{9.9}$$

Due to the symmetry property $H_g(\tilde{x},\xi;t) = H_g(-\tilde{x},\xi;t)$, integration over $\tilde{x}$ in this case can be restricted to the $0 \le \tilde{x} \le 1$ region. Note, that in the forward limit $\xi=0$, $t=0$, the function $H_g(\tilde{x},\xi;t)$ reduces to $\tilde{x}f_g(\tilde{x})$ (cf.

Eq.(4.11)). To get an off-forward distribution reducing to $f_g(\tilde{x})$, Ji [13] uses the definition equivalent to adding a factor of $\tilde{x}$ in the integrand on the rhs of Eq.(9.9): $H_g(\tilde{x},\xi;t) \to xH_g^{Ji}(\tilde{x},\xi;t)$. However, $\tilde{x}=0$ corresponds to $X=\zeta/2$ or to the middle-point $Y=1/2$ of the distribution amplitude $\Psi_\zeta^g(Y)$ (see Eq.(9.5)), *i.e.*, to a situation when the gluons carry equal fractions $\zeta p/2$ of the original momentum p. Since $H_g(\tilde{x},\xi;t)$ is an even function of $\tilde{x}$, there are no evident reasons that it vanishes for $\tilde{x}=0$. Hence, dividing $H_g(\tilde{x},\xi;t)$ by $\tilde{x}$ produces an artificial singularity of $H_g^{Ji}(\tilde{x},\xi;t)$ for $\tilde{x}=0$.

Another parametrization for the non-forward matrix element of the gluon operator was proposed by Collins, Frankfurt and Strikman [16]. Their definition of the *non-diagonal* gluon distribution $f_g(x_1,x_2;t)$ is also based on the Fourier transformation. For positive values, their variables x_1, x_2 correspond to our fractions X and $X-\zeta \equiv X'$, respectively. In our notations, the function $f_g(x_1=X, x_2=X-\zeta;t)$ can be written as

$$f_g(X, X-\zeta;t) = \frac{1}{X(X-\zeta)}\widetilde{\mathcal{F}}_\zeta^g(X;t)\,. \tag{9.10}$$

The factor $1/X(X-\zeta)$ was motivated by the necessity to cancel the inverse factor which may emerge from the derivatives present in the field-strength tensor $G_{\mu\nu}$. Actually, this expectation is not supported by perturbative calculations. Take, *e.g.*, the evolution kernel $P_\zeta^{gQ}(X,Z)$. It can be treated as a perturbative, leading-logarithm approximation for the gluon distribution inside a quark (cf. [17]). According to Eq.(5.28), $P_\zeta^{gQ}(X,Z)$ does not vanish for $X=\zeta$. If $\mathcal{F}_\zeta^g(X)$ does not vanish for $X=\zeta$, the function $\widetilde{\mathcal{F}}_\zeta^g(X;t)$ does not vanish both for $X=0$ and $X=\zeta$ and $f_g(x_1,x_2)$ is singular both for $x_1=0$ and $x_2=0$.

In fact, the combination $\mathcal{F}_\zeta^g(X)/(X-i\epsilon)(X-\zeta+i\epsilon)$ appears in our parametrization (7.14) for the matrix element of the operator constructed from two vector potentials A_μ, A_ν taken in the light-cone gauge. In this sense, $f_g(x_1,x_2)$ or, what is the same, $\mathcal{F}_\zeta^g(X)/X(X-\zeta)$ can be treated as a basic gluon distribution given by the matrix element of the product of fundamental gluonic fields $A_\mu A_\nu$ rather than by that of the secondary fields $G_{\mu\rho}G_\nu^\rho$. Note, however, that if $f_g(x_1,x_2)$, *i.e.* $\mathcal{F}_\zeta^g(X)/X(X-\zeta)$, has no singularities, then the meson electroproduction amplitude has no imaginary part at leading twist. Since this is impossible, $f_g(x_1,x_2)$ *must* have singularities, and one may wish to explicitly display them specifying their nature, *e.g.*, $1/(x_2-i\epsilon)$, $1/(x_1+i\epsilon)$. This goal is achieved automatically if $\mathcal{F}_\zeta^g(X)$ is used as the basic distribution.

In our approach, the starting point is the double distribution $F_g(x,y;t)$ defined through the non-forward matrix element of the gauge-invariant gluonic operator

$$\begin{aligned}&\langle p' \,|\, z_\mu z_\nu G^a_{\mu\alpha}(0)E^{ab}(0,z;A)G^b_{\alpha\nu}(z)\,|\,p\rangle|_{z^2=0} \\ &= \frac{1}{2}\bar{u}(p')\hat{z}u(p)\,(pz)\int_0^1 dx\int_0^1 \frac{1}{2}\left(e^{-ix(pz)-iy(rz)} + e^{ix(pz)-i\bar{y}(rz)}\right)F_g(x,y;t)\,\theta(x+y\le 1)\,dy\,.\end{aligned} \tag{9.11}$$

As explained earlier, in perturbation theory the spectral properties $0 \le \{x, y, x+y\} \le 1$ can be proved to any order with the help of the α-representation. Furthermore, the function $F_g(x,y;t)$ does not depend on the ζ-parameter. The family of ζ-dependent nonforward gluon distributions $\mathcal{F}_\zeta^g(X;t)$ is obtained from $F_g(x,y;t)$ by integration over y (see (2.45)):

$$\mathcal{F}_\zeta^g(X;t) = \int_0^{\min\{X/\zeta, \bar{X}/\bar{\zeta}\}} F_g(X-y\zeta, y;t)\,dy\,. \tag{9.12}$$

Recall that the double distribution $F_g(x,y;t)$ can be treated as a distribution function with respect to x and as a distribution amplitude with respect to y. This physical interpretation suggests that $F_g(x,y;t)$ is a regular function for all values of y and for at least nonzero values of x. We made this reservation because the evolution asymptotically makes $F_g(x,y;t;\mu)$ (we added the dependence on the factorization scale μ) proportional to $\delta(x)$ as $\mu\to\infty$. In this situation, $F_g(x,y;t;\mu)$ is singular at $x=0$. However, the $\delta(x)$-term still produces a regular nonforward distribution $\mathcal{F}_\zeta^g(X;t)$, though confined to the restricted region $0 \le X \le \zeta$.

Assuming that the double distribution $F_g(x,y;t;\mu)$ is finite everywhere, we conclude that the nonforward distribution $\mathcal{F}_\zeta^g(X;t;\mu)$ in this case is also finite for all $0 \le X \le 1$ and, moreover, that it vanishes for $X=0$. As discussed earlier, the latter property is vital for factorization. If it is not fulfilled, the X-integral in the lowest-order expression diverges at the end-point $X = 0$, where the $1/(X - i\epsilon)$ prescription is of no help. One may think that this problem can be avoided if one uses the function $\tilde{\mathcal{F}}_\zeta^g(\tilde{X};t)$ defined through the Fourier transformation with the variable $\tilde{X}$ changing from $-1+\zeta$ to 1. Since the point $\tilde{X} = 0$ is inside the integration region, the $1/(\tilde{X} - i\epsilon)$ prescription apparently may help. Note, however, that if our function $\mathcal{F}_\zeta^g(X;t)$ does not vanish for $X = 0$, the Fourier transform $\tilde{\mathcal{F}}_\zeta^g(\tilde{X};t)$ is not continuous both for $\tilde{X} = 0$ and $\tilde{X} = \zeta$. As a result, the singularities of $\tilde{\mathcal{F}}_\zeta^g(\tilde{X};t)/(\tilde{X} - i\epsilon)(\tilde{X} - \zeta + i\epsilon)$ are not integrable.

X. CONCLUSIONS

In this paper, we discussed basic properties of nonforward parton distributions, a new type of functions accumulating nonperturbative information about hadron dynamics. We demonstrated that there are two basic ways to describe asymmetric matrix elements $\langle p'|\mathcal{O}(0,z)|p\rangle$ of quark and gluon light-cone operators $\mathcal{O}(0,z)$. One possibility is to introduce double distributions $F(x,y;t)$ which are independent of the longitudinal momentum asymmetry parameter $\zeta = 1 - (p'z)/(pz)$ of the matrix element and refer to the light-cone fractions xp, yr of the original hadron momentum p and momentum transfer $r = p' - p$ carried by the active parton. Another approach is to use nonforward distribution functions $\mathcal{F}_\zeta(X;t)$ which specify the light-cone projection of the total momentum $Xp = xp + yr$ carried by the parton. These functions $\mathcal{F}_\zeta(X;t)$ explicitly depend on ζ. Both types of distributions have hybrid properties, in some aspects resembling usual parton distribution functions and in other ones the distribution amplitudes. Their t-dependence is analogous to that of hadronic form factors. The use of $\mathcal{F}_\zeta(X;t)$ is more convenient for ultimate applications to hard pQCD processes, resulting in a formalism that is very similar to the standard pQCD parton picture. On the other hand, the double distributions $F(x,y;t)$ have more transparent spectral properties which has serious advantages at the foundation stages of the pQCD analysis. In this paper, we concentrated on general aspects of the theory of nonforward distributions and their uses. There are many interesting applications to deeply virtual Compton scattering and hard exclusive electroproduction processes which require further, more specific studies of the nonforward distribution functions including modeling their nonperturbative low-energy shape, logarithmic pQCD evolution, calculation of nonlogarithmic higher-order corrections, *etc.* Work in this direction has already been started [11]- [21].

XI. ACKNOWLEDGEMENTS

I thank A. Afanasev, I. Balitsky, L. Frankfurt, I. Halperin, P. Hoodbhoy, C. Hyde-Wright, X. Ji, G. Korchemsky, I. Musatov, A. Szczepaniak, G. Sterman, M. Strikman and A. Zhitnitsky for useful discussions and correspondence. I am grateful to N. Isgur for continued encouragement and advice. This work was supported by the US Department of Energy under contract DE-AC05-84ER40150.

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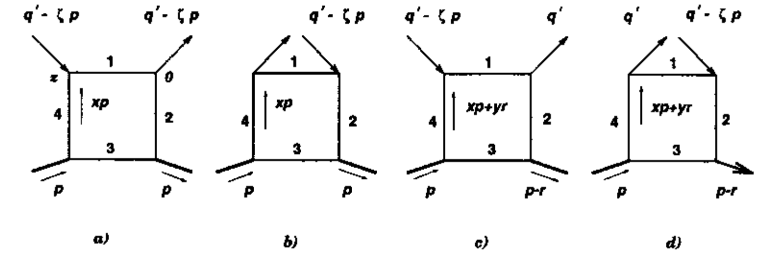

FIG. 1. Scalar model analogs of a), b) virtual forward Compton amplitude and c), d) deeply virtual Compton scattering.

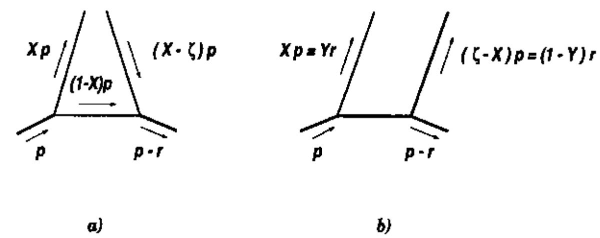

FIG. 2. Longitudinal momentum flow for two components of the asymmetric distribution function $\mathcal{F}_\zeta(X)$: a) $X > \zeta$ and b) $X < \zeta$.

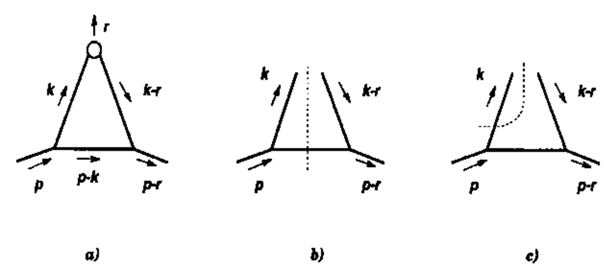

FIG. 3. a) Structure of momentum integral defining the asymmetric distribution function $\mathcal{F}_\zeta(X)$. b) Cut of parton-hadron amplitude corresponding to the residue for the region $X > \zeta$. c) Cut of parton-hadron amplitude corresponding to the residue for the region $X < \zeta$.

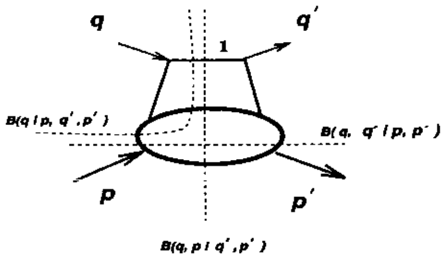

FIG. 4. Handbag diagram for deeply virtual Compton scattering.

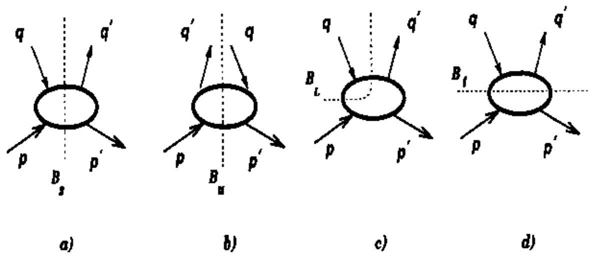

FIG. 5. Four-point amplitude corresponding to the deeply virtual Compton scattering.

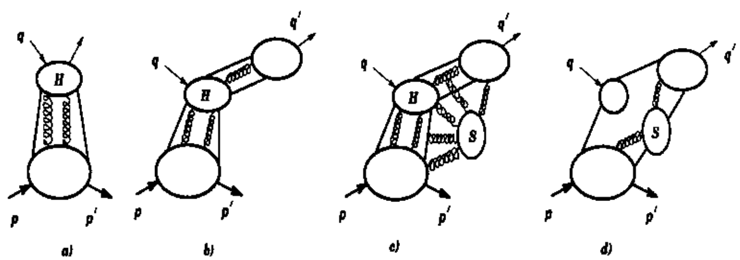

FIG. 6. Some regimes responsible for powerlike contributions to the DVCS amplitude.

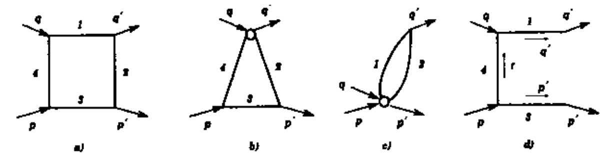

FIG. 7. *a)* Scalar one-loop analog of the DVCS amplitude. Reduced graphs corresponding to *SD*-regimes *b)* $\alpha_1 \sim 0$, *c)* $\alpha_3 + \alpha_4 \sim 0$ and *d)* *IR*-regime $\alpha_2 \sim \infty$.

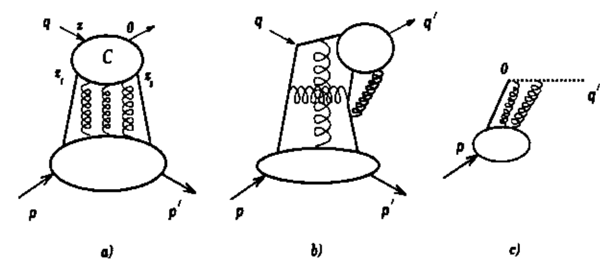

FIG. 8. *a)* General structure of the leading *SD* contribution to the DVCS amplitude in QCD. *b)* *SD* configuration with two long-distance parts.*c)* Matrix element with double-logarithmic dependence on the *UV* cut-off parameter μ.

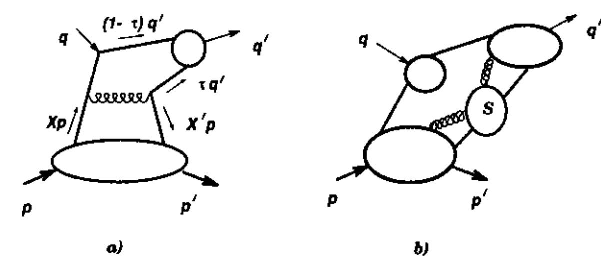

FIG. 9. Hard exclusive meson electoproduction process: *a)* Leading *SD*-contribution with quark nonforward distribution; *b)* Soft contribution.

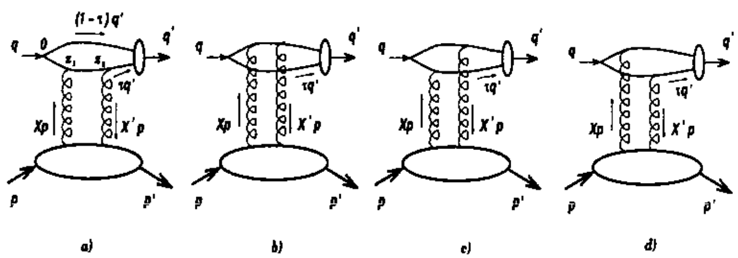

FIG. 10. Gluon contribution to hard exclusive meson electoproduction amplitude.

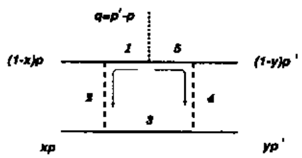

FIG. 11. Double-flow regime for the scalar analog of a meson form factor.