Supersymmetry on the Lattice: Theory and Applications of $\mathcal{N} = 4$ Yang–Mills

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OUTLINE

- Construction of $4d$ $\mathcal{N} = 4$ SYM theory on the lattice
- One-loop renormalization of lattice theory
- Restoration of supersymmetries
- Sign problem on the lattice
- Applications: Gauge-gravity dualities
- Conclusions/Further explorations

Motivation: Why is SUSY interesting?

SUSY is a natural extension of Poincaré symmetry.

Poincaré symmetry group plays a central role in field theories (including the Standard Model) on flat spacetime.

Generators: P_μ, $\Sigma_{\mu\nu}$.

P: 4-vector, Σ: anti-symmetric tensor

Algebra: $[P, P] = 0$, $[P, \Sigma] \sim P$, $[\Sigma, \Sigma] \sim \Sigma$.

We can extend Poincaré algebra to include fermionic generators Q and $\overline{Q}$.

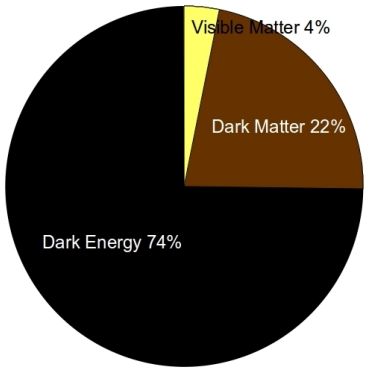
$$\{Q, Q\} = 0,\ [P, Q] = 0,\ [Q, \Sigma] \sim Q,\ \{Q, \overline{Q}\} \sim P.$$

Super Poincaré algebra.

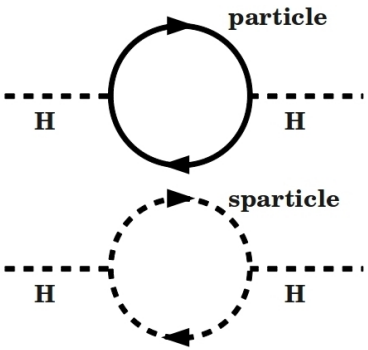
Motivation: Why is SUSY interesting? [contd.]

Supersymmetric theories come with new particles.

New types of bosons and fermions. (SUSY is a symmetry connecting bosons and fermions.)



Dark matter particles?



SUSY could solve the hierarchy problem. (Why weak force is much stronger than gravity?)

Supersymmetric extensions of the Standard Model are being tested now (at the LHC...)

SUSY is an important ingredient of string theory.

Some low energy theories with extra-dimensions include SUSY as part of their symmetry group.

Supersymmetric Yang–Mills Theory

In this talk I will focus on Supersymmetric Yang–Mills (SYM) theories.

(QCD is a Yang–Mills theory with SU(3) gauge group.)

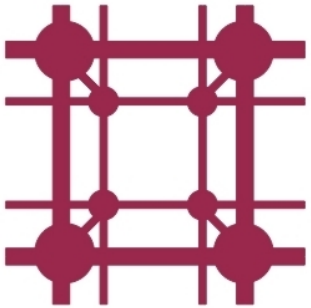
- Supersymmetric Yang–Mills: Many interesting features/results
  - Confinement
  - Spontaneous chiral symmetry breaking
  - Strong coupling - weak coupling duality
  - Electric-magnetic duality
  - Conformal field theory (CFT)
- There are intriguing connections between $\mathcal{N} = 4$ SYM theory and string theory. (AdS/CFT correspondence.)
- Interesting features at strong coupling. Non-perturbative definition is needed: lattice construction.

SUSY and Lattice: Are they compatible?

- SUSY has spinor generators Q:

$$\{Q, \overline{Q}\} = \gamma \cdot P.$$

P: generator of infinitesimal translations.



  - Above relation is broken on the lattice.
  - No discreet subgroup of SUSY.
- Folklore: Impossible to put SUSY on the lattice exactly.
- Leads to (very) difficult fine tuning - lots of relevant SUSY breaking counter-terms.
- $\mathcal{N} = 4$ SYM theory is particularly difficult - contains scalar fields - scalar mass terms are relevant operators.

Can SUSY and Lattice co-exist?

SUSY on the Lattice: Options

OPTION 1:

Let SUSY emerge as "accidental symmetry" in the continuum limit.

Examples: $\mathcal{N} = 1$ SYM in $4d$, SQCD. [Curci and Veneziano 1987], [Kaplan and Schmaltz 2000], [Huet, Narayanan, Neuberger 1996]

OPTION 2:

Preserve a subset of SUSY algebra exactly on the lattice: Exact lattice SUSY.

Examples: $\mathcal{N} = 2$ SYM in $2d$, $\mathcal{N} = 4$ SYM in $4d$, dimensional reductions of these theories... [Catterall, Sugino, Kawamoto, d'Adda, Matsuura, Giedt 2000 -], [Kaplan, Cohen, Damgaard, Matsuura, Ünsal 2002 -]

SUSY on the Lattice: Exact Lattice SUSY

I will focus on the second option - Exact lattice SUSY.

Recent reviews: D. B. Kaplan, [Nucl. Phys. Proc. Suppl. 129, 109 (2004)],
S. Catterall, D. B. Kaplan, M. Ünsal, [Phys. Rept. 484, 71 (2009)],
A.J., [Int. J. Mod. Phys. A 26, 5057 (2011)]

There are two approaches.

- Topological Twisting. [Catterall, Sugino, Kawamoto, Matsuura, Giedt, ...]
  Inspired by techniques in topological field theory.

  E. Witten [Commun. Math. Phys. 117 (1988) 353]

- Orbifolding/deconstruction. [Kaplan, Ünsal, Cohen, Damgaard, Matsuura, Giedt, ...]
  Inspired by the method of "Deconstruction" by Arkani-Hamed, Cohen, Georgi (AHCG).

  N. Arkani-Hamed, A. G. Cohen, H. Georgi [Phys. Rev. Lett. 86 (2001) 4757]

These two approaches produce identical lattice theories!

S. Catterall, D. B. Kaplan and M. Ünsal [Phys. Rept. 484 (2009) 71]

The Idea of Twisting

SYM theories that allow twisting in d (Euclidean) spacetime dimensions have the property:

$$SO(d)_E \times SO(d)_R \subset SO(d)_E \times G_R.$$

G_R : R-symmetry group of the theory.

Idea of twisting: Replace $SO(d)_E$ by another subgroup - the diagonal subgroup - of $SO(d)_E \times SO(d)_R$.

New rotation group - twisted rotation group - $SO(d)'$:

$$SO(d)' = \text{Diagonal Subgroup } [SO(d)_E \times SO(d)_R].$$

Twisted rotation group acts the same way on spacetime.

Note: Whenever we make a rotation in spacetime $\mathbb{R}^d$, we accompany this by $SO(d)_R$ (internal) transformation.

Example: $\mathcal{N} = 2$ SYM in $2d$

Obtained by dimensional reduction of $4d$ $\mathcal{N} = 1$ SYM theory.

The $4d$ theory has symmetry group: $SO(4)_E \times U(1)$.

After dimensional reduction it splits into:

$$G = SO(2)_E \times SO(2)_{R_1} \times U(1)_{R_2}.$$

$SO(2)_E$: Euclidean Lorentz symmetry,
$SO(2)_{R_1}$: rotational symmetry along reduced dimensions,
$U(1)_{R_2}$: chiral $U(1)$ symmetry of the original theory.

Twist gives:

$$SO(2)' = \text{Diag Subgroup } [SO(2)_E \times SO(2)_{R_1}].$$

$\mathcal{N} = 2$ SYM in $2d$ [contd.]

Untwisted theory has a two component Dirac fermion Ψ, a gauge field A_μ and a complex scalar $s = (s_1 + is_2)$.

Counting the number of degrees of freedom: 4 fermionic and 4 bosonic.

Decompose the Dirac fermion in the following way

$$\Psi_{\alpha i} = \left[\eta I + \psi_\mu \gamma_\mu + \tfrac{1}{2}\chi_{[12]}(\gamma_1\gamma_2 - \gamma_2\gamma_1)\right]_{\alpha i}.$$

The components η, ψ_μ, $\chi_{\mu\nu}$ transform under $SO(2)'$ as anti-symmetric tensors (p-forms).

They are called twisted fermions. They form components of a Dirac-Kähler field.

We can place twisted fermions on sites, links and plaquettes of the unit cell of the lattice.

$\mathcal{N} = 2$ SYM in $2d$ [contd.]

We can place the two scalars (s_1, s_2) on links too.

Under $SO(2)'$, gauge field and scalars transform as vectors.

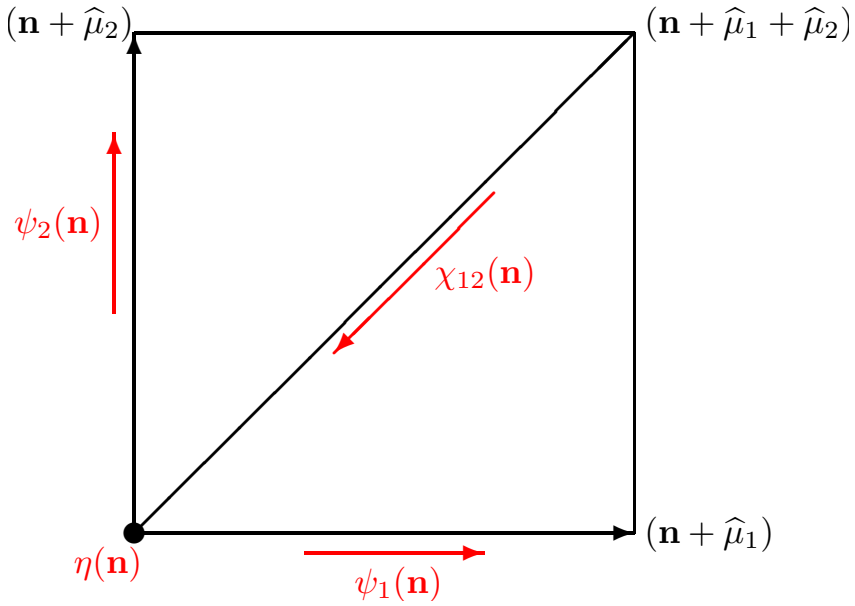
$$A_\mu = (A_1, A_2), B_\mu = (s_1, s_2)$$

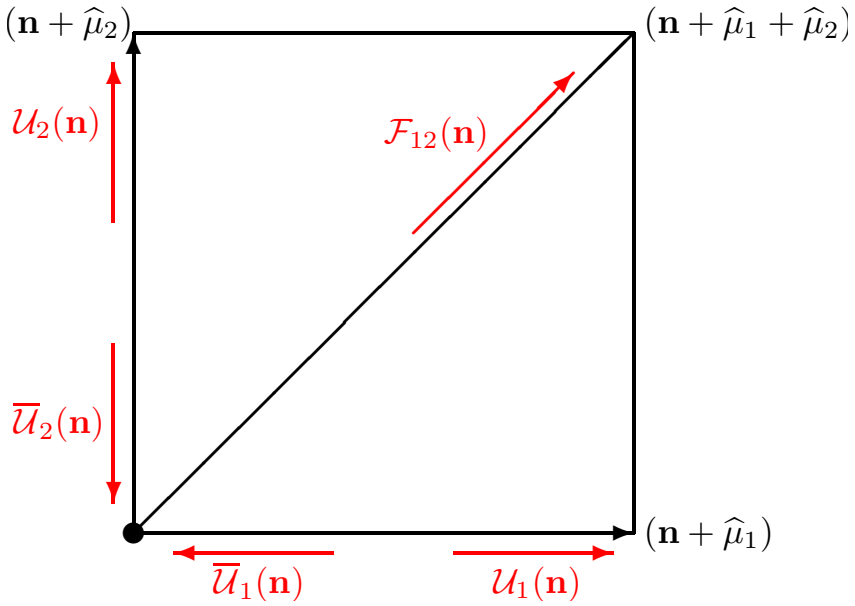
We can package the gauge field and scalars to obtain a complexified gauge field.

$$\mathcal{A}_\mu = A_\mu + iB_\mu \text{ and } \overline{\mathcal{A}}_\mu = A_\mu - iB_\mu.$$

We can place these complexified gauge fields on links of the unit cell of the lattice.

$\mathcal{N} = 2$ SYM in $2d$ [contd.]





$\mathcal{N} = 2$ SYM in $2d$ [contd.]

Supersymmetric charges transform as p-forms under twisted rotation group.

In the untwisted $\mathcal{N} = 2$ SYM we have

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\delta_{ij}\gamma^{\mu}_{\alpha\beta}P_{\mu}.$$

$Q_{\alpha i}$ is supercharge.

$\alpha(= 1, 2)$: Lorentz spinor index, $i(= 1, 2)$: label two different $\mathcal{N} = 2$ supercharges.

$$\mathcal{Q}_{\alpha i} = \left[\mathcal{Q}I + \mathcal{Q}_{\mu}\gamma_{\mu} + \tfrac{1}{2}\mathcal{Q}_{[12]}(\gamma_1\gamma_2 - \gamma_2\gamma_1)\right]_{\alpha i}.$$

Twisted $\mathcal{N} = 2$, $2d$ supersymmetry algebra:

Define $\tilde{\mathcal{Q}} = \epsilon_{\mu\nu}\mathcal{Q}_{\mu\nu}$

$$\{\mathcal{Q}, \mathcal{Q}\} = 0, \quad \{\tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}\} = 0, \quad \{\mathcal{Q}, \tilde{\mathcal{Q}}\} = 0, \quad \{\mathcal{Q}_{\mu}, \mathcal{Q}_{\nu}\} = 0,$$

$$\{\mathcal{Q}, \mathcal{Q}_{\mu}\} = P_{\mu}, \quad \{\tilde{\mathcal{Q}}, \mathcal{Q}_{\mu}\} = \epsilon_{\mu\nu}P_{\nu}.$$

$\mathcal{N} = 2$ SYM in $2d$ [contd.]

The scalar supersymmetry $\mathcal{Q}$ is nilpotent (similar to BRST charge):

$$\mathcal{Q}^2 = 0.$$

It does not generate any translations on the lattice.

That means we can implement a subalgebra of the twisted SUSY algebra on the lattice - exact lattice SUSY!

$$\begin{aligned} \mathcal{Q}\mathcal{A}_\mu &= \psi_\mu, & \mathcal{Q}\psi_\mu &= 0, \\ \mathcal{Q}\overline{\mathcal{A}}_\mu &= 0, & \mathcal{Q}\chi_{\mu\nu} &= -[\overline{\mathcal{D}}_\mu, \overline{\mathcal{D}}_\nu], \\ \mathcal{Q}\eta &= d, & \mathcal{Q}d &= 0. \end{aligned}$$

d: auxiliary field.
Complexified covariant derivatives: $\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu$, $\overline{\mathcal{D}}_\mu = \partial_\mu + \overline{\mathcal{A}}_\mu$.

$\mathcal{N} = 2$ SYM in $2d$ [contd.]

Action of the theory takes a $\mathcal{Q}$-exact form:

$$S = \mathcal{Q}\Lambda,$$

$$\Lambda = \int \mathrm{Tr}\left(\chi_{\mu\nu}\mathcal{F}_{\mu\nu} + \eta[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] - \tfrac{1}{2}\eta d\right).$$

Integrating out the auxiliary field:

$$S = \int \mathrm{Tr}\left(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}_\mu\psi_\mu\right).$$

Action is $\mathcal{Q}$ invariant:

$$\mathcal{Q}S = 0.$$

$2d\ \mathcal{N} = 2$ SYM on the Lattice

Can be discretized on a hypercube.

Fermions on sites, links and plaquettes... Complex bosons on links...

Complex bosons $\rightarrow$ complexified Wilson links: $\mathcal{A}_\mu \rightarrow \mathcal{U}_\mu$

Covariant derivatives $\rightarrow$ covariant differences: $\mathcal{D}_\mu^{(+)}$ and $\mathcal{D}_\mu^{(-)}$.

$$\mathcal{D}_\mu^{(+)} f_\nu(\mathbf{n}) = \mathcal{U}_\mu(\mathbf{n}) f_\nu(\mathbf{n} + \widehat{\boldsymbol{\mu}}_\mu) - f_\nu(\mathbf{n}) \mathcal{U}_\mu(\mathbf{n} + \widehat{\boldsymbol{\mu}}_\nu),$$

$$\mathcal{D}_\mu^{(-)} f_\mu(\mathbf{n}) = \mathcal{U}_\mu(\mathbf{n}) f_\mu(\mathbf{n}) - f_\mu(\mathbf{n} - \widehat{\boldsymbol{\mu}}_\mu) \mathcal{U}_\mu(\mathbf{n} - \widehat{\boldsymbol{\mu}}_\mu).$$

Use of forward and backward difference operators $\Longrightarrow$
Solutions of the lattice theory map one-to-one with that of the continuum theory.

Fermion doubling problems are hence evaded.

T. Banks, Y. Dothan, D. Horn [Phys. Lett. B117, 413 (1982)]

$2d\ \mathcal{N} = 2$ SYM on the Lattice [contd.]

Orientation of fields on the lattice ensures gauge invariance.

$$\begin{aligned}
\eta(\mathbf{n}) &\rightarrow G(\mathbf{n})\eta(\mathbf{n})G^\dagger(\mathbf{n}) \\
\psi_\mu(\mathbf{n}) &\rightarrow G(\mathbf{n})\psi_\mu(\mathbf{n})G^\dagger(\mathbf{n}+\widehat{\boldsymbol{\mu}}_\mu) \\
\chi_{\mu\nu}(\mathbf{n}) &\rightarrow G(\mathbf{n}+\widehat{\boldsymbol{\mu}}_\mu+\widehat{\boldsymbol{\mu}}_\nu)\chi_{\mu\nu}(\mathbf{n})G^\dagger(\mathbf{n}) \\
\mathcal{U}_\mu(\mathbf{n}) &\rightarrow G(\mathbf{n})\mathcal{U}_\mu(\mathbf{n})G^\dagger(\mathbf{n}+\widehat{\boldsymbol{\mu}}_\mu) \\
\overline{\mathcal{U}}_\mu(\mathbf{n}) &\rightarrow G(\mathbf{n}+\widehat{\boldsymbol{\mu}}_\mu)\overline{\mathcal{U}}(\mathbf{n})G^\dagger(\mathbf{n})
\end{aligned}$$

Terms in the action form closed loops: gauge-invariant.

$$S = \sum_{\mathbf{n}} \mathrm{Tr}\left(\mathcal{D}_\mu^{(+)}\mathcal{U}_\nu(\mathbf{n})\right)^\dagger\left(\mathcal{D}_\mu^{(+)}\mathcal{U}_\nu(\mathbf{n})\right) + \tfrac{1}{2}\left(\mathcal{D}_\mu^{\dagger(-)}\mathcal{U}_\mu(\mathbf{n})\right)^2$$
$$- \chi_{\mu\nu}(\mathbf{n})\mathcal{D}_{[\mu}^{(+)}\psi_{\nu]}(\mathbf{n}) - \eta(\mathbf{n})\mathcal{D}_\mu^{\dagger(-)}\psi_\mu(\mathbf{n}).$$

$2d\ \mathcal{N} = 2$ SYM on the Lattice [contd.]

Scalar SUSY transformations on the lattice:

$$
\begin{aligned}
\mathcal{Q}\mathcal{U}_\mu(\mathbf{n}) &= \psi_\mu(\mathbf{n}), \\
\mathcal{Q}\psi_\mu(\mathbf{n}) &= 0, \\
\mathcal{Q}\overline{\mathcal{U}}_\mu(\mathbf{n}) &= 0, \\
\mathcal{Q}\chi_{\mu\nu}(\mathbf{n}) &= -\overline{\mathcal{D}}_\mu^{(+)}\overline{\mathcal{U}}_\nu(\mathbf{n}), \\
\mathcal{Q}\eta(\mathbf{n}) &= d(\mathbf{n}), \\
\mathcal{Q}d(\mathbf{n}) &= 0.
\end{aligned}
$$

Only scalar supercharge is unbroken by discretization.

Transformations interchange bosons and fermions at the same place on the lattice.

$4d$ $\mathcal{N} = 4$ SYM

Obtained by dimensional reduction of $10d$ $\mathcal{N} = 1$ SYM down to $4d$.

The theory has a $4d$ gauge field A_μ, 4 Majorana fermions and six scalars.

The action is

$$S = \int d^4x \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu A_i D^\mu A_i + \frac{1}{4}[A_i, A_j]^2\right)$$
$$-\frac{i}{2}\mathrm{Tr}\left(\bar{\lambda}\Gamma^\mu D_\mu \lambda + i\bar{\lambda}\Gamma_i[A_i, \lambda]\right).$$

It has a Euclidean Lorentz rotation symmetry group $SO_E(4)$ and an R-symmetry group $SO_R(6)$.

Twist of $4d$ $\mathcal{N} = 4$ SYM

- Appropriate twist due to Marcus. N. Marcus [Nucl. Phys. B452 (1995) 331-345]

$$SO(4)' = \text{Diag Subgroup } [SO_E(4) \times SO_R(4)].$$

- After twisting we have a Dirac-Kähler (DK) fermion field with 16 components:

  $(\eta,\ \psi_\mu,\ \chi_{\mu\nu},\ \theta_{\mu\nu\rho},\ \kappa_{1234})$. p-forms, p=0,1,2,3,4
- A $4d$ gauge field A_μ transforming as a vector.
- Six scalars decompose as a vector B_μ and two scalars. ($SO(6) \to SO(4) \times SO(2)$.)
- A_μ and B_μ transform the same way under twisted rotation group. We can combine them: $\mathcal{A}_\mu = A_\mu + iB_\mu$.

Twist of $4d$ $\mathcal{N} = 4$ SYM [contd.]

- A more compact expression - package these fields as dimensional reduction of a 5d theory!

  Fermions: $\Psi = (\eta, \psi_m, \chi_{mn})$, $m, n = 1, \cdots 5$

  Bosons: $\mathcal{A}_m$, $m = 1, \cdots 5$.

- Action of the twisted theory

$$S = \mathcal{Q} \int \mathrm{Tr} \left(\sum_{mn} \chi_{mn} \mathcal{F}_{mn} + \sum_m \eta [\overline{\mathcal{D}}_m, \mathcal{D}_m] - \frac{1}{2} \eta d \right) + S_{\text{closed}},$$

  where

$$S_{\text{closed}} = -\frac{1}{4} \epsilon_{abcde} \chi_{de} \overline{\mathcal{D}}_c \chi_{ab}.$$

- Same as $2d$ example but with extra $\mathcal{Q}$-closed piece in the action.

$4d\ \mathcal{N} = 4$ SYM on the Lattice

- Discretization on a hypercube. The scalar supercharge is preserved.
- Fundamental cell of the hypercubic lattice: one site, four links, six faces, four cubes and one hypercube.
- Place a p-form tensor fermion on p-cell of the hypercube.
- Place complexified bosons on the links.
- Basis vectors:
  $\{\widehat{\boldsymbol{\mu}}_1 = (1,0,0,0),\ \widehat{\boldsymbol{\mu}}_2 = (0,1,0,0),\ \ \widehat{\boldsymbol{\mu}}_3 = (0,0,1,0),$
  $\widehat{\boldsymbol{\mu}}_4 = (0,0,0,1),\ \ \widehat{\boldsymbol{\mu}}_5 = (-1,-1,-1,-1)\}$, with $\sum_i \widehat{\boldsymbol{\mu}}_i = 0$.
- There is a more symmetrical lattice arrangement: A_4^* lattice. Unit cell has 5 basis vectors correspond from center of a 4-simplex to vertices.

$4d\ \mathcal{N} = 4$ SYM on the Lattice [contd.]

Lattice action:

$$S = (S_1 + S_2)$$

$$\begin{aligned} S_1 &= \sum_{\mathbf{n}} \mathrm{Tr}\left(\mathcal{F}^{\dagger}_{ab}\mathcal{F}_{ab} + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_{a}\mathcal{U}_a\right)^2 - \chi_{ab}\mathcal{D}^{(+)}_{[a}\psi_{b]} - \eta\overline{\mathcal{D}}^{(-)}_{a}\psi_a\right) \\ S_2 &= -\frac{1}{4}\sum_{\mathbf{n}} \mathrm{Tr}\ \epsilon_{abcde}\chi_{de}(\mathbf{n} + \widehat{\boldsymbol{\mu}}_a + \widehat{\boldsymbol{\mu}}_b + \widehat{\boldsymbol{\mu}}_c)\overline{\mathcal{D}}^{(-)}_{c}\chi_{ab}(\mathbf{n} + \widehat{\boldsymbol{\mu}}_c) \end{aligned}$$

- Bosonic action is just Wilson plaquette if $\mathcal{U}^{\dagger}_a\mathcal{U}_a = 1$. Fermions: Dirac-Kähler action - no doublers (staggered).
- Well defined prescription for difference operators: $\mathcal{D}^{(+)}_a$, $\mathcal{D}^{(-)}_a$.
- The action is $\mathcal{Q}$-supersymmetry invariant: $\mathcal{Q}S = 0$.

$4d$ $\mathcal{N} = 4$ SYM: One-loop Renormalization

Dangerous operators can be generated on the lattice through radiative corrections.

But they all must respect symmetries of the underlying lattice theory:

- Gauge invariance.
- Lattice point group symmetry - S^5 (subgroup of $SO(4)'$).
- $\mathcal{Q}$-supersymmetry.

S^5 point group symmetry guarantees that twisted rotation group $SO(4)'$ is restored in the continuum limit.

Power counting shows that only relevant counter-terms correspond to renormalizations of existing terms in the action:

$$S = \int \mathrm{Tr}\ \mathcal{Q}\Big(\alpha_1 \chi_{mn}\mathcal{F}_{mn} + \alpha_2 \eta[\overline{\mathcal{D}}_a, \mathcal{D}_a] - \alpha_3 \tfrac{1}{2}\eta d\Big) + \alpha_4 S_{\text{closed}}.$$

S. Catterall, E. Dzienkowski, J. Giedt, **A.J.**, R. Wells [JHEP 1104 (2011) 074]

$4d$ $\mathcal{N} = 4$ SYM: One-loop Renormalization [contd.]

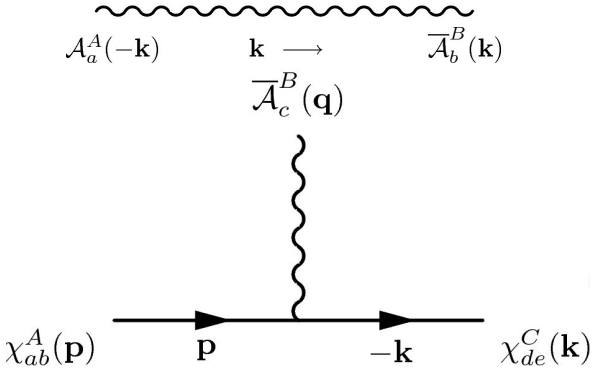
Propagators and vertices

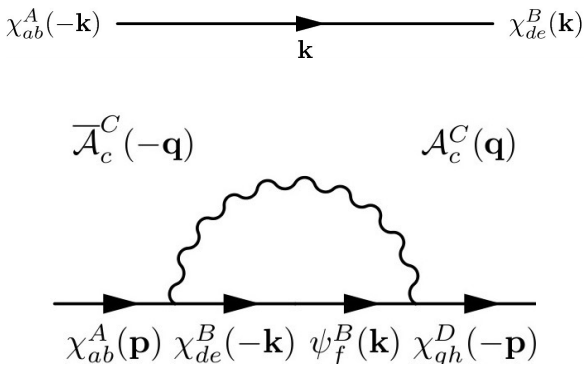
- Bosonic propagator (Feynman gauge):
  $\langle \overline{\mathcal{A}}_m^C(k)\mathcal{A}_n^D(-k)\rangle = \frac{1}{\sum_c \widehat{k}_c^2}\delta_{mn}\delta^{CD}$, $\widehat{k}_c = 2\sin(\frac{k_c}{2})$.
- Fermionic propagator:
  $M_{16\times16}^{-1}(k) = \frac{1}{\sum_c \widehat{k}_c^2} M_{16\times16}(k)$, where $M_{16\times16}$ is a block-matrix acting on fermions $(\eta, \psi_m, \chi_{mn})$.
- Vertices: $\psi_m - \eta$, $\psi_m - \chi_{mn}$ and $\chi_{mn} - \chi_{rs}$.

Four one-loop Feynman diagrams are needed to renormalize the three fermion propagators. They give three α's.

The remaining α is obtained from one-loop correction to the bosonic propagator.

$4d$ $\mathcal{N} = 4$ SYM: One-loop Renormalization [contd.]





Contributions of amputated diagrams all vanish in the limit $p \to 0$. Mass counter-terms are absent in the lattice theory at one-loop.

In fact, no mass counter-terms are needed at any finite order of perturbation theory.

Note: Tadpole diagrams do not appear in this particular construction of the lattice theory.

$4d$ $\mathcal{N} = 4$ SYM: One-loop Renormalization [contd.]

- All these diagrams possess identical logarithmic divergences of the form $\ln(\mu a)$.
  a: lattice spacing,
  μ: a mass scale introduced to regulate the small momentum behaviour of the integrands.
- This divergence can be absorbed by a common wavefunction renormalization Z of the twisted fermions and bosons.
- It is expected, since lattice theory is in one-to-one correspondence with the continuum twisted theory. Untwisting the fields require a common wavefunction renormalization.
- Similar arguments indicate that $\beta_{\text{lattice}}^{\mathcal{N}=4\ \text{SYM}}(g) = 0$ at one-loop.

Restoration of full SUSY

- There are 15 other SUSYs - $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ - that are broken on the lattice.
- What about restoration of the full set of SUSY as we take the continuum limit?
- Apparent solution: Construct SUSY Ward-Takahashi (WT) identities and examine the restoration of SUSYs as we approach the continuum limit.

$$\langle \partial_m \mathcal{S}_A^m(x) O(y) \rangle = \delta^{(4)}(x-y) \langle \mathcal{Q}_A O(y) \rangle.$$

- Check how strongly these WT relations hold on the lattice for a given operator O.
- A way of measuring the amount of SUSY breaking by the lattice.

Restoration of full SUSY [contd.]

Need to know supercurrents $\mathcal{S}_A^m(x)$ and Q_A supersymmetries. (A corresponds to 0-form, 1-form or 2-form)
For example, we have the $\mathcal{Q}_a$ transformations:

$$\mathcal{Q}_a \mathcal{A}_b = \frac{1}{2}\delta_{ab}\eta, \qquad \mathcal{Q}_a \eta = 0,$$

$$\mathcal{Q}_a \overline{\mathcal{A}}_b = -\chi_{ab}, \qquad \mathcal{Q}_a \chi_{bc} = -\frac{1}{2}\epsilon_{abcgh}\mathcal{F}_{gh},$$

$$\mathcal{Q}_a d = 0, \qquad \mathcal{Q}_a \psi_b = \frac{1}{2}\delta_{ab} d + (1-\delta_{ab})[\overline{\mathcal{D}}_a, \mathcal{D}_b].$$

On the lattice WT identities take the form:

$$\partial_m \langle \mathcal{S}_A^m(\mathbf{x}) O(0)\rangle + \langle I_A^m(\mathbf{x}) O(0)\rangle = \delta^{(4)}(\mathbf{x})\langle \mathcal{Q}_A O(0)\rangle$$

Symmetry breaking terms: $I_0^m(\mathbf{x})$, $I_a^m(\mathbf{x})$, $I_{ab}^m(\mathbf{x})$. They are $\mathcal{O}(a)$ artifacts: $\langle I_a^m(\mathbf{x})\rangle \to 0$, $\langle I_{ab}^m(\mathbf{x})\rangle \to 0$ in the continuum limit.

S. Catterall, J. Giedt and **A.J.** [Work in progress]

Simulating SYM Theories on the Lattice

- The basic algorithms we use to simulate them are borrowed directly from lattice QCD.
- Rational Hybrid Monte Carlo (RHMC) algorithm
  - To compute the (non-local) fermion determinant with fractional power.
- Leapfrog algorithm
  - To evolve the system of equations in discrete (simulation) time steps.
- Metropolis test
  - To accept or reject the configurations.

S. Catterall and **A.J.** [Comput.Phys.Commun. 183 (2012) 1336-1353]

Sign problem in $4d$ $\mathcal{N} = 4$ SYM

Integration measure involves only the fields $(\eta, \psi_a, \chi_{ab})$.

Not their complex conjugates.

Thus we have a Pfaffian rather than a determinant: $\mathrm{Pf}(M(\mathcal{U}))$.

There can be a phase in general: $\mathrm{Pf} = |\mathrm{Pf}(M(\mathcal{U})|e^{i\alpha(\mathcal{U})}$.

Positive definite measure is needed for Monte Carlo. We can reweight:

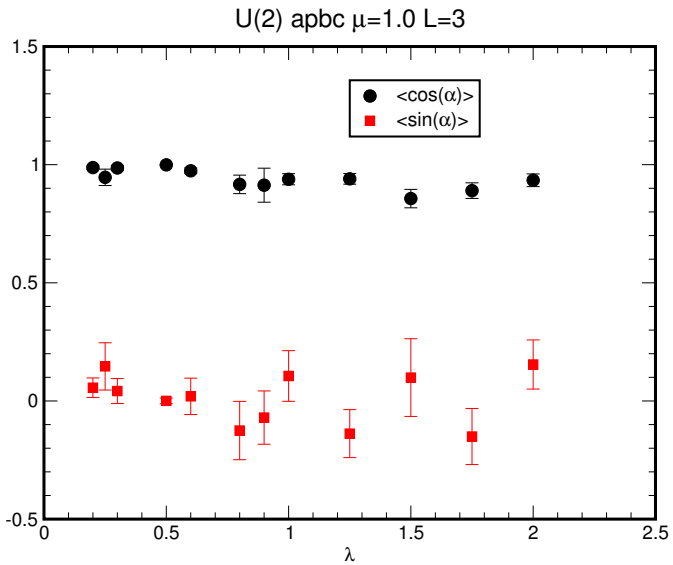
$$\langle O(\mathcal{U})\rangle = \frac{\langle O(\mathcal{U})e^{i\alpha(\mathcal{U})}\rangle_{\mathrm{pq}}}{\langle e^{i\alpha(\mathcal{U})}\rangle_{\mathrm{pq}}}.$$

Fluctuations in α large $\implies$ statistical error in such measurement grows exponentially with volume: sign problem.

Sign problem in $4d$ $\mathcal{N} = 4$ SYM [contd.]

Observed phase small in phase quenched simulations of $4d$ $\mathcal{N} = 4$ SYM.

The phase of the Pfaffian can be ignored in actual simulations.



S. Catterall, P. Damgaard, T. Degrand, R. Galvez and D. Mehta [JHEP 1211 (2012) 072]

Same is true for $\mathcal{Q} = 4$ and $\mathcal{Q} = 16$ SYM theories in $2d$.

S. Catterall, R. Galvez, **A.J.** and D. Mehta [JHEP 1201 (2012) 108]

Applications: Gauge-gravity Dualities

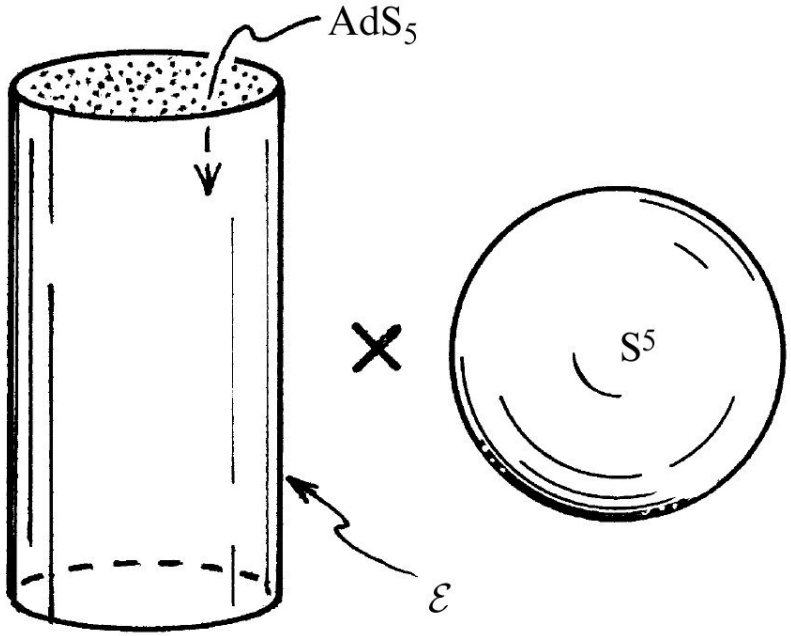


Fig. taken from: *The Road to Reality: A Complete Guide to the Physical Universe*, R. Penrose.

AdS/CFT conjecture: String theory on $\mathcal{M} = AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ SYM on $\mathcal{E}$, the conformal boundary of AdS_5.

Applications: Gauge-gravity Dualities [contd.]

Gravitational theory

- Low energy string theory (SUGRA)
- Contains semi-classical black Dp-branes
- N units of charge
- At temperature T

Gauge theory

- 16 supercharge YM theory in $d = (p + 1)$ dimensions
- SU(N) gauge group
- Strongly coupled
- At temperature T

Black Holes from YM

Case $p = 0$:

Gauge theory - Strongly coupled 16 supercharge SYM in $1d$ taken at large N and temperature T.

Gravitational theory - Black holes with N units of D0-charge at temperature T.

The energy of this black hole can be precisely computed in SUGRA

$$\epsilon = cN^2 t^{14/5}, \quad c \simeq 7.41,$$

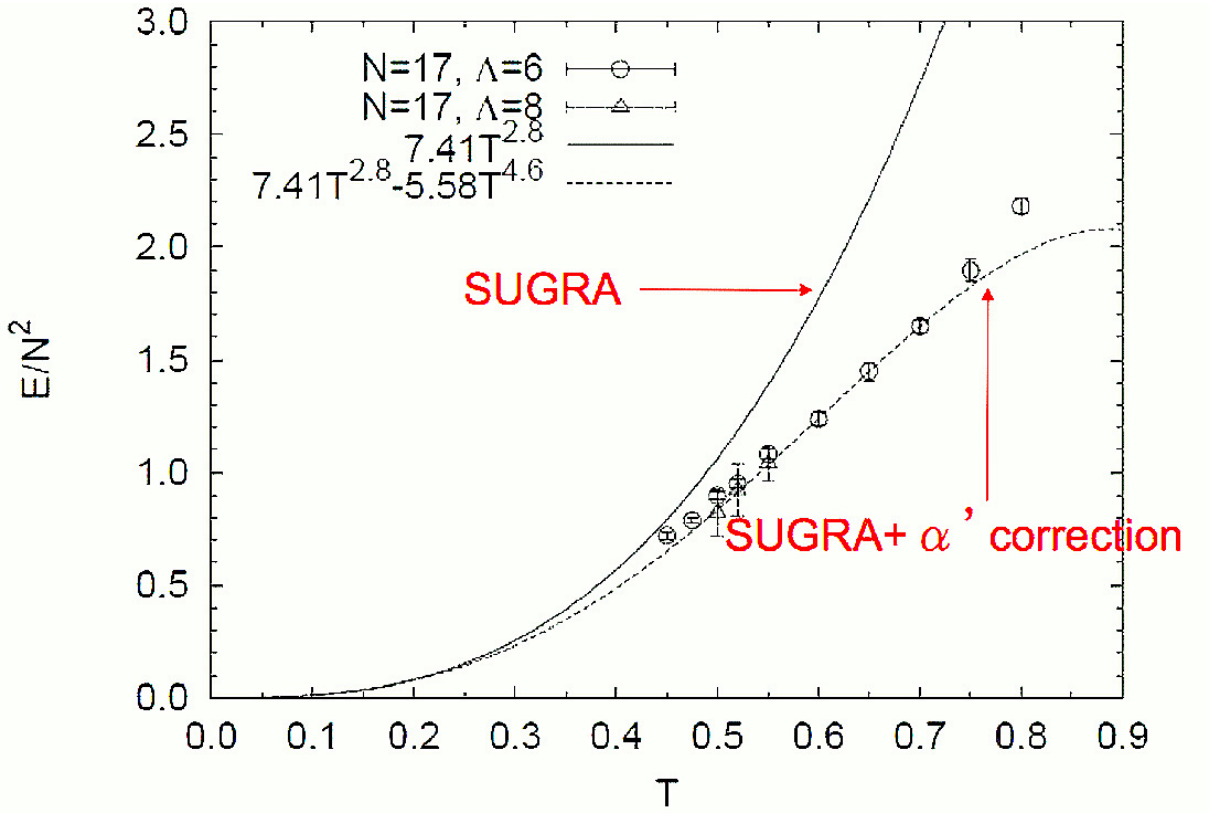
Dimensionless energy $\epsilon = E\lambda^{-1/3}$ and temperature $t = T\lambda^{-1/3}$.

N. Itzhaki, J. M. Maldacena, J. Sonnenschein, S. Yankielowicz [Phys.Rev. D58, 046004 (1998)]

Can we reproduce the thermodynamics of this black hole in dual SYM theory?

Black Holes from YM [contd.]

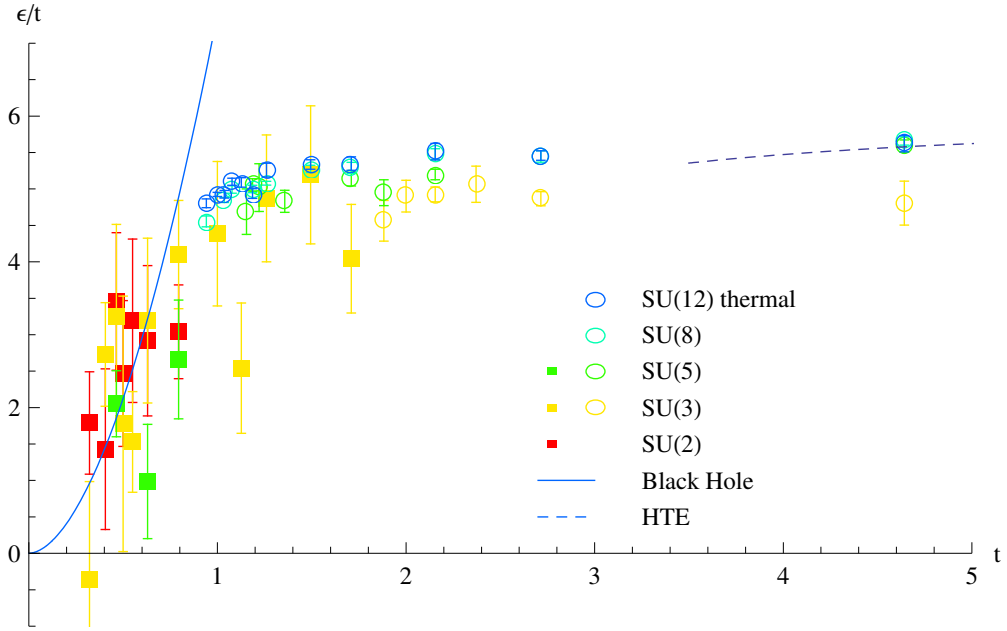
Numerical study using a non-lattice formulation of SYMQM (BFSS model).



K. N. Anagnostopoulos, M. Hanada, J. Nishimura, S. Takeuchi [Phys.Rev.Lett. 100 (2008) 021601]

Black Holes from YM [contd.]

Numerical study using the twisted formulation of SYMQM.



S. Catterall, T. Wiseman [Phys. Rev. D78 (2008) 041502]

Non-vanishing Polyakov line even at low $T \implies$ no phase transition in the SUSY case, as predicted by the gauge/gravity correspondence. A single deconfined phase.

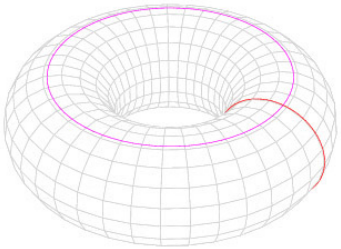
Black Strings from YM [contd.]

Case $p = 1$:

Gauge theory: Strongly coupled 16 supercharge SYM in $2d$ taken at large N and temperature T. Spatial direction compactified on a circle.

Gravitational theory: N D1-branes wrapping on a circle at temperature T.

AdS/CFT correspondence maps charged black solutions of SUGRA on $R^{8,1} \times S^1$ to thermal phases of 16 supercharge SYM on S^1.



Radii $r_x = \sqrt{\lambda} R$ and $r_\tau = \sqrt{\lambda}\beta$. t' Hooft coupling, $\lambda = g_{YM}^2 N$

Black Strings from YM [contd.]

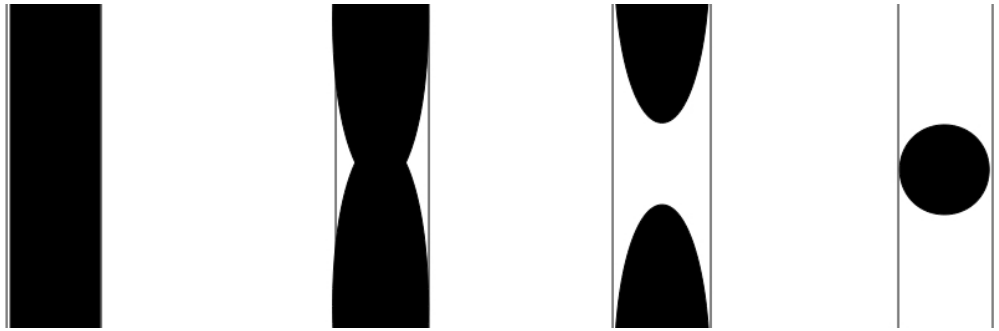
- Depending on r_x, r_τ black string solution may become less stable than black hole.
- Supergravity analysis predicts first order thermal phase transition between uniform black string and localized black holes.
- Line of first order phase transition

$$r_x^2 = c_{crit} r_\tau$$

- c_{crit} not yet known in SUGRA.
- SUGRA predicts: c_{crit} should be of order 1 and greater than 2.29.

Black Strings from YM [contd.]

- Black string solutions are unstable!



- Develops Gregory-Laflamme instability.

R. Gregory and R. Laflamme [Phys.Rev.Lett. 70 (1993) 2837-2840]

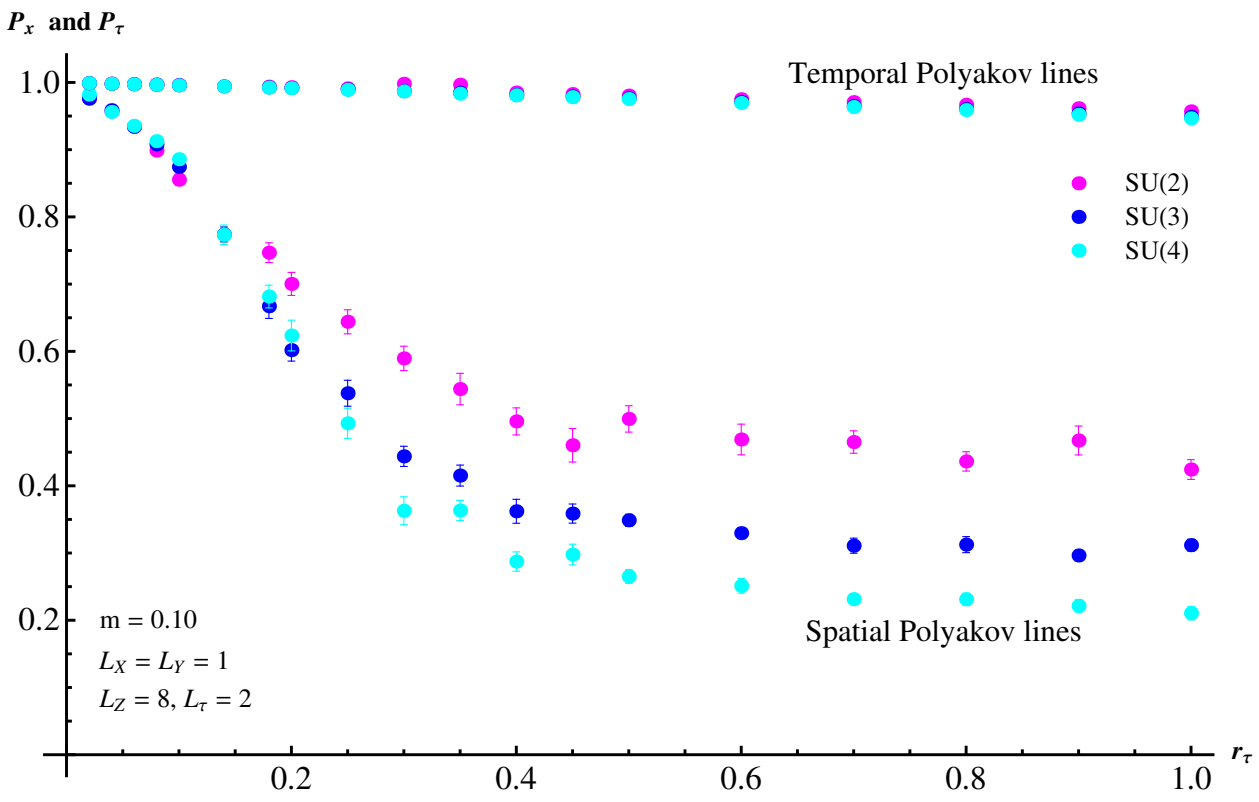
- In dual gauge theory:

  Thermal phase transition associated with breaking of center symmetry

  Order parameter spatial Polyakov line P_s.

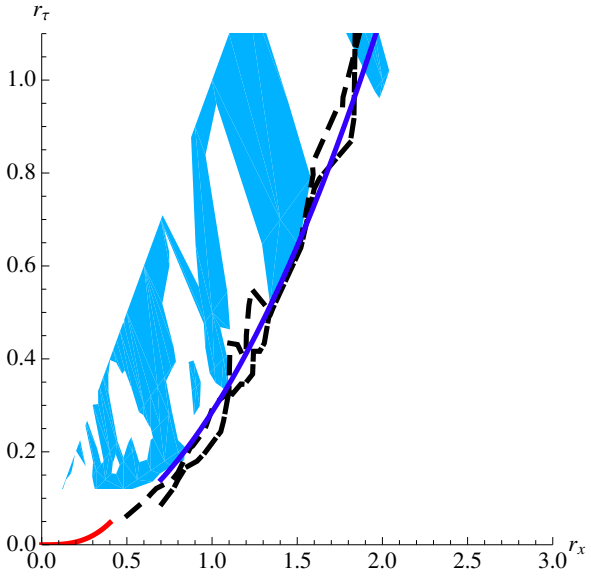
Black Strings from YM [contd.]

$$P_x = \frac{1}{N}\left\langle \left| \mathrm{Tr}\, \Pi_{a_x=0}^{L-1} U_{ax} \right| \right\rangle,\ P_\tau = \frac{1}{N}\left\langle \left| \mathrm{Tr}\, \Pi_{a_\tau=0}^{T-1} U_{a\tau} \right| \right\rangle.$$



S. Catterall, **A.J.**, T. Wiseman [JHEP 1012 (2010) 022]

Black Hole-Black String Phase Transition



Boundary between confined/deconfined phases of SYM corresponds to $\frac{1}{N}|P_s| = 0.5$ S. Catterall, **A.J.**, T. Wiseman [JHEP 1012 (2010) 22]
Good agreement with supergravity - blue curve - $r_x^2 = c_{crit} r_\tau$ with fitted $c_{crit} \sim 3.5$
Good agreement with high temperature prediction (red curve).

Conclusions/Further Explorations

- Exciting time for lattice SUSY - much activity, many developments.
- Lattice actions retaining some exact SUSY possible. They describe topologically twisted SYMs in continuum limit.
- Possibility for non-perturbative exploration of $\mathcal{N} = 4$ SYM. Tests of AdS/CFT. Beyond SUGRA using lattice?
- Dimensional reductions - duality between strings with Dp-branes and (p + 1)-SYM. eg: thermal D0/D1/D2 branes.
- For $d = 4$ SYM theory - most pressing question - what residual (fine) tuning needed to get full SUSY as $a \to 0$? Nonperturbative study (Ward identities) required.