Consistency constraints on m_s from QCD dispersion relations and chiral perturbation theory in $K_{\ell 3}$ decays

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Abstract

We use both old and new theoretical developments in QCD dispersion relation constraints on the scalar form factor in the decay $K \to \pi \ell \nu_\ell$ to obtain constraints on the strange quark mass. The perturbative QCD side of the calculation incorporates up to four-loop corrections, while the hadronic side uses a recently developed parameterization constructed explicitly to satisfy the dispersive constraints. Using chiral perturbation theory (χPT) as a model for soon-to-be measured data, we find a series of lower bounds on m_s increasing with the accuracy to which one believes χPT to represent the full QCD result.

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I. INTRODUCTION

The level of interest in using dispersion relations to study the analytic properties of Green functions has ebbed and flowed for decades. Recently, attention has increased due to studies of bounds obtained on form factors from heavy hadron semileptonic decays, which can be used in concert with Heavy Quark Effective Theory to isolate the CKM elements $|V_{cb}|$ and $|V_{ub}|$ (see [1] for a compilation of references). Such studies were initially motivated by the 1981 paper of Bourrely, Machet, and de Rafael [2], which first applied the basic method of equating the dispersion integral over hadronic form factors to its perturbative QCD evaluation in the deep Euclidean region, in that paper for the case of $K_{\ell 3}$ decays. Here we revisit $K_{\ell 3}$ decays, armed with new improvements in technique and the promise of vastly more data to be produced at DAΦNE [3]. Because the dispersive bound uses QCD to constrain the shape of each hadronic form factor (as a function of momentum transfer), one can use the experimentally measured form factors to obtain limits on QCD parameters such as the quark masses; however, until this data is available, one can use precisely the same method to take the form factor from a given model or calculation as being correct, and discover limits on values of the QCD parameters for which the shape of the form factor is consistent with the dispersive bounds.[1]

In particular, we use the one-loop corrected chiral perturbation theory (χPT) result of Gasser and Leutwyler [4] for the scalar form factor in $K_{\ell 3}$ decays. The scalar form factor is especially interesting because it contributes to the scalar current correlator, which is proportional to $(m_s - m_u)^2$ (*i.e.*, vanishing for flavor-conserving processes), and is thus more sensitive to quark mass values. Moreover, it satisfies low-energy theorems in χPT to a high degree of precision. Our results may also be used for the vector form factor to obtain interesting information, but our thrust in this paper is to show how experiment will be able to teach us about limits on m_s, and specifically, that the consistency of χPT and QCD places substantial lower limits on the strange quark mass.

This paper is organized as follows: In Sec. II we summarize the most important parts of the dispersion relation approach with references to papers explaining the fine points in more detail, as well as describe our χPT inputs. Section III presents the QCD result for the scalar current correlator to four loops, including leading nonperturbative effects. In Sec. IV we present improved versions of the analytic functions used to parameterize form factors consistent with QCD and describe how to judge the quality of fits to this form. In Sec. V we present our results and conclude.

II. USING THE DISPERSION RELATIONS AND χPT

By means of Cauchy's theorem, the dispersion integral simply relates an integral of a Green function (in our case, a current two-point correlator) expressed over a kinematic region

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[1]Alternately, one could insist that particular QCD inputs are correct and use the bounds to obtain limitations on the consistency of the model with QCD. Since we are using the model as a proxy for real data, we do not follow this approach here.

of momentum transfer q^2 where hadronic quantities conveniently describe the physics, to the Green function itself evaluated at a point deep in the Euclidean region, where perturbative QCD is reliable. In the present case, the current is $V^\mu = \bar{s}\gamma^\mu u$, whose divergence $\partial \cdot V = i(m_s - m_u)(\bar{s}u)$ is just the scalar current multiplied by an explicit factor of quark masses; it is this feature that leads us to interesting bounds on m_s. The scalar correlator, related to the vector correlator by means of a Ward identity, is given by

$$\psi(Q^2) = i \int d^4x \, e^{iqx} \langle 0 | \mathrm{T} \partial_\mu V^\mu(x) \partial_\nu V^\nu(0)^\dagger | 0 \rangle, \tag{2.1}$$

where $Q^2 \equiv -q^2$. The unsubtracted dispersion relation reads

$$\psi(Q^2) = \frac{1}{\pi} \int_0^\infty dt \, \frac{\mathrm{Im}\,\psi(t)}{(t + Q^2)}. \tag{2.2}$$

The n-times subtracted form of (2.2) is readily obtained by taking Q^2 derivatives on both sides, and in the remainder of this section we take $\psi(Q^2)$ to mean the perturbatively computed two-point function with any fixed number of subtractions.

The l.h.s. of (2.2) is evaluated at a point Q^2 far from the resonant region of V^μ, which for $s \to u$ transitions is satisfied when $Q^2 \gg \Lambda^2_{\mathrm{QCD}}$. In practice, we choose $Q^2 = 4\,\mathrm{GeV}^2$, which is popular in lattice simulations. The r.h.s. of (2.2) is expressed as a sum of integrals over hadronic quantities. It is a manifestly nonnegative quantity, and so neglecting any subset of contributions produces a strict inequality, in that any subset of hadronic form factors in any kinematic region connected by crossing symmetry is limited by an expression computable in QCD. In particular, integrals of the form factors for $K \to \pi$ in the crossed kinematic region of (vacuum $\to \bar{K}\pi$) are bounded, leading to restrictions on the shape of form factors permitted by QCD. The tightness of these bounds is of course regulated by numerical inputs to the QCD calculation, and particularly in this case by m_s. It is precisely this sensitivity that is useful to us: Given the shape of a $K \to \pi$ form factor obtained from data or a model, one may ask which values of m_s permit this shape to be consistent with QCD.

The derivation of the bounds on the shapes of semileptonic decay form factors and its expression in terms of a well-defined parameterization draws upon techniques developed in Refs. [2], and [5–7], and the approach is explained in detail in [6]. Here we present expressions only to establish notation for later use. Beginning with the kinematic points

$$t_\pm \equiv (m_K \pm m_\pi)^2, \tag{2.3}$$

it is convenient in such decays to use the kinematic variable

$$z(t; t_s) \equiv \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_s}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_s}}, \tag{2.4}$$

where $t_s < t_+$ is a kinematic scale fixed for convenience, as described below. For each form factor $F(t)$ there is a computable function $\phi_F(t; t_s; Q^2)$ obtained from (2.2), and in terms of these quantities all functional forms for the form factors allowed by QCD may be parameterized by

$$F(t) = \frac{\sqrt{\psi(Q^2)}}{\phi_F(t; t_s; Q^2)} \sum_{n=0}^\infty a_n \, z(t; t_s)^n \quad , \tag{2.5}$$

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where the coefficients a_n are unknown parameters obeying

$$\sum_{n=0}^{\infty} |a_n|^2 \leq 1 . \tag{2.6}$$

This result employs analyticity of the two-point function away from hadronic thresholds, crossing symmetry between the matrix elements for $\bar{K}\pi$ production and $K \to \pi$, and knowledge of QCD at Q^2. To lowest order in G_F this form is exact, for there are no resonant poles, multiparticle continuum states, or anomalous thresholds in the scalar channel below the threshold $t = t_+$ of (vacuum $\to \bar{K}\pi$). To be precise, one must also distinguish the two isospin channels (vacuum $\to \bar{K}^0\pi^-$) and (vacuum $\to K^-\pi^0$). Apart from a small explicit isospin breaking in the form factors, whose inclusion is described below, the two channels have slightly different values of $t_\pm$. Since nonanalytic q^2 behavior in the correlator, which arises from physical and anomalous particle thresholds, is what determines the form of the parameterization (2.5), we take t_+ to be the smaller of the two possibilities, $(m_{K^+} + m_{\pi^0})^2$, so that no threshold is ever crossed in the region where (2.5) is used.

The two form factors relevant to $K_{\ell 3}$ decays are defined by

$$\langle \pi(p')|V^\mu|K(p)\rangle = 2f_+(q^2)\left(p^\mu - \frac{p \cdot q}{q^2}q^\mu\right) + d(q^2)\frac{q^\mu}{q^2} \tag{2.7}$$

where $q^\mu = (p - p')^\mu$. This decomposition separates the vector f_+ and scalar d form factors, and consequently the bounds studied in this work apply to d. In the differential width, $|d(q^2)|^2$ appears multiplied by a helicity suppression factor of m_ℓ^2.

In lieu of very precise experimental information, we use the results [4] of one-loop chiral perturbation theory for the scalar form factor as "data". At two kinematic points in particular, $q^2 = 0$ and $q^2 = (m_K^2 - m_\pi^2)$, predictions of amazing precision have been made. At $q^2 = 0$, the authors of [4] find

$$f_+^{K^0\pi^-}(0) = 0.977, \qquad f_+^{K^+\pi^0}(0) = 0.998. \tag{2.8}$$

By the Ademollo-Gatto theorem [8], the symmetry relation $f_+(0) = 1$ is violated by terms of order m_s^2, and we see these percent-level deviations in the predictions (2.8). Uncertainties on (2.8) are even smaller, perhaps a few parts per 10^3. The relation to the scalar form factor at this point is

$$d(0) = (m_K^2 - m_\pi^2)f_+(0). \tag{2.9}$$

At $q^2 = (m_K^2 - m_\pi^2)$, the value of $d(q^2)$ is given by the Callan-Treiman relation [9],

$$d(m_K^2 - m_\pi^2) = (m_K^2 - m_\pi^2)\left(\frac{f_K}{f_\pi} + \Delta_{\rm CT}\right), \tag{2.10}$$

where $f_K/f_\pi = 1.22 \pm 0.01$, and $\Delta_{\rm CT}$ measures deviations from the exact Callan-Treiman limit. As calculated in [4], $\Delta_{\rm CT} = -3.5 \cdot 10^{-3}$. Since no strange 0^+ resonances appear until the relatively high mass $K_0^*(1430)$ and $K_0^*(1950)$, the form factor in the decay region should be very smooth (and indeed, almost linear) to a very good approximation. It is therefore very reasonable to assume that the scalar form factor is given in the intermediate

region, *i.e.,* between $q^2 = 0$ and $q^2 = (m_K^2 - m_\pi^2)$, by chiral perturbation theory to the same accuracy as at the endpoints. It is, of course, difficult to estimate this error, although some indication may be obtained from a forthcoming two-loop χPT calculation [10], despite the possible appearance there of currently undetermined renormalization constants. At any rate, since the one-loop corrections to the low-energy theorems are already quite small, the whole framework of χPT would crumble if new experiments showed deviations from the predictions by more than a few percent.

III. THE SCALAR CORRELATOR IN QCD

Instead of the divergent scalar correlator $\psi(Q^2)$, we consider its second derivative $\psi''(Q^2)$, which is free of renormalization point dependence and therefore satisfies a homogeneous renormalization group (RG) equation. $\psi''(Q^2)$ is calculated in perturbative QCD for large Q^2 using an expansion in powers of the quark masses and the operator product expansion. The leading term has been calculated to four loops [11], and the $O(m_s^2/Q^2)$ correction to three loops [12]. The $O(m_s^4/Q^4)$ contribution is best considered together with the nonperturbative condensate terms. Collecting these results,

$$
\begin{aligned}
\psi''(Q^2) = {} & \frac{6(m_s - m_u)^2}{(4\pi)^2 Q^2} \left\{ 1 + \frac{11}{3}\left(\frac{\alpha_s}{\pi}\right) + \left(\frac{\alpha_s}{\pi}\right)^2 \left(\frac{5071}{144} - \frac{35}{2}\zeta(3)\right) \right. \\
& \left. + \left(\frac{\alpha_s}{\pi}\right)^3 \left[-\frac{4781}{9} + \frac{1}{6}\left(\frac{4748953}{864} - \frac{\pi^4}{6} - \frac{91519}{36}\zeta(3) + \frac{715}{2}\zeta(5)\right) + \frac{475}{4}\zeta(3) \right] \right\} \\
& - \frac{12(m_s - m_u)^2 m_s^2}{(4\pi)^2 Q^4} \left\{ 1 + \frac{28}{3}\left(\frac{\alpha_s}{\pi}\right) + \left(\frac{\alpha_s}{\pi}\right)^2 \left(\frac{8557}{72} - \frac{77}{3}\zeta(3)\right) \right\} \\
& + \frac{(m_s - m_u)^2}{Q^6} \left\{ 2\langle m_s \bar{u}u \rangle \left[1 + \frac{23}{3}\left(\frac{\alpha_s}{\pi}\right) \right] - \frac{1}{9}I_G \left[1 + \frac{121}{18}\left(\frac{\alpha_s}{\pi}\right) \right] \right. \\
& \left. + I_s \left[1 + \frac{64}{9}\left(\frac{\alpha_s}{\pi}\right) \right] - \frac{3}{7\pi^2}m_s^4 \left[\left(\frac{\pi}{\alpha_s}\right) + \frac{155}{24} \right] \right\},
\end{aligned}
\tag{3.1}
$$

where the RG-invariant condensate combinations I_s and I_G are given by

$$
I_s = m_s\langle \bar{s}s \rangle + \frac{3}{7\pi^2}m_s^4 \left[\left(\frac{\pi}{\alpha_s}\right) - \frac{53}{24} \right],
\tag{3.2}
$$

$$
\begin{aligned}
I_G = {} & -\frac{9}{4}\left\langle \frac{\alpha_s}{\pi}G^2 \right\rangle \left[1 + \frac{16}{9}\left(\frac{\alpha_s}{\pi}\right) \right] + 4\left(\frac{\alpha_s}{\pi}\right)\left[1 + \frac{91}{24}\left(\frac{\alpha_s}{\pi}\right) \right] m_s\langle \bar{s}s \rangle \\
& + \frac{3}{4\pi^2}\left[1 + \frac{4}{3}\left(\frac{\alpha_s}{\pi}\right) m_s^4 \right],
\end{aligned}
\tag{3.3}
$$

where we have set $n_f = 3$ and omitted logarithms that vanish when taking $\mu = Q \equiv \sqrt{Q^2}$. Note that the peculiar π/α_s terms cancel. The full result depends logarithmically on the renormalization point μ and on the parameters of the theory, like α_s, m_s, and condensates, which are renormalized at μ; however, as has been advocated in [13], we implement the RG improvement for the case of the scalar correlator in the following way: $\psi''(Q^2)$ is evaluated at $\mu = Q$, and the parameters α_s and m_s are extrapolated from a chosen reference point (in our case $\Lambda_{\overline{\text{MS}}}$) to $\mu = Q$ using the four-loop beta functions (compiled in [13]). The

condensates are so poorly known and their effect at the chosen scale $Q^2 = 4$ GeV^2 so small that their μ dependence may be ignored.

The numerical values chosen for the QCD inputs are $Q^2 = 4$ GeV^2, $\Lambda_{\overline{\text{MS}}}^{n_f=3} = 380 \pm 60$, $m_u = m_s/25$, $\langle m_s \bar{u}u \rangle = \langle m_s \bar{s}s \rangle = -f_K^2 m_K^2 = -0.031$ GeV^4, and $\langle \alpha_s G^2/\pi \rangle = 0.02$–$0.06$ GeV^4, although the sensitivity of the analysis to these nonperturbative parameters is small. The scale Q is arbitrary subject to the constraints that if it is too small, perturbative QCD is unreliable, while if it is too large, the dispersive bounds thus obtained are weak.

IV. PARAMETERIZATION AND QUALITY OF FIT

In order to use the parameterization of Eq. (2.5), one requires expressions for the function ϕ for each form factor. In notation designed to be similar to that of [2], we define

$$\beta_0 \equiv \sqrt{\frac{t_+}{t_+ - t_s}}, \qquad \beta_1 \equiv \sqrt{\frac{t_+ - t_-}{t_+ - t_s}}, \qquad \beta_2 \equiv \sqrt{\frac{t_+ + Q^2}{t_+ - t_s}}. \tag{4.1}$$

Note that, in our notation, one must set the parameter $t_s = -Q^2$ to obtain the limit of Ref. [2], in which case $\beta_2 = 1$. The ϕ for each form factor defined in Eq. (2.7) is given by

$$\phi_d(z) = \sqrt{\frac{\eta^2}{2\pi} \frac{(1+z)}{(1-z)^2} \frac{1}{t_+ - t_s}} \left[\beta_0 + \frac{1+z}{1-z} \right]^{-1} \left[\beta_1 + \frac{1+z}{1-z} \right]^{1/2} \left[\beta_2 + \frac{1+z}{1-z} \right]^{-3}, \tag{4.2}$$

$$\phi_{f_+}(z) = \sqrt{\frac{\eta^2}{48\pi} \frac{(1+z)^2}{(1-z)^3}} \sqrt{\frac{Q^2}{t_+ - t_s}} \left[\beta_0 + \frac{1+z}{1-z} \right]^{-3} \left[\beta_1 + \frac{1+z}{1-z} \right]^{3/2} \left[\beta_2 + \frac{1+z}{1-z} \right]^{-2}. \tag{4.3}$$

These expressions for ϕ differ from those in [1], because the dispersion integrals are formulated using different linear combinations of the two polarization tensor component functions than used in the other work, leading to a different pattern of subtractions. The factors η represent Clebsch-Gordan coefficients based on isospin symmetry; that is, both charged and neutral K processes contribute to the dispersion relation, so we exploit the near equality of their contributions. For the decay $K_L \to \pi^{\pm} \ell^{\mp} \nu_\ell$ (whose width equals, by CPT, either that for K^0 or $\bar{K}^0$ semileptonic decays), $\eta^2 = 3/2$, while for $K^+ \to \pi^0 \ell^+ \nu_\ell$, $\eta^2 = 3$. If isospin breaking between charged and neutral K processes is significant, one can incorporate this difference into η^2, or even include a factor with q^2 dependence (written in terms of z) if the pattern of isospin breaking is known. In practice, we obtain a conservative correction to $\eta_{K^+}^2 = 3$ by supposing that the form factor ratio represented by (2.8) at $q^2 = 0$ persists for all values of q^2 in the $\bar{K}\pi$ production region, to obtain an effective $\eta_{K^+}^2 = 2.92$.

In the original work [2], the bounds are expressed as the determinant of an $n \times n$ semipositive-definite matrix, where the form factor is assumed known at $(n-2)$ points. This produces an envelope of allowed form factors resembling a chain of sausages, since the form factor is required to pass through each of the $(n-2)$ points. Although presented explicitly in [2] for only 1 or 2 fixed points, the determinant method can of course be generalized to an arbitrary number of points, with a corresponding increase in the complexity of algebraic expressions appearing (see, *e.g.*, [14]). However, once the envelope of points allowed by the dispersion relation and chosen fixed points is established, it is *not* true that any curve lying within this envelope still satisfies the dispersive bound. In contrast, the parameterization of Eq. (2.5) subject to (2.6) always satisfies the determinant bound of arbitrarily high degree.

The central features that make the parameterization (2.5) useful are the bound (2.6) on the coefficients a_n and the smallness of the kinematic variable z over the full range for allowed semileptonic decay. It follows that one may express the form factor over the full range using only the first $N + 1$ parameters $\{a_0, a_1, \ldots, a_N\}$, with the remaining infinite set bounded in magnitude and forming a theoretical *truncation error* [7], δ_N:

$$\delta_N \equiv \frac{\sqrt{\psi''(Q^2)}}{|\phi(z)|} \sqrt{1 - \sum_{n=0}^{N} |a_n|^2 \frac{|z|^{N+1}}{\sqrt{1 - |z|^2}}}, \tag{4.4}$$

which means that the form factor fit to these parameters using (2.5) has a *theoretical* uncertainty no larger than δ_N.

The kinematic parameter t_s is used to minimize the size of this already small truncation error [1,5]. Equation (4.4) makes it clear that this minimization occurs when $z = 0$ lies within the kinematic range chosen for the fit, $t \in [t_{\min}, t_{\max}]$; from (2.4) one sees that this can occur only if t_s also assumes some value in this range. Therefore, the truncation error is minimized by plotting (4.4) as a function of $t, t_s \in [t_{\min}, t_{\max}]$, and finding that value of t_s for which the maximum value over all $t \in [t_{\min}, t_{\max}]$ is smallest.

Now suppose that the "data", in our case the χPT expression for the form factor, has an "experimental" uncertainty Δ. In order to state that χPT "data" agrees with QCD to within Δ, it must be possible to expand the QCD fit to an order N such that $\sum_{n=0}^{N} |a_n|^2 \leq 1$ *and* $\delta_N \leq \Delta$. In other words, in order to be certain that the *exact all-orders* QCD form factor F_{exact} (which is not accessible to us) lies within an uncertainty Δ of the data $F_{\chi\text{PT}}$, we require that the deviation δ_N of the *truncated* QCD form factor F_{trunc} from F_{exact} is smaller than[2] the experimental uncertainty Δ. This is the central argument of our reasoning.

The requirement that both $\delta_N \leq \Delta$ and (2.6) are satisfied is the key to obtaining bounds on parameters in $\psi''(Q^2)$, in particular m_s. A curve such as $F_{\chi\text{PT}}$, which does not *à priori* satisfy (2.6), when expanded as a power series in z eventually produces a value of a_N for some N so large that (2.6) is violated. This does not necessarily mean that $F_{\chi\text{PT}}$ violates QCD, because it is only required to equal the F_{exact} within an uncertainty of Δ. However, the value of N must be large enough that $\delta_N \leq \Delta$, or else F_{trunc} expanded to $n = N$ is not necessarily a good enough approximation to use in place of F_{exact}; if it is not possible to carry out this fit and maintain Eq. (2.6), then the chosen inputs are not consistent with QCD bounds. When the fit is successful, one concludes that the true QCD form factor is actually given by the fit of the χPT form factor to the parameterization up to order N, and the higher-order terms cannot be probed with χPT since they lie within our stated uncertainty Δ.

The particular nature of the fit is irrelevant to us, since one requires additional physical input to distinguish two curves that otherwise satisfy all the requisite conditions. For example, we use (2.5) and choose to Taylor expand the χPT form factor in z as an expression for $F(z)\phi_F(z)/\sqrt{\psi''(Q^2)}$, to obtain a_n values until (2.6) is violated and throw away all higher

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[2]If one prefers to combine the theoretical and experimental uncertainties, then the requirement becomes $|F_{\text{trunc}} - F_{\chi\text{PT}}| < \sqrt{\Delta^2 - \delta_N^2}$.

a_n. Alternately, one may choose a highest-order N and perform a χ^2 fit to $\{a_0, a_1, \ldots, a_N\}$ subject to the constraint (2.6); this would be the preferred method of analysis with binned data.

Now consider the dependence of the bounds on the strange quark mass, which enters approximately linearly in $\sqrt{\psi''(Q^2)}$ as seen in (3.1). If one fixes all other quantities and decreases m_s, it is clear from (2.5) that one obtains the same form factor by increasing each a_n by the same factor. However, eventually the a_n's are large enough to saturate (2.6), and thus one finds a minimal allowed value for m_s. There is no corresponding maximal value, for one can certainly make each a_n as small as desired and still satisfy (2.6). Furthermore, as we increase the precision Δ to which we believe the χPT form factor holds, δ_N must decrease, so that we must go to a higher order N in the parameterization expansion. It becomes increasingly more difficult to avoid large values of a_n unless m_s is increased, so the lower bound on m_s becomes larger as Δ decreases.

V. RESULTS AND CONCLUSIONS

We present lower bounds on $m_s \equiv m_s^{\overline{\text{MS}}}(1 \text{ GeV})$ from the form factor $F(t) \equiv d(t)/(m_K^2 - m_\pi^2)$ of $K^+ \to \pi^0 \ell^+ \nu_\ell$, since its bound (due to the isospin factor η) is roughly $\sqrt{2}$ tighter than that from K_L decay, as described above. The natural tendency for the truncation errors δ_N is to decrease an order of magnitude with each unit of N, since $z_{\max} \approx -z_{\min} \approx 0.1$ for optimal values of t_s over the range $t_{\min} = 0$, $t_{\max} = (m_K^2 - m_\pi^2)$; however, this is somewhat counteracted by the fact that $\delta_N \propto \sqrt{\psi''}$, which increases when the smallest allowed value of m_s increases with N. Moreover, δ_N can be much smaller than the numbers we present, since we do not include the effects of the $\sqrt{1 - \sum_n |a_n|^2}$ term in (4.4); near the saturation of (2.6) by the first few terms, this is a large suppression.

Our numerical results are summarized in Table I. The conclusion we draw is that, if the χPT form factor calculation is believed only to a precision of 2–5%, then we conclude only that $m_s > 40$ MeV. At a precision of $\frac{1}{2}$–1%, $m_s > 90$ MeV, and at 1/20%, $m_s > 140$ MeV. The same sort of analysis will be possible with data, although the details of the fit will differ somewhat.

We conclude by comparing briefly with other recent approaches [15–17] that bound m_s using dispersion relations. Typically, what is done is to saturate the hadronic side as much as possible with phenomenological input from expected pole contributions, and/or a continuum contribution in the deep Minkowski region of the hadronic integral modeled by the perturbative QCD result. Often, multiple constraints are obtained by taking moments of both sides. We view this work as complementary to those approaches. On one hand, it is minimal in the sense that only $\bar{K}\pi$, which is presumably but a small portion of the total hadronic result, is used in the bounding inequalities; of course, modelers may obtain stronger bounds by including $\bar{K}_0^*$ poles or continuum contributions at the cost of introducing model-dependent inputs. Although we considered only one moment in our calculation, certainly additional moments also give constraints, although then Q^2 must be adjusted to larger values in order to make sure that $\psi^{(n)}(Q^2)$ is calculable perturbatively. On the other hand, it includes data from semileptonic form factors measured over their entire kinematic range, and so is expected to provide a substantial amount of additional input to such constraints.

We expect that combined program of calculations from all of these approaches will deliver rather strong constraints on m_s.

N	$\{a_n\}$	$t_s(\mathrm{GeV}^2)$	$\psi'' \cdot 10^5$	$\delta_{N,\max}$	$\Delta F_{\max}$	$m_s(\mathrm{MeV})$
1	$a_0 = +0.910$ $a_1 = +0.414$	0.144	1.16	$1.4 \cdot 10^{-2}$	$2.8 \cdot 10^{-2}$	> 41
2	$a_0 = +0.418$ $a_1 = +0.184$ $a_2 = -0.890$	0.142	5.54	$3.1 \cdot 10^{-3}$	$3.4 \cdot 10^{-3}$	> 90
3	$a_0 = +0.273$ $a_1 = +0.118$ $a_2 = -0.584$ $a_3 = -0.756$	0.140	13.01	$5.0 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$	> 139

Table 1. *Bounds on $m_s \equiv m_s^{\overline{\mathrm{MS}}}$ (1 GeV), fit parameters, and truncation errors based on the saturation (2.6) of the parameterization (2.5). ΔF is an abbreviation for $|F_{\mathrm{trunc}} - F_{\chi\mathrm{PT}}|$. The numerical value of $\delta_{N,\max}$ here neglects the $\sqrt{1 - \sum_n |a_n|^2}$ term in (4.4), which can be a large suppression when (as here) (2.6) is nearly saturated by the first N terms.*

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