Excited hadrons from data

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The intermediate energy region...
The intermediate energy region... provides a key to our understanding of QCD.
How can ab-initio QCD predictions be matched to the rich phenomenology of excited baryons?

Are there explicit experimental signatures of gluon dynamics at intermediate energies? Do exotics exist?

- Analysis of excited baryons
- Analysis of excited mesons
- Analysis of lattice data
The baryon spectrum: $N^*$ and $\Delta$ resonances

- Many resonances predicted in lattice calculations — Missing resonance problem from quark model reappears
  

$$\begin{align*}
J^P &= \frac{1}{2}^+ \quad \frac{3}{2}^+ \quad \frac{5}{2}^+ \quad \frac{7}{2}^+ \\
&\quad \frac{1}{2}^- \quad \frac{3}{2}^- \quad \frac{5}{2}^- \quad \frac{7}{2}^-
\end{align*}$$

$N^*$ and $\Delta^*$

$m_{\pi} = 396 \text{ MeV} (!)$

- Search for these states in dedicated experimental programs
Photoproduction experiments: Jefferson Lab, MAMI, ELSA, ...
Photoproduction cross sections

\[ \gamma + p \rightarrow X \]

\[ \gamma + p \rightarrow p + \pi^- + \]

\[ \gamma + p \rightarrow p + \pi^0 + \]

\[ \gamma + p \rightarrow K^+ + \Lambda \]

\[ \gamma + p \rightarrow p + \eta \]

Multipole/PW analysis required

[data: JLab, ELSA, MAMI]
Principles for scattering: 2-body unitarity

\[ \text{Im } T^{-1} = \sigma, \quad \sigma \sim p_{\text{cms}} \sim \sqrt{z - (m + M)} \]

\[ \rightarrow \text{One branchpoint at } z = m + M, \text{ one cut, two Riemann sheets.} \]

Implementation:

- **K-matrix:** \( T = K - i K \sigma T \)

- **On-shell:** \( T = V + V G T, \text{ Re } G \neq 0, \text{ e.g., from dispersion relation.} \)

- **Off-shell:** \( T(q'', q') = V(q'', q') + \int dq \ V(q'', q) g(q) T(q, q') \)
Coupled channels
Known large inelasticities
\[ \pi\pi N \ [\pi\Delta, \sigma N, \rho N, \ldots] \]

\[ \pi\pi / \pi N \] boosted subsystems constructed to fit the respective phase shifts.

Is it enough to include (boosted) 2-particle subsystems in the propagator?
No.

Three-body \( s \)-channel dynamics requires meson exchange transitions. \( \Rightarrow \) Three-body unitarity
(also applicable for meson analysis).

[Aaron, Almado, Young, PR 174 (1968) 2022,
Aitchison, Brehm, PLB 84 (1979) 349, PRD 25 (1982) 3069,\ldots]
Consequence: Threshold openings in the complex plane
Existence shown model-independently in [S. Ceci, M.D., C. Hanhart et. al., PRC 84 (2011)]
Further consequences
[S. Ceci, M.D., C. Hanhart et. al., PRC 84 (2011)]

Amplitude $A$ with complex $\rho N$ branchpoint at $E = 1700 - i 70$ MeV

Fit to $A$ using model $B$ without this branchpoint
→ $B$ needs fake resonance at $E = 1700 - i 130$ MeV

→ Erroneous resonance signal induced by wrong analytic structure
Definitely... 

...the wrong analytic structure.
Coupled channels
Circular cut: Induced by partial wave projection, NOT present in the amplitude in plane-wave basis.

\[ V_u = \int_{-1}^{1} dx \frac{P_\ell(x)}{u - m_N^2 + i\epsilon}, \quad V_t = \int_{-1}^{1} dx \frac{P_\ell}{t - m_t^2 + i\epsilon} \]

Branch points given where integration border hits pole:

\[ m_N^2 - u |_{x=\pm 1} = 0, \quad u = (p_f^2 - p_N^2)^2 \]

→ u-channel exchanges to obtain and approximate left-hand & other cuts

(Excited) t- and u-channel exchanges relate all partial waves and, at the same time, provide the background of s-channel resonances.
Approximations of crossing symmetry at the level of the potential (not the amplitude)

For $\sigma(600)$ and $\rho(770)$ quantum numbers: $\pi N$ $t$-channel interaction from $\bar{N}N \rightarrow \pi\pi$ (analytically continued) data.

Use of crossing symmetry and dispersion techniques [Schütz et al. PRC 49 (1994) 2671].
Analytic structure in the $P_{11}$ partial wave
The scattering equation

\[ V = \sum_{\gamma, \lambda''} \int d^3 q V_{\mu \gamma}^{I}(\vec{k}', \lambda', \vec{q}, \lambda'') \frac{1}{Z - E_{\gamma}(q) + i\epsilon} T_{\gamma \nu}^{I}(\vec{q}, \lambda'', \vec{k}, \lambda) + \sum_{\gamma, \lambda'} V_{\mu \nu}^{I}(\vec{k}', \lambda', \vec{k}, \lambda) \]
Data base
Analysis of the world data base on $\eta N$, $K\Lambda$, $K\Sigma$

<table>
<thead>
<tr>
<th>Reaction</th>
<th>PWA</th>
<th>$\sigma_{tot}$</th>
<th>$\frac{d\sigma}{d\Omega}$</th>
<th>$P$</th>
<th>$\beta$</th>
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<tbody>
<tr>
<td>$\pi N \rightarrow \pi N$</td>
<td>GWU/SAID 2006</td>
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<td>up to $J=9/2$</td>
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<td>$\pi^- p \rightarrow \eta n$</td>
<td>62 data points</td>
<td>38 energy points</td>
<td>12 energy points</td>
<td>E=1489 to 2235 MeV</td>
<td>1740 to 2235 MeV</td>
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<td></td>
<td>66 data points</td>
<td>46 energy points</td>
<td>27 energy points</td>
<td>1626 to 1405 MeV</td>
<td>1633 to 2208 MeV</td>
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<td>29 energy points</td>
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<td>15 energy points</td>
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<td>1739 to 2405 MeV</td>
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<tr>
<td>$\pi^+ p \rightarrow K^+ \Sigma^+$</td>
<td>18 data points</td>
<td>32 energy points</td>
<td>32 energy points</td>
<td>1729 to 2318 MeV</td>
<td>2021 and 2107 MeV</td>
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<td>2021 and 2107 MeV</td>
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plus $\sim 6000$ data points

plus $\sim 2000$ data points for photoproduction: differential cross section and spin asymmetry for

$\gamma p \rightarrow \pi^+ n$, $\gamma p \rightarrow \pi^0 p$, $\gamma n \rightarrow \pi^- p$, $\gamma n \rightarrow \pi^0 n$
$\pi N \to \pi N$: Partial wave amplitudes $l=1/2$ PWA data: SAID [Phys.Rev. C 74 (2006)]

[Rönchen, M.D. et al., arXiV: 1211.6998, accepted for publication in EPJA]
\( \pi N \rightarrow \pi N: \text{Partial wave amplitudes } I=3/2 \)

[Rönchen, M.D. et al., arXiV: 1211.6998, accepted for publication in EPJA]
Selected results for $\pi^- p \rightarrow K^0 \Lambda$

This is a complete experiment in the sense that it fixes the amplitude up to one energy-dependent, overall phase.
\( \pi^- p \rightarrow K^0 \Lambda \): Total cross section

Result: Partial wave content.
The $P_{11}$ partial wave

Nucleon
Dynamical Roper $P_{11}(1440)$
Genuine $P_{11}(1710)$
(Dynamical $P_{11}(1750)$)
The $P_{11}$ partial wave

Nucleon Dynamical Roper $P_{11}(1440)$
Genuine $P_{11}(1710)$
(Dynamical $P_{11}(1750)$)

Genuine $P_{11}(1710)$:

- Input for our fit (blue dashed): energy-dependent SAID solution (black solid)
- Inclusion of $P_{11}(1710)$ necessary from $K\Lambda$ data
The $P_{11}$ partial wave

Nucleon Dynamical Roper $P_{11}(1440)$
Genuine $P_{11}(1710)$
(Dynamical $P_{11}(1750)$)

Input for our fit (blue dashed):
energy-dependent SAID solution
(black solid)

Inclusion of $P_{11}(1710)$ necessary from $K\Lambda$ data

But: Our solution matches single-energy solution (“data” points)

Coupled-channel dynamics at work
Gauge invariance: Generalized Ward-Takahashi identity (WTI)

\[
k_\mu M^\mu = -|F_s \tau\rangle S_{p+k} Q_i S_p^{-1} + S_{p'}^{-1} Q_f S_{p'-k} |F_u \tau\rangle + \Delta_{p-p'+k}^{-1} Q_{pi} \Delta_{p-p'} |F_t \tau\rangle
\]

Photoproduction amplitude:

\[
M^\mu = \underbrace{M_s^\mu + M_u^\mu + M_t^\mu}_{\gamma \text{ to external legs}} + \underbrace{M_{int}^\mu}_{\gamma \text{ inside hadronic vertex}}
\]

Strategy: Replace \(\text{by phenomenological contact term such that the generalized WTI is satisfied}\)
The importance of Gauge Invariance

Huang, M.D., Haberzettl, Haidenbauer, Hanhart, Krewald, Meißner, Nakayama, PRC 85 (2011)

[data: GWINS/DAC data base; JLab, ELSA, MAMI]
Differential cross section for $\gamma p \rightarrow \pi^+ n$

Photon spin asymmetry for $\gamma p \rightarrow \pi^+ n$

Data: CNS Data analysis center [CBELSA/TAPS, JLAB, MAMI,...]
Pion photoproduction: $d\sigma/d\Omega$ and $\Sigma_\gamma$ for $\gamma n \rightarrow \pi^- p$


Differential cross section for $\gamma n \rightarrow \pi^- p$

Photon spin asymmetry for $\gamma n \rightarrow \pi^- p$

Data: CNS Data analysis center [CBELSA/TAPS, JLAB, MAMI,...]
Pion photoproduction: $d\sigma/d\Omega$ and $\Sigma_\gamma$ for $\gamma p \rightarrow \pi^0 p$


Differential cross section for $\gamma p \rightarrow \pi^0 p$

Photon spin asymmetry for $\gamma p \rightarrow \pi^0 p$

Data: CNS Data analysis center [CBELSA/TAPS, JLAB, MAMI,...]
Multipole amplitudes $l=3/2$
not included in the fit

Wealth of new data arriving from JLab, MAMI, ELSA,…
Resonance content: Nucleon-like resonances

- N(1440) $1/2^+$
- N(1520) $3/2^-$
- N(1535) $1/2^-$
- N(1570) $1/2^-$
- N(1650) $1/2^-$
- N(1675) $5/2^-$
- N(1680) $5/2^+$
- N(1710) $1/2^+$
- N(1720) $3/2^+$
- N(1750) $1/2^+$
- N(1990) $7/2^+$
- N(2190) $7/2^-$
- N(2200) $9/2^-$
- N(2220) $9/2^+$

Physical axis: Im $E=0$
Perspectives

Meson spectroscopy @ GlueX

Hadronic approaches for the lattice
From 2 to 3 particles in the final state: an example
[M. D., E. Oset, Ulf-G. Meißner, EPJA 46 (2010)]

- 3-bodies in the final state: complex kinematics
- Many observables
- Illustrated for \( \vec{\gamma}p \to \pi^0 \eta p \)

First measurement of \( I^S, I^C \) (fixed \( \phi^* \))

\[
\frac{d\sigma}{d\phi} = \sigma_0[1 + I^S \sin(2\phi) + I^C \cos(2\phi)]
\]

Predictions for $I^C$


Red solid lines: Predicted results.

Black dotted lines: Without the strong $\Delta(1700) \rightarrow \eta \Delta(1232)$ coupling predicted from U$\chi$PT.
Hybrids and exotics in recent lattice calculations
Dudek PRD 83 (2011) 111502, Dudek PRD 84 (2011) 074023

\[ J^{PC}=1^{-+} \] cannot be made of quark spins and angular momentum \( \rightarrow \) \( q\bar{q}g \)

normal mesons

exotics

\[ q\bar{q} S \]
\[ q\bar{q} P \]
\[ q\bar{q} D \]
\[ q\bar{q} F \]
\[ q\bar{q} G \]
\[ q\bar{q}g \]
Binning in $m_{3\pi}$ and $t$.

→ No assumption on resonances.

Event-based log-likelihood fit.

Future: effective Regge-parameterization.
Transfer of methods from baryon to meson analysis

Ensures 3-body unitarity

+ pions from N* and Δ's
Beyond the isobar model

1) no FSI, no 3-body unitarity, commonly used.

2) 3-body unitary isobars, off or on-shell reduction.

Match to $[X \rightarrow \pi$ isobar] natural

3) “Isobars” from unitarized $\chi$PT

Match to $[X \rightarrow \pi$ isobar] not straightforward

4) Faddeev calculation.

- Usually integral equations $\rightarrow$ on-shell simplifications possible?
- Effective isobar parameterizations (Pelaez, Caprini, . . .):
  take account of Adler zero, increased convergence by complex mapping techniques, . . .
Perspectives

Meson spectroscopy @ GlueX

Hadronic approaches for the lattice
The world as lattice sees it

Finite volume $L^3$

Finite lattice spacing $a$

$m_q(\text{Lattice}) \neq m_q$
Resonances decaying on the lattice

Eigenvalues in the finite volume

Avoided level crossing

Energy

Resonance energy

L (box size)
Resonances in the finite volume

- Unitarized chiral interaction with NLO contact terms

Data: SAID (2006)
Chiral extrapolation to a QCDSF lattice setup

The graph shows the real and imaginary parts of the S-matrix, $S_{11}$, as a function of the square root of the energy, $\sqrt{s}$, plotted on the real and imaginary axes. The energy is given in MeV. The graph highlights the contributions of various meson-nucleon and baryon-baryon channels.

- $\pi N$ channel
- $K\Sigma$ channel
- $\eta N$ channel
- $K\Lambda$ channel
- $K\Sigma$ channel
- $\eta N$ channel
- $N_1$ channel
- $N_2$ channel

The graph displays the resonance structure and the phase shifts at different energies, showing the complex behavior of the S-matrix in the chiral limit.
The inverse process:

Analyze existing lattice spectra to extract hadronic resonances!
Extracting resonances from lattice data
[M.D., Meißner, Oset/Rusetsky, EPJA 47 (2011)]

Three unknown potentials

- $V(\pi\pi \to \pi\pi)$
- $V(\pi\pi \to \bar{K}K)$
- $V(\bar{K}K \to \bar{K}K)$

Expand a two-channel potential $V$ in energy $(i, j: \pi\pi, \bar{K}K)$:

$$V_{ij}(E) = a_{ij} + b_{ij}(E^2 - 4M_K^2)$$

to extract phase shifts/resonances

![lattice data & fit](image1)

![extracted phase shift](image2)

![$f_0(980)$ pole position](image3)
\(\kappa/K^*\) in moving frames: complexity of data at physical masses

Lattice spectrum simulated from NLO \(U\chi PT\) [M.D. et al., EPJ A 48 (2012)]
Mixing of partial waves, coupled channels.
Eventually three particles on-shell (large finite-volume corrections to the Roper?), isobar schemes including rearrangement for 3-body unitarity,…

Principles and tools from data analysis become useful.
The study of resonances at intermediate energies provides a key to our understanding of Quantum Chromodynamic.

JLab, MAMI, ELSA, … provide high-precision photoproduction data.

The analysis allows to extract the resonance spectrum to test QCD predictions.

Tools and principles used in baryon analysis can boost the analysis of meson data (GlueX) and lattice eigenvalues.
**s-, t- and u-channel exchanges**

- **s-channel states coupling to** $\pi N, \eta N, K\Lambda, K\Sigma, \pi \Delta, \rho N$.
- **t- and u-channel exchanges:**

<table>
<thead>
<tr>
<th></th>
<th>$\pi N$</th>
<th>$\rho N$</th>
<th>$\eta N$</th>
<th>$\pi \Delta$</th>
<th>$\sigma N$</th>
<th>$K\Lambda$</th>
<th>$K\Sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi N$</td>
<td>$N, \Delta, (\pi \pi)<em>\sigma, (\pi \pi)</em>\rho$</td>
<td>$N, \Delta, \text{Ct.}, \pi, \omega, a_1$</td>
<td>$N, a_0$</td>
<td>$N, \Delta, \rho$</td>
<td>$N, \pi$</td>
<td>$\Sigma, \Sigma^<em>, K^</em>$</td>
<td>$\Lambda, \Sigma, \Sigma^<em>, K^</em>$</td>
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<td>$\rho N$</td>
<td>$N, \Delta, \text{Ct.}, \rho$</td>
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<td>$N, \pi$</td>
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<td>$\eta N$</td>
<td>$N, f_0$</td>
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<td>$K^*, \Lambda$</td>
<td>$\Sigma, \Sigma^<em>, K^</em>$</td>
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<td>$N, \Delta, \rho$</td>
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<td>$K\Lambda$</td>
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<td>$\Xi, \Xi^*, f_0, \omega, \phi$</td>
<td>$\Xi, \Xi^*, \rho$</td>
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<td>$K\Sigma$</td>
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<td>$\Xi, \Xi^*, f_0, \omega, \phi, \rho$</td>
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</tbody>
</table>
$\pi^- p \rightarrow \eta N$: Cross section and Polarization

**Selected results**

**Difficult data situation**

- Inconsistencies among different data sets
- Data at higher energies questionable
- Polarization data questionable
$\pi^- p \rightarrow \eta N$: Total cross section

Partial wave content:
\[ \pi^- p \rightarrow K^0 \Sigma^0: \frac{d\sigma}{d\Omega} \& Polarization \]

Selected results
\[ \pi^- p \rightarrow K^0 \Sigma^0: \] Total cross section

Partial wave content \( I = 1/2: \)

\[
\begin{align*}
\sigma \text{ [mb]} & \quad \text{vs} \quad z \text{ [MeV]} \\
S_{11} & \\
P_{11} & \\
P_{13} & \\
D_{13} & \\
D_{15} & \\
P_{15} & \\
F_{17} & \\
F_{19} & \\
G_{17} & \\
G_{19} & \\
H_{19} & \\
\end{align*}
\]
\( \pi^- p \rightarrow K^0 \Sigma^0 \): Total cross section

**Partial wave content** \( I = 3/2 \):
\[ \pi^- p \rightarrow K^+ \Sigma^- : \frac{d\sigma}{d\Omega} \]

Selected results

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>No Polarization Data</th>
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<tbody>
<tr>
<td>1763</td>
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<td>1818</td>
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<td>2025</td>
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<td>2305</td>
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</table>

No polarization data
$\pi^+ p \rightarrow K^+ \Sigma^+: d\sigma/d\Omega$, polarization & spinrotation angle

Selected results
Exciting CLAS data to come from FROST, HD-ICE, ...  
from: Eugene Pasyuk @ Meson 2012

<table>
<thead>
<tr>
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<th>σ</th>
<th>Σ</th>
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“Neutron” target

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Two-body scattering
Scattering in the infinite volume limit

- Unitarity of the scattering matrix $S$: $SS^\dagger = 1$ 
  
\[
S = 1 - i \frac{p}{4\pi E} \, T.
\]

\[
\text{Im } T^{-1}(E) = \sigma \equiv \frac{p}{8\pi E}
\]

- Generic (Lippman-Schwinger) equation for unitarizing the $T$-matrix:

\[
T = V + V \, G \, T \quad \text{Im } G = -\sigma
\]

$V$: (Pseudo)potential, $\sigma$: phase space.

- $G$: Green’s function:

\[
G = \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2 + i\epsilon},
\]

\[
\omega_{1,2}^2 = m_{1,2}^2 + \vec{q}^2
\]
Discretized momenta in the finite volume with periodic boundary conditions:

\[ \Psi(x) = \Psi(x + \hat{e}_i L) = \exp(i L q_i) \Psi(x) \implies q_i = \frac{2\pi}{L} n_i, \quad n_i \in \mathbb{Z}, \quad i = 1, 2, 3 \]

\[
\int \frac{d^3 \vec{q}}{(2\pi)^3} g(|\vec{q}|^2) \rightarrow \frac{1}{L^3} \sum_{\vec{n}} g(|\vec{q}|^2), \quad \vec{q} = \frac{2\pi}{L} \vec{n}, \quad \vec{n} \in \mathbb{Z}^3
\]

\[
G \to \tilde{G} = \frac{1}{L^3} \sum_{\vec{q}} \frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2}
\]

- \( E > m_1 + m_2 \): \( \tilde{G} \) has poles at free energies in the box, \( E = \omega_1 + \omega_2 \)
- \( E < m_1 + m_2 \): \( \tilde{G} \to G \) exponentially with \( L \) (regular summation theorem).
- Formalism can be mapped to Lüsher’s \( \mathcal{Z}_{\ell m} \).
Measured eigenvalues of the Hamiltonian (tower of lattice levels $E(L)$) → Poles of scattering equation $\tilde{T}$ in the finite volume → determines $V$:

$$\tilde{T} = (1 - V \tilde{G})^{-1} V \rightarrow V^{-1} - \tilde{G} = 0 \rightarrow V^{-1} = \tilde{G}$$

The interaction $V$ determines the $T$-matrix in the infinite volume limit:

$$T = (V^{-1} - G)^{-1} = (\tilde{G} - G)^{-1}$$

Re-derivation of Lüscher's equation ($T$ determines the phase shift $\delta$):

$$p \cot \delta(p) = -8\pi \sqrt{s} \left( \tilde{G}(E) - \text{Re} G(E) \right)$$

$V$ and dependence on renormalization have disappeared (!)
Twisting the boundary conditions (B.C.)

[M.D., Meißner, Oset, Rusetsky, 47 (2011)]

- Periodic B.C.:
  \[\Psi(\vec{x} + \hat{e}_i L) = \Psi(\vec{x})\]
- Periodic in 2 dim.:

\[\begin{align*}
\theta_1 &= 0 \\
\theta_2 &= 0
\end{align*}\]

- Twisted B.C.:
  \[\Psi(\vec{x} + \hat{e}_i L) = e^{i\theta_i} \Psi(\vec{x})\]
- Periodic/antiperiodic:

\[\begin{align*}
\theta_1 &= 0 \\
\theta_2 &= \pi
\end{align*}\]

Example: the \(f_0(980)\)

- \(S\)-wave, coupled-channels \(\pi\pi, \bar{K}K\).
- Twisted B.C. for the \(s\)-quark:
  \[\begin{align*}
u(\vec{x} + \hat{e}_i L) &= u(\vec{x}) \\
d(\vec{x} + \hat{e}_i L) &= d(\vec{x}) \\
s(\vec{x} + \hat{e}_i L) &= e^{i\theta_i} s(\vec{x})
\end{align*}\]
- Three unknown potentials
  \[\begin{align*}
V(\pi\pi \rightarrow \pi\pi) \\
V(\pi\pi \rightarrow \bar{K}K) \\
V(\bar{K}K \rightarrow \bar{K}K)
\end{align*}\]

\[\begin{align*}
E [\text{MeV}] \\
2 & 2.5 & 3 & 3.5 \\
L [M_{\pi}^{-1}] \\
\theta_i = 0 & \theta_i = \pi/2 & \theta_i = \pi
\end{align*}\]
Mixing of partial waves

Example: $S$- and $P$-waves

Infinite volume limit: Rotational symmetry

\[ I \int \frac{d^3 \vec{q}}{(2\pi)^3} g(|\vec{q}|) \ Y_{\ell m}(\theta, \phi) \ Y^{*}_{\ell' m'}(\theta, \phi) \sim \delta_{\ell \ell'} \delta_{m m'}. \]

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Mixing of partial waves
Example: $S$- and $P$-waves

- Infinite volume limit: Rotational symmetry

$$
\int \frac{d^3 \vec{q}}{(2\pi)^3} g(|\vec{q}|) \ Y_{\ell m}(\theta, \phi) \ Y_{\ell' m'}^*(\theta, \phi) \sim \delta_{\ell \ell'} \delta_{mm'}.
$$

- Wigner-Eckart theorem:

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$\rightarrow$ $P_1$
Mixing of partial waves
Example: $S$- and $P$-waves

Finite volume: Rotational symmetry → Cubic symmetry

$$\frac{1}{L^3} \sum_{\vec{n}} g(|\vec{q}|) Y_{\ell m}(\theta, \phi) Y^{*}_{\ell' m'}(\theta, \phi) \sim A_{\ell \ell' m m'}.$$ 

$S - G$-wave mixing, but $S - P$ waves still orthogonal:

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Breaking of cubic symmetry through boost

Example: Lattice points $\vec{q}^*$ boosted with $P = (0, 0, 0) \rightarrow \frac{2\pi}{L} (0, 0, 2)$:
Mixing of partial waves
Example: $S$- and $P$-waves

Finite volume & boost: Cubic symmetry $\rightarrow$ subgroups of cubic symmetry

\[
\frac{1}{L^3} \sum_{\vec{n} \bar{\vec{n}}} g(|\vec{q}|) \ Y_{\ell m}(\theta, \phi) Y^*_{\ell' m'}(\theta, \phi) \sim A_{\ell \ell' m m'}.
\]

For boost $P = \frac{2\pi}{L} (0,1,1)$:

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Mixing of partial waves
Example: $S$- and $P$-waves

- **Finite volume & boost:** Cubic symmetry $\rightarrow$ subgroups of cubic symmetry

\[
\frac{1}{L^3} \sum_{\vec{n}} g(|\vec{q}|) \ Y_{\ell m}(\theta, \phi) \ Y_{\ell' m'}^*(\theta, \phi) \sim A_{\ell \ell' mm'}.
\]

- **More complicated boosts:**

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Disentanglement of partial waves
Example: $S$- and $P$-waves for the $\kappa(800)/K^*(892)$ system

Knowledge of $P$-wave (from separate analysis of lattice data) allows to disentangle the $S$-wave:

$$p \cot \delta_S = -8\pi E \frac{p \cot \delta_P \hat{G}_{SS} - 8\pi E (\hat{G}_{SP}^2 - \hat{G}_{SS} \hat{G}_{PP})}{p \cot \delta_P + 8\pi E \hat{G}_{PP}}$$

- $\delta_S \equiv \delta_{1/2}^0 (\pi K \rightarrow \pi K)$
- Red solid: Actual $S$-wave phase shift.
- Dash-dotted: Reconstructed $S$-wave phase shift, PW-mixing ignored.
- Dashed: Reconstructed $S$-wave phase shift, PW-mixing disentangled.
- Small $p$-wave: Level shift
$$\Delta E \simeq \frac{6\pi E_S \delta_P}{L^3 p \omega_1 \omega_2}$$
Mixing of partial waves in boosted multiple channels: $\sigma(600)$

- $\pi\pi$ & $\bar{K}K$ in $S$-wave, $\pi\pi$ in $D$-wave.
- Organization in Matrices ($A_1^+$), e.g. 

$$\vec{P} = (2\pi/L)(0, 0, 1), (2\pi/L)(1, 1, 1),$$

and $(2\pi/L)(0, 0, 2)$:

$$V = \begin{pmatrix}
V_S^{(11)} & V_S^{(12)} & 0 \\
V_S^{(21)} & V_S^{(22)} & 0 \\
0 & 0 & V_D^{(22)}
\end{pmatrix}$$

$$\tilde{G} = \begin{pmatrix}
\tilde{G}^{R(1)}_{00,00} & 0 & 0 \\
0 & \tilde{G}^{R(2)}_{00,00} & \tilde{G}^{R(2)}_{00,20} \\
0 & \tilde{G}^{R(2)}_{20,00} & \tilde{G}^{R(2)}_{20,20}
\end{pmatrix}$$

- Phase extraction ($\kappa$): Expand and fit $V_S$, $V_P$ simultaneously to different representations instead of

1. $P$-wave from $B_1$, $B_2$, $E$
2. $S$-wave from $P$ and $A_1$

(reduction of error).

Solid: Levels from $A_1^+$. 
Non-solid: Neglecting the $D$-wave.
Phase shifts from a moving frame: the $\sigma(600)$

Comparison: Variation of $L$ vs moving frames

The first two levels for the first five boosts:

- Pseudo-data [10 MeV error]

Moving frame: 2 L's

Vary volume $L^3$: 6 L's

Actual position

Central, $(s_0, s_1, s_2, s_3)$ fit

Central, $(s_0, s_1, s_2)$ fit

Central, $(s_0, s_1)$ fit

20 pts., $\Delta E=10$ MeV

11 pts., $\Delta E=10$ MeV
Unitary isobars in the finite volume: the $a_1(1260)$

3 × 3 self energy:

$$
\tilde{\Pi} = \begin{pmatrix}
\tilde{\Pi}_{1,1} & \tilde{\Pi}_{1,0} & \tilde{\Pi}_{1,-1} \\
\tilde{\Pi}_{0,1} & \tilde{\Pi}_{0,0} & \tilde{\Pi}_{0,-1} \\
\tilde{\Pi}_{-1,1} & \tilde{\Pi}_{-1,0} & \tilde{\Pi}_{-1,-1}
\end{pmatrix},
$$

Dressed propagator

$$
\tilde{S}^D_{\pi\rho} = \frac{1}{2\omega_1} \left( (S^{B}_{\pi\rho} \mathbb{1})^{-1} - \tilde{\Pi} \right)^{-1}
$$

Sum over boosts $\vec{P}$:

$$
\tilde{G}_{\pi\rho} = \frac{1}{L^3} \sum_{\vec{P}} \tilde{S}^D_{\pi\rho}.
$$

$\pi\rho$ scattering equation:

$$
\tilde{T}_{\pi\rho} = (\mathbb{1} - \hat{V}_{\pi\rho} \tilde{G}_{\pi\rho})^{-1} \hat{V}_{\pi\rho}, \quad \hat{V}_{\pi\rho} = V_{\pi\rho} \mathbb{1}
$$

Lattice levels:

$$
\det(\mathbb{1} - \hat{V}_{\pi\rho} \tilde{G}_{\pi\rho}) = 0.
$$

(\text{+ pion exchange, required from 3-body unitarity; certain 3-body singularities cancel [see also K. Polejaeva and A. Rusetsky, EPJA 48 (2012)]})