- Brief introduction to small-$x$ physics.
- DIS from nucleus at high energy and Wilson line.
- Leading order: BK equation.
- Quark and gluon loop in the shock-wave.
- NLO kernel in QCD.
- Conformal invariance of the leading BK kernel.
- NLO kernel in $\mathcal{N} = 4$ SYM.
- Conclusions.
- Outlook.
Deep Inelastic Scattering at Small-$x$

$$s = (P + q)^2 \quad Q^2 = -q^2 > 0$$

- Because of confining property, quarks and gluons cannot be measured
- DIS processes are used to study the quarks and gluons distributions inside a hadron
- Experiments have shown that the structure functions grow very rapidly with decreasing value of $x_B$
DGLAP vs. BFKL

\[ Y = \ln \frac{1}{x_B} \]

\[ \ln \Lambda^2_{QCD} \]

\[ \ln Q^2 (Y) \]

\[ \ln Q^2 \]

SATURATION REGION

BK

JIMWLK

BFKL

DGLAP

G. A. Chirilli (JLAB & ODU)

NLO evolution of color dipole

JLAB November 5, 2008
BFKL: Leading Logarithmic Approximation

\[ \alpha_s \ll 1 \quad (\alpha_s \ln s)^n \sim 1 \]

pQCD at LLA: \[ A(s, t) \propto s^{\Delta} \]
BFKL: Leading Logarithmic Approximation \( \alpha_s << 1 \) \( (\alpha_s \ln s)^n \sim 1 \)

- pQCD at LLA: \( A(s, t) \propto s^\Delta \)

- Froissart-Martin theorem: \( A(s, t) \propto \ln^2 s \)
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\[ \alpha_s \ll 1 \quad (\alpha_s \ln s)^n \sim 1 \]

- DIS cross section

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- At very high energy recombination begins to compensate gluon production. Gluon density reaches a limit and does not grow anymore. So does the total DIS cross sections. Unitarity is restored!
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At very high energy recombination begins to compensate gluon production. Gluon density reaches a limit and does not grow anymore. So does the total DIS cross sections. **Unitarity is restored!**

In order to take into account recombination of gluons the evolution equation for the structure function has to be non-linear.
Consider the amplitude of $\gamma^* A \rightarrow \gamma^* A$
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At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \rightarrow \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):
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the amplitude of $\gamma^* A \rightarrow \gamma^* A$ scattering reduces to the matrix element of
a two-Wilson-line operator (color dipole):

\[
A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \text{Tr} \{U(k_\perp)U^\dagger(-k_\perp)\} | B \rangle
\]

\[
U(x_\perp) = Pe^{ig \int_{-\infty}^{\infty} du \, n^\mu A_\mu(un+x_\perp)}
\]

Wilson line
The structure function of a hadron is proportional to a matrix element of this color dipole operator

\[ \hat{U}^\eta(x_\perp, y_\perp) = 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}^\eta(x_\perp)\hat{U}^{\eta\dagger}(y_\perp)\} \]

switched between the target’s states \((N_c = 3\) for QCD). The gluon parton density is approximately:

\[ x_B G(x_B, \mu^2 = Q^2) \approx \langle p|\hat{U}^\eta(x_\perp, 0)|p\rangle\bigg|_{x_\perp^2 = Q^{-2}} \]

where \(\eta = \ln \frac{1}{x_B}\)
Propagation in the shock wave: Wilson line
Propagation in the shock wave: Wilson line

Boosted Field
Propagation in the shock wave: Wilson line

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Propagation in the shock wave: Wilson line

Each path is weighted with the gauge factor $Pe^{ig} \int dx_\mu A^\mu$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.
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Each path is weighted with the gauge factor $P e^{i g \int d x_{\mu} A^{\mu}}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.
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\[
U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]
\]

\[
[x, y] = Pe^{ig} \int_0^1 du (x-y)^\mu A_\mu (ux+(1-u)y)
\]

\[
p^{\mu} = \alpha p_1^{\mu} + \beta p_2^{\mu} + p_\perp^{\mu}
\]
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$[U^{ab}(z_\perp) \text{ - instantaneous interaction with the } \eta < \eta_2 \text{ shock wave}] \times$

G. A. Chirilli (JLAB & ODU)  
NLO evolution of color dipole  
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Propagation in the shock wave: Wilson line

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$$[x \rightarrow z$: free propagation$] \times [U_{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$] \times [z \rightarrow y$: free propagation$]$$
Propagation in the shock wave: Wilson line

Each path is weighted with the gauge factor $P e^{ig \int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.
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The interaction with the shock wave is given by the Wilson line in the adjoint representation.
To get the evolution equation, consider the dipole with the rapidities up to $\eta_1$ and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to $\eta_2$).
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$$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \otimes$$
Leading order: BK equation

\[
\frac{d}{d \eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \ldots \quad \Rightarrow
\]

\[
\frac{d}{d \eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}
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\[ x_\bullet = \sqrt{\frac{s}{2}} x^- \quad x_* = \sqrt{\frac{s}{2}} x^+ \]
\[ U_{z}^{ab} = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s (\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2} \]
Non linear evolution equation

\[ U_{z}^{ab} = \text{Tr}\{t^{a}U_{z}t^{b}U_{z}^{\dagger}\} \Rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{1}} \rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{2}} + \alpha_{s}(\eta_{1} - \eta_{2})(U_{x}U_{y}^{\dagger}U_{y}U_{x}^{\dagger})^{\eta_{2}} \]

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_{c}}\text{Tr}\{\hat{U}(x_{\perp})\hat{U}^{\dagger}(y_{\perp})\} \]


\[ \frac{d}{d\eta}\hat{U}(x, y) = \frac{\alpha_{s}N_{c}}{2\pi^{2}} \int \frac{d^{2}z (x - y)^{2}}{(x - z)^{2}(y - z)^{2}} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\} \]

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LLA for DIS in pQCD \(\Rightarrow\) BFKL

(LLA: \(\alpha_s \ll 1, \alpha_s \eta \sim 1\)
Non linear evolution equation

\[ U_{z}^{ab} = \text{Tr}\{t^{a}U_{z}t^{b}U_{z}^{\dagger}\} \Rightarrow (U_{x}U_{y}^{\dagger})^{\eta_1} \rightarrow (U_{x}U_{y}^{\dagger})^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_{x}U_{z}^{\dagger}U_{z}U_{y}^{\dagger})^{\eta_2} \]

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\[ \frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_sN_c}{2\pi^2} \int \frac{d^2z}{(x - z)^2(y - z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\} \]


LLA for DIS in pQCD \( \Rightarrow \) BFKL \( \quad \) (LLA: \( \alpha_s \ll 1, \alpha_s \eta \sim 1 \))

LLA for DIS in sQCD \( \Rightarrow \) BK eqn \( \quad \) (LLA: \( \alpha_s \ll 1, \alpha_s \eta \sim 1, \alpha_s^2 A^{1/3} \sim 1 \))

(s for semiclassical)
Why NLO correction?
Why NLO correction?

- To get the region of application of the leading order evolution equation.
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- To determine the argument of the coupling constant:
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  - Theoretical viewpoint: Whether the coupling constant is determined by the size of the original dipole or by the size of the parent dipole, we have different behavior of the solutions.
Why NLO correction?

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  - **Theoretical view point**: Whether the coupling constant is determined by the size of the original dipole or by the size of the parent dipole, we have different behavior of the solutions.
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  - **Experimental view point:** The cross section is proportional to some power of the coupling constant, so the argument of the coupling constant determined how big or how small the cross section is.

- To check conformal invariance in $\mathcal{N} = 4$ SYM.
Non linear evolution equation in the NLO

\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^{\dagger}\} = \\
\int \frac{d^2z}{2\pi^2} \left( \frac{(x - y)^2}{(x - z)^2(z - y)^2} + \alpha_s^2 K_{NLO}(x, y, z) \right) \left[ \text{Tr}\{U_x U_z^{\dagger}\} \text{Tr}\{U_z U_y^{\dagger}\} - N_c \text{Tr}\{U_z U_y^{\dagger}\} \right] + \\
\alpha_s^2 \int d^2z d^2z' \left( K_4(x, y, z, z') \{U_x, U_z^{\dagger}, U_z, U_y^{\dagger}\} + K_6(x, y, z, z') \{U_x, U_z^{\dagger}, U_z', U_y^{\dagger}\} \right)
\]

\( K_{NLO} \) is the next-to-leading order correction to the dipole kernel and \( K_4 \) and \( K_6 \) are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.
\[ \frac{d}{d\eta} \text{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + K_{\text{NLO}} \text{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \]
Definition of the NLO kernel

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + K_{\text{NLO}} \text{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots
\]

\[
\langle K_{\text{NLO}} \text{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} \rangle - \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} \rangle
\]
Definition of the NLO kernel

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\]

\[
\langle K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle - \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle
\]

Where \(\langle \ldots \rangle\) is evaluated in the background of the shock wave
Definition of the NLO kernel

\[ \frac{d}{d\eta} \text{Tr}\{\hat{U}_x\hat{U}^+_y\} = K_{LO} \text{Tr}\{\hat{U}_x\hat{U}^+_y\} + K_{NLO} \text{Tr}\{\hat{U}_x\hat{U}^+_y\} + \ldots \]

\[ \langle K_{NLO} \text{Tr}\{\hat{U}_x\hat{U}^+_y\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x\hat{U}^+_y\} \rangle - \langle K_{LO} \text{Tr}\{\hat{U}_x\hat{U}^+_y\} \rangle \]

Where \( \langle \ldots \rangle \) is evaluated in the background of the shock wave

Subtraction of \( BK^2 \) contribution \( \implies \left[ \frac{1}{u} \right] + \) prescription
Definition of the NLO kernel

\[ \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + K_{NLO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \ldots \]

\[ \langle K_{NLO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle - \langle K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle \]

Where \( \langle \ldots \rangle \) is evaluated in the background of the shock wave

Subtraction of \( BK^2 \) contribution \[ \Rightarrow \quad \left[ \frac{1}{u} \right]_+ \text{ prescription} \]

\[ \int_0^1 du f(u) \left[ \frac{1}{u} \right]_+ \equiv \int_0^1 du \frac{f(u) - f(0)}{u}, \quad \int_0^1 du f(u) \left[ \frac{1}{\bar{u}} \right]_+ \equiv \int_0^1 du \frac{f(u) - f(1)}{\bar{u}} \]
Regularization of the rapididity divergence

For light-like Wilson lines loop integrals are divergent in the longitudinal direction

\[ \int_{0}^{\infty} \frac{d\alpha}{\alpha} = \int_{-\infty}^{\infty} d\eta = \infty \]
Regularization of the rapididity divergence

For light-like Wilson lines loop integrals are divergent in the longitudinal direction

\[
\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty
\]

Regularization by: slope

\[
U^n(x_\perp) = \text{Pexp}\left\{ig \int_{-\infty}^{\infty} du \ n_\mu \ A^\mu(u n + x_\perp) \right\} \quad n^\mu = p_1^\mu + e^{2\eta} p_2^\mu
\]
Regularization by: slope

\[ U^n(x_\perp) = \text{Pexp}\left\{ ig \int_{-\infty}^{\infty} du \ n_\mu \ A^\mu (u n + x_\perp) \right\} \]
Regularization of the rapiditidy divergence

**Regularization by: slope**

$$U_\eta(x_\perp) = P\exp\left\{ig \int_{-\infty}^{\infty} du \, n_\mu \, A^\mu(u n + x_\perp)\right\}$$

**Regularization by: Rigid cut-off**

$$U_\eta^\chi = P\exp\left[ig \int_{-\infty}^{\infty} du \, p_1^\mu A_\mu^\eta(up_1 + x_\perp)\right]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$
Regularization of the rapididty divergence

Regularization by: slope

\[ U^\eta(x_\perp) = \text{Pexp}\left\{ ig \int_{-\infty}^{\infty} du \, n_\mu \, A^\mu(un + x_\perp) \right\} \]

Regularization by: Rigid cut-off

\[ U_{x_\perp}^\eta = \text{Pexp}\left[ ig \int_{-\infty}^{\infty} du \, p_{1\mu}^\eta A_\mu^\eta(up_1 + x_\perp) \right] \]

\[ A^\eta_\mu(x) = \int \frac{d^4k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu(k) \]

The rigid cut-off leads to (almost) conformal result
Argument of coupling constant

\[ \frac{d}{d \eta} \hat{U}(x, y) = \frac{\alpha_s(?) N_c}{2 \pi^2} \int \frac{dz (\vec{x} - \vec{y})^2}{(\vec{x} - \vec{z})^2 (\vec{z} - \vec{y})^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z) \hat{U}(z, y) \right\} \]

I. Balitsky (2006)  
**Renormalon-based approach: summation of quark bubbles**

\[ -\frac{2}{3} n_f \rightarrow b = \frac{11}{3} N_c - \frac{2}{3} n_f \]
Argument of coupling constant

Bubble chain sum:

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \frac{\alpha_s((x - y)^2)}{2\pi^2} \int d^2 z \left[ \text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_y^\dagger\} - N_c \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \right]
\times \left[ \frac{(x - y)^2}{X^2 Y^2} + \frac{1}{X^2} \left( \frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1 \right) + \frac{1}{Y^2} \left( \frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1 \right) \right] + \ldots
\]

Argument of coupling constant

Bubble chain sum:

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^{\dagger}\} = \frac{\alpha_s((x-y)^2)}{2\pi^2} \int d^2z \left[ \text{Tr}\{\hat{U}_x \hat{U}_z^{\dagger}\}\text{Tr}\{\hat{U}_z \hat{U}_y^{\dagger}\} - N_c \text{Tr}\{\hat{U}_x \hat{U}_y^{\dagger}\} \right] \\
\times \left[ \frac{(x-y)^2}{X^2Y^2} + \frac{1}{X^2} \left( \frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1 \right) + \frac{1}{Y^2} \left( \frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1 \right) \right] + \ldots
\]


When the sizes of the dipoles are very different the kernel reduces to:

\[
\frac{\alpha_s((x-y)^2)}{2\pi^2} \frac{(x-y)^2}{X^2Y^2} \ , \ |x - y| \ll |x - z| , |y - z|
\]
\[
\frac{\alpha_s(X^2)}{2\pi^2X^2} \ , \ |x - z| \ll |x - y| , |y - z|
\]
\[
\frac{\alpha_s(Y^2)}{2\pi^2Y^2} \ , \ |y - z| \ll |x - y| , |x - z|
\]
Argument of coupling constant

Bubble chain sum:

\[
\frac{d}{d \eta} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} = \frac{\alpha_s((x - y)^2)}{2\pi^2} \int d^2z \left[ \text{Tr}\{\hat{U}_x \hat{U}^\dagger_z\} \text{Tr}\{\hat{U}_z \hat{U}^\dagger_y\} - N_c \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} \right] \\
\times \left[ \frac{(x - y)^2}{X^2 Y^2} + \frac{1}{X^2} \left( \frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1 \right) + \frac{1}{Y^2} \left( \frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1 \right) \right] + ... 
\]


When the sizes of the dipoles are very different the kernel reduces to:

\[
\frac{\alpha_s((x-y)^2)}{2\pi^2} \frac{(x-y)^2}{X^2 Y^2} \quad |x - y| \ll |x - z|, |y - z| \\
\frac{\alpha_s(X^2)}{2\pi^2 X^2} \quad |x - z| \ll |x - y|, |y - z| \\
\frac{\alpha_s(Y^2)}{2\pi^2 Y^2} \quad |y - z| \ll |x - y|, |x - z| 
\]

⇒ the argument of the coupling constant is given by the size of the smallest dipole.
Diagrams with 2 gluons interaction

(i) (II) (III) (IV) (V)

(VI) (VII) (VIII) (IX) (X)

(I)

(XIV)

(XI)

(XIII)

(XII)

(XV)
Diagrams with 2 gluons interaction
Diagrams with 2 gluons interaction

(XXXI)  (XXXII)  (XXXIII)  (XXXIV)
Extracting the UV divergencies

\[ \text{Tr}\{t^a U_z t^b U_{z'}^\dagger\} = \text{Tr}\{t^a U_z t^b U_{z'}^\dagger - t^a U_{z'} t^b U_z^\dagger\} + \text{Tr}\{t^a U_{z'} t^b U_z^\dagger\} \]

I. Balitsky 2006; Y. Kovchegov, H. Weigert 2006
"Running coupling" diagrams

G. A. Chirilli (JLAB & ODU)
1 → 2 dipole transition diagrams

(a) (b) (c) (d) (e) (f) (g) (h) (i) (j)
Quark and gluon loop in the shock wave

**typical diagram: gluon in the shock wave**
Quark loop in the shock wave

Quark loop diagram + QCD counterterms

QCD counterterms
If we simply subtract the divergence using \( \overline{MS} \), we could be missing some extra finite terms.
A way to get potential extra terms is to consider the exact calculation of the light-cone expansion of $U_x U_y^\dagger$ at $(x - y)^2_\perp \to 0$ in QCD and compare it with the expansion of the sum of the diagrams:
Typical diagrammatic expression of the light-cone expansion $(x - y)^2 \rightarrow 0$ of the quark-loop contribution to the gluon propagator in the external field is
light-cone expansion

Typical diagrammatic expression of the light-cone expansion \((x - y)_{\bot}^2 \rightarrow 0\) of the quark-loop contribution to the gluon propagator in the external field is

Which coincides with the expansion of
Typical diagrammatic expression of the light-cone expansion \((x - y)^2 \rightarrow 0\) of the quark-loop contribution to the gluon propagator in the external field is

\[
(x - y)^2 \rightarrow 0
\]

Which coincides with the expansion of

\[
\Rightarrow \text{no additional local operator at one loop level.}
\]
\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2z \left( \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\} \right) \\
\times \left\{ \frac{(x - y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{11}{3} \ln(x - y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \\
- \frac{11 \alpha_s N_c}{3} \frac{X^2 - Y^2}{4\pi} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c (x - y)^2}{2\pi} \frac{X^2}{X^2 Y^2} \ln \frac{X^2}{(x - y)^2} \ln \frac{Y^2}{(x - y)^2} \right\} \\
+ \frac{\alpha_s}{4\pi^2} \int d^2z' \left\{ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
- (z' \to z)] \left[ \frac{1}{(z - z')^4} \left[ - 2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x - y)^2 (z - z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
\left. + \text{Tr}\{U_x U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} \{U_{z'} U_{z'}^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_{z'} U_y^\dagger U_{z'} U_{z'}^\dagger\} - (z' \to z) \right\] \\
\times \left[ \frac{(x - y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x - y)^2}{(z - z')^2 X^2 Y'^2} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right] \right\}
\]
\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2 z \left[ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\} \right] \\
\times \left\{ \frac{(x - y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{11}{3} \ln(x - y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \\
- \frac{11 \alpha_s N_c}{3} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c (x - y)^2}{2\pi \frac{X^2 Y^2}{X^2 Y^2}} \ln \frac{X^2 (x - y)^2}{(x - y)^2} \ln \frac{Y^2}{(x - y)^2} \right\} \\
+ \frac{\alpha_s}{4\pi^2} \int d^2 z' \left\{ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \left\{ U_{z'} U_y^\dagger \right\} - \text{Tr}\{U_x U_{z'}^\dagger U_y^\dagger U_{z'} U_z^\dagger\} \right\} \\
- (z' \to z) \left[ \frac{1}{(z - z')^4} \left[ -2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x - y)^2(z - z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
+ \text{Tr}\{U_x U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} \left\{ U_{z'} U_z^\dagger \right\} - \text{Tr}\{U_x U_{z'}^\dagger U_y^\dagger U_{z'} U_z^\dagger\} \right] \\
\times \left[ \frac{(x - y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x - y)^2}{(z - z')^2 X^2 Y'^2} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right] \right) =
\]

Running coupling part
\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2 z \left[ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\} \right] \\
\times \left\{ \frac{(x - y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{11}{3} \ln(x - y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \\
- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{4\pi} \frac{(x - y)^2}{X^2 Y^2} \ln \frac{X^2}{(x - y)^2} + \frac{Y^2}{(x - y)^2} \right\} \\
+ \frac{\alpha_s}{4\pi^2} \int d^2 z' \left[ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_y U_{z'}^\dagger U_z U_{z'}^\dagger\} \right] \\
- (z' \to z) \left[ \frac{1}{(z - z')^4} \left[ -2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x - y)^2 (z - z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \right. \\
+ \left. \text{Tr}\{U_x U_{z'}^\dagger\} \text{Tr}\{U_z U_y^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_y U_{z'}^\dagger U_z U_{z'}^\dagger\} \right] - (z' \to z) \right] \\
\times \left[ \frac{(x - y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x - y)^2}{(z - z')^2 X^2 Y'^2} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right) \right) =
\]

Running coupling part + Non-conformal part
\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2 z \left(\text{Tr}\{U_x U_z^\dagger\}\text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}\right)
\]
\[
\times \left\{ \frac{(x - y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{11}{3} \ln(x - y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right]
\right.
\]
\[
- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x - y)^2}{X^2 Y^2} \ln \frac{X^2}{(x - y)^2} \ln \frac{Y^2}{(x - y)^2} \right\}
\]
\[
+ \frac{\alpha_s}{4\pi^2} \int d^2 z' \left\{ \text{Tr}\{U_x U_z^\dagger\}\text{Tr}\{U_{z'} U_y^\dagger\}\{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right.
\]
\[
- (z' \rightarrow z) \frac{1}{(z - z')^4} \left[ -2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x - y)^2(z - z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right]
\]
\[
+ \text{Tr}\{U_x U_z^\dagger\}\text{Tr}\{U_{z'} U_y^\dagger\}\{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \left. - (z' \rightarrow z) \right\}
\]
\[
\times \left[ \frac{(x - y)^4}{X^2 Y'^2(X^2 Y'^2 - X'^2 Y^2)} + \frac{(x - y)^2}{(z - z')^2 X^2 Y'^2} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right) \}
\]

Running coupling part + Non-conformal part + Conformal

"non-analytic" part
\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2z \left[ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right.
\]
\[
\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right. \right.
\]
\[
- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right\}
\]
\[
+ \frac{\alpha_s}{4\pi^2} \int d^2z' \left[ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\}]\{U_z' U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_z' U_y^\dagger U_z U_y^\dagger\} \right]
\]
\[
- (z' \rightarrow z) \frac{1}{(z-z')^4} \left[ -2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2(z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X^2 Y^2}{Y'^2 X^2} \right]
\]
\[
+ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_z' U_y^\dagger\}]\{U_z' U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_z' U_y^\dagger U_z U_y^\dagger\} - (z' \rightarrow z) \right]
\]
\[
\times \left[ \frac{(x-y)^4}{X^2 Y^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y^2} \ln \frac{X^2 Y'^2}{X^2 Y^2} \right] \right\}
\]

Running coupling part + Non-conformal part + Conformal
"non-analytic" part + "conformal-analytic" (\(\mathcal{N} = 4\)) part
\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2z \left( [\text{Tr}\{U_x U_z^\dagger\}] \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\} \right] \\
\times \left\{ \frac{(x - y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{11}{3} \ln(x - y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \\
- \frac{11 \alpha_s N_c}{3} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{4\pi} \frac{(x - y)^2}{X^2 Y^2} \ln \frac{X^2}{(x - y)^2} \ln \frac{Y^2}{(x - y)^2} \right\} \\
+ \frac{\alpha_s}{4\pi^2} \int d^2z' \left\{ [\text{Tr}\{U_x U_z^\dagger\}] \text{Tr}\{U_z' U_y^\dagger\} \{U_z' U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_z' U_y^\dagger U_z' U_y^\dagger\} \right\} \\
- (z' \rightarrow z) \frac{1}{(z - z')^4} \left[ - 2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x - y)^2(z - z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
+ [\text{Tr}\{U_x U_z^\dagger\}] \text{Tr}\{U_z U_z'^\dagger\} \{U_z' U_y^\dagger\} - \text{Tr}\{U_x U_z'^\dagger U_z U_y^\dagger U_z' U_y^\dagger\} - (z' \rightarrow z) \right] \\
\times \left[ \frac{(x - y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x - y)^2}{(z - z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right\} \right\} + \frac{\alpha_s^2 N_c^2}{4\pi^2} \zeta(3) \text{Tr}\{U_x U_y^\dagger\} =
\]

Our result + Extra term \Rightarrow Agrees with NLO BFKL

(Comparing the eigenvalue of the forward kernel)
\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dag\} = \frac{\alpha_s}{2\pi^2} \int d^2z \left( [\text{Tr}\{U_x U_y^\dag\}] \text{Tr}\{U_z U_z^\dag\} - N_c \text{Tr}\{U_x U_y^\dag\} \right)
\]

\[
\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( \frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] 
- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} 
- \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right\}
\]

\[
+ \frac{\alpha_s}{4\pi^2} \int d^2z' \left\{ [\text{Tr}\{U_x U_z^\dag\}] \text{Tr}\{U_z U_z' \} \{U_z' U_y^\dag\} - \text{Tr}\{U_x U_z^\dag U_z' U_y^\dag U_z' U_z^\dag\} 
- (z' \rightarrow z) \right\] \frac{1}{(z-z')^4} \left[ - 2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2 (z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right]
\]

\[
+ [\text{Tr}\{U_x U_z^\dag\}] \text{Tr}\{U_z U_z'\} \{U_z' U_y^\dag\} - \text{Tr}\{U_x U_z^\dag U_z' U_y^\dag U_z' U_z^\dag\} - (z' \rightarrow z) \right\] \times \left[ \frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X'^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right\} + \frac{\alpha_s^2 N_c^2}{4\pi^2} \zeta(3) \text{Tr}\{U_x U_y^\dag\}
\]

However the term \( \frac{\alpha_s^2 N_c^2}{4\pi^2} \zeta(3) \text{Tr}\{U_x U_y^\dag\} \) contradicts the requirement \( \frac{d}{d\eta} U_x U_y^\dag = 0 \) at \( x = y \).
Conformal and non-conformal diagrams

Conformal

Non–Conformal
\[ \langle K_{\text{NLO}} \text{Tr}\{ \hat{U}_x \hat{U}_y^\dagger \} \rangle^\text{rigid} = \frac{d}{d\eta} \langle \text{Tr}\{ \hat{U}_x \hat{U}_y^\dagger \} \rangle^\text{rigid} - \langle K_{\text{LO}} \text{Tr}\{ \hat{U}_x \hat{U}_y^\dagger \} \rangle^\text{rigid} \]
\[ \equiv \left[ \frac{d}{\eta} \langle \text{Tr}\{ \hat{U}_x \hat{U}_y^\dagger \} \rangle \right]^\text{rigid} \]
\[ \equiv \left[ \frac{d}{\eta} \langle \text{Tr}\{ \hat{U}_x \hat{U}_y^\dagger \} \rangle \right]^\text{rigid} + \]
Slope vs. rigid cut-off regularization

\[
\langle K_{NLO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\text{rigid}} = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\text{rigid}} - \langle K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\text{rigid}} \\
\equiv \left[ \frac{d}{\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle \right]_{\text{rigid}}^{\text{"+"}}
\]

But

\[
\langle K_{NLO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\text{slope}} = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\text{slope}} - \langle K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\text{slope}} \\
\neq \left[ \frac{d}{\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle \right]_{\text{slope}}^{\text{"+"}}
\]

\[
\langle K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\text{rigid}} - K_{LO} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\text{slope}} \neq 0
\]

The regularization by slope adds an extra term to the \( K_{NLO} \) calculated using rigid cut-off regularization. This extra term is not conformal invariant.
\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2z \frac{(x-y)^2}{X^2Y^2} \left\{ 1 + \frac{\alpha_s}{4\pi} \left[ b \ln(x-y)^2 \mu^2 - b \frac{X^2 - Y^2}{(x-y)^2} \ln \frac{X^2}{Y^2} + \left( \frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} n_f \right] - 2N_c \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \right\} \left[ \text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_y^\dagger\} - N_c \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \right]
\]

\[
+ \frac{\alpha_s^2}{16\pi^4} \int d^2z d^2z' \left[ - \frac{4}{(z-z')^4} + \left\{ \frac{2X^2Y'^2 + X'^2Y'^2 - 4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2 - X'^2Y^2]} \right\} \ln \frac{X^2Y'^2}{X'^2Y^2} \right]
\times \left[ \text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_y^\dagger\} \text{Tr}\{\hat{U}_y \hat{U}_z^\dagger\} \right] \left[ \text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_y^\dagger\} \text{Tr}\{\hat{U}_y \hat{U}_z^\dagger\} \right] - (z' \rightarrow z)
\]

\[
+ \left\{ \frac{(x-y)^2}{(z-z')^2} \left[ \frac{1}{X^2Y'^2} + \frac{1}{Y'^2X'^2} \right] - \frac{(x-y)^4}{X^2Y'^2X'^2Y^2} \right\} \ln \frac{X^2Y'^2}{X'^2Y^2} \text{Tr}\{\hat{U}_x \hat{U}_z^\dagger\} \text{Tr}\{\hat{U}_z \hat{U}_y^\dagger\} \text{Tr}\{\hat{U}_y \hat{U}_z^\dagger\}
\]

\[
+ 4n_f \left\{ \frac{4}{(z-z')^4} - 2 \frac{X'^2Y^2 + Y'^2X^2 - (x-y)^2(z-z')^2}{(z-z')^4(X^2Y'^2 - X'^2Y^2)} \ln \frac{X^2Y'^2}{X'^2Y^2} \right\} \text{Tr}\{t^a \hat{U}_x t^b \hat{U}_z^\dagger\} \text{Tr}\{t^a \hat{U}_z t^b \hat{U}_y^\dagger\} - (z' \rightarrow z) \right] \]

From NLO BK kernel

\[
\frac{s \, d}{ds} \langle \hat{U}(n, \gamma) \rangle = \frac{\alpha_s N_c}{\pi} \left\{ \left[ 1 - \frac{b \alpha_s}{4 \pi} \frac{d}{d\gamma} \right] + \left( \frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} \frac{n_f}{N_c^2} \right\} \chi(n, \gamma) \\
+ \frac{\alpha_s b}{4 \pi} \left[ \frac{1}{2} \chi^2(n, \gamma) - \frac{1}{2} \chi'(n, \gamma) - \frac{2\gamma}{\gamma^2 - n_f^2} \chi(\gamma) \right] \\
+ \frac{\alpha_s N_c}{4 \pi} \left[ - \chi''(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma) + 4\zeta(3) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \right] \langle \hat{U}(n, \gamma) \rangle
\]
From NLO BK kernel

\[
\frac{d}{ds} \langle \hat{U}(n, \gamma) \rangle = \frac{\alpha_s N_c}{\pi} \left\{ 1 - \frac{b \alpha_s}{4\pi} \frac{d}{d\gamma} + \left( \frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} \frac{n_f}{N_c^2} \right\} \chi(n, \gamma) \\
+ \frac{\alpha_s b}{4\pi} \left[ \frac{1}{2} \chi^2(n, \gamma) - \frac{1}{2} \chi'(n, \gamma) - \frac{2\gamma}{\gamma^2 - \frac{n^2}{4}} \chi(\gamma) \right] \\
+ \frac{\alpha_s N_c}{4\pi} \left[ - \chi''(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma) + 4\zeta(3) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \right] \right\} \langle \hat{U}(n, \gamma) \rangle
\]

From NLO BFKL kernel

\[
\frac{d}{ds} \langle \hat{U}(n, \gamma) \rangle = \frac{\alpha_s N_c}{\pi} \left\{ 1 - \frac{b \alpha_s}{4\pi} \frac{d}{d\gamma} + \left( \frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} \frac{n_f}{N_c^2} \right\} \chi(n, \gamma) \\
+ \frac{\alpha_s b}{4\pi} \left[ \frac{1}{2} \chi^2(n, \gamma) - \frac{1}{2} \chi'(n, \gamma) - \frac{2\gamma}{\gamma^2 - \frac{n^2}{4}} \chi(\gamma) \right] \\
+ \frac{\alpha_s N_c}{4\pi} \left[ - \chi''(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma) + 6\zeta(3) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \right] \right\} \langle \hat{U}(n, \gamma) \rangle
\]
Gluon-loop in the shock wave

From NLO BK kernel

\[ s \frac{d}{ds} \langle \hat{U}(n, \gamma) \rangle = \frac{\alpha_s N_c}{\pi} \left\{ \left[ 1 - \frac{b \alpha_s}{4\pi} \frac{d}{d\gamma} \right] + \left( \frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} \frac{n_f}{N_c^2} \right\} \chi(n, \gamma) \]
\[ + \frac{\alpha_s b}{4\pi} \left[ \frac{1}{2} \chi^2(n, \gamma) - \frac{1}{2} \chi'(n, \gamma) - \frac{2\gamma}{\gamma^2 - \frac{n^2}{4}} \chi(\gamma) \right] \]
\[ + \frac{\alpha_s N_c}{4\pi} \left[ - \chi''(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma) + 4\zeta(3) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \right] \} \langle \hat{U}(n, \gamma) \rangle \]

From NLO BFKL kernel

\[ s \frac{d}{ds} \langle \hat{U}(n, \gamma) \rangle = \frac{\alpha_s N_c}{\pi} \left\{ \left[ 1 - \frac{b \alpha_s}{4\pi} \frac{d}{d\gamma} \right] + \left( \frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} \frac{n_f}{N_c^2} \right\} \chi(n, \gamma) \]
\[ + \frac{\alpha_s b}{4\pi} \left[ \frac{1}{2} \chi^2(n, \gamma) - \frac{1}{2} \chi'(n, \gamma) - \frac{2\gamma}{\gamma^2 - \frac{n^2}{4}} \chi(\gamma) \right] \]
\[ + \frac{\alpha_s N_c}{4\pi} \left[ - \chi''(n, \gamma) - 2\chi(n, \gamma)\chi'(n, \gamma) + 6\zeta(3) + F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) \right] \} \langle \hat{U}(n, \gamma) \rangle \]

The coincidence of terms with the nontrivial \( \gamma \) dependence proves that there is no additional \( O(\alpha_s) \) correction to the vertex of the gluon-shock wave interaction coming from the small loop inside the shock wave.
About the discrepancy

From BK NLO side

The extra term \( \frac{\alpha_s^2 N_c^2}{4\pi^2} \zeta(3) \text{Tr} U_x U_y^\dagger \) would contradict the requirement
\[
\frac{d}{d\eta} U_x U_y^\dagger = 0 \text{ at } x = y
\]

From BFKL NLO side

The coefficient \( 6\zeta(3) \) agrees with the \( j \to 1 \) asymptotics of the three-loop anomalous dimensions of leading-twist gluon operators
Formally, a light-like Wilson line

$$[\infty p_1 + x_{\perp}, -\infty p_1 + x_{\perp}] = P e^{ig \int_{-\infty}^{+\infty} dx^+ A_+(x^+,x_{\perp})}$$

is invariant under inversion (with respect to the point with $x^- = 0$).
Formally, a light-like Wilson line

\[
[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = Pe^{ig \int_{-\infty}^{+\infty} d x^+ A_+(x^+, x_\perp)}
\]

is invariant under inversion (with respect to the point with \(x^- = 0\)). Indeed,

\[(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow \text{after inversion } x_\perp \to \frac{x_\perp}{x_\perp^2} \text{ and } x^+ \to \frac{x^+}{x_\perp^2} \Rightarrow \]
Formally, a light-like Wilson line

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\[ [\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \rightarrow Pe^{ig \int_{-\infty}^{+\infty} \frac{dx^+}{x_\perp^2} A_+\left(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}\right)} = [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}] \]
Conformal invariance of the BK equation

Formally, a light-like Wilson line

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\[ [\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \rightarrow Pe^{ig \int_{-\infty}^{+\infty} \frac{dx^+}{x_\perp^2} A_+\left(\frac{x^+}{x_\perp}, \frac{x_\perp}{x_\perp^2}\right)} = [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}] \]

⇒ The dipole kernel is invariant under the inversion \( V(x_\perp) = U(x_\perp / x_\perp^2) \)

\[ \frac{d}{d\eta} \text{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int \frac{d^2 z}{z^4} \frac{(x - y)^2 y^4}{(x - z)^2 (y - z)^2} [\text{Tr}\{V_x V_z^\dagger\}\text{Tr}\{V_z V_y^\dagger\} - N_c \text{Tr}\{V_x V_y^\dagger\}] \]
Conformal invariance of the BK equation

Formally, a light-like Wilson line

\[
[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = Pe^{ig \int_{-\infty}^{+\infty} dx^+ A_+(x^+, x_\perp)}
\]

is invariant under inversion (with respect to the point with \(x^- = 0\)). Indeed,

\[(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow \text{after inversion } x_\perp \rightarrow \frac{x_\perp}{x_\perp^2} \text{ and } x^+ \rightarrow \frac{x^+}{x_\perp^2} \Rightarrow\]

\[
[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \rightarrow P e^{ig \int_{-\infty}^{+\infty} \frac{dx^+_\perp}{x_\perp^2} A_+(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2})} = [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]
\]

\(\Rightarrow\) The dipole kernel is invariant under the inversion \(V(x_\perp) = U(x_\perp/x_\perp^2)\)

\[
\frac{d}{d\eta} \text{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int \frac{d^2 z}{z^4} \frac{(x - y)^2 z^4}{(x - z)^2(y - z)^2} \left[\text{Tr}\{V_x V_z^\dagger\}\text{Tr}\{V_z V_y^\dagger\} - N_c \text{Tr}\{V_x V_y^\dagger\}\right]
\]

In the leading order we have conformal invariance. What about the next-to-leading order?
$\mathcal{N} = 4$ diagrams (scalar and gluino loops)
$\mathcal{N} = 4$ diagrams (scalar and gluino loops)
Evolution equation in $\mathcal{N} = 4$ (I. Balitsky and G.A.C)

\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2z \left( [\text{Tr}\{U_x U_y^\dagger\}\text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \times \left\{ \frac{(x - y)^2}{X^2Y^2} \left[ 1 + \frac{\alpha_s N_c \pi}{12} \right] - \frac{\alpha_s N_c (x - y)^2}{2\pi X^2Y^2} \ln \frac{X^2}{(x - y)^2} \ln \frac{Y^2}{(x - y)^2} \right\} + \frac{\alpha_s}{16\pi^2} \int d^2z' \left[ \text{Tr}\{U_x U_z^\dagger\}\text{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z\} - (z' \rightarrow z) \right] \times \left[ \frac{(x - y)^4}{X^2Y'^2(X^2Y'^2 - X'^2Y^2)} + \frac{(x - y)^2}{(z - z')^2X^2Y'^2} \ln \frac{X^2Y'^2}{X'^2Y^2} \right] \right)
\]
Evolution equation in $\mathcal{N} = 4$ \cite{Balitsky:2008zza, Balitsky:2008zza, Balitsky:2008zza}

$$\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2 z \left( \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\} \right)$$

$$\times \left\{ \frac{(x - y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c \pi}{12} \right] - \frac{\alpha_s N_c}{2\pi} \frac{(x - y)^2}{X^2 Y^2} \ln \frac{X^2}{(x - y)^2} \ln \frac{Y^2}{(x - y)^2} \right\}$$

$$+ \frac{\alpha_s}{16\pi^2} \int d^2 z' \left[ \text{Tr}\{U_x U_{z'}^\dagger\} \text{Tr}\{U_{z'} U_y^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_{z'} U_y^\dagger U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_{z'} U_y^\dagger U_{z'} U_y^\dagger\} \right]$$

$$\times \left[ \frac{(x - y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x - y)^2}{(z - z')^2 X^2 Y'^2} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right]$$

Conformal scheme-dependent part
\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2 z \left( \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\} \right) \\
\times \left\{ \frac{(x - y)^2}{X^2 Y^2} \left[ 1 - \frac{\alpha_s N_c \pi}{12} \right] - \frac{\alpha_s N_c (x - y)^2}{2\pi} \frac{X^2}{X^2 Y^2} \ln \frac{X^2}{(x - y)^2} \ln \frac{Y^2}{(x - y)^2} \right\} \\
+ \frac{\alpha_s}{16\pi^2} \int d^2 z' \left[ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_{z'} U_z^\dagger\} \\
- \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} \right] \left( z' \to z \right) \\
\times \left[ \frac{(x - y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x - y)^2}{(z - z')^2 X^2 Y'^2} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right]
\]

Conformal scheme-dependent part  +  Non-conformal part
Evolution equation in $\mathcal{N} = 4$ \cite{Balitsky:2008zza} (I. Balitsky and G.A.C)

\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int d^2 z \left[ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\} \right] \\
\times \left\{ \frac{(x - y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c \pi}{12} \right] - \frac{\alpha_s N_c}{2\pi} \frac{(x - y)^2}{X^2 Y^2} \ln \frac{X^2}{(x - y)^2} \ln \frac{Y^2}{(x - y)^2} \right\} \\
+ \frac{\alpha_s}{16\pi^2} \int d^2 z' \left[ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z^\dagger U_y^\dagger U_{z'}^\dagger\} \\
- \text{Tr}\{U_x U_{z'}^\dagger U_z^\dagger U_{z'}^\dagger U_y^\dagger\} \right] - (z' \rightarrow z) \\\n\times \left[ \frac{(x - y)^4}{X^2 Y^2 (X^2 Y^2 - X'^2 Y'^2)} + \frac{(x - y)^2}{(z - z')^2 X^2 Y^2} \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right]
\]

Conformal scheme-dependent part + Non-conformal part
+ Conformal analytic part
Evolution equation in $\mathcal{N} = 4$ (I. Balitsky and G.A.C)

\[
\frac{d}{d\eta} \text{Tr}\{U_x U^\dagger_y\} = \frac{\alpha_s}{2\pi^2} \int d^2 z \left( \left[ \text{Tr}\{U_x U^\dagger_z\}\text{Tr}\{U_z U^\dagger_y\} - N_c \text{Tr}\{U_x U^\dagger_y\} \right] \ight.
\]

\[
\times \left\{ \frac{(x - y)^2}{X^2 Y^2} \left[ 1 + \frac{\alpha_s N_c \pi}{12} \right] - \frac{\alpha_s N_c (x - y)^2}{2\pi X^2 Y^2} \ln \frac{X^2}{(x - y)^2} \ln \frac{Y^2}{(x - y)^2} \right\}
\]

\[
+ \frac{\alpha_s}{16\pi^2} \int d^2 z' \left[ \text{Tr}\{U_x U^\dagger_z\}\text{Tr}\{U_z U^\dagger_{z'}\}\{U_{z'} U^\dagger_y\} - \text{Tr}\{U_x U^\dagger_{z'} U_{z'} U^\dagger_y U_{z'} U^\dagger_z\} \ight.
\]

\[
- \text{Tr}\{U_x U^\dagger_{z'} U_{z'} U^\dagger_y U_{z'} U^\dagger_z\} - (z' \rightarrow z) \right\] 

\[
\times \left[ \frac{(x - y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x - y)^2}{(z - z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right) + \frac{\alpha_s^2 N_c^2}{4\pi^2} \zeta(3) \text{Tr}\{U_x U^\dagger_y\} =
\]

Our result + Extra term $\Rightarrow$ Agrees with NLO BFKL in $\mathcal{N} = 4$

Lipatov and Kotikov, 2004

(Comparing the forward kernel)
\[
\int d^2 p K_{\text{evol}}(q, p) f(p) = 4\alpha_s N_c \int d^2 p \left\{ \frac{1}{(p-q)^2} \left( 1 - \frac{\alpha_s N_c \pi}{12} \right) f(p) - \frac{q^2}{2p^2} f(q) \right\} \\
+ \frac{\alpha_s N_c}{4\pi} \left( - \frac{2 \ln q^2 / p^2 \ln \left( \frac{q-p}{p} \right)^2}{(q-p)^2} + F(q, p) \right) \right\} f(p) + \frac{3\alpha_s^2 N_c^2}{2\pi^2} \zeta(3) f(p)
\]
\[
\int d^2 p K_{\text{evol}}^q(p) f(p) = 4 \alpha_s N_c \int d^2 p \left\{ \frac{1}{(p - q)^2} \left( 1 - \frac{\alpha_s N_c \pi}{12} \right) [f(p) - \frac{q^2}{2p^2} f(q)] \right. \\
\left. + \frac{\alpha_s N_c}{4\pi} \left( - \frac{2 \ln q^2 / p^2 \ln (q-p)^2}{p^2} + F(q, p) \right) \right\} f(p) + \frac{3\alpha_s^2 N_c^2}{2\pi^2} \zeta(3) f(p)
\]

In coordinate space

\[
\int d^2 p K_{\text{evol}}^x(z) f(z) = 4 \alpha_s N_c \int d^2 z \left\{ \frac{1}{(x - z)^2} \left( 1 + \frac{\alpha_s N_c}{4\pi} \right) [f(z) - \frac{q^2}{2p^2} f(x)] \right. \\
\left. + \frac{\alpha_s N_c}{4\pi} \left( - \frac{\pi^2}{3} - \frac{2 \ln z^2 / x^2 \ln (x-z)^2}{x^2} + F(q, p) \right) \right\} f(z) + \frac{3\alpha_s^2 N_c^2}{2\pi^2} \zeta(3) f(x)
\]
\[
\int d^2pK^{\text{evol}}(q,p)f(p) = 4\alpha_sN_c\int d^2p \left\{ \frac{1}{(p-q)^2} \left( 1 - \frac{\alpha_sN_c\pi}{12} \right) f(p) - \frac{q^2}{2p^2}f(q) \right\} \\
+ \frac{\alpha_sN_c}{4\pi} \left( -2 \ln \frac{q^2}{p^2} \ln \frac{(q-p)^2}{p^2} + F(q,p) \right) \right\} f(p) + \frac{3\alpha_s^2N_c^2}{2\pi^2} \zeta(3)f(p)
\]

In coordinate space

\[
\int d^2pK^{\text{evol}}(x,z)f(z) = 4\alpha_sN_c\int d^2z \left\{ \frac{1}{(x-z)^2} \left( 1 + \frac{\alpha_sN_c}{4\pi} \right) f(z) - \frac{q^2}{2p^2}f(x) \right\} \\
+ \frac{\alpha_sN_c}{4\pi} \left( -\frac{\pi^2}{3} - \frac{2 \ln \frac{z^2}{x^2} \ln \frac{(x-z)^2}{x^2}}{(q-p)^2} + F(q,p) \right) \right\} f(z) + \frac{3\alpha_s^2N_c^2}{2\pi^2} \zeta(3)f(x)
\]

While we have

\[
\frac{d}{d\eta} \langle \mathcal{V}(x) \rangle = 4\alpha_sN_c\int d^2z \left\{ \frac{1}{(x-z)^2} \left( 1 + \frac{\alpha_sN_c}{4\pi} \right) \right\} \left[ \langle \mathcal{V}(z) \rangle - \frac{q^2}{2p^2} \langle \mathcal{V}(x) \rangle \right] \\
+ \frac{\alpha_sN_c}{4\pi} \left( -\frac{\pi^2}{3} - \frac{2 \ln \frac{z^2}{x^2} \ln \frac{(x-z)^2}{x^2}}{(q-p)^2} + F(q,p) \right) \right\} \langle \mathcal{V}(z) \rangle
\]
\[ \int d^2p K_{\text{evol}}(q, p)f(p) = 4\alpha_s N_c \int d^2p \left\{ \frac{1}{(p - q)^2} \left( 1 - \frac{\alpha_s N_c \pi}{12} \right) [f(p) - \frac{q^2}{2p^2} f(q)] \right\} \]

\[ + \frac{\alpha_s N_c}{4\pi} \left( - \frac{2 \ln q^2 / p^2 \ln \frac{(q-p)^2}{p^2}}{(q-p)^2} + F(q, p) \right) \left\{ f(p) + \frac{3\alpha_s^2 N_c^2}{2\pi^2} \zeta(3) f(p) \right\} \]

In coordinate space

\[ \int d^2p K_{\text{evol}}(x, z)f(z) = 4\alpha_s N_c \int d^2z \left\{ \frac{1}{(x - z)^2} \left( 1 + \frac{\alpha_s N_c}{4\pi} \right) [f(z) - \frac{q^2}{2p^2} f(x)] \right\} \]

\[ + \frac{\alpha_s N_c}{4\pi} \left( - \frac{\pi^2}{3} - \frac{2 \ln z^2 / x^2 \ln \frac{(x-z)^2}{x^2}}{(q-p)^2} + F(q, p) \right) \left\{ f(z) + \frac{3\alpha_s^2 N_c^2}{2\pi^2} \zeta(3) f(x) \right\} \]

While we have

\[ \frac{d}{d\eta} \langle \mathcal{V}(x) \rangle = 4\alpha_s N_c \int d^2z \left\{ \frac{1}{(x - z)^2} \left( 1 + \frac{\alpha_s N_c}{4\pi} \right) [\langle \mathcal{V}(z) \rangle - \frac{q^2}{2p^2} \langle \mathcal{V}(x) \rangle] \right\} \]

\[ + \frac{\alpha_s N_c}{4\pi} \left( - \frac{\pi^2}{3} - \frac{2 \ln z^2 / x^2 \ln \frac{(x-z)^2}{x^2}}{(q-p)^2} + F(q, p) \right) \left\{ \langle \mathcal{V}(z) \rangle \right\} \]

⇒ Same kernel up to a term proportional to \( \frac{\alpha_s^2 N_c^2}{2\pi^2} \zeta(3) \)
Checking for the missing term

We have checked the possible missing term using gauge/scalar links at infinity

\[
\text{Tr}\{U_x U_y^\dagger\} = \lim_{L \to \infty} \text{Tr}\{[L p_1 + x_\perp, -L p_1 + x_\perp][-L p_1 + x_\perp, -L p_1 + y_\perp]\times[-L p_1 + y_\perp, L p_1 + y_\perp][L p_1 + y_\perp, L p_1 + x_\perp]\}
\]
The NLO kernel for the evolution of the color dipole has been calculated. It consists of three parts: the running-coupling part proportional to $\beta$-function, the conformal part describing $1 \rightarrow 3$ dipoles transition and the non-conformal term.

The result agrees with the forward NLO BFKL kernel up to a term proportional $\alpha_s^2 \zeta(3)$ times the original dipole.

For the creation of dipoles in the small-$x$ evolution, the coupling constant is determined by the size of the smallest dipole.

The NLO evolution kernel depends on the precise definition of the cutoff in the longitudinal momenta.

With $|\alpha| < \sigma$ cutoff, the NLO-BK and the NLO-BFKL for $\mathcal{N} = 4$ is (almost) conformally invariant in the transverse plane.
Although Wilson lines are formally conformal invariant, they are divergent. The rigid cutoff does not preserve conformal invariance.

It is not known how to regularize the Wilson line in such a way that conformal invariance is preserved.

Checking the conformal invariance for the NLO amplitude of the SYM $\mathcal{N} = 4$, and therefore the calculation of the NLO scalar impact factor.