**T-Odd Effects From Color Gauge Inv. via Wilson Line**

Gauge link determined re-summing gluon interactions btwn soft and hard


\[ \Phi_{ij}(x, \vec{p}_T) = \int \frac{dz^- d^2 z_T}{(8\pi^3)} e^{ip_z z} \langle P|\bar{q}_j(0)U_{[0,\xi]}^+[q_i(z)]|P\rangle|_{z^+=0} \]

- **The path [C]** is fixed by hard subprocess within hadronic process.

\[ \int d^4 p d^4 k \delta^4(p + q - k) \text{Tr} \left[ \Phi[U_{[\infty;\xi]}^C(p)H_\mu^+(p, k)\Delta(k)H_\nu(p, k)] \right] \]
We now explain factorization for the semi-inclusive deep inelastic cross section when the incoming hadron $A$ is transversely polarized but the lepton remains unpolarized. (It is left as an exercise to treat the most general case.) The factorization theorems, eq. (12) and eq. (14), continue to apply when we include polarization for the incoming hadron, but with the insertion of helicity density matrices for in and out quarks; this is a simple generalization of the results in refs. [10,23].

\[ \frac{d\sigma}{d^3l' d^3p_B} = \sum_a \int d\xi \frac{d\zeta}{\zeta} \int d^2k_a \int d^2k_{b\perp} \hat{f}_{a/A}(\xi, k_a) \times E'E_{k_b} \hat{D}_{B/a}(\zeta, k_{b\perp}) + Y(x_{Bj}, Q^2, z, q_{\perp}/Q) \]

The function $\hat{f}_{a/A}$ defined earlier gives the intrinsic transverse-momentum dependence of partons in the initial-state hadron. Similarly, $\hat{D}_{B/a}$ gives the distribution of hadrons in a parton, with $k_{b\perp}$ being the transverse momentum of the parton relative to the hadron.
How to motivate the PDFs and FFs from SIDIS at moderate $P_T$

Consider the limit where $Q^2$ is large and $\gamma^*$ assumed to scatter \textit{incoherently} off constituents. Currents treated as in free field theory. Interactions between struck quark and target remnant are neglected at least most of the time---color gauge links will modify this statement.
1) A “DIS” reaction where hadron in current region is detected in the final state-rapidity sep.

2) Parton model assumption: virtual photon strikes quark inside nucleon.

3) In case of SIDIS the tagged final state hadron comes from fragmentation of struck quark.

4) The scattering process can be “factorized” into two soft hadronic parts connected by a hard scattering piece.
Kinematics and “Geometry” of SIDIS

\[ \ell(l) + N(P) \rightarrow \ell(l') + h(P_h) + X, \]

\[ x_B = \frac{Q^2}{2P \cdot q}, \quad y = \frac{P \cdot q}{P \cdot l}, \quad z_h = \frac{P \cdot P_h}{P \cdot q} \]

1) Interested in limit where \( Q^2, p \cdot q, P_h \cdot q \) and \( P \cdot P_h \) become large while \( x_B \) and \( z_h \) remain finite

2) In case of SIDIS the tagged final state hadron comes from the fragmentation of the struck quark
• Interactions btwn. struck quark and target remnant are neglected at least most of the time---color gauge links will modify this statement

• Conserved quantities Helicity and Transversity (will come back to this & “transversity”)

• Tree level factorization-”diagramatic”

• Agrees with Landshoff-Polkinghorne-hints at non-perturbative dynamics

"Covariant parton model” where non-perturbative dynamics are treated in terms of cuts and poles of soft factors. Will come back to this in seminar in May
Light-cone kinematics Photon-Hadron Frame

Light Cone Vectors

\[ V^+ = \frac{V^0 + V^3}{\sqrt{2}}, \quad V^- = \frac{V^0 - V^3}{\sqrt{2}}, \quad V_T \]

\[ V^\mu = (V^+, V^-, V_T) \]

Photon – Hadron Frame

\[ P^\mu = \left( P^+, \frac{M^2}{2P^+}, 0 \right), \quad q^\mu = \left( -x_B P^+, \frac{Q^2}{2x_B P^+}, 0 \right) \quad P_h = \left( \frac{m_h^2 + P_{h\perp}^2}{2P_h^-}, P_h^-, P_{h\perp} \right) \]

\[ P^\mu = P^+ n^\mu_+ + \frac{M^2}{2P^+} n^\mu_- \]

note from \[ z_h = \frac{P \cdot P_h}{P \cdot q} \approx \frac{P_h^-}{q^-} \]

\[ q^\mu = \left( -x_B P^+, \frac{P_h^-}{z_h}, 0 \right) \Rightarrow \frac{Q^2}{2x_B P^+} = \frac{P_h^-}{z_h} \]
Cross Section for SIDIS

- Sum over spin of outgoing electron
- Differential in phase space of final detected electron and hadron
- Sum over all unobserved final states “X”

\[
d\sigma = \frac{1}{4} \sum_{s} \sum_{P_h} \sum_{X} \left\{ \int \frac{d^3 P_X}{(2\pi)^3} \frac{d^3 P}{(2\pi)^3} \frac{d^3 \ell}{(2\pi)^3} \frac{d^3 P_h}{(2\pi)^3} \frac{d^3 \ell'}{(2\pi)^3} \delta^4(P + \ell - P_X - P_h - \ell') |M|^2 \right\} \frac{(2\pi)^3}{2E_X E_h} \frac{d^3 P}{dE_h dP_h d\Omega}
\]

\[
|M|^2 = \frac{e^4}{q^4} L_{\mu\nu} W^{\mu\nu}
\]

\[
W^{\mu\nu} = \frac{1}{(2\pi)^4} \sum_X \int \frac{d^3 P_X}{(2\pi)^3} \frac{d^3 P}{(2\pi)^3} \frac{d^3 \ell}{(2\pi)^3} \frac{d^3 P_h}{(2\pi)^3} \frac{d^3 \ell'}{(2\pi)^3} \delta^4(P + \ell - P_X - P_h - \ell')
\]

\[
\times \langle PS|J^\mu(0)|X, P_h S_h\rangle \langle X, P_h S_h|J^{\nu}(0)|PS\rangle
\]

\[
d\sigma = \frac{1}{4} \frac{e^4}{P} \frac{L_{\mu\nu} W^{\mu\nu}}{q^4} (2\pi)^4 \frac{d^3 \ell'}{(2\pi)^3} \frac{d^3 P_h}{(2\pi)^3} \frac{2E_h d\sigma}{2MQ^4 E L_{\mu\nu} W^{\mu\nu}}
\]
• Interactions btwn. struck quark and target remnant are neglected at least most of the time---color gauge links will modify this statement

• Conserved quantities Helicity and Transversity (will come back to this & “transversity”)

• Tree level factorization-”diagramatic”

• Agrees with Landshoff-Polkinghorne-hints at non-perturbative dynamics

”Covariant parton model” where non-perturbative dynamics are treated in terms of cuts and poles of soft factors. Will come back to this in seminar in May
Cross Section for SIDIS in terms of invariants

$$x_B = \frac{Q^2}{2 P \cdot q}, \quad y = \frac{P \cdot q}{P \cdot l}, \quad z_h = \frac{P \cdot P_h}{P \cdot q}$$

In terms of invariants

$$2E_h \frac{d\sigma}{dx_B dy d^3 P_h} = \frac{\pi \alpha_{em}^2 y}{Q^4} \frac{L_{\mu\nu} W^{\mu\nu}}{z_h}$$

When $P_{h \perp} \ll P_h \sim E_h \implies$

$$\frac{d^3 P_h}{2 E_h} = \frac{dz_h d^2 P_{h \perp}}{2 z_h}$$

Compact expression

$$\frac{d\sigma}{dx_B dy dz_h d^2 P_{h \perp}} = \frac{\pi \alpha_{em}^2}{2 Q^4} \frac{y}{z_h} \frac{L_{\mu\nu} W^{\mu\nu}}{z_h}$$
Comment on DIS from SIDIS

\[
W^{\mu\nu}(q, P, S, P_h) = \frac{1}{(2\pi)^4} \sum_X \int \frac{d^3P_X}{(2\pi)^3 2E_X} (2\pi)^4 \delta^4(P + \ell - P_X - P_h - \ell') \\
\times \langle PS | J^\mu(0) | X, P_h S_h \rangle \langle X, P_h S_h | J^\nu(0) | PS \rangle
\]

Summing over all possible hadrons and integrating over the final state hadron recover inclusive DIS result

\[
\sum_h \int \frac{d^3P_h}{(2\pi)^3 2E_h} W^{\mu\nu}(q, P, S, P_h) = W^{\mu\nu}(q, P, S)
\]
\[ W^{\mu\nu} = \frac{1}{(2\pi)^4} \sum_a e^2 \sum_X \int \frac{d^3P_X}{(2\pi)^3 2E_X} \sum_{X'} \int \frac{d^3P_{X'}}{(2\pi)^3 2E_{X'}} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \times \int \frac{d^4k}{(2\pi)^4} (2\pi)^4 \delta^4(P - p - P_X)(2\pi)^4 \delta^4(p + q - k)(2\pi)^4 \delta^4(k - Ph - P_{X'}) \]
\[ \times [\bar{\chi}(k; Ph, S_h)\gamma^\mu \phi(p; P, S)]^\dagger [\bar{\chi}(k; Ph, S_h)\gamma^\nu \phi(p; P, S)] \]

**Hadron Matrix element of quark fields**

\[ \chi(k; Ph, S_h) = \langle 0|\psi(0)|PhS_h; X \rangle \]
\[ \phi(p; P, S) = \langle X|\psi(0)|PS \rangle \]
Non-perturbative quark-quark Correlators
from Hadron matrix elements of quark fields

\[ \Phi_{ji}(p; P, S) = \sum_X \int \frac{d^3P_X}{(2\pi)^3 2E_X} (2\pi)^4 \delta^4(P - p - P_X) \phi_i(p; P, S) \bar{\phi}_j(p; P, S) \]

exponentiate delta function

\[ = \int d^4\xi e^{ip \cdot \xi} \langle PS| \bar{\psi}_j(0) \psi_i(\xi) |PS\rangle \]

and

\[ \Delta_{ij}(k; P_hS_h) = \sum_{X'} \int \frac{d^3P_{X'}}{(2\pi)^3 2E_{X'}} (2\pi)^4 \delta^4(P_h + P_X - k) \chi_i(k; P_hS_h) \bar{\chi}(k; P_hS_h) \]

\[ = \sum_{X'} \int \frac{d^3P_{X'}}{(2\pi)^3 2E_{X'}} \int d^4\xi e^{ik \cdot \xi} \langle 0| \psi_i(\xi) |P_hS_h, X\rangle \langle P_hS_h, X| \bar{\psi}_j(0)|0\rangle \]

\[ W^{\mu\nu}(q, P, S, P_h) = \sum_a e^2 \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \delta^4(p + q - k) \mathrm{Tr} \left[ \Phi^{\mu} \Delta^{\nu} \right] \]
Light-cone kinematics quark momenta

QUARK Momenta

\[ p^\mu = (p^+, \frac{p^2 + p_\perp^2}{2p^+}, p_\perp) \]

initial quark

\[ k^\mu = (\frac{k^2 + k_\perp^2}{2k^-}, k^-, k_\perp) \]

fragmenting quark

Parton assumptions

\[ p^+ \propto P^+, \quad k^- \propto P_h^- \quad \iff \text{LARGE} - \text{Momenta} \sim Q \]

therefore \[ p^- \propto 1/P^+, \quad k^+ \propto 1/P_h^- \quad \iff \text{SMALL} - \text{Momenta} \sim 1/Q \]
Factorization of Long. Momenta

\[ W^{\mu\nu}(q, P, S, P_h) = \sum_a e^2 \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} \delta^4(p + q - k) \text{Tr} [\Phi \gamma^\mu \Delta \gamma^\nu] \]

\[ \delta^4(p + q - k) = \delta(p^+ + q^+ - k^+) \delta(p^- + q^- - k^-) \delta^2(p_T + q_T - k_T) \]

Parton model assumptions

\[ p^+ \propto P^+, \quad k^- \propto P_h^- \quad \iff \text{LARGE} - \text{Momenta} \sim Q \]

therefore

\[ p^- \propto 1/P^+, \quad k^+ \propto 1/P_h^- \quad \iff \text{SMALL} - \text{Momenta} \sim 1/Q \]

\[ \Rightarrow \delta^4(p + q - k) \approx \delta(p^+ - x_B P^+) \delta(k^- - \frac{P_h}{z_h}) \delta^2(p_T + q_T - k_T) \]
Message: the $P_T$ of the hadron is small!

$$W^{\mu\nu}(q, P, S, P_h) \approx \sum_a e^2 \int \frac{d^2p_T dp^- dp^+}{(2\pi)^4} \int \frac{d^2k_T dk^- dk^+}{(2\pi)^4} \delta(p^+ - x_B P^+) \delta(k^- - \frac{P_h}{z_h}) \delta^2(p_T + q_T - k_T)$$

$$\times \text{Tr} [\Phi(p, P, S) \gamma^\mu \Delta(k, P_h) \gamma^\nu]$$

$$W^{\mu\nu}(q, P, S, P_h) = \int \frac{d^2p_T}{(2\pi)^4} \int \frac{d^2k_T}{(2\pi)^4} \delta^2(p_T - \frac{P_h}{z_h} - k_T) \text{Tr} [\Phi(x, p_T) \gamma^\mu \Delta(z, k_T) \gamma^\nu]$$

$$\Phi(x, p_T) = \int \frac{dp^-}{2} \Phi(p, P, S)|_{p^+=x_B P^+}, \quad \Delta(z, k_T) = \int \frac{dk^-}{2} \Delta(k, P_h)|_{k^- = \frac{P^-}{z_h}}$$
From soft quark-quark Correlators to TMDs

\[ \Phi_{ji}(p; P, S) = \sum_X \int \frac{d^3P_X}{(2\pi)^3 2E_X} (2\pi)^4 \delta^4(P - p - P_X) \phi_i(p; P, S) \bar{\phi}_j(p; P, S) \]

exponentiate delta function

\[ = \int d^4\xi e^{ip\cdot\xi} \langle PS|\bar{\psi}_j(0)\psi_i(\xi)|PS\rangle \]

and

\[ \Delta_{ij}(k; P_hS_h) = \sum_{X'} \int \frac{d^3P_{X'}}{(2\pi)^3 2E_{X'}} (2\pi)^4 \delta^4(P_h + P_X - k) \chi_i(k; P_hS_h) \bar{\chi}(k; P_hS_h) \]

\[ = \sum_{X'} \int \frac{d^3P_{X'}'}{(2\pi)^3 2E'_{X'}} \int d^4\xi e^{ik\cdot\xi} \langle 0|\psi_i(\xi)|P_hS_h, X\rangle \langle P_hS_h, X|\bar{\psi}_j(0)|0\rangle \]

\[ W^{\mu\nu}(q, P, S, P_h) = \sum_a e^2 \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \delta^4(p + q - k) \text{Tr} [\Phi_{ij} \Delta_{jk}] \]

Factorization proceeds by Sudakov/L.C. decomp of quark momenta ... next time
\[ F_{UU,T}(x, z, P_{h \perp}^2, Q^2) = C[f_1 D_1] \]
\[ = \int d^2 p_T \, d^2 k_T \, d^2 l_T \, \delta^{(2)}(p_T - k_T + l_T - P_{h \perp}/z) \]
\[ x \sum_a e_a^2 f_1^a(x, p_T^2, \mu^2) D_1^a(z, k_T^2, \mu^2) U(l_T^2, \mu^2) H(Q^2, \mu^2) \]

**FIG. 8 (color online).** The leading region for SIDIS after soft and collinear factorizations.

Collins, Soper, NPB 193 (81)
Ji, Ma, Yuan, PRD 71 (05)

And Bacchetta Boer Diehl Mulders JHEP 08
Few extra Slides on Transversity should come back to this
Conserved quantities, mass & Polarization

\[ W^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma - \text{Pauli – Lubanski} \]

\[ P^\mu - \text{Momentum,} \]

\[ J_{\nu\rho} - \text{AngularMomentum} \]

where,

\[ W^2 = -m^2 s(s + 1) - \text{Casimir} \]

\[ P^2 = m^2 - \text{Casimir} \]

Eigen States of polarization

\[ P^\mu |p, s > = p^\mu |p, s > \]

\[ -\frac{W \cdot n}{m} |p, s > = \mp \frac{1}{2} |p, s > \]

where \[ [H, P^\mu] = [H, W \cdot n] = 0, \text{ and } n \cdot p = 0, n^2 = -1 \]
- Polarization Longitudinal and Transverse helicity and transversity states

When $\vec{n}||\vec{p}$,

$$-\frac{W \cdot n}{m} = \frac{1}{2m} \gamma_5 \gamma_\perp \vec{p} = \frac{\vec{\Sigma} \cdot \vec{p}}{2p}$$

helicity spinors

$$\frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|} u_\pm(p) = \pm u_\pm(p),$$

$$\frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|} v_\pm(p) = \mp v_\pm(p).$$

When $\vec{n} \cdot \vec{p} = 0$,

$$-\frac{W \cdot n}{m} = \frac{\gamma_5 \gamma_\perp \cdot \vec{n}_\perp \vec{p}}{2m} = \frac{\gamma_0 \vec{\Sigma}_\perp \cdot \vec{n}_\perp \vec{p}}{2m}$$

transversity spinors

$$\frac{\gamma_0 \vec{\Sigma}_\perp \cdot \vec{n}_\perp}{2} u_{\uparrow\downarrow}(p) = \pm u_{\uparrow\downarrow}(p),$$

$$\frac{\gamma_0 \vec{\Sigma}_\perp \cdot \vec{n}_\perp}{2} v_{\uparrow\downarrow}(p) = \mp v_{\uparrow\downarrow}(p).$$
• Classification of quark spin states depends on Dirac Matrices that
  • 1) commute with each other and
  • 2) Hamiltonian
• Helicity is a good quantum number
• Transversity is a good quantum number in the parton model
- **Parton model-nucleon at leading twist yields 3 leading quark distribution functions. Introducing projection ops for good component of Dirac spinor reveals meaning.**

\[
P_{[+/−]} = \frac{1}{2} (1 \mp γ₅)
\]

\[
f_1(x) \sim \int d^2k_{\perp} \langle P| b_+^\dagger(xp, k_{\perp})b_+(xp, k_{\perp}) + b_-^\dagger(xp, k_{\perp})b_-(xp, k_{\perp})|P\rangle
\]

\[
g_1(x) \sim \int d^2k_{\perp} \langle PS_z| b_+^\dagger(xp, k_{\perp})b_+(xp, k_{\perp}) - b_-^\dagger(xp, k_{\perp})b_-(xp, k_{\perp})|PS_z\rangle
\]

\[
h_1(x) \sim \int d^2k_{\perp} \text{Re}\langle PS_\perp| b_+^\dagger(xp, k_{\perp})b_+(xp, k_{\perp})|PS_\perp\rangle
\]

\[
Q_{[⊥/⊤]} = \frac{1}{2} (1 \pm γ₅γ_{⊥}) \rightarrow γ_{⊥} = γ^y
\]

\[
f_1(x) \sim \int d^2k_{\perp} \langle P| b_-^\dagger(xp, k_{\perp})b_-(xp, k_{\perp}) + b_+^\dagger(xp, k_{\perp})b_+(xp, k_{\perp})|P\rangle
\]

\[
g_1(x) \sim \int d^2k_{\perp} \text{Re}\langle PS_z| b_-^\dagger(xp, k_{\perp})b_-(xp, k_{\perp})|PS_z\rangle
\]

\[
h_1(x) \sim \int d^2k_{\perp} \langle PS_\perp| b_-^\dagger(xp, k_{\perp})b_-(xp, k_{\perp}) - b_+^\dagger(xp, k_{\perp})b_+(xp, k_{\perp})|PS_\perp\rangle
\]
Need another chiral odd soft factor, i.e., a fragmentation function to flip the helicity of the initial quark, e.g., SIDIS.