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New Constraints on Dispersive Form Factor Parameterizations
from the Timelike Region

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Abstract

We generalize a recent model-independent form factor parameterization derived from rigorous dispersion relations to include constraints from data in the timelike region. These constraints dictate the convergence properties of the parameterization and appear as sum rules on the parameters. We further develop a new parameterization that takes into account finiteness and asymptotic conditions on the form factor, and use it to fit to the elastic π electromagnetic form factor. We find that the existing world sample of timelike data gives only loose bounds on the form factor in the spacelike region, but explain how the acquisition of additional timelike data or fits to other form factors are expected to give much better results. The same parameterization is seen to fit spacelike data extremely well.

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I. INTRODUCTION

Dispersion relations in field theory represent nothing more than the application of Cauchy's theorem to Green functions, and yet information obtained from these identities continues to provide fascinating new insights into systems difficult or impossible to probe with other techniques. The natural advantage of the dispersive approach within QCD is that its incorporation of nonperturbative information is fully rigorous and model independent, since it works directly with hadronic Green functions in the contour integral. Thanks to Cauchy's theorem, the same quantity may be evaluated at a kinematic point where perturbative QCD provides the best physical description of the Green function.

And yet the physical input is minimal. Nothing has been included from the fundamental theory except for the value of the Green function at a single point and the existence of hadronic bound states of given quantum numbers. Clearly, QCD dispersion relations provide only a bare framework of conditions that Green functions must satisfy in order to be consistent with QCD (or any well defined field theory), namely unitarity in the form of quark-hadron duality, crossing symmetry, and analyticity outside of poles and cuts with locations dictated by physical particle thresholds. There still remains a large space of possible hadronic Green functions, each of which satisfies the dispersive constraints. Nevertheless, the constraints can be surprisingly restrictive, especially when a small amount of additional physical input, such as data or the result of some model calculation, is included. It follows that constructing models manifestly satisfying dispersive constraints is an economical way of enforcing consistency with the basic features of QCD.

The particular formulation of dispersion relations presented here focuses upon obtaining bounds on both the size and shape of electroweak form factors; this is a very old game, dating back into the early 1960s [1], long before the advent of QCD. The essential notion is the computation of the current two-point correlator in two ways: at a point deep in the Euclidean region of momentum transfer t , and as an integral over the cut generated by on-shell hadronic states produced by the current. Components of the correlator are chosen so that both sides are positive definite, implying that the neglect of contributions from some of the hadronic states leads to a strict inequality on the hadronic amplitudes, *i.e.*, form factors.

The inequality thus obtained is expressed as a weighted integral of squared form factors over the kinematic cut in t corresponding to production of these hadrons. Assuming only that the branch point t_+ of this cut is the lowest real value of t where the Green function exhibits non-analytic behavior, analytic continuation to the rest of the complex t plane carries the inequality to other kinematic regions. The inclusion of perturbative QCD Green functions in these studies was first made in Ref. [2]. Subsequently, it was observed that the most general space of analytic functions satisfying the dispersive bounds obeys a simple parameterization [3], which led to a number of applications in the study of semileptonic decays, in which it is the weak current that appears in the correlator.

Recent applications use only data below t_+ to restrict possible form factors. Surely data from $t \geq t_+$ must provide additional restrictions; however, for the semileptonic case this would require data from pair production via a weak current, which is beyond current experimental capabilities. Instead, we focus here on electromagnetic form factors, where data in the timelike region is abundant. Nevertheless, the mathematical techniques are the same. We show that data from the timelike region applied to the parameterization of Ref. [3]

leads to a number of new sum rules on the parameters, which may be incorporated in an improved parameterization. These sum rules permit a more complete characterization of the form factor at all values of t .

As an explicit example, we consider the elastic π form factor and show that fits to the world sample of timelike data lead to predictions on the spacelike behavior of the form factor, particularly its normalization at $t = 0$ and the pion charge radius. These predictions turn out to be very loose, but we shall see that this is the result of gaps in the data resulting from the fact little data has been taken near the pair production threshold $t = 4m_\pi^2$. The same parameterization using spacelike data yields an excellent fit, in particular to the normalization of the form factor at $t = 0$ and the pion charge radius.

In Sec. II we review the derivation of the parameterization from dispersive bounds, with an eye toward its application to electromagnetic form factors. Section III demonstrates the extension of the dispersive bound to the inclusion of timelike data, and its effect on the parameterization. In Sec. IV we discuss issues of convergence of the parameterization in the timelike region, and observations made in this inquiry lead us, in Sec. V, to develop an improved parameterization. Section VI presents the result of fits to the elastic π form factor and discusses at length the quality of the fits and extrapolations from timelike to spacelike regions. We also compare the results to direct fits to spacelike data using the same parameterization, as well as fits from the literature. Section VII summarizes possible improvements and concludes.

II. REVIEW OF THE DISPERSIVE APPROACH

Much of this discussion is patterned on that in Ref. [4] and references therein, but appears here again for clarity in understanding what is to follow. We begin with the QCD two-point function of a vectorlike current J^μ ; in the electromagnetic case, this is simply the conserved vector current $\bar{q}\gamma^\mu q$. The polarization tensor is defined by

$$\Pi_J^{\mu\nu}(q) = \frac{1}{q^2}(q^\mu q^\nu - q^2 g^{\mu\nu})\Pi_J^T(q^2) + \frac{q^\mu q^\nu}{q^2}\Pi_J^L(q^2) \equiv i \int d^4x e^{iqx} \langle 0 | T J^\mu(x) J^{\nu\dagger}(0) | 0 \rangle, \quad (2.1)$$

and in the case of a conserved current J , the polarization function Π^L vanishes. The remaining function Π^T is rendered finite in QCD by making two subtractions, leading to the dispersion relation

$$\chi_J^T(q^2) \equiv \frac{1}{2} \frac{\partial^2 \Pi_J^T}{\partial (q^2)^2} = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im} \Pi_J^T(t)}{(t - q^2)^3}. \quad (2.2)$$

The function $\chi_J^T(q^2)$ may be computed reliably in the perturbative QCD operator product expansion for values of q^2 far from the kinematic region where the current J can create on-shell hadronic states; for any light quark current and $Q^2 \equiv -q^2$, this condition reads $Q^2 \gg \Lambda_{\text{QCD}}^2$.

Inserting a complete set of states X into the two-point function relates Π_J^T to the production rate of hadrons from a virtual photon,

$$\text{Im} \Pi_J^T = \frac{1}{2} \sum_X (2\pi)^4 \delta^4(q - p_X) | \langle 0 | J | X \rangle |^2, \quad (2.3)$$

where the sum is over all hadronic states X with the same quantum numbers as the current J , weighted by phase space. It follows from the dispersion relation (2.2) that the perturbatively evaluated $\chi_J^T(q^2)$ equals the integrated production rate of $\gamma^* \rightarrow X$ weighted with a smooth function of momentum transfer squared t . Since the sum is positive semidefinite, one may restrict attention to a subset of hadronic states to obtain a strict inequality. In the case discussed in this paper, $J^\mu = \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d$, and we restrict to $X = \pi^+\pi^-$.¹ This places an upper bound on the electromagnetic π form factor $F(t)$ in the pair-production region that takes the form

$$\frac{1}{\pi\chi^T(q^2)} \int_{t_+}^{\infty} dt \frac{W(t) |F(t)|^2}{(t-q^2)^3} \leq 1, \quad (2.4)$$

from the dispersion relation Eq. (2.2). Here $W(t)$ is a computable function of t that depends on phase space and the quantum numbers of the particular form factor under consideration.

Using analyticity to turn (2.4) into a constraint in the spacelike region in t requires that the integrand is analytic below the pair production threshold $t < t_+$. To do this, we introduce a parameter $t_s < t_+$ and a function

$$z(t; t_s) = \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_s}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_s}} \quad (2.5)$$

that is real for $t < t_+$, zero at $t = t_s$, and a pure phase for $t \geq t_+$. Any poles in the integrand of Eq. (2.4) can be removed by multiplying the integrand by various powers of $z(t; t_s)$, provided the positions t_s of the sub-threshold poles in $F(t)$ are known. Each pole has a distinct value of t_s , and the product $z(t; t_{s1})z(t; t_{s2})\dots$ serves to remove all of them. Such poles arise as the contribution of resonances with masses below $\sqrt{t_+}$ to the form factor $F(t)$, as well as singularities in the kinematic part of the integrand. After determining these positions from the hadronic mass spectrum, the upper bound on $F(t)$ becomes

$$\frac{1}{\pi} \int_{t_+}^{\infty} dt \left| \frac{dz(t; t_0)}{dt} \right| \cdot |\phi(t; t_0)P(t)F(t)|^2 \leq 1, \quad (2.6)$$

where the weight function $\phi(t; t_0)$ (known as an *outer function* in complex analysis) is given by

$$\phi(t; t_0) = \tilde{P}(t) \left[\frac{W(t)}{|dz(t; t_0)/dt| \chi^T(q^2)(t-q^2)^3} \right]^{\frac{1}{2}}. \quad (2.7)$$

The factor $\tilde{P}(t)$ is a product of $z(t; t_s)$'s and $\sqrt{z(t; t_s)}$'s, with t_s chosen to remove the sub-threshold singularities and cuts in the kinematic part of the integrand, while the *Blaschke factor* $P(t)$ is a product of $z(t; t_p)$'s with t_p chosen to be the positions of sub-threshold poles in $F(t)$. The functions $\phi(t; t_0)$ and $\tilde{P}(t)$ also depend on q^2 , which we leave implicit for notational simplicity, while t_0 is the (yet to be chosen) value of t for which $z(t; t_0) = 0$.

The dispersion inequality expressed in terms of z reads

$$\frac{1}{2\pi i} \int_C \frac{dz}{z} |\phi(z)P(z)F(z)|^2 \leq 1, \quad (2.8)$$

where $F(z)$ means $F[z(t; t_0)]$ and so on. Lacking poles, the quantity $\phi(t; t_0)P(t)F(t)$, is expected to be analytic within the unit disc,² and may be expanded in a set of orthonormal functions that are simply powers of $z(t; t_0)$. The result of expanding in $z(t; t_0)$ is an expression for $F(t)$ valid even in the spacelike region,

$$F(t) = \frac{1}{P(t)\phi(t; t_0)} \sum_{n=0}^{\infty} a_n z(t; t_0)^n, \quad (2.9)$$

where, as a result of Eq. (2.8), the coefficients a_n are unknown constants obeying

$$\sum_{n=0}^{\infty} |a_n|^2 \leq 1. \quad (2.10)$$

The functions $P(t)$, $\phi(t; t_0)$, and $z(t; t_0)$ are real by construction for $t < t_+$. Moreover, physical cuts in the form factor, which lie on the real axis, generate discontinuities only in the imaginary part of F . Since there is no physical distinction between the upper and lower complex half-planes, one must have $|F(t+i\epsilon)| = |F(t-i\epsilon)|$ for all real t , and consequently the form factor satisfies the Schwarz reflection principle.³ It follows from analytic continuation away from the cut that the form factor is real on the real t axis below threshold, which maps to the real z axis, and therefore that the coefficients a_n are real.

III. CONSTRAINTS FROM THE TIMELIKE REGION

The constraints derived in the previous section require no input from the value of the form factor $F(t)$ for $t \geq t_+$. Clearly, if such information is supplied from another source such as direct measurement, the bound provided by, e.g., (2.4) should be strengthened. In terms of the equivalent bound of Eq. (2.8), such data appears on a segment of the unit circle in z . Suppose that one is given data for $F(t)$ from threshold up to some t_u . According to (2.5), the segment $(t_+ \mp i\epsilon, t_u \mp i\epsilon)$ written in terms of angles from $z = e^{i\theta}$ occupies the circular segments $\theta \in (\theta_u, \pi)$ and $(-\pi, -\theta_u)$, where

$$\theta_u = \cos^{-1} \left[\frac{(t_u - t_+) - (t_+ - t_0)}{t_u - t_0} \right]. \quad (3.1)$$

²Of course, subthreshold singularities due to multiparticle or anomalous thresholds must be considered. In many cases [4–6] these singularities treated as cuts are numerically unimportant, and in the $K\pi$ [7], $\pi^+\pi^-$, or $K\bar{K}$ cases they are absent. Moreover, we here correct an oversight of these previous works: For on-shell multiparticle resonances of the correlator or triangle-type anomalous threshold diagrams, the kinematics of the internal particles is completely fixed, meaning that the singularity in t is a pole, not a cut.

³Alternately, the form factor is initially defined only in the upper t half plane, and the Schwarz reflection principle continues it into the lower half plane.

¹The neutral pions do not appear here because of charge conjugation.

The additional information tells us about the integrand of Eq. (2.8) directly on the unit circle; here the Blaschke factor $P(z)$ is unimodular, and so we may express this input as follows:

$$\begin{aligned} |\phi(z = e^{i\theta_u})F(z = e^{i\theta_u})|^2 &= \left| \sum_{n=0}^{\infty} a_n z^n \right|^2 \\ &= \left(\sum_{n=0}^{\infty} a_n z^n \right) \left(\sum_{m=0}^{\infty} a_m^* \bar{z}^m \right) \\ &= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} a_m^* z^{n-m} \\ &= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} a_m^* [\cos(n-m)\theta_u + i \sin(n-m)\theta_u]. \end{aligned} \quad (3.2)$$

Since the value of the modulus $|\phi(z)F(z)|$ is invariant under reflection about the cut (which corresponds to $\theta_u \rightarrow -\theta_u$), the left hand side of the corresponding expression for $z = e^{-i\theta_u}$ is identical, while the right hand side flips the sign of the sine term; this term must therefore vanish. Equivalently, noting that the left hand side is real and recalling from Sec. II that the a_n 's are real, the purely imaginary sine term must again vanish. Yet another way of seeing the same result is by using $\sin(n-m)\theta_u = \sin n\theta_u \cos m\theta_u - \cos n\theta_u \sin m\theta_u$ and writing the full term as

$$\left(\sum_{n=0}^{\infty} a_n \sin n\theta_u \right) \left(\sum_{m=0}^{\infty} a_m^* \cos m\theta_u \right) - \left(\sum_{n=0}^{\infty} a_n \cos n\theta_u \right) \left(\sum_{m=0}^{\infty} a_m^* \sin m\theta_u \right), \quad (3.3)$$

which is pure imaginary and vanishes trivially if all the a_n 's are real. We are therefore left with one nontrivial sum rule:

$$|\phi(z = e^{i\theta_u})F(z = e^{i\theta_u})|^2 = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} a_m \cos(n-m)\theta_u, \quad (3.4)$$

We arrive at the mathematical result that allows for analysis of timelike data. Its derivation has been exceptionally straightforward, but a number of comments are in order before proceeding.

First, the derivation is completely consistent with the constraint (2.10). Indeed, integrating $\cos(m-n)\theta$ over the unit circle gives $2\pi\delta_{mn}$, and thus (3.4) integrates to (2.10), using (2.8).

Second, although the first line of (3.2) is written as a perfect square, exploiting the reality of the a_n 's means that it is more convenient to combine the two sums as in (3.4). This is because the sum rule applies to the magnitude of F for $t > t_+$. Although phase data on above-threshold form factors exists in some cases, typically it is data for $|F|$ above threshold that is presented in experimental papers. Indeed, the very nature of our dispersion relations [see (2.4)] precludes our use of such phase information.

Third, we immediately see from taking particular values of θ_u in (3.2), or even from its first line, two very interesting special cases:

$$\theta_u = \pi \quad (t = t_+) : \quad |\phi(-1)F(-1)| = \left| \sum_{n=0}^{\infty} a_n (-1)^n \right|, \quad (3.5)$$

$$\theta_u = 0 \quad (t \rightarrow +\infty) : \quad |\phi(+1)F(+1)| = \left| \sum_{n=0}^{\infty} a_n \right|. \quad (3.6)$$

Clearly, conditions on ϕ or F at these special points provide strong constraints on the parameters. We explore this next.

IV. CONVERGENCE OF THE PARAMETERIZATION

In the last section we manipulated infinite series without regard to the fine points of convergence. Does the derivation of our sum rules suffer from the possibility that some of these actions are ill-defined? In this section we show that this is not the case.

The central convergence problem may be posed as follows: In deriving (2.10), we use the analyticity of the combination $P(z)\phi(z)F(z)$ inside the unit circle to expand it in the Taylor series $\sum_n a_n z^n$. Since physical information “along the cut” actually appears at values of momentum transfer an infinitesimal distance from the genuine cut $[t \pm i\epsilon]$, it follows that this data is actually infinitesimally inside the unit circle in z , where the geometric series $\sum_n z^n$ is still (barely) convergent. The finiteness of analytic complex functions on a compact region such as the disc $|z| \leq 1 - \epsilon$, $\epsilon > 0$, gives one sense in which the sum $\sum_n a_n z^n$ has meaning as a finite number, through geometric convergence. On the other hand, for $|z| = 1$ the expression $\sum_n a_n z^n$ is defined as the value of the quantity $P(z)\phi(z)F(z)$ along (the appropriate side of) the cut, which exists since the only place where this combination is ill defined, the cut $t \geq t_+$, has been mapped so that its two sides are sent to the two separated halves of the unit circle in z . These two expressions for $\sum_n a_n z^n$ at $|z| = 1$ are equal due to a theorem of Abel's: In words, since analyticity demands that the power series $\sum_n a_n z^n$ converges for all $|z| < 1$, and since its value at $|z| = 1$ is defined, the $|z| \rightarrow 1$ limit of the former equals the latter. Therefore, the expansion as an infinite power series in z with $|z| = 1$ is true as a formal statement, but does it have meaning as a useful series with rapidly converging partial sums when we do not invoke the technicality that we are “just inside” the unit circle?

The utility of the expansion in z , when continued to the sub-threshold region for semileptonic decays, depends on its geometric convergence, using $|z| < 1$ inside the unit circle and the boundedness of the a_n 's *via* (2.10). It follows in that case that only the first few a_n 's are required to describe the form factor over the entire semileptonic region [3–6]. However, $|z| = 1$ on the unit circle, and so geometric convergence fails completely. *A priori* it seems that an arbitrarily large number of a_n 's is required to describe the form factor in the region $t > t_+$.

Yet things are not so bleak. We now show that the analyticity structure plus some mild physical assumptions tells us much about the convergence properties of the parameterization, even on the unit circle $|z| = 1$.

To begin with, in order to use Parseval's theorem rigorously to prove (2.10), one must be able to exchange the order of sum $\sum_n a_n z^n$ and integral in z ; for this purpose, we demand the sufficient condition that $\sum_n a_n z^n$ is uniformly convergent in z for $|z| = 1$. Now, $|P(z)\phi(z)F(z)|$ is by physical assumption a bounded, smooth, continuous function on the compact region represented by either half of the unit circle.⁴ However, even with these

⁴The functions $|\phi(z)|$ and $|P(z)| = 1$ have no singularities on $|z| = 1$, as discussed below, so our

assumptions, uniform convergence could be spoiled by a region of z where the form factor is not smooth in z even though it is smooth in the original kinematic variable t . The only region fitting this description is $t \rightarrow +\infty$, which is compressed into the finite region $z \rightarrow +1$, $|z| = 1$. That is, a form factor could continue to oscillate smoothly in t all the way out to infinity, but mapped to z this behavior would appear to oscillate wildly. We therefore make the additional physical assumption that $|F(z)|$ becomes *featurelessly* smooth for $t \rightarrow +\infty$ in order to guarantee uniform convergence in z and thus the proof of (2.10).

Now we may consider the full power of (2.10). Rather than using only the boundedness of each term as in the geometric case, we recognize that the absolute convergence of the sum of $|a_n|^2$ implies that for sufficiently large n , $|a_n|$ falls off faster than $1/\sqrt{n}$.⁵ Notice that, although this new criterion restricts the pattern of a_n 's, the condition of convergence of $\sum_n a_n z^n$ is still stronger. For, one could imagine, for example, that the true form factor has $a_n = +1/\sqrt{\zeta(3/2)n^{3/2}}$, which satisfies $\sum_n a_n^2 = 1$ but clearly leads to divergence of $\sum_n a_n z^n$ at $z = +1$. To proceed further, we need additional input.

Up to this point, we have ignored the specific form of the function $\phi(t; t_0)$. In the notation of [4], one begins with

$$\text{Im } \Pi^T \geq \frac{n_I}{K\pi} (t - t_+)^{\frac{1}{2}} (t - t_-)^{\frac{1}{2}} t^{-c} |F(t)|^2 \theta(t - t_+), \quad (4.1)$$

where a , b , c , and K are integers specific to the form factor under consideration, and n_I is an isospin Clebsch-Gordan factor. In the electromagnetic case for pseudoscalars, only the form factor analogous to f_+ in $\bar{B} \rightarrow D$ appears, for which $a = 3$, $b = 3$, $c = 2$, and $K = 48$. The kinematic factors are simply an expression of two-body phase space, with $t_{\pm} = (M \pm m)^2$ for $M \rightarrow m$ decays, which for the elastic case simplifies to $t_+ = 4M^2$, $t_- = 0$. The manipulations described in Sec. II lead to

$$\begin{aligned} \phi(t; t_0) &= \sqrt{\frac{n_I}{48\pi\chi^T}} \left(\frac{t_+ - t}{t_+ - t_0} \right)^{\frac{1}{2}} (\sqrt{t_+ - t} + \sqrt{t_+ - t_0}) (t_+ - t)^{\frac{1}{2}} \\ &\times (\sqrt{t_+ - t} + \sqrt{t_+})^{-\frac{1}{2}} (\sqrt{t_+ - t} + \sqrt{t_+ + Q^2})^{-3}. \end{aligned} \quad (4.2)$$

A more convenient expression for $|\phi(t; t_0)|$ with $t > t_+$ reads

$$|\phi(t; t_0)| = \sqrt{\frac{n_I}{48\pi\chi^T}} t^{-1/4} (t - t_+)(t_+ - t_0)^{-1/4} \sqrt{t - t_0} (t + Q^2)^{-3/2}. \quad (4.3)$$

Note in particular that $\phi \rightarrow (t - t_+)^1$ as $t \rightarrow t_+$, while $\phi \rightarrow t^{-1/4}$ as $t \rightarrow +\infty$. The vanishing of the first limit indicates powers of the spatial momentum $[|p|^3$ for the vector form factor]

statements refer to the behavior of $|F(z)|$. Resonances above threshold have finite widths and residues, so $|F(z)|$ is also smooth, and we assume that $|F(z)|$ is finite as $t \rightarrow +\infty$, i.e., $z \rightarrow +1$. Finally, the one-sided limit $z \rightarrow -1$ along the circle is assumed to exist.

⁵Of course, such statements here and below refer to the magnitudes of a_n in a statistical sense. Individual terms may deviate above or below the given n dependence, but the overall pattern of the terms is bounded by the stated behaviors.

of the pair created near threshold, while the vanishing of the second limit indicates the asymptotic unitarizing behavior of $|p|^3/\sqrt{t}$. Making only the mild physical assumptions that the form factor F is not infinite at threshold and does not grow as $t \rightarrow +\infty$ [already assumed in proving (2.10)], (3.5) and (3.6) give the sum rules

$$\sum_{n=0}^{\infty} a_n = 0, \quad \sum_{n=0}^{\infty} a_n (-1)^n = 0, \quad (4.4)$$

or equivalently,

$$\sum_{n=0}^{\infty} a_{2n} = 0, \quad \sum_{n=0}^{\infty} a_{2n+1} = 0. \quad (4.5)$$

Further improvements are possible if one considers the particular nature of the vanishing of $\phi(t; t_0)$ as $t \rightarrow t_+$. Since the parameters a_n are defined with respect to the variable z , this is most obvious if one considers ϕ as a function of z rather than t , in which one sees [4] that $\phi(z) \propto (1+z)^{(a+1)/2}$. For all form factors studied in that paper, $a = 1$ or 3 , which originates from the suppression of pair production by $|p|^a$ at threshold. For $a = 1$, not only does $\phi(z)$ vanish at $t = t_+$, but $\phi'(z = -1)$ is finite, while for $a = 3$, $\phi'(z = -1)$ vanishes and $\phi''(z = -1)$ is finite. To be explicit, $\phi(z)$ for the vector form factor written in z assumes the form

$$\begin{aligned} \phi(z; t_0) &= \frac{1}{\sqrt{12\pi t_+ \chi^T}} (1+z)^2 (1-z)^{1/2} \left(1 - \frac{t_0}{t_+}\right)^{5/4} \left[\sqrt{1 - \frac{t_0}{t_+}} (1+z) + (1-z) \right]^{-1/2} \\ &\times \left[\sqrt{1 + \frac{Q^2}{t_+}} (1-z) + \sqrt{1 - \frac{t_0}{t_+}} (1+z) \right]^{-3}. \end{aligned} \quad (4.6)$$

Writing $\sum_n a_n z^n = P(z)\phi(z)F(z)$, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} a_n (-1)^n &= (P\phi F)(-1), \\ \sum_{n=0}^{\infty} a_n n (-1)^n &= (P\phi F)'(-1), \\ \sum_{n=0}^{\infty} a_n n(n-1) (-1)^n &= (P\phi F)''(-1). \end{aligned} \quad (4.7)$$

Since one may confirm that P , P' , and P'' are finite on the circle, making the physical assumption that the n th z derivative of $F(z)$ near $z = -1$ is no more singular than $(1+z)^{-n}$ gives

$$\begin{aligned} \sum_{n=0}^{\infty} a_n (-1)^n &= 0, \\ \sum_{n=0}^{\infty} a_n n (-1)^n &= 0 \quad (a = 3); \quad \neq \infty \quad (a = 1), \\ \sum_{n=0}^{\infty} a_n n(n-1) (-1)^n &\neq \infty \quad (a = 3). \end{aligned} \quad (4.8)$$

The second statement improves upon the constraint of the previous sum rule by implying $|a_n| \rightarrow 0$ faster than $1/n$, rather than $1/\sqrt{n}$: Now we may state that $\sum_n a_n z^n$ is absolutely convergent. Thus one may reorder terms in the sums, as was implicit in the proof of Eq. (3.4).

Moreover, in the case of the vector form factor ($a = 3$), the second and third expressions together tell us that in fact $a_n \rightarrow 0$ at least as fast as $1/n^2$. This leads to a tremendous improvement in our knowledge of convergence of $\sum_n a_n z^n$ on the circle.⁶ Where before we knew only that convergence occurred, now we can quantify how fast. The relative error of the partial sum $\sum_{n=1}^N 1/n^2$ from its exact value of $\zeta(2) = \pi^2/6$ is bounded by $6/(\pi^2 N)$. We find that only 3 terms are required for a 20% relative error, 6 for 10%, and 60 for 1%. In principle, it should be very a simple matter to fit the coarse structure of the form factor over its entire kinematic range with very few parameters; even such a structure as the large resonant ρ peak may be accommodated by this quasi-Fourier analysis. However, it turns out that this expectation is unfulfilled, for reasons that we now discuss.

The problem is that we do not know which of the a_n 's are most important in this convergence. That is, they are not necessarily the first few $\{a_0, a_1, \dots, a_N\}$, as for the semileptonic decays or more generally for any spacelike factor, where moments about $z = 0$ ($t = t_0$) dominate. However, the constraint (2.10) tells us that, if this set is appreciable in magnitude, there is little room for higher a_n 's; this leads to a rapid convergence in n , and consequently, one obtains a unified picture of the form factor spanning both spacelike and timelike regions. The alternate possibility is that one could have many a_n 's much smaller than the saturation paradigm $1/n^2$, so that convergence is painfully slow in n ; in this case, the fit to timelike data voraciously demands ever higher a_n 's and is relatively insensitive to the lowest coefficients, so that the spacelike data is relatively unconstrained by the timelike data. A detailed fit to data is required to determine which scenario is realized.

We pause momentarily to review the assumptions made to obtain appropriate convergence properties of the parameterization. Convergence of the series $\sum_n a_n z^n$ for $t \geq t_+$ is obtained by requiring $|F(z)|$ to be a bounded, smooth, continuous function, while in obtaining the sum rule (2.10), we additionally assume that $|F(z)|$ becomes *featurelessly* smooth as $t \rightarrow +\infty$. Absolute convergence of the series uses the particular forms of P and ϕ , and the assumption that $F'(z)$ near $z = -1$ is no more singular than $(1+z)^{-1}$. Finally, the convergence of the series like $1/n^2$ depends on the form of $\phi(z)$ for the vector form factor and the assumption that $F''(z)$ near $z = -1$ is no more singular than $(1+z)^{-2}$.

V. A NEW PARAMETERIZATION

After the detailed discussion of the series $\sum_n a_n z^n$ and its properties, it may seem incongruous to introduce a new parameterization for the form factor. Yet it is entirely appropriate to do so, since it is more natural to incorporate the sum rules (4.4) and (4.8) directly into the parameterization than to impose them by hand. The sort of trouble one might encounter by using the parameterization in a_n 's becomes clear with reflection upon the content of the

⁶A similar treatment with powers of $(1-z)$ about $t = +\infty$ leads to the same convergence for the scalar form factor in processes with non-conserved currents.

sum rules (4.8). One may impose explicit constraints on a truncated set $\{a_0, a_1, \dots, a_N\}$ to enforce the vanishing of the first two of these as well as the first expression in (4.4), for example by fixing a_0, a_1 , and a_2 by means of the sum rules and the values of $\{a_3, a_4, \dots, a_N\}$. But then these sum rules, which came from certain limits of the parameterization in z , might be accomplished in perverse ways. For example, the final expression in (4.8) may be finite but exceptionally large, leaving $|F(t_+)|$ essentially unbounded.

The cure for such phenomena is straightforward. One simply factors out the appropriate behavior determined by $\phi(z)$ from the parameterization:

$$\sum_{n=0}^{\infty} a_n z^n \equiv (1+z)^2 (1-z)^{1/2} \sum_{n=0}^{\infty} b_n z^n, \quad (5.1)$$

defining the series $\sum_n b_n z^n$. Since the original series $\sum_n a_n z^n$ and also the prefactor $(1+z)^{-2} (1-z)^{-1/2}$ are analytic in z inside the unit circle, the same holds for $\sum_n b_n z^n$. The requirements on the singularity structure of $F(z)$ as $z \rightarrow \pm 1$ discussed in the previous section are carried verbatim to $\sum_n b_n z^n$ since $1/\phi(z)$ has no singularities on the unit disc except those removed by the prefactor, while the sum rules (4.4), (4.8) are automatically satisfied by the inclusion of the z -dependent prefactor. The parameters b_n , like a_n , are real since they in particular describe the form factor in the spacelike region, where $F(z)$ and z are real. On the other hand, the b_n 's no longer satisfy any particular constraints except that $\sum_n b_n z^n$ is analytic.⁷ The extra prefactor is just that appearing explicitly in $\phi(z)$, so one obtains the effective weight function

$$\begin{aligned} \tilde{\phi}(z; t_0) &= \phi(z) (1+z)^{-2} (1-z)^{-1/2} \\ &= \frac{1}{\sqrt{12\pi t_+} \chi^T} \left(1 - \frac{t_0}{t_+}\right)^{5/4} \left[\sqrt{1 - \frac{t_0}{t_+}} (1+z) + (1-z) \right]^{-1/2} \\ &\quad \times \left[\sqrt{1 + \frac{Q^2}{t_+}} (1-z) + \sqrt{1 - \frac{t_0}{t_+}} (1+z) \right]^{-3}, \end{aligned} \quad (5.2)$$

so that

$$|F(z)| = \frac{1}{|\tilde{\phi}(z)|} \cdot \left| \sum_n b_n z^n \right|. \quad (5.3)$$

The sum rule that relates timelike data points to the parameterization, Eq. (3.4), thus assumes the parallel form

$$\left| \tilde{\phi}(z = e^{i\theta_u}) F(z = e^{i\theta_u}) \right|^2 = \sum_{n=0}^{\infty} b_n \sum_{m=0}^{\infty} b_m \cos(n-m)\theta_u. \quad (5.4)$$

Nevertheless, it is $|\phi(z)F(z)|$ and not $|\tilde{\phi}(z)F(z)|$ that appears in the dispersion integral, and so the original sum rule (2.10) in the new basis is *not* just $\sum_n b_n^2 \leq 1$, but assumes the more complicated form

⁷An additional constraint occurs if a particular $t \rightarrow +\infty$ or $t \rightarrow t_+$ behavior of the form factor is assumed.

$$\sum_{n=0}^{\infty} b_n \sum_{m=0}^{\infty} b_m \delta(m, n) \leq 1, \quad (5.5)$$

where

$$\delta(m, n) = \frac{64}{\pi} (4p^2 - 22p + 45) \prod_{i=0}^2 [(2i+1)^2 - p]^{-1}, \quad (5.6)$$

with $p \equiv 4(m-n)^2$. In the original sum rule (2.10), where the analogue of $\delta(m, n)$ is just δ_{mn} , one clearly requires each $|a_n| \leq 1$, and every nonzero a_n serves to limit the size of succeeding terms. However, it is not quite so simple to determine at a glance whether the parameters b_n obtained from fitting to data or a model satisfy the original dispersion relation (2.8). Moreover, without the sum rules that produced a strongly convergent series in a_n , the convergence of the series in b_n follows no guaranteed pattern but still may give an adequate fit. We will see that this is in fact true for the π form factor.

An important aspect of the transformation between the parameters a_n and b_n represented by (5.1) is that both series require only a small number of parameters to describe the form factor in the vicinity of $z = 0$; such behavior is necessary for a useful and minimal description of the spacelike form factor around $t = t_0$. Specifically, one finds

$$\begin{aligned} a_0 &= b_0, \\ a_1 &= b_1 + \frac{3}{2}b_0, \\ a_2 &= b_2 + \frac{3}{2}b_1 - \frac{1}{4}b_0, \end{aligned} \quad (5.7)$$

and so forth. The essential conundrum in obtaining a parameterization valid in both spacelike and timelike regions is that, in the spacelike region, or at least near $t = t_0$ where data is abundant, it cannot be very different from the parameterizations in a_n or b_n discussed above. That is, it must have a rapidly converging Taylor series about $z = 0$. On the other hand, it must be able to recognize strongly localized structures in the timelike region $|z| = 1$, such as the ρ resonance in the present case. We return to this point in the next section, once we have exhibited our empirical results.

To summarize, the parameter basis a_n admits a number of sum rules from timelike constraints which give rise to a strongly convergent behavior. It also admits the very simple dispersive bound (2.10), but may exhibit pathological behavior in fits to timelike data. The parameter basis b_n has all of the sum rules built in and is much more stable in timelike fits, but its dispersive bound (5.5) is much more complicated, and in general may converge much more slowly. Both bases are useful in the spacelike region, while it requires a fit to data to determine whether either is useful in the timelike region.

VI. THE ELASTIC π FORM FACTOR

Our intent is to study the possibility of using the considerable $\pi^+\pi^-$ production data in the timelike region $t \geq t_+$, from which the form factor $|F(t)|^2$ is extracted, to fit to the parameters b_n as determined from the analytic structure of the QCD dispersion relation

(2.2) and expressed by (5.2)–(5.6). Of particular interest is the number N of parameters necessary to give a good accounting of the data, and what these parameters tell us about the shape of the form factor in the spacelike region, where there exists considerable data from π^+ elastic scattering.

A. Inputs

Let us first consider the calculation of the current two-point correlator, as represented by its subtracted form $\chi_V^T(Q^2)$ in (2.2). As noted in Sec. II, this quantity possesses a well defined operator product expansion in inverse powers of Q^2 . The first few terms are particularly simple in the case of the light quarks u and d . Neglecting subleading mass corrections,

$$\chi_V^T(Q^2)_{m=0} = \frac{1}{8\pi^2 Q^2} \left(1 + \frac{\alpha_s}{\pi}\right) - \frac{1}{12Q^6} \left\langle \frac{\alpha_s}{\pi} G_{\mu\nu}^a G^{a\mu\nu} \right\rangle - \frac{2}{Q^6} \langle m\bar{q}q \rangle + O\left(\frac{1}{Q^8}\right) + O\left(\frac{\alpha_s^2}{\pi^2}\right). \quad (6.1)$$

Note that the $\bar{q}q$ condensate has been included, despite the vanishing of the quark mass m , to indicate its place in the expansion. The electromagnetic current bilinear $J^\mu = \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d$ induces an extra factor of $(+2/3)^2 + (-1/3)^2 = 5/9$ in χ_V^T through the quark charges. Our goal is to choose Q^2 as small as possible in order to maximize the stringency of the dispersive bound through perturbative QCD input, but large enough that the perturbative expansion remains valid.

Although numerical estimates for the first few condensates certainly exist in the literature, we use them principally to establish that region in Q^2 for which convergence of the expansion is satisfactory and to obtain a numerical uncertainty on the lowest-order result. To be specific, we use $\langle \alpha_s G^2/\pi \rangle = 0.02\text{--}0.06 \text{ GeV}^4$ and the expression for $\alpha_s(Q^2)/\pi$ from the three-loop beta function with $n_f = 3$ and $\Lambda_{\overline{\text{MS}}}^{n_f=3} = 380 \pm 60 \text{ MeV}$. Values for the α_s correction and the gluon condensate relative to leading order are presented in Table I for various values of Q^2 . We learn that corrections become quite large for $Q^2 < 2 \text{ GeV}^2$. Although the $O(\alpha_s^2)$ correction and the gluon condensate have opposite signs, large uncertainties on the latter prevent one from knowing how complete this cancellation might be. Moreover, even with a realistically small but finite quark mass, the $\bar{q}q$ condensate becomes of relative size $\sim 10\%$ by $Q^2 = 1 \text{ GeV}^2$. We therefore conservatively choose $Q^2 = 2 \text{ GeV}^2$, estimating corrections to the lowest order result to be no more than 15%. We thus obtain $\chi_V^T(2.0 \text{ GeV}^2) = (3.52 \pm 0.53) \cdot 10^{-3} \text{ GeV}^{-2}$.

$Q^2 \text{ (GeV}^2\text{)}$	$\alpha_s(Q^2)/\pi$	$\langle \alpha_s G^2/\pi \rangle$
4.0	+0.101 ± 0.013	−(0.008 − 0.025)
3.0	+0.109 ± 0.015	−(0.015 − 0.044)
2.0	+0.126 ± 0.020	−(0.033 − 0.099)
1.0	+0.175 ± 0.037	−(0.132 − 0.396)

Table 1. Values of corrections to $\chi_V^T(Q^2)$ as appearing in Eq. (6.1) relative to leading order. Coefficients have been suppressed for simplicity.

The isospin factor n_I in Eq. (4.2) for the pion case is set to unity; in previous cases, more than one isospin channel could couple to the current J^μ in the same dispersion relation so that $n_I > 1$, thus strengthening the expression (2.9). In the π case this is no longer possible. As we have pointed out, π^0 pairs cannot couple to a vector current due to charge conjugation. Moreover the electromagnetic current possesses both $I = 0$ and 1 amplitudes, so an improvement through n_I with other contributions to the dispersive bound would first require disentangling these amplitudes.

The dispersive bounds may also be strengthened by the inclusion of perturbative QCD information along the cut in the deep Minkowski region $t \rightarrow +\infty$, as in [8]. However, we opt not to do so in this work, for in this approach one must not only select a point $t = t_*$ for the onset of this region, but employ the perturbative expression over the whole interval $(t_*, +\infty)$ for inclusion in the dispersion integral. The uncertainties are therefore those of the integrated perturbative result, rather than those of just one point in the Euclidean case. For our purposes here, we prefer the more minimal approach of using only deep Euclidean QCD calculations.

We select the parameter $t_0 = 0$. This natural choice means that $z = 0$ occurs where current conservation normalizes the form factor $F(t = 0)$ to unity, the charge of the pion. It further means that the form factor near this point is well determined by the first few b_n 's, owing to the geometric convergence of $\sum_n b_n z^n$. Of course, such convergence is contingent upon the small size of $|b_n|$, which is not so obvious as that of the $|a_n|$ [compare (2.10) and (5.5)] but is empirically true in our fits.

Explicitly, the normalization of the form factor and the pion charge radius $\langle r^2 \rangle \equiv 6 (\partial F / \partial q^2)|_{q^2=0}$ are given by

$$|F(t = 0)| = \frac{|b_0|}{|\tilde{\phi}(z = 0)|},$$

$$\langle r^2 \rangle = \frac{3}{2m_\pi} \sqrt{6\pi\chi^T} \left(\sqrt{1 + Q^2/4m_\pi^2} + 1 \right)^3 \left[3 \frac{(\sqrt{1 + Q^2/4m_\pi^2} - 1)}{(\sqrt{1 + Q^2/4m_\pi^2} + 1)} b_0 - b_1 \right]. \quad (6.2)$$

The experimental data for the form factor in the timelike region is collected from a number of sources [9], and each data point is regarded as having significance entirely determined by its stated uncertainty. This set contains many dozens of points stretching from $q^2 = 0.1$ to almost 10 GeV² and covers decades of experiment, although the most recent published measurements are already over ten years old.⁸ To be precise, our sample consists of 145 timelike data points, which gives the absolute upper limit of the number N of parameters b_n one may hope to extract from the data; more on this in a moment.

⁸Recent spacelike results from E93-021 at Jefferson Lab are not yet in print and thus have not included in the analysis.

B. The Fit

We minimize χ^2 with regard to the unknown parameters b_n subject to the sum rule (5.4) by using the Levenberg-Marquardt method [11]. Naturally, the topology described by a function with scores of parameters utterly escapes intuitive expectations about where the minimum might lie, so several choices of parameters for the initial iteration are selected to test the proper convergence of the algorithm to the global minimum. Typically, on each given run the program finds the same χ_{\min}^2 to within a few tenths of a percent. Uncertainties on the parameters are estimated by means of the covariance matrix, despite the fact that (5.4) is quadratic rather than linear in the parameters b_n . Indeed, the two sets $\{\pm b_n\}$ trivially give the same fit, but we select $b_0 > 0$ to guarantee the positivity of the form factor at $t = t_0 = 0$. The program is permitted to wander freely in $\{b_n\}$ parameter space without imposing the constraint (5.4); the degree of saturation of this bound, which expresses quark-hadron duality, is computed at the end of the fit and is observed to satisfy the bound in all cases. The 15% uncertainty in the perturbative value of χ^T appears as a systematic uncertainty in the parameters b_n since $\tilde{\phi}(z) \propto (\chi^T)^{-1/2}$, so that the dispersive bound (5.5) has a linear uncertainty in $\Delta\chi^T$, meaning that the 1 on the r.h.s. is to be replaced by 1.15.

The figure of merit in these fits is χ^2 per degree of freedom (d.o.f.), where the number d.o.f. is the number of data points minus the number of fit parameters N . Obviously the average variance of the fit value of the form factor from data decreases as N is increased, but the d.o.f. denominator eventually compensates for this advantage, and so there exists a particular value of N such that $\chi^2/\text{d.o.f.}$ achieves a minimum. For our sample of 145 data points, this occurs for about 60 parameters b_n .

There is another reason to choose N substantially smaller than the number of data points. If we think of the fit as being essentially a decomposition into Fourier modes (as evidenced by the fact that our basis functions are $z^n = e^{in\theta}$ on the unit circle), then extracting a number of parameters comparable to the number of data points is analogous to probing modes with wavelengths as small as the spacing between these points. Spurious high-frequency oscillations then appear in the fit, and the same phenomenon is observed in the present case. If one fits not to isolated data points but a model with a continuous prediction for the form factor, then one is free to expand to arbitrarily high harmonics; the oscillations are an effect of finite experimental resolution.

In fact, these spurious oscillations tend to be compounded with even more dramatic effects when we attempt to extrapolate from the data-rich regions around the ρ peak down to t_+ : The oscillations assume huge proportions, many times larger than the ρ peak itself, with the few data points far below the peak nestling themselves in the minima of these oscillations (Fig. 1). The cause of this particular misfortune has a very simple origin: Analyticity in z is precisely not the same as analyticity in t , and the map (2.5) is not conformal at the point $t = t_+$, *i.e.*, $z = -1$. This is obvious from geometric considerations: whereas in t space segments of the real axis below and above t_+ are parallel, the corresponding segments of the unit disk in z are the real axis and the two halves of the unit circle, which make 90° angles in antiparallel directions at $z = -1$. Indeed, the Jacobian of the map is given by

$$\frac{\partial z}{\partial t} = -\frac{(1-z)^3}{4(t_+ - t_0)(1+z)}, \quad (6.3)$$

meaning that *any* parameterization in the variable z has problems as $t \rightarrow t_+$. And yet this is a necessary feature of the dispersive analyticity constraints: It is z , and not t , that is the natural kinematic variable for describing the analyticity of the form factor. Additional physical requirements or a greater density of data are necessary in order to impose smoothness on the form factor in the region near t_+ .

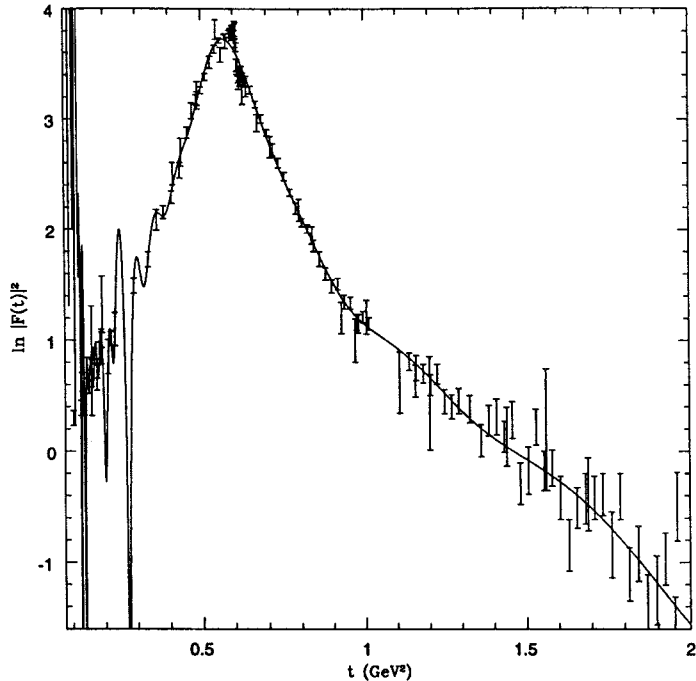


FIG. 1. Best fit of the timelike π elastic form factor data $|F(t)|^2$ to the parameterization of (5.3) with parameters $\{b_0, b_1, \dots, b_{60}\}$. The range $t_+ \leq t \leq 2 \text{ GeV}^2$ is shown, although some data exists out to 10 GeV^2 , in order to emphasize the ρ peak. Note especially the difficulty of the fit in accommodating the ω shoulder, oscillations of the fit below the peak, and the tendency of the fit to become unhinged for t near threshold where there are gaps in the data.

C. Results

Our best fit to the form factor over the timelike region appears in Fig. 1. As stated above, the fit uses $N = 61$ parameters $\{b_0, b_1, \dots, b_{60}\}$ and has a $\chi^2_{\text{min}}/\text{d.o.f.}$ of 3.20. This large value, despite the visual goodness of the fit in the figure, can be explained by the mild inconsistency of data from numerous different experiments, as well as the famous “ ω shoulder” due to the presence of destructive interference between the narrow ω and broad ρ peaks. The shoulder represents a sudden, nearly discontinuous change in the form factor, which is difficult to simulate using only the lower harmonics. Indeed, eliminating the 15 data points in the immediate vicinity of this shoulder from the fit diminishes $\chi^2_{\text{min}}/\text{d.o.f.}$ by more than 1/3; of course, the oscillations in the fit below the ρ peak remain. A third possible explanation is that χ^2 is large because $N \gg 60$ might be required to fit the data adequately. However, suppose one considers not $\chi^2_{\text{min}}/\text{d.o.f.}$ but rather $\chi^2_{\text{min}}/\text{datum}$ and fit to 200 parameters (approaching the limitations of modern workstations). Ignoring the numerous spurious oscillations produced by this stretch, we find that $\chi^2_{\text{min}}/\text{datum}$ decreases from 1.85 to 0.75, i.e., much slower than linearly with N .

The degree of saturation in the best fit, namely, the fit value for the l.h.s. of (5.4) divided by the loosened bound of 1.15, turns out to be 0.466, meaning that charged pions and resonances coupled to them account for about half the dual perturbative result at $Q^2 = 2 \text{ GeV}^2$.

Using Eq. (6.2), the values for the spacelike parameters extracted from the $N = 61$ fit are given by

$$|F(t=0)| = 2.56 \pm 2.00, \quad \langle r^2 \rangle = 68.2 \pm 88.3 \text{ GeV}^{-2}. \quad (6.4)$$

On the surface, this is not a very impressive set of results, considering the values extracted directly from the spacelike data [10]:

$$|F(t=0)| = 0.995 \pm 0.002, \quad \langle r^2 \rangle = 10.26 \pm 0.26 \text{ GeV}^{-2}. \quad (6.5)$$

What has gone wrong? To sharpen this complaint, we observe that even after fitting to a full 60 parameters, one still cannot do a very good job extrapolating the timelike data into the spacelike region. After all, the naive behavior of the data plotted in t seems to suggest that the lower edge of the ρ peak extrapolates smoothly into the spacelike region to give much better figures for the normalization and slope of the form factor at $t = 0$. However, as we have pointed out above, the dispersive bound uses analyticity in z , not t . Models based upon the expected theoretical shape of the ρ peak or chiral perturbation theory implicitly assume that no peculiar behavior afflicts the form factor near t_+ . For the dispersive bounds to do the same, one would have to absorb the $(1+z)^{-1}$ factor in (6.3) into the parameterization in order to make $\partial F/\partial t$ finite at t_+ .⁹ However, since we have insisted upon z as the natural variable of analyticity, for the remainder of this work we maintain that such a modification takes us too far from the original motivation of rigorous, minimal

⁹In fact, one might expect a discontinuity in $\partial F/\partial t$ at t_+ , indicating the threshold of absorptive processes entering through $\text{Im}F$.

bounds based on quark-hadron duality and analyticity. Certainly such modifications are straightforward to implement, but we forgo them for now for the sake of minimality of assumptions.

The next issue is how one can believe the extrapolation from the timelike to the spacelike region if indeed the best fit is pathological in the neighborhood of t_+ . Again, the answer is that it is z and not t that is the relevant variable of analyticity. An extrapolation along the real t axis, expressed in z coordinates, consists of following the contour of the unit circle in z until reaching t_+ ($z = -1$), and then moving along the real z axis to $t = 0$ ($z = 0$). Such a path takes us straight through the eye of the storm at t_+ , and should be avoided if possible. In fact, since the form factor is analytic on the entire unit disk in z , one may choose a more direct and less contentious path; for example, start at the point on the circle where data is plentiful, such as $t_\rho \equiv t(m_\rho^2)$ (corresponding to $\theta = 42.5^\circ$), and proceed directly along a radius to $t = 0$. The fit for $|F|^2$ along this contour is presented in Fig. 2. This contour expressed in complex t space resembles a cardioid with its outer edge at t_ρ and its cusp at $t = 0$. Since it does not pass near t_+ , it does not exhibit strong oscillations, as is apparent from the figure. Clearly, analyticity in z has no trouble with this region, and so the extrapolated results have meaning.

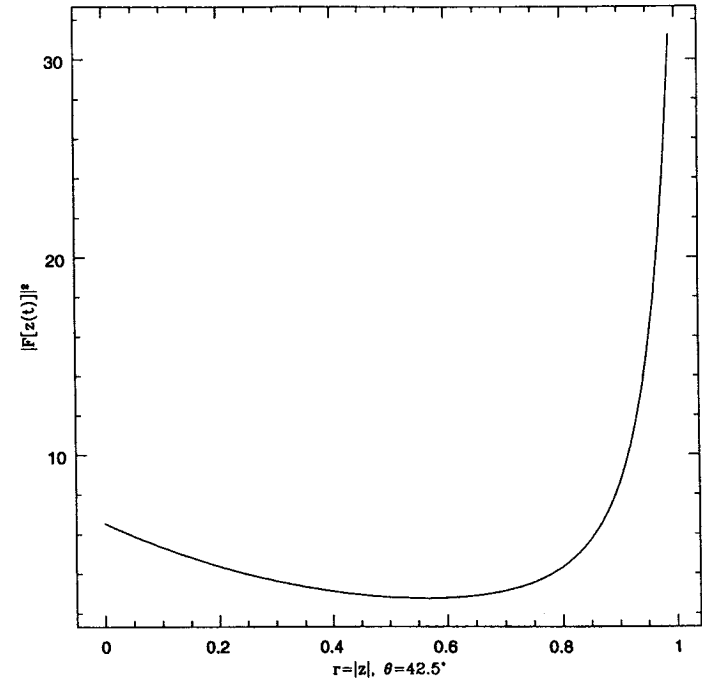


FIG. 2. Elastic π form factor $|F(t)|^2$ as predicted by analyticity in $z = re^{i\theta}$ along the radius connecting the ρ peak $t = m_\rho^2$ ($\theta = 42.5^\circ$) to $t = 0$ ($z = 0$). Note the absence of sudden oscillations seen in Fig. 1.

So our fit values are meaningful, and the numbers extracted for $|F(t=0)|$ and $\langle r^2 \rangle$ are certainly consistent with those from the spacelike fit, but why are the uncertainties so large? Again, the interpretation becomes clear in z space. The position of points $z(t, t_0)$ in the unit disk is completely determined by the ratios t/t_+ and t_0/t_+ , as is clear from (2.5). With our choice $t_0 = 0$, the timelike data becomes compressed almost entirely into the half-circle $\text{Re } z > 0$: See Fig. 3. As mentioned above, the ρ peak data is clustered about $\theta \approx \pi/4$, while the lowest measured points are at $t \approx 0.1 \text{ GeV}^2 \rightarrow \theta \approx 2\pi/3$. This is a direct result of $t_\rho \gg (t_+ - t_0)$. If one takes $t_0 < 0$, the extrapolation to t_0 should be much more precise. Unfortunately, $t_0 = 0$ is exceptionally convenient for the extraction of $|F(t=0)|$ and $\langle r^2 \rangle$, as we have discussed above. We expect these problems would become much less severe if more

near-threshold pion production data were measured. That this is true may be confirmed by performing the fit after including additional Monte Carlo data points in this region. The extrapolation uncertainties would also decrease substantially if we considered form factors where much more data naturally falls just above threshold; this is precisely the situation for the K form factor, where the ϕ peak occurs very close to the threshold $4m_K^2$, although in that case one faces the prospect of depleted data for $\theta < \pi/2$.

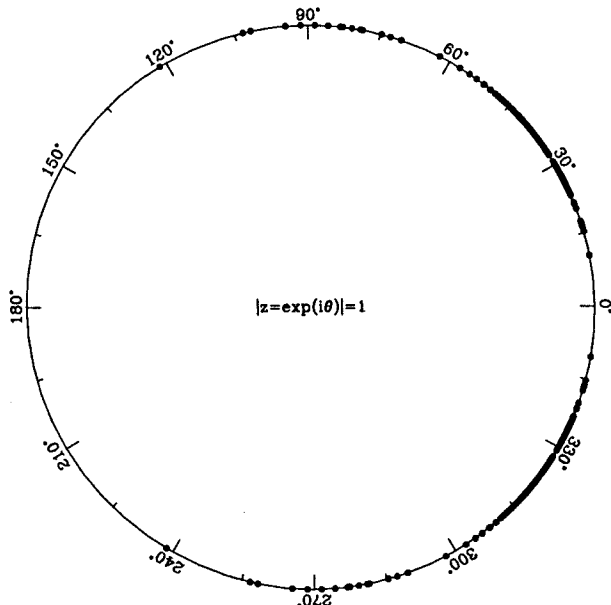


FIG. 3. A density plot of timelike data on the unit circle in z . Note the preponderance of data near the ρ peak ($\theta = 42.5^\circ$) and the paucity of data near threshold t_+ ($\theta = 180^\circ$), or even beyond $\theta = 90^\circ$.

The distribution of the parameters b_n in magnitude is an interesting test of the convergence of the series $\sum_n b_n z^n$. Recall that we no longer have specific mathematical information on the rate of the convergence of this series. In our best fit with $N = 61$ we find only 3 parameters with $|b_n| > 0.05$, 12 with $0.02 < |b_n| < 0.05$, 11 with $0.01 < |b_n| < 0.02$, and

35 with $|b_n| < 0.01$. The largest single parameter is b_0 , and the largest 8 parameters all occur in $\{b_0, \dots, b_{10}\}$. Although the series is not monotonic, it appears to exhibit behavior consistent with fairly rapid convergence in n .

It is also possible to perform the fit to data fixing the normalization $|F(t=0)| = 1$ by choosing b_0 via Eq. (6.2). In a fit with $|F(t=0)| \equiv 1$ using spacelike data, [10] gives

$$\langle r^2 \rangle = +11.07 \pm 0.26 \text{ GeV}^{-2}, \quad (6.6)$$

whereas our fit with timelike data gives

$$\langle r^2 \rangle = -13.3 \pm 27.9 \text{ GeV}^{-2}. \quad (6.7)$$

Again, our fit value is consistent with the other determination. As expected, the uncertainty in $\langle r^2 \rangle$ decreases once $|F(t=0)|$ is fixed, since an additional constraint has been placed on the set of allowed analytic functions that satisfied the original fit. The value of $\chi_{\text{min}}^2/\text{d.o.f.}$ for this fit is again 3.20, and the level of saturation is now 0.52.

If enough data existed that the uncertainties on the parameters b_n were much smaller, one could extrapolate throughout the entire spacelike region $t \in (-\infty, 0)$, which is $z \in (0, +1)$. The prediction for $|F|^2$ for our best fit values turns out to fall smoothly and monotonically to zero as $t \rightarrow \infty$, and we expect this paradigm to persist in similar fits with more constraints. Because of the large uncertainties in our extrapolations, we decline to exhibit just the central value fit; however, we expect this paradigm to persist in similar fits with more restrictive constraints.

Finally, the model-independent parameterization of (5.3) may be used directly in a fit to spacelike data. This application is entirely analogous to the semileptonic fits of [3–7], except that we pick $t_0 = 0$ and do not here seek to minimize the “truncation error” of between the fit and true form factor at points with $t \ll 0$: After all, we are interested in moments about $t = 0$. A collection of 45 data points in the spacelike region taken from [9] may be fit with just $\{b_0, b_1, b_2\}$ to give $\chi_{\text{min}}^2/\text{d.o.f.}$ of only 0.97 and a saturation of the dispersion relation of only 0.018 (as expected, fewer parameters lead to less saturation of the dispersive bound). From the parameter fit values we obtain

$$|F(t=0)| = 0.997 \pm 0.005, \quad \langle r^2 \rangle = 11.28 \pm 1.86 \text{ GeV}^{-2}, \quad (6.8)$$

or taking $|F(t=0)| = 1$,

$$\langle r^2 \rangle = 12.33 \pm 0.51 \text{ GeV}^{-2}, \quad (6.9)$$

with $\chi_{\text{min}}^2/\text{d.o.f.} = 0.98$ and a degree of saturation 0.022. As a parting shot we point out that this model-independent determination of $\langle r^2 \rangle$ differs from that obtained by a naive linear extrapolation in t (6.6) by over 2σ . The difference lies in the fact that while the usual determination assumes F linear in t near $t = 0$, the model-independent form factor has a natural curvature associated with the factor $1/\phi(z)$, which in turn arose from analyticity constraints. The larger size of the uncertainty in our determination takes into account theoretical uncertainties in the shape of the form factor, which of course are absent when one assumes linearity in t .

VII. CONCLUSIONS

We have explored some of the constraints that timelike data and dispersive constraints place on the shape of form factors, particularly in the spacelike region. Much can be learned from little more than the analytic structure of the two-point function to which the form factor contributes and a few mild physical assumptions from finiteness. The sum rules and improved parameterization derived above arise directly from these minimal considerations.

Nevertheless, the extrapolation of a finite amount of timelike data to the spacelike region is seen to lead to huge uncertainties in the case of the pion elastic form factor $F(t)$. This means that the bare assumptions of analyticity and the applicability of QCD in the deep Euclidean region are not enough to obtain what seems to be a natural extrapolation of the ρ peak to the spacelike region. We have discussed the dangers of extrapolating in the kinematic variable t rather than the natural kinematic variable of analyticity, z : Namely, one must pass around a branch point $t_+ = 4m_\pi^2$ or $z = -1$, near which the smooth behavior of the form factor may be compromised. In terms of z , the assumption of a resonant pole in the timelike region places extremely tight constraints on the allowed parameter space of b_n .

On the other hand, the z contour from timelike to spacelike data may be distorted to avoid the troublesome point $z = -1$. The large uncertainties in our fit are seen to arise from the density and placement of the data set in z ; basically, the problem is that most of the data, clustered near the ρ peak, is far above the pair production threshold t_+ . Since this is not the case for the K form factor, the extrapolation may be much more reliable in that case.

This is not to indicate that the technique is hopelessly weak or intrinsically flawed, however. Rather, we have endeavored to exhibit what is obtained from the *absolute minimum* of assumptions. We have also suggested at various points possible improvements to the program presented above. It is useful to collect them here for the convenience of the reader. First, the calculation of the perturbative quantity $\chi^T(Q^2)$ may be improved by a careful analysis of multiloop effects and condensates. We have evaluated the perturbative result only at the single value $Q^2 = 2 \text{ GeV}^2$. Certainly the behavior of the dispersion relation as a function of Q^2 , which is equivalent to a moment analysis, provides additional constraints.

Second, the stringency of the dispersive bound is determined by both $\chi^T(Q^2)$ and the level of saturation by all contributing states excepting those being probed by the parameterization. In the present case, this means any states like $K\bar{K}$ that receive contributions from the electromagnetic current but have not already been counted in $\pi^+\pi^-$ production (such as the ρ). A calculation or estimation of this contribution effectively decreases the dispersive saturation limit allotted to the pion form factor. Also falling into this category is the deep Minkowski region $t \gg m_\rho^2$ of the pion form factor, for which data has not yet been collected. Here, one may either use its integrated weight to increase the stringency of the dispersive bound, or retain its functional form to obtain another sum rule on the form factor parameterization.

The latter suggestion leads us to the third possible improvement, that conditions on the shape of the form factor lead to constraints on the form of the parameterization. We have endeavored to make as few assumptions as possible along these lines, lest the model-independency of the parameterization is lost. Nevertheless, modelers are free to start with the dispersive bounds as a starting point for building form factors that automatically satisfy

nontrivial QCD constraints.

Another possibility along these lines not yet mentioned is the choice of the series used to parameterize the unknown physics of the form factor. We have always used a simple Taylor series in z^n to express the analyticity of the form factor since the set $\{z^n\}$ is a complete and orthonormal basis on the unit circle for analytic functions. Although this basis proved immensely useful (indeed, optimal) for spacelike fits, perhaps a better basis exists for accommodating the relatively rapid variations in the form factor due to structure in resonant peaks, such as the ω shoulder. However, the constraints on this new parameterization are manifold: If it is to be adequate for both timelike and spacelike fits, it must admit a Taylor series rapidly convergent in z^n as $z \rightarrow 0$, all of its Taylor coefficients must be real (so that the form factor is real in the spacelike region, where z is real), but it cannot converge *too* quickly in z^n as $|z| \rightarrow 1$, or else our polynomial fits in z^n would already have been adequate. Moreover, since the proposed functions are analytic in z and not \bar{z} on the unit disk, natural choices such as $\cos n\theta$ and $\sin n\theta$ are forbidden.

Clearly, many aspects of this program yet remain to be fully explored, but the variety of directions for improvements are quite astounding. Dispersive techniques possess the rare ability to encapsulate a great deal of nontrivial physics with true simplicity and elegance.

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