Confinement through a relativistic generalization of the linear interaction*

Çetin Şavkhl, Franz Gross$^{1,2}$

$^1$Department of Physics, College of William and Mary, Williamsburg, Virginia 23187

$^2$Jefferson Lab, 12000 Jefferson Avenue, Newport News, VA 23606

Abstract

Generalizing a covariant framework previously developed, it is shown that confinement insures that meson $\rightarrow q + q$ decay amplitudes vanish when both quarks are on-shell. Regularization of singularities in a covariant linear potential associated with nonzero energy transfers (i.e. $q^2 = 0, q^\mu \neq 0$) is improved.

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I. INTRODUCTION

Even for a simple system such as a quark-antiquark (referred to collectively “quarks”) bound state, color confinement implies that the matrix element \( \mu \to q + \bar{q} \) must vanish whenever this decay is kinematically possible. This trivial statement can be realized by two possible mechanisms: i) the quark propagators are free of timelike mass poles, as is usually summed in Euclidean metric based studies \[1\], or ii) the vertex function of the und state vanishes when both quarks are on-shell. In an earlier work \[2\] was assumed that the quark propagators have mass poles, but it was not own that relativistic generalization of the linear confining potential guaranteed a correct vanishing of the vertex function. In this work it is shown that the vanishing of the vertex function is a general feature of any confining interaction, d that insisting on the correct nonrelativistic limit leads naturally to a second option.

II. THE NONRELATIVISTIC LINEAR POTENTIAL

It was shown earlier \[2\] that the nonrelativistic linear interaction

\[
V(r) = -C + \sigma r,
\]

\[\tag{2.1}\]

a be expressed in momentum space by

\[
V(q) = \lim_{\epsilon \to 0} \left[ V_A(q) - \delta^3(q) \int d^3 q' V_A(q') \right] - (2\pi)^3 \delta^3(q) C,
\]

\[\tag{2.2}\]

where

\[
V_A(q) \equiv -8\pi \sigma \frac{1}{(q^2 + \epsilon^2)^2}.
\]

\[\tag{2.3}\]

coordinate space

\[
V_A(q) = \int \frac{d^3 q}{(2\pi)^2} e^{-i q \cdot r} V_A(q) = -\frac{\epsilon}{\epsilon} \lim_{\epsilon \to 0} \sigma (r - \frac{1}{\epsilon}),
\]

\[\tag{2.4}\]

d hence the delta function subtraction cancels the infinite \(1/\epsilon\) term, insuring at the limit \(\epsilon \to 0\) exists and that the potential confines. It also insures that a linear potential vanishes at the origin. In fact, the delta function subtraction should be used to construct confining potentials for any monotonically increasing \(V_A(r) - V_A(0) = \infty\) for some \(r\). For example, after the delta function subtraction even the coulomb potential \(V_A(r) = -1/r\) would result in a confining interaction.

Using the linear potential Eq. (2.2) the Schrödinger equation for two quarks of equal mass \(m\) becomes

\[
\left[ \frac{p^2}{m} - E \right] \Psi(p) = -\int \frac{d^3 k}{(2\pi)^3} V_A(p - k) \left[ \Psi(k) - \Psi(p) \right] + C \Psi(p),
\]

\[\tag{2.5}\]

where \(E\) is the binding energy.

III. THE RELATIVISTIC GENERALIZATION

A seemingly natural candidate for a relativistic generalization of the linear interaction (2.2) is to use the Bethe-Salpeter equation with a kernel given by

\[
V(q) \to \lim_{\epsilon \to 0} \left[ V_A(q) - \delta^4(q) \int d^4 q' V_A(q') \right] - (2\pi)^4 \delta^4(q) C,
\]

\[\tag{3.1}\]

where \(q^2 \to -q^2\) in Eq. (2.3). Unfortunately this form does not have the correct nonrelativistic limit. Therefore we rephrase the question: Can we find a covariant equation that reduces to the Schrödinger equation with linear interaction in the nonrelativistic limit? In order to motivate the relativistic equation, start by defining the Schrödinger vertex function \(\Phi(p)\)

\[
\Phi(p) = \left[ \frac{p^2}{m} - E \right] \Psi(p).
\]

\[\tag{3.2}\]

The Schrödinger equation with linear interaction is then

\[
\Phi(p) = -\int \frac{d^3 k}{(2\pi)^3} m V_A(p - k) \left[ \frac{\Phi(k)}{(k^2 - mE)} - \frac{\Phi(p)}{(p^2 - mE)} \right] + m C \frac{\Phi(p)}{(p^2 - mE)}.
\]

\[\tag{3.3}\]

Introducing the four-vectors \(P\), \(k\), and \(p\), with \(P = (2m + E, \bar{0})\) and \(p^2 = k^2 = m^2\) (so that \(E_k = \sqrt{m^2 + k^2}\)), the relativistic generalization of (3.3) which reduces to it in the nonrelativistic limit (when \(m \to \infty\)) is

\[
\Phi(p) = -\int \frac{d^3 k}{(2\pi)^3} \frac{2m^2}{E_k} V_A(p - k) \left[ \frac{\Phi(k)}{m^2 - (P - k)^2} - \frac{\Phi(p)}{m^2 - (P - p)^2} \right] + 2m C \frac{\Phi(p)}{m^2 - (P - p)^2}.
\]

\[\tag{3.4}\]
FIG. 1. The Gross equation. The × indicate that the particle is on-shell.

\( s \) is the Gross equation (see Fig. 1) and the discussion shows that this is the natural relativistic equation obtained from a generalization of the nonrelativistic Schrödinger equation for confined particles.

In a previous application of the Gross equation [2] the kernel (2.3), with \( \rightarrow -q^2 \), was used. This choice is undesirable for use with the two channel solution of this equation, where the mass shell constraints which fix \( q_0 \) introduce singularities for non-zero momentum transfers (i.e. \( q^2 = 0 \), but \( q^\mu \neq 0 \)). To rect this problem the kernel \( V_A(q) \) is defined to be

\[
V_A(q) \equiv -8\pi\sigma \frac{1}{q^4 + (P \cdot q)^4 / P^4}
\]  

(3.5)

vantages of this form are many: i) singularities are restricted to \( q^\mu = 0 \), ii) reaction strength does not depend on the bound state momentum \( P \) in the \( \hat{\imath} \) frame, iii) it has the correct nonrelativistic dependence on \( q^2 \), and iv) the violet regularization used previously [2] is no longer needed.

IV. PROOF OF CONFINEMENT

It is sufficient to consider the case when the constant term \( C = 0 \). If the mass \( \mu > 2m \), then there exists a value of three momentum \( |\vec{p}| = p_c \) in both quarks can be on-shell. In this case \( (P - p)^2 = \mu^2 - 2\mu E_p + m^2 = m^2 \), if the subtraction term in Eq. (3.4) appears to be singular. This singularity is cancelled by the first term, and hence, if the equation is to have a solution, vertex function must be zero at \( p_c \). In particular, as \( |\vec{p}| \to p_c \pm \epsilon \),

\[
(P - p)^2 - m^2 \to \mp \frac{2\mu p_c \epsilon}{E_p},
\]  

(4.1)

the vertex function would have an infinite discontinuity at \( p_c \) unless it were there. We conclude that \( \Phi(p) = 0 \), when \( 2E_p = \mu \) (as illustrated in Fig. 2).

A numerical confirmation of this result is shown in Fig. 3. In the actual calculations to be presented in an upcoming paper, the bound state equation is with the pole contributions of both constituents. The average over positive and negative energy contributions leads to a two channel equation with a two component vertex function. In Fig. 3 the solid line is the (large) component in which the off-shell quark has positive energy and the dashed line is the (small) component in which the off-shell quark has negative energy. Since a physical decay must produce two positive energy quarks, only the large component must have the “confinement” node. Fig. 3 shows the vertex functions for an excited pion with mass \( \mu = 1.2 \text{ GeV} \) composed of two quarks with masses \( m = 0.34 \text{ MeV} \). The large vertex function must have two nodes: one due to the excitation and another at \( p^2 = 0.244 \text{ GeV}^2 \), exactly where both particles are on-shell.

FIG. 2. When both constituents are on-shell, the vertex function vanishes.

FIG. 3. The pion 1st excited state Gross vertex functions are shown. The first node is due to the excited state. The second node assures that the bound state does not decay.
V. CONCLUSIONS

The relativistic equation obtained from a generalization of the nonrelativistic Dirac equation for confined particles is of the Gross equation type rather than the Bethe-Salpeter equation type. The confinement mechanism arises not on the lack of quark mass poles, but through the vanishing of the vertex function when both quarks are on their positive energy mass-shell. Because of this confinement mechanism, the vertex functions (ground or excited state) have one ditional node if the bound state is heavy enough.

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