Relativistic Dynamics in Nuclear and Particle Physics

Ben Bakker

Vrije Universiteit
Faculty of Sciences
Department of Physics and Astronomy
Amsterdam, The Netherlands

May 2011

Outline

Motivation

Relativistic Frameworks

Interactions

Forms of dynamics
- Kinematics in Front Form (LFD)
- Spin

Variations on a theme in ϕ^3 Theory

Summary and outlook

Motivation

Definitions

What is meant by relativity?	*Einstein*
What is meant by dynamics?	*Hamilton*
What is meant by relativistic dynamics?	*Dirac*

Does it matter?

General relativity and GPS	*Precision*
Atomic physics and chemistry	*Precision*
Nuclear physics	*Precision*
Particle physics	*Essential*

Where to find common ground?

Quantum Field Theory

Specify the Lagrangean and choose the approach

Manifestly covariant: Bethe-Salpeter Equation (BSE)

Quasipotential reduction of the BSE

- Blankenbecler-Sugar
- Kadyshevsky
- Thompson
- Covariant Spectator Theory (Gross)

Hamiltonian Dynamics

- Instant-Form Dynamics
- Front-Form Dynamics
- Point-Form Dynamics

Relativistic quantum-field theory

Features

Particle creation and annihilation of
photons, gluons, W and Z bosons, pions ...

"Natural" basis for interaction understood
à la Yukawa

Problems

Quantum field theories do not "exist"

Regularization and renormalization

Spectrum may be unbounded from below
Example: collapse in φ^3 theory.

Digression

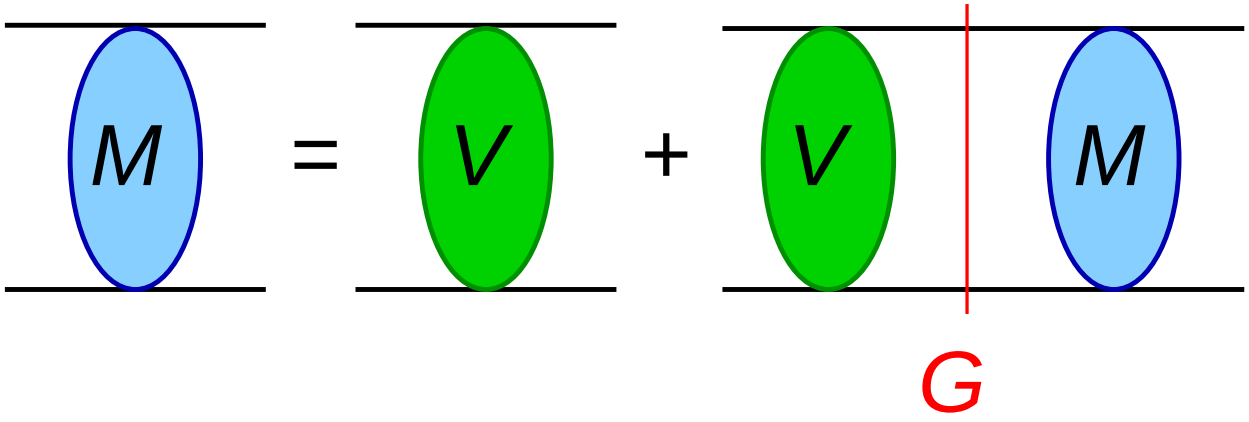
Effective Field Theory

Effective field theory is limited to low energy and small momentum transfers.

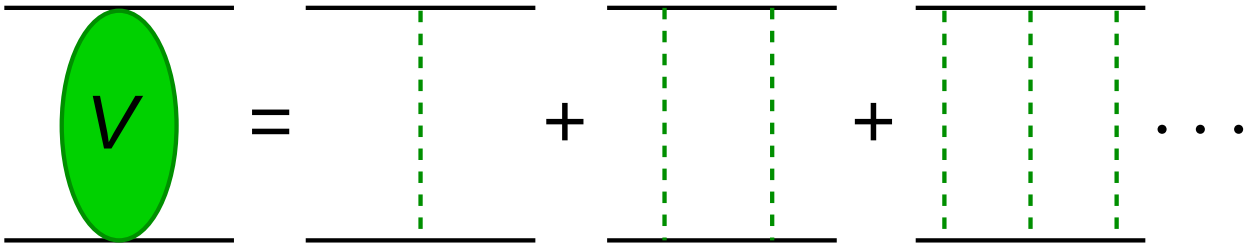
Despite its successes in few-body nuclear systems, it is difficult to see how effective field theory could accommodate the description of intrinsically relativistic processes.

Bethe-Salpeter Equation

In a relativistic approach to non-perturbative physics, one may use the Beth-Salpeter equation.



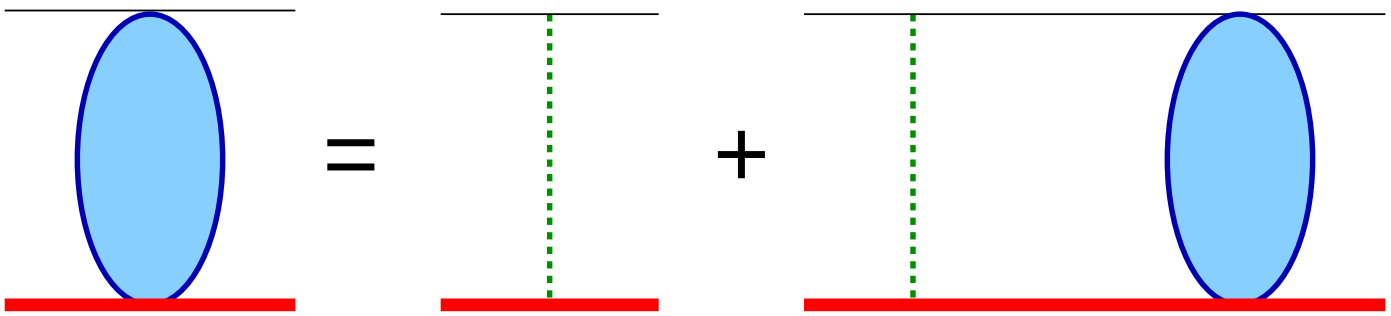
How to construct the kernel V?



Include all one-particle irreducible diagrams?

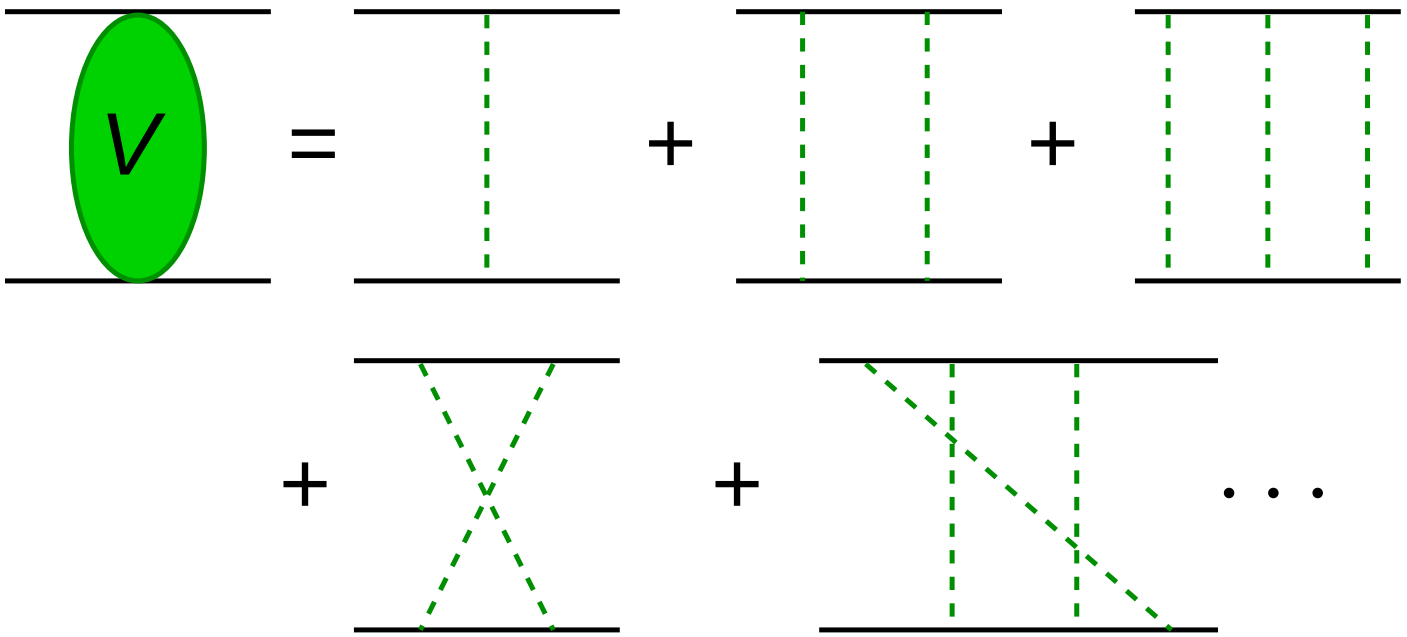
Ladder Bethe-Salpeter equation

Problem: The one-body limit for the ladder equation is incorrect.
BSeq $\not\to$ Klein-Gordon or Dirac equation.



$$M(E) = V + V\,G(E)\,M(E)$$

Including crossed-ladder diagrams can cure the problem with the on-body limit (Gross).



An exact calculation becomes more complicated if one uses such a kernel.

Quasi-Potential equations

Projection

Replace the free propagator with a three-dimensional one by changing one component of the four-momenta of the particles in the intermediate state using a Dirac δ-function in such a way that the one-body limit exists.

For instance, the Blankenbecler-Sugar propagator can be written as

$$G_{\mathrm{BbS}}(P,p) = \frac{\delta(2P\cdot p)}{p^2 - k^2} g(p,k)$$

$$P = p_1 + p_2, \quad p = \frac{1}{2}(p_1 - p_2), \quad g(k,k) = 1.$$

The *parametric energy* E corresponds to $E = 2\sqrt{\mathbf{k}^2 + m^2}$.

Potentials in nuclear physics

Developments

Purely phenomenological $\lesssim$ 1960 (Hamada-Johnston, Reid)
Pion exchange always built in as a local Yukawa potential.

Boson-exchange potentials, still in use
Paris, Bonn, Nijmegen, Gross

Potentials or mass operators constructed in a form of relativistic Hamiltonian dynamics
Instant form, front form, and point form

The idea goes back to Dirac 1949, but realistic models were only recently
developed.

Motivation

S. Weinberg, Phys. Rev. **150**, 1313 (1966)

> *"The Feynman rules provide a perturbation theory in which the Lorentz invariance of the S matrix is kept visible at every step. However, this is accomplished only at the cost of manifest unitarity, by lumping together intermediate states with different numbers of particles and antiparticles. Thus, when we try to sum Feynman diagrams to obtain integral equations like the Bethe-Salpeter equation, it proves very difficult to justify the omission of any particular diagrams, since there is no one-to-one relation between internal lines and intermediate states."*

The Poincaré group and algebra (reminder)

Generators

P^μ	space-time translations
$M^{\mu\nu}$	Lorentz transformations

Commutation relations

$$
\begin{aligned}
[P^\mu, P^\nu] &= 0 \\
[M^{\mu\nu}, P^\sigma] &= i\left(P^\mu g^{\nu\sigma} - P^\nu g^{\mu\sigma}\right) \\
[M^{\mu\nu}, M^{\rho\sigma}] &= i\left(g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} + g^{\mu\sigma} M^{\nu\rho} - g^{\nu\sigma} M^{\mu\rho}\right)
\end{aligned}
$$

The well known physical interpretation of these operators is

$$
\begin{aligned}
J^i &= \tfrac{1}{2}\epsilon_{ijk} M^{jk} \\
K^i &= M^{0i}
\end{aligned}
$$

Dirac's 1949 approach: two requirements

Relativity

"General relativity requires that physical laws expressed in terms of a system of curvilinear coordinates in space-time, shall be invariant under transformations from one such coordinate system to another."

Dynamics

"A second general requirement for dynamical theory has been brought to light through the discovery of quantum mechanics by Heisenberg and Schrödinger, namely the requirement that the equations of motion shall be expressible in the Hamiltonian form."

Dirac discussed three forms

Instant Form	$x^0 = 0,$
Point Form	$x^2 = \kappa^2 > 0,\ x^0 > 0,$
Front Form	$x^0 + x^3 = 0.$

Instant form

$$
\begin{aligned}
P^0 &= \sum \sqrt{\mathbf{p}^2 + m^2} + V, \\
M^{0r} &= \sum x^r \sqrt{\mathbf{p}^2 + m^2} + V^r,
\end{aligned}
$$

where V is a three-dimensional scalar, independent of the origin of the coordinates $\mathbf{x}$, and $\mathbf{V}$ is a three-dimensional vector, such that

$$
\mathbf{V} = \mathbf{x}\, V + \mathbf{V}',
$$

where $\mathbf{V}'$ is again independent of the origin of the coordinates. The real difficulty is to satisfy the commutators $[V, \mathbf{V}]$ and $[V^i, V^j]$ that follow from the Poincaré algebra.

Point form

The quantization surface is the hyperboloid $x^2 = \kappa^2$. The dynamical operators are (p^μ is conjugate to x^μ)

$$P^\mu = \sum [p^\mu + x^\mu B(p^2 - m^2)] + V^\mu,$$

with

$$B(p^2 - m^2) = \frac{1}{x^2}\left[\sqrt{(p\cdot x)^2 - x^2(p^2 - m^2)} - p\cdot x\right].$$

The interaction V^μ must be a four-vector and the real difficulty is to satisfy the commutators $[V^\mu, V^\nu]$ that follow from $[P^\mu, P^\nu] = 0$.

Front form; $x^{\pm} = (x^0 \pm x^3)/\sqrt{2}$

$$
\begin{aligned}
P^- &= \sum \frac{\mathbf{p}^{\perp\,2} + m^2}{2p^+} + V, \\
M^{-\,i} &= \sum \left[x^i \frac{\mathbf{p}^{\perp\,2} + m^2}{2p^+} - x^- p^i \right] + V^i.
\end{aligned}
$$

The interaction V must be invariant under all transformations of $\mathbf{x}^{\perp}$ and x^-, except those of the form $x^- \to \lambda x^-$, in which case $V \to \lambda V$. The interactions $\mathbf{V}^{\perp}$ can be written as

$$
\mathbf{V}^{\perp} = \mathbf{x}^{\perp} V + \mathbf{V}'^{\perp},
$$

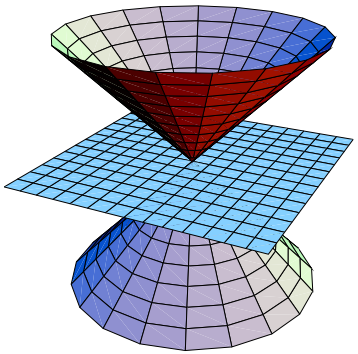
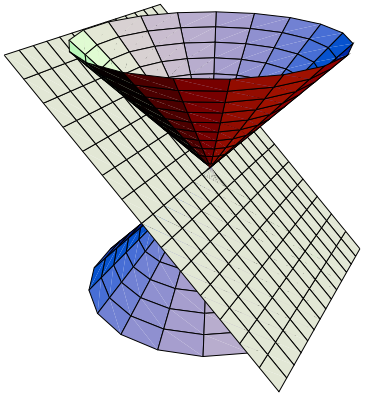
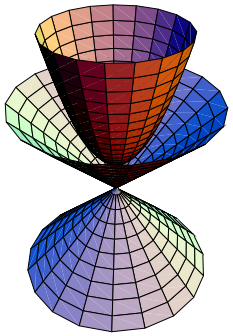
where $\mathbf{V}'^{\perp}$ is subject to the same limitations as V, and in addition transforms as a vector under rotations about the z-axis.

A complete construction of the generators was given by Bakamjian and Thomas (1953) starting from an invariant mass operator. Their method is peculiar in that all interaction dependence is introduced solely through this operator.

It was proven by Sokolov an Shatny (1978) that this leads to equivalent forms of dynamics. These authors consider two forms equivalent if their Hamiltonians are related by a unitary similarity transformation, which guarantees that S-matrix elements calculated in these two forms coincide.

Summary of forms of dynamics

Comparison of three different forms of dynamics

Instant Form	**Front Form**	**Point Form**
		
Quantization Surface		
$x^0 = 0$	$\frac{x^0+x^3}{\sqrt{2}} = 0$	$x^2 = \kappa^2 > 0,\ x^0 > 0$

Summary, continued

Instant Form	**Front Form**	**Point Form**
Kinematical Generators		
$\mathbf{P}$ $\mathbf{J}$	P^+, $\mathbf{P}^\perp$ $E^1 = M^{+1} = \frac{K_x+J_y}{\sqrt{2}}$ $E^2 = M^{+2} = \frac{K_y-J_x}{\sqrt{2}}$ $J_z = M^{12}$ $K_z = M^{-+}$	$M^{\mu\nu}$
Dynamical Generators		
P^0 $\mathbf{K}$	P^- $F^1 = M^{-1} = \frac{K_x-J_y}{\sqrt{2}}$ $F^2 = M^{-2} = \frac{K_y+J_x}{\sqrt{2}}$	P^μ

Comment: *In the front form, the boost operators from a subalgebra of the Poincaré algebra.*

Summary, continued

Instant Form	**Front Form**	**Point Form**
Plane-wave Representation		
$\lvert\mathbf{p}\rangle$ $p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$ $p^0 > 0$ and $p^0 < 0$	$\lvert p^+, \mathbf{p}^\perp\rangle$ $p^- = \frac{\mathbf{p}^{\perp\,2} + m^2}{2p^+}$ $p^- > 0 \leftrightarrow p^+ > 0$	$\lvert\mathbf{u}\rangle$ $u^\mu = p^\mu/m,\ u^2 = 1$ $u^0 = \pm\sqrt{\mathbf{u}^2 + 1}$
Measure		
$\int \frac{\mathrm{d}^3 p}{2p^0}$	$\int \frac{\mathrm{d}^2 p^\perp \mathrm{d}p^+}{2p^+}$	$\int \frac{\mathrm{d}^3 u}{2u^0}$

Comments

In the front form the energy-dispersion relation allows for a kinematical separation of positive and negative energy particles.

States with $p^+ = 0$ may cause singularities (zero modes).

Lorentz transformations of momenta

Notation (reminder)

$$p^\mu = (p^+, \mathbf{p}_\perp, p^-),\ p^\pm = \frac{p^0 \pm p^3}{\sqrt{2}},\ \mathbf{p}_\perp = (p^1, p^2),\ p^2 = 2p^+p^- - \mathbf{p}_\perp^2$$

$$E^1 = M^{+1} = \frac{K_x + J_y}{\sqrt{2}}, \quad E^2 = M^{+2} = \frac{K_y - J_x}{\sqrt{2}}, \quad K_z = M^{-+}$$

A kinematical boost is given by

$$B_{\mathrm{LF}}(\mathbf{v}_\perp; \chi) = \exp(-i\sqrt{2}\mathbf{v}_\perp \cdot \mathbf{E}_\perp) \exp(-i\chi K_z)$$

A boost from the rest frame to a frame where a massive particle with mass m has momentum p^μ is given by

$$B_{\mathrm{LF}}(\mathbf{v}_\perp; \chi)(m/\sqrt{2}, 0, 0, m/\sqrt{2}) = (p^+, p_x, p_y, p^-)$$

$$\chi = \log \frac{\sqrt{2}p^+}{m}, \quad \mathbf{v}_\perp = \frac{\mathbf{p}_\perp}{\sqrt{2}p^+} \longleftrightarrow p^+ = e^\chi \frac{m}{\sqrt{2}}, \quad \mathbf{p}_\perp = e^\chi m\mathbf{v}_\perp$$

Kinematics

Matrix form

$$B_{\mathrm{LF}}(\mathbf{v}_\perp;\chi) = \begin{pmatrix} e^{\chi} & 0 & 0 & 0 \\ v_x & 1 & 0 & 0 \\ v_y & 0 & 1 & 0 \\ \frac{\mathbf{v}_\perp^2}{2}e^{-\chi} & v_x e^{-\chi} & v_y e^{-\chi} & e^{-\chi} \end{pmatrix}$$

This boost acting on an arbitrary four momentum p^μ gives

$$B_{\mathrm{LF}}(\mathbf{v}_\perp;\chi)p^\mu = \left(e^{\chi}p^+, p^+\mathbf{v}_\perp + \mathbf{p}_\perp, e^{-\chi}\left[\frac{\mathbf{v}_\perp^2}{2}p^+ + \mathbf{v}_\perp\cdot\mathbf{p}_\perp + p^-\right]\right)$$

There does not exist an LF boost that changes the plus momentum of a particle from 0 to a finite value or *vice versa*.

This fact is in agreement with the *spectrum condition* $p^+ \geq 0$

$p^+ = 0$ occurs either for massless particles, or for massive particles in the limit where the IF momentum component p_z goes to $-\infty$

Goodness[1]

An operator A has goodness g, if $[K_z, A] = -igA$

In practice it means that $B(\mathbf{v}_\perp; \chi)A \sim e^{g\chi}A$

It is related to the infinite-momentum limit $\chi \to \infty$

E^1, E^2, and P^+ have goodness $+1$ (good)
K^3, J^3, and $\mathbf{P}_\perp$ have goodness 0, (bad)
F^1, F^2, and P^- have goodness -1, (terrible)

Momentum p^μ

p^+ has goodness $+1$, $\mathbf{p}_\perp$ has goodness 0, and p^- has goodness -1

similarly, current J^μ

J^+ has goodness $+1$, $\mathbf{J}_\perp$ has goodness 0, and J^- has goodness -1

and the scalar $q \cdot J$ has goodness 0

[1] H. Leutwyler and J. Stern, Ann. Phys. **112**, 94 (1978)

States

Single particle states $|p^+, \mathbf{p}_\perp\rangle$ are normalized invariantly

$$\langle p'^+, \mathbf{p}'_\perp | p^+, \mathbf{p}_\perp\rangle = (2\pi)^3 2p^+ \delta(p'^+ - p^+)\delta(\mathbf{p}'_\perp - \mathbf{p}_\perp)$$

which corresponds to the phase-space integral

$$\int \mathrm{d}^4 p\, \delta(p^2 - m^2) = \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2} \int \frac{\mathrm{d}p^+}{(2\pi)2p^+} \int \mathrm{d}p^- \delta(p^- - p^-_{\mathrm{on}})$$

Two-particle states: total momentum P and individual momenta p_1 and p_2, $P = p_1 + p_2$

$$\begin{aligned} p_1^+ &= xP^+, & p_2^+ &= (1-x)P^+, \\ \mathbf{p}_{1\perp} &= x\mathbf{P}_\perp + \mathbf{q}_\perp, & \mathbf{p}_{2\perp} &= (1-x)\mathbf{P}_\perp - \mathbf{q}_\perp \end{aligned}$$

If the momenta p_i are on mass shell, their minus components are of course given by

$$p_i^- = \frac{\mathbf{p}_{i\perp}^2 + m_i^2}{2p_i^+}$$

Four momentum conservation gives

$$P^- = \frac{\mathbf{P}_\perp^2 + M^2}{2P^+}, \quad M^2 = \frac{\mathbf{q}_\perp^2 + m_1^2}{x} + \frac{\mathbf{q}_\perp^2 + m_2^2}{1-x}$$

The quantities x and $\mathbf{q}_\perp$ are invariant under LF boosts

We write a two-particle state as $|P^+, \mathbf{P}_\perp, x, \mathbf{q}_\perp\rangle$ and find the invariant two-body phase space

$$\mathrm{d}^2 p_{1\perp} \frac{\mathrm{d}p_1^+}{2p_1^+} \mathrm{d}^2 p_{2\perp} \frac{\mathrm{d}p_2^+}{2p_2^+} = \mathrm{d}^2 P_\perp \mathrm{d}P^+ \, \mathrm{d}^2 q_\perp \frac{\mathrm{d}x}{2x(1-x)}$$

We can write LF relative two-body wave functions (vertices) in terms of the LF boost invariant quantities x and $\mathbf{q}_\perp$ only, separating out the "center-of-momentum" part that depends on P^+ and $\mathbf{P}_\perp$

This property is peculair to LFD, since only in this form of dynamics there exists a kinematic subgroup of the Poincaré group consisting of the boosts B_{LF}. Wigner rotations do not occur.

LF helicity

In interacting LFD one cannot use the total angular momentum $\mathbf{J}$, because only J_z is kinematical. One may use the LF helicity defined as follows for a state that is related to the rest state by LF boost $B_{\mathrm{LF}}(\mathbf{v}_\perp;\chi)$

$$\begin{aligned} h_{\mathrm{LF}} &= \exp(-i\sqrt{2}\mathbf{v}_\perp\cdot\mathbf{E}_\perp)\,J_z\,\exp(i\sqrt{2}\mathbf{v}_\perp\cdot\mathbf{E}_\perp) \\ &= B_{\mathrm{LF}}(\mathbf{v}_\perp;\chi)\,J_z\,B_{\mathrm{LF}}^{-1}(\mathbf{v}_\perp;\chi) \\ &= J_z - \frac{P_xE_y - P_yE_x}{P^+} \end{aligned}$$

For particles of different spin, the helicity takes of course different forms. We shall discuss spin-1/2 and spin-1 states.

Spinors

The helicity operator for a spin-1/2 particle with momentum $(p^+, \mathbf{p}_\perp, p^-)$ and mass m is

$$h_{\mathrm{LF}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{p_x + ip_y}{\sqrt{2}p^+} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{p_x - ip_y}{\sqrt{2}p^+} \\ 0 & 0 & 0 & -\frac{1}{2}. \end{pmatrix}$$

The spinors are in chiral representation.

$$u(\uparrow) = \frac{1}{\sqrt{2m\sqrt{2}p^+}} \begin{pmatrix} \sqrt{2}p^+ \\ p_x + ip_y \\ m \\ 0 \end{pmatrix}, u(\downarrow) = \frac{1}{\sqrt{2m\sqrt{2}p^+}} \begin{pmatrix} 0 \\ m \\ -p_x + ip_y \\ \sqrt{2}p^+ \end{pmatrix}$$

Notice that $p^+ = 0$ renders the spinors undefined.

Vector-particle polarization

For a vector particle with mass m the polarization vectors are obtained by LF boosting the (circular) polarization vectors

$$\varepsilon(\pm 1) = \left(0, \mp\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 0\right), \quad \varepsilon(0) = (1, 0, 0, -1)$$

to the frame where the particle has momentum p^μ

$$\begin{aligned} \varepsilon(p^+, \mathbf{p}_\perp; \pm 1) &= \left(0, \mp\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, \mp\frac{p_x \pm i p_y}{\sqrt{2}p^+}\right) \\ \varepsilon(p^+, \mathbf{p}_\perp; 0) &= \left(\frac{p^+}{m}, \frac{p_x}{m}, \frac{p_y}{m}, \frac{\mathbf{p}_\perp^2 - m^2}{2mp^+}\right). \end{aligned}$$

These polarization vectors are transverse: $p \cdot \varepsilon(p^+, \mathbf{p}_\perp; h) = 0$

Note that these polarization vectors are undefined if $p^+ = 0$ and that moreover $\varepsilon(p^+, \mathbf{p}_\perp; 0)$ is undefined if $m = 0$

Partial summary

Light-Front Dynamics (LFD) is ideally suited for a description of relativistic processes because:

(i) A Fock-space expansion of many-particle states is valid owing to the simplicity of the Fock vacuum owing to the spectrum condition $q^+ \geq 0$.

(ii) In LFD one works with physical degrees of freedom only. No negative-energy particles are included. Antiparticles can of course be described in LFD.

(iii) LFD treats physical systems at the amplitude level: LF wave functions are defined independently of the reference frame. They are boost invariant.

Two approaches to Light-Front Dynamics (LFD):

Kogut and Soper [2]*: project on the light front* $\int dk^-$

Brodsky, Pauli, and Pinsky [3]*: construct the Hamiltonian*

There exist many pitfalls, treacherous points, ... in both approaches.

A particular category is formed by light-front singularities, i.e., singularities that exist in the LF formulation, but do not occur in a manifestly-covariant approach.

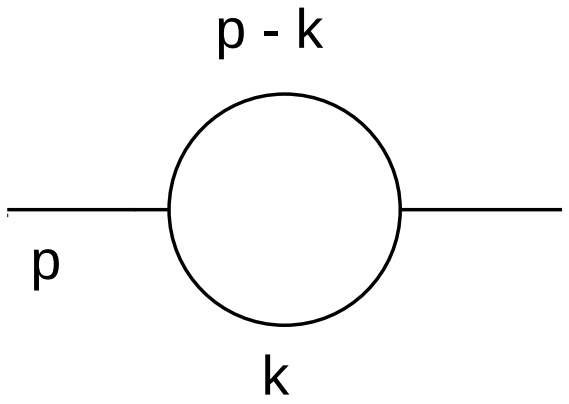
We shall discuss them in the second lecture.

---

[2] J.B. Kogut and D.E. Soper, Phys. Rev. D **1**, 2901 (1970).

[3] S.J. Brodsky, H.-C. Pauli, and S.S. Pinsky, Phys. Rept. **301**, 299 (1998).

Variations on a theme in ϕ^3 Theory



Boson loop in ϕ^3 theory

Consider the simplest diagram with one loop in ϕ^3 theory. The particle mass is m. The amplitude is proportional to

$$A(p^2) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 k \frac{1}{(k^2 - m^2 + i\epsilon)((k-p)^2 - m^2 + i\epsilon)}.$$

Using one Feyman parameter x and making the substitution $k - xp \to k$ we find

$$A(p^2) = \frac{1}{(2\pi)^4} \int_0^1 \mathrm{d}x \int \mathrm{d}^4 k \frac{1}{(k^2 + x(1-x)p^2 - m^2 + i\epsilon)^2}.$$

After performing a Wick rotation and using dimensional regularization in $D = 4 - 2\epsilon$ dimensions we find

$$\begin{aligned} A(p^2) &= i \int_0^1 \mathrm{d}x \frac{\mu^{4-D}}{(2\pi)^D} \int \mathrm{d}^D k \frac{1}{(k^2 + M_{\mathrm{cov}}^2)^2}, \\ M_{\mathrm{cov}}^2(x; p^2) &= m^2 - x(1-x)p^2. \end{aligned}$$

To keep it simple, we discuss this amplitude for the range $0 < p^2 < 4m^2$, where $M_{\mathrm{cov}}^2(x; p^2) > 0$. The final result is

$$\begin{aligned} A(p^2) &= \frac{i}{(4\pi)^2} \int_0^1 \mathrm{d}x\, \Gamma(\epsilon) \left(\frac{4\pi\mu^2}{M_{\mathrm{cov}}^2} \right)^2 \\ &= \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma + \log 4\pi + \int_0^1 \mathrm{d}x \log \frac{\mu^2}{M_{\mathrm{cov}}^2} \right) \end{aligned}$$

We write the final formula for two renormalizations:

on-shell renormalization, $A^{\text{ren}}(m^2) = 0$:

$$A^{\text{ren}}(p^2) = A(p^2) - A(m^2) = -\frac{i}{(4\pi)^2} \int_0^1 \mathrm{d}x \log\left(\frac{m^2 - x(1-x)p^2}{m^2(1 - x(1-x))}\right).$$

and

$p^2 = 0$ *renormalization*, $A^{\text{ren}}(0) = 0$:

$$A^{\text{ren}}(p^2) = A(p^2) - A(0) = -\frac{i}{(4\pi)^2} \int_0^1 \mathrm{d}x \log\left(\frac{m^2 - x(1-x)p^2}{m^2}\right).$$

In both cases we have to keep in mind that the argument of the logarithm is positive for $p^2 < 4m^2$, which means that $A(p^2)$ is real below the two-body threshold.

The integral is known

$$\int_0^1 \mathrm{d}x \log[m^2 - x(1-x)p^2] = 2\left(\frac{\sqrt{4m^2 - p^2}}{p} \arctan \frac{p}{\sqrt{4m^2 - p^2}} + \log m - 1\right).$$

LF calculation

In the LF approacht, we work strictly in Minkowski space; the integral over de momentum component k^- is performed first (Kogut-Soper approach)

$$\begin{aligned} A(p^2) &= \frac{1}{(2\pi)^4}\int \mathrm{d}k^+ \int \mathrm{d}^2k_\perp \int \mathrm{d}k^- \\ &\times \frac{1}{2k^+(k^- - k^-_{1\,\mathrm{on}})2(k^+ - p^+)(k^- - k^-_{2\,\mathrm{on}})}. \end{aligned}$$

If $0 < k^+ < p^+$, the two poles $k^-_{1\,\mathrm{on}}$ and $k^-_{2\,\mathrm{on}}$ lie at different sides of the real k^--axis. Thus, picking up the residue of pole $k^-_{1\,\mathrm{on}}$ we find the result

$$A(p^2) = -\frac{i}{(2\pi)^3}\int_0^{p^+} \mathrm{d}k^+ \int \mathrm{d}^2k_\perp \frac{1}{4k^+(k^+ - p^+)(k^-_{2,\mathrm{on}} - k^-_{1\,\mathrm{on}})},$$

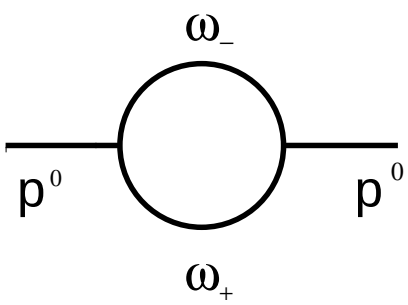
$$k^-_{2,\mathrm{on}} - k^-_{1\,\mathrm{on}} = p^- - \frac{\mathbf{k}_\perp^2 + m^2}{2k^+} - \frac{(\mathbf{p}_\perp - \mathbf{k}_\perp)^2 + m^2}{2(p^+ - k^+)}.$$

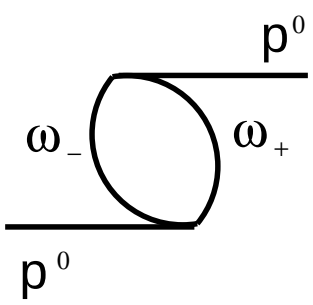
Now we may without loss of generality work in a reference frame where $\mathbf{p}_\perp = 0$; then $p^- = p^2/2p^+$. Moreover, we scale the plus momentum: $k^+ = xp^+$. Then we get

$$A(p^2) = \frac{i}{4\pi} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2} \, \frac{1}{\mathbf{k}_\perp^2 + M_{\mathrm{cov}}^2}.$$

Remarks
(i) We find the same mass variable M_{cov}^2 as in the manifestly covariant calculation;
(ii) the integral over $\mathbf{k}_\perp$ can be treated again by dimensional regularization;
(iii) The denominator is interpreted as an LF energy denominator;
(iv) Only one LF time-ordered diagram appears.

The final result is identical with the manifestly covariant result.





Other variations on the calculation of the same diagram can be tried.[4]

In IFD, the poles in the energies ω of the two particles in the loop are always in different halfplanes, which corresponds to two IF time-ordered diagrams. The calculation simplifies if one takes for the external particle $\mathbf{p} = 0$ and $p^0 = \sqrt{p^2}$. The left-hand diagram corresponds to both energies positive, in the right-hand diagram they are negative.

[4] BLGB in "Methods of Quantization", H. Latal and W. Schweiger (Eds.), Lect. Notes in Phys. LNP 572, Springer-Verlag, 2001, p 1-54

The difference with the LF calculation arises from the spectrum condition, namely the LF energies satisfy $p^- \geq 0$, which preclude the occurrence of vacuum diagrams, i.e., the r.h. one.

In the limit to the infinite-momentum frame, i.e., take the external momentum $p^\mu = (\sqrt{\mathbf{p}^2 + m^2}, p_x, p_y, p_z)$ and perform the limit $p_z \to \infty$, the diagram with negative energies vanishes and the one with positive energies reproduces the covariant result.

This argument was used by Weinberg[5] to advocate using Dynamics at Infinite Momentum. As will be explaned in the second lecture, this argument is entirely correct only in the case of scalar particles (fields), the propagators of particles with spin spoil the infinite momentum limit.

[5] S. Weinberg, Phys. Rev. **150**, 1313 (1966)

Could one use Feyman's trick to write integrals with several denominators as one with a single denominator raised to some power, and combine it with the use of LF variables to work exclusively in Minkowsk space?
The answer is, that double poles in k^- will appear (see LNP **572**).

$$\int \mathrm{d}k^+ \int \mathrm{d}k^- \frac{1}{(2k^+k^- - M^2 + i\epsilon)^2}$$

If one writes the integral over k^- as ($z = k^2 = 2k^+k^-$) one gets

$$\int \mathrm{d}k^- \frac{1}{(2k^+k^- - M^2 + i\epsilon)^2} = \frac{1}{2|k^+|} \int \mathrm{d}z \frac{1}{(z - M^2 + i\epsilon)^2}.$$

This integral can only be evaluated by regularizing the distribution

$$\frac{1}{(z + i\epsilon)^2} = \frac{1}{z^2} + i\pi\delta'(z), \ \left(\frac{1}{z^2}, f(z)\right) = \int \mathrm{d}z \frac{f(z) - 2f(0) + f(-z)}{z^2}.$$

In the case shown above, it turns out that the imaginary part can be shown to produce the covariant result, while the real part vanishes.

Summary

- Light-front dynamics (LFD) as a way to quantize theories has interesting properties: while remaining truly relativistic, it avoids negative-energy states. In theories where the vacuum is trivial, LFD has the advantage that the perturbative vacuum is the true Fock vacuum. (Spectrum condition)
- The kinematical subgroup of the Poincaré group allows for a construction of many-particle basis states where all particles are on-mass-shell. Connecting different basis states avoids Wigner rotations, which are so difficult to treat in instant-form dynamics.
- The kinematics of a multiparticle system can be split into a part that describes the LF center of momentum and a part that describes the relative motions.

Outlook

- The number of different diagrams to be computed is larger in a Hamiltonian approach than in a Langrangean approach. Still, in LFD the spectrum condition reduces the number of LF time-ordered diagrams.
- The loss of manifest covariance in LFD makes checks of invariance more difficult to perform in Hamiltonian dynamics than in Lagrangean approaches.
- Expressions like

$$p^- = \frac{\mathbf{p}^2 + m^2}{2p^+}, \quad \varepsilon(p^+, \mathbf{p}_\perp; 0) = \left(\frac{p^+}{m}, \frac{p_x}{m}, \frac{p_y}{m}, \frac{\mathbf{p}_\perp^2 - m^2}{2mp^+} \right)$$

  are potentially a source of LF singularities that must be tamed.
- There is no guarantee that a Hamiltonian approach will produce the same results as a manifestly covariant one if regularization of the amplitudes is involved, even after renormalization.

Thanks to

Norbert Ligterink
Nico Schoonderwoerd
Leonid Kondratyuk
Cheung Ji
Ho-Meoyng Choi