Scattering amplitudes from lattice QCD

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Based on work in collaboration with J.J. Dudek, R.G. Edwards and C.E. Thomas.
Jefferson lab theory center
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Two talks

Today:
Introductory stuff
The methods we are using
Elastic scattering

Next Week:
Jo Dudek
Resonances in coupled channel scattering from lattice QCD

Introduction

Lagrangian of QCD

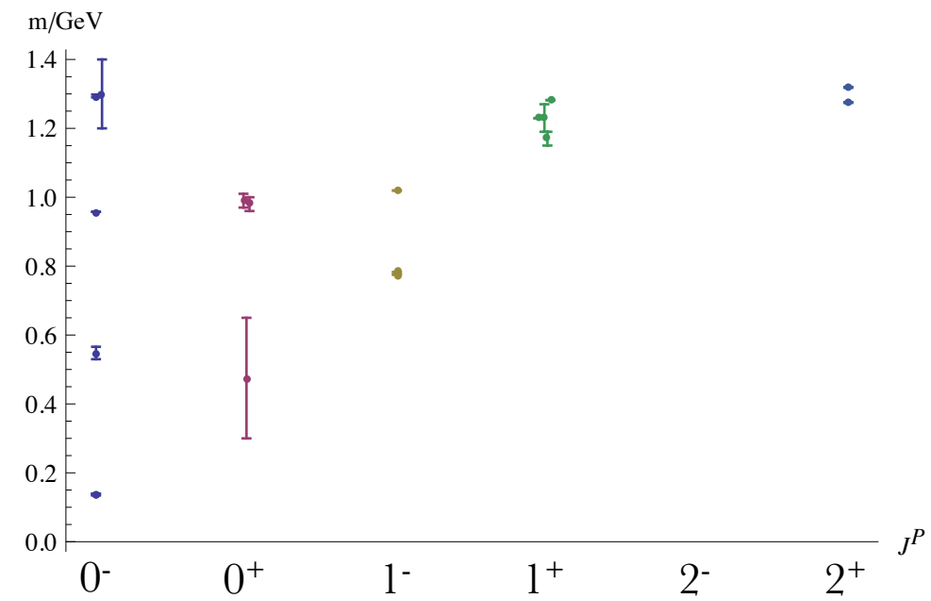
$$\mathcal{L}_{\mathrm{QCD}} = \sum_q \bar{\psi}_q \left(i\not{D} - m_q\right)\psi_q - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$$

$$\not{D} = \gamma^\mu \left(\partial_\mu - igA_\mu\right)$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + g\left[A_\mu, A_\nu\right]$$

Coloured quark and gluon degrees of freedom.

Spectrum of hadrons



Only colourless states, no asymptotic quarks and gluons.

- Excited state spectrum contains many interesting open questions.
- Interesting effects near thresholds: tetraquarks? meson-meson bound states? $f_0(980)$ in $\pi\pi$ scattering and new charmonium states, eg: Z(4430).
- Hybrid states, containing explicit gluonic degrees of freedom. Could be seen in new experiments, like Glue-X.

Strong coupling

Lagrangian of QCD

$$\mathcal{L}_{\mathrm{QCD}} = \sum_q \bar{\psi}_q \left(i\not{D} - m_q\right)\psi_q - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$$

$$\not{D} = \gamma^\mu \left(\partial_\mu - igA_\mu\right)$$

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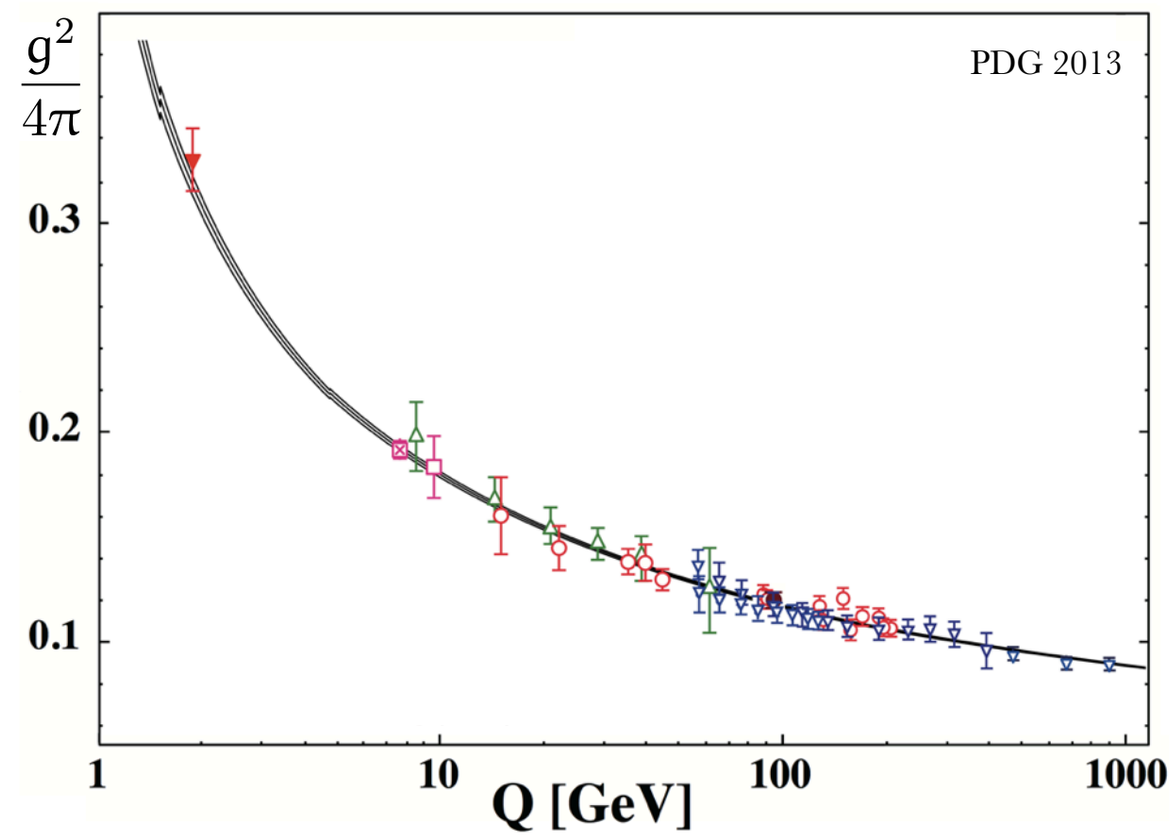


Many interesting consequences:
Confinement - no asymptotic quarks or gluons.
Dynamical chiral symmetry breaking.
Light physical pion ~ goldstone boson of the symmetry breaking.
Cannot use perturbation theory: Non-perturbative methods needed.

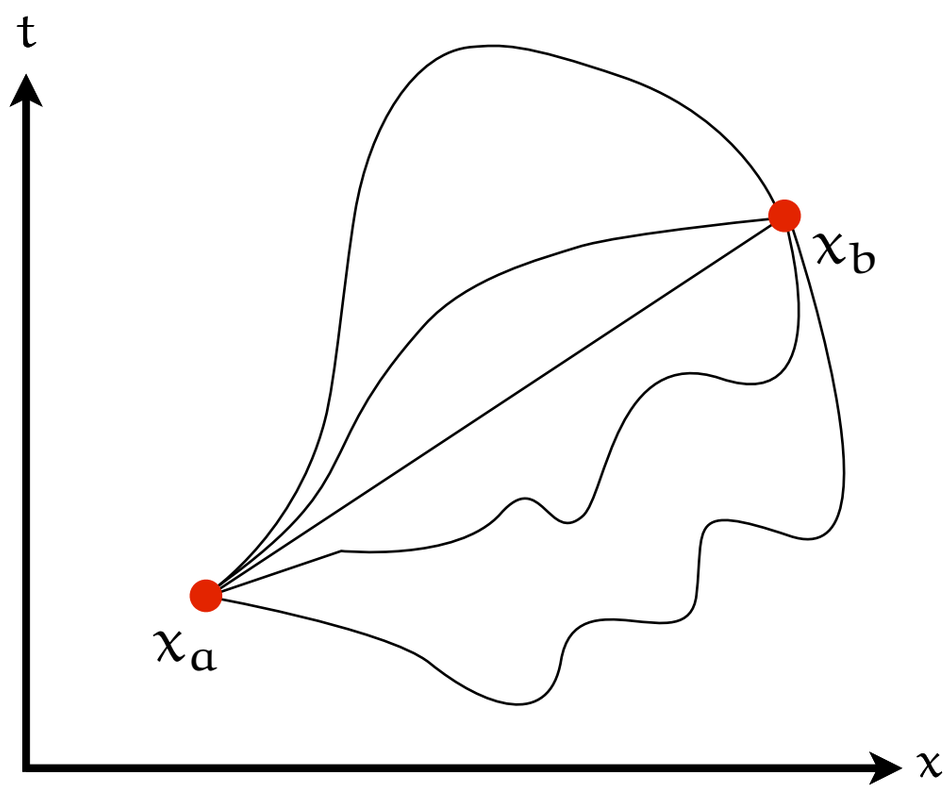
Several options including:

Models
Schwinger-Dyson+Bethe-Salpeter
Effective Field Theories
Lattice QCD

Path integrals

The starting point is the path integral:

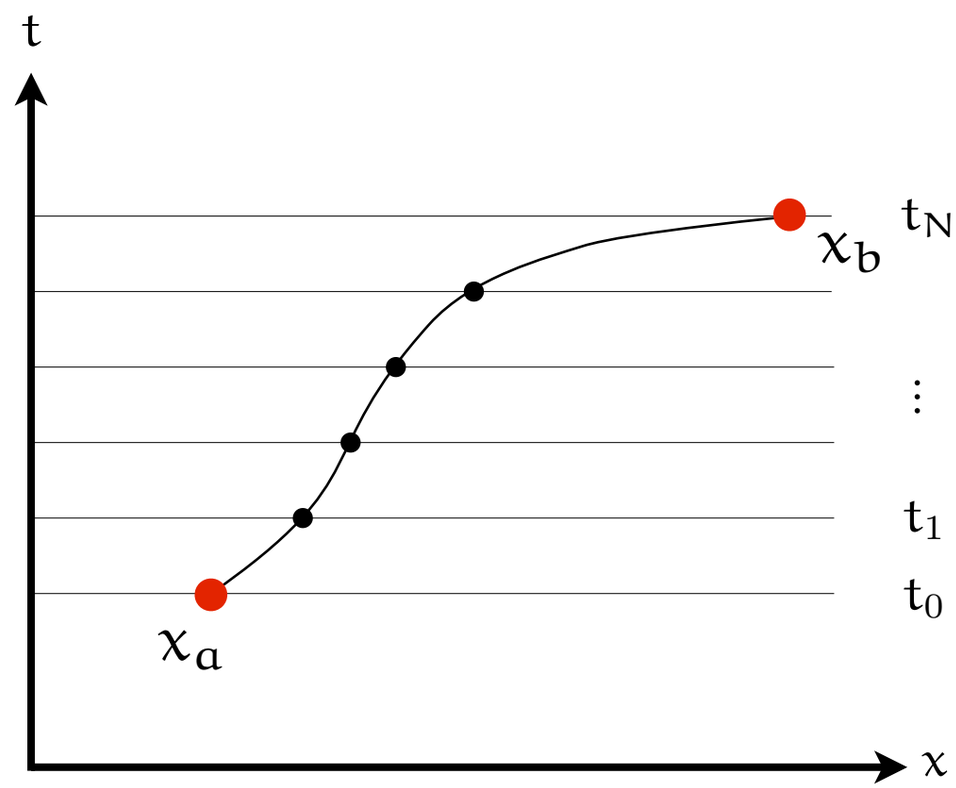
$$\langle x_b | e^{-iHT} | x_a \rangle = \int \mathcal{D}x(t) e^{iS[x(t)]}$$



Path integrals

To solve numerically, consider the discretised version

$$\langle x_b | e^{-iHT} | x_a \rangle = \Pi_{x_{t_i}} \int dx_{t_i} \, e^{iS(x_{t_i})}$$



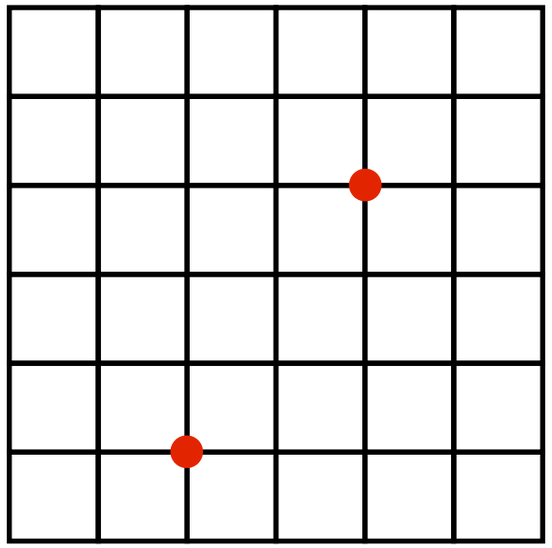
Path integrals

Evaluate correlation functions from the path integral:

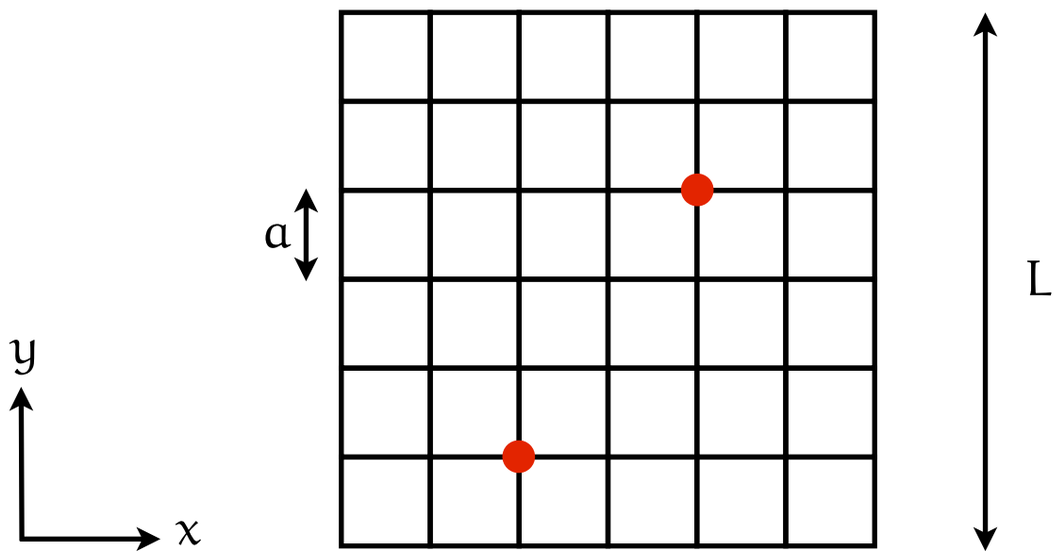
$$\langle 0| \mathcal{O}_i(t)\mathcal{O}_j^\dagger(0) |0\rangle = \frac{1}{Z_0}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\ \mathcal{O}_i(t)\mathcal{O}_j^\dagger(0)\ e^{iS[\bar{\psi},\psi,A]}$$

In order to deal with strong coupling: Solve the QCD path integral numerically.

Integrate over gauge field configurations: Infinitely many possibilities.
➜ Store field values on a discrete set of points



Lattice QCD

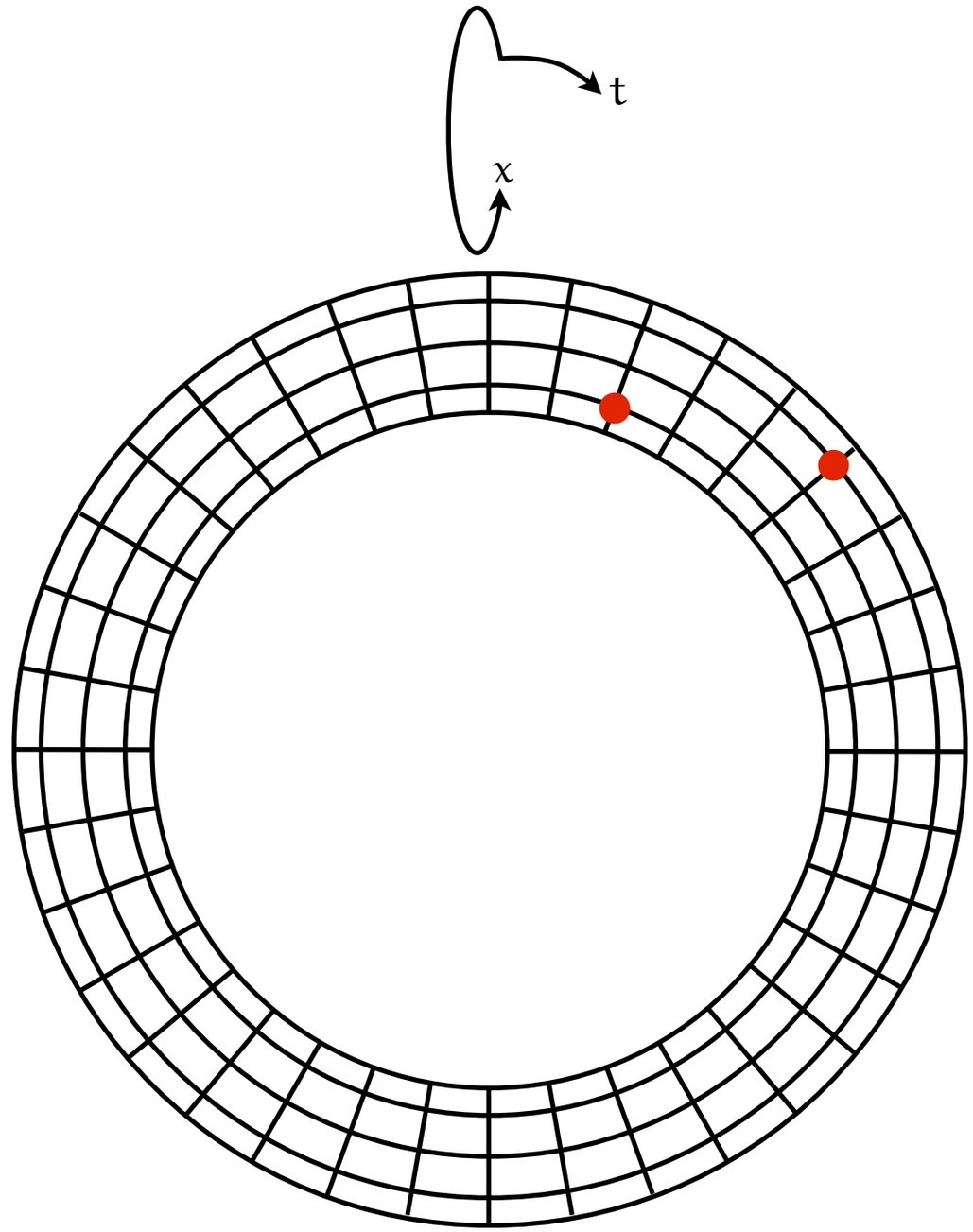


Use a finite spacetime volume, $L^3 \times t$. (L Roughly 2-3fm in these studies).
Use a finite number of points, with separation $a \sim 0.1$fm. ($L/a = 16, 20, 24$)

Quarks live on discrete points and the gluons live on the links between them.

Use periodic boundary conditions: Volume becomes a torus.

Lattice QCD



Use a finite spacetime volume, $L^3 \times t$. (L Roughly 2-3fm in these studies).
Use a finite number of points, with separation $a \sim 0.1$fm ($L/a = 16, 20, 24$).
Change variables to Euclidean spacetime to simplify integration.

Quarks live on discrete points and the gluons live on the links between them.

Use periodic boundary conditions: Volume becomes a torus.

Correlation functions

Evaluate correlation functions from the path integral:

$$C_{ij}(t) = \langle 0| \mathcal{O}_i(t)\mathcal{O}_j^\dagger(0) |0\rangle$$
$$= \frac{1}{Z_0}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\ \mathcal{O}_i(t)\mathcal{O}_j^\dagger(0)\ e^{-S[\bar{\psi},\psi,A]}$$

Leads to the ground state energy for large t:

$$C_{ij}(t) = \sum_n \frac{1}{2E_n}\langle 0| \mathcal{O}_i |n\rangle \langle n| \mathcal{O}_j^\dagger |0\rangle\, e^{-E_n t}$$
$$= \frac{Z_i^\star Z_j}{2E_n} e^{-E_n t}$$

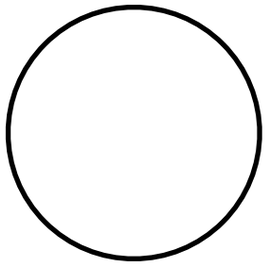
The symmetries of the operators dictate which states can be extracted

	Γ	
$\overline{\psi}\Gamma\psi$	$\gamma_5 \sim \pi$	$J^P = 0^-$
	$\gamma_i \sim \rho$	$J^P = 1^-$

Symmetry on the lattice

The lattice has a cubic symmetry.
It does not have the O(3) symmetry of continuous space.

Eg: 2D QM



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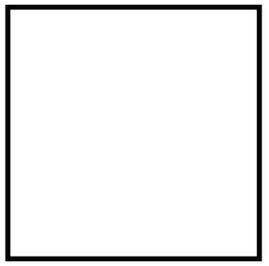


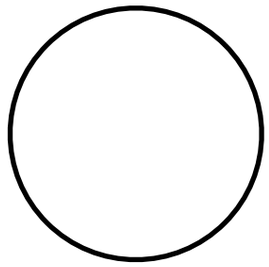
Continuous rotational spatial symmetry

$$e^{i\phi} \to e^{i\phi + i\alpha}$$

$$e^{i\phi} \to e^{i\phi + in\pi/2}$$

Only symmetric at discrete angles

Symmetry on the lattice



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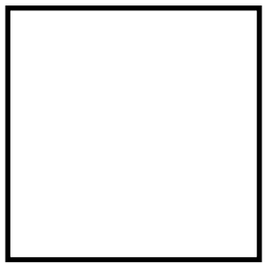

Continuous rotational spatial symmetry

$$e^{i\phi} \to e^{i\phi + i\alpha}$$

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Only symmetric at discrete angles

Cubic symmetry groups mix the continuum angular momentum:

Irrep	J^P
A_1^+	$0^+, 4^+, \ldots$
T_1^-	$1^-, 3^-, \ldots$

Operators with overall momentum

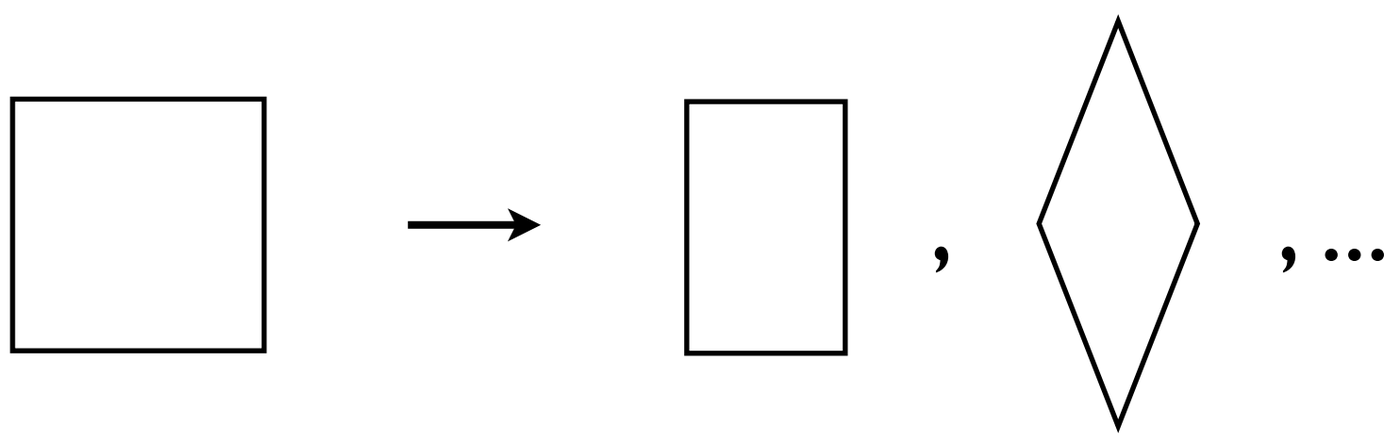
Because momentum is quantised, different energies can be accessed by considering operators with an overall momentum

Overall zero momentum: $\pi(0,0,0)\pi(0,0,0)$, $\pi(1,0,0)\pi(-1,0,0)$, ...
One unit: $\pi(1,0,0)\pi(0,0,0)$, $\pi(1,1,0)\pi(-1,0,0)$, ...

$$\vec{p} = \frac{2\pi}{\xi L}\vec{n}$$

$$E_{\mathsf{lat}}^2 = E_{\mathsf{cm}}^2 + \left(\frac{2\pi}{\xi L}|\vec{n}|\right)^2$$

Useful to consider systems with $\vec{n} = (0,0,0),\ (1,0,0),\ (1,1,0),\ (1,1,1),\ (2,0,0)$



Less symmetry: More mixing of angular momentum!

Extracting a spectrum

Getting the ground state is useful, but we want to extract the whole spectrum in a finite volume.

Fitting subleading exponentials doesn't get very far:
With very precise data, sometimes a second state can be found.

A solution: The variational method.

$$C_{ij}(t)v_j^n = \lambda_n(t)C_{ij}(t_0)v_j^n$$

If more than one operator overlaps onto the same state represented by some eigenvector v_i^n the generalised eigenvalue problem can be solved and then as many states as operators may be extracted.

$$\lambda_n(t) \sim e^{-E_n(t-t_0)}$$

... a large basis of operators are needed

Operators and the variational method

$$C(t)v^n = \lambda_n(t)C(t_0)v^n$$
$$\lambda_n(t) \sim e^{-E_n(t-t_0)}$$

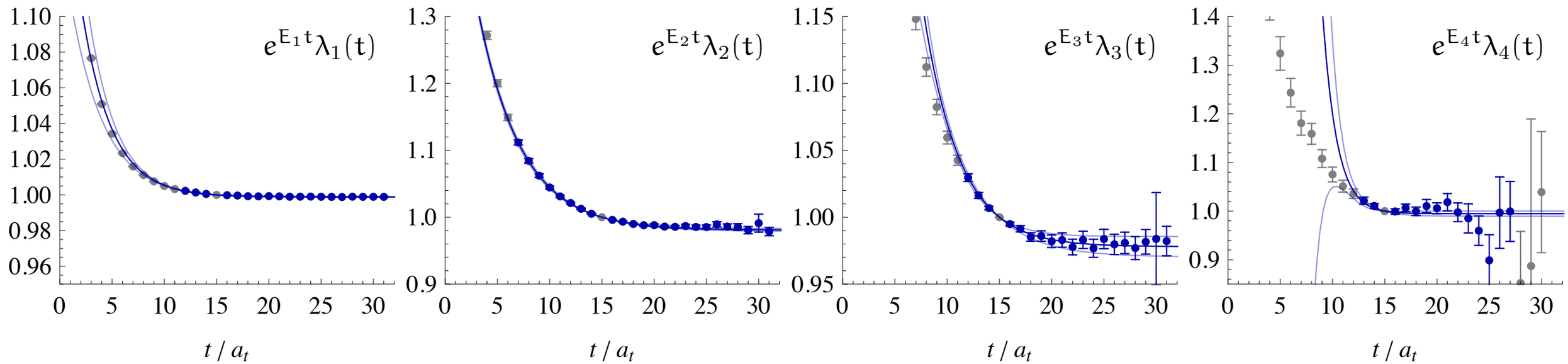
Use a large basis of operators

$$\mathcal{O}_i = \bar{\psi}\Gamma\psi$$
$$\mathcal{O}_i = \bar{\psi}\Gamma \overleftrightarrow{D} \ldots \overleftrightarrow{D}\psi$$

$$\Gamma_i = \{1, \gamma_0, \gamma_5, \gamma_0\gamma_5, \gamma_i, \gamma_0\gamma_i, \gamma_5\gamma_i, [\gamma_i, \gamma_j]\}$$

Use the variational method with a large correlation matrix to obtain an optimal spectrum.

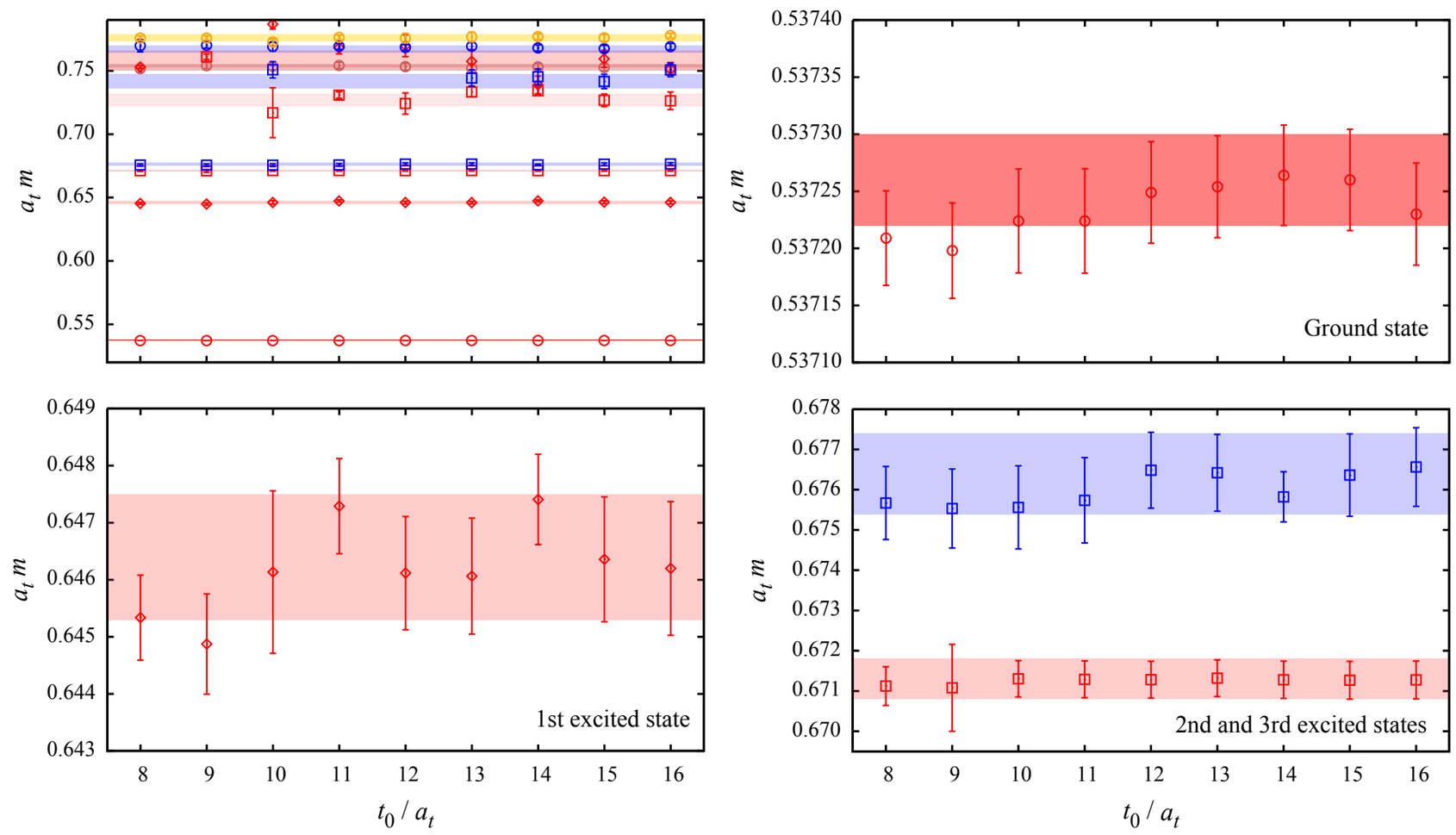
Operators and the variational method



T_1^- irrep ($\sim J^P = 1^-$) using $c\bar{c}$ operators

$$C(t)v^n = \lambda_n(t)C(t_0)v^n$$

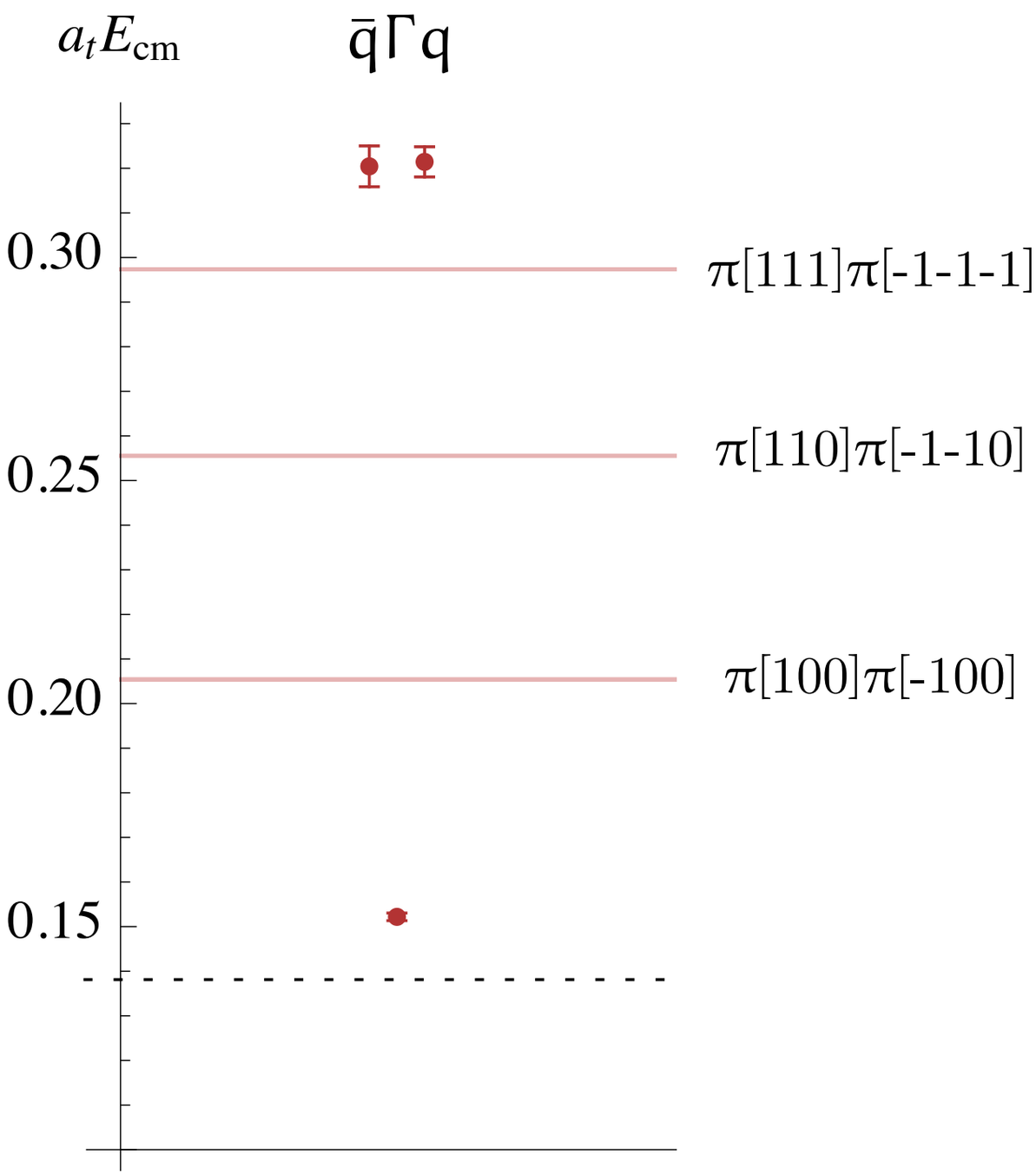
$$\lambda_n(t) \sim e^{-E_n(t-t_0)}$$



L. Liu *et al*

Meson-meson energy levels

T_1^- irrep: contains states with $J^P=1^-, 3^-, 4^-, \ldots$



The spectrum should also contain multiparticle states: $\pi(\vec{p}_1)\pi(\vec{p}_2)$

No continuum of energies: allowed momentum is quantised

Two meson states do not appear to overlap well onto the "single-particle" operators we have used.

Meson-meson energy levels

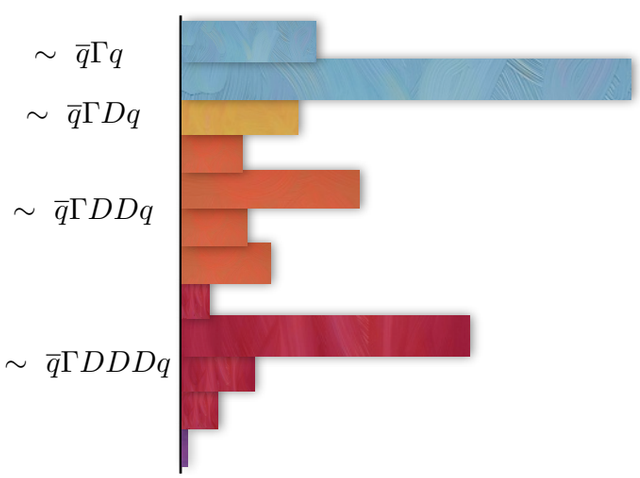
We could construct something simple to overlap on to two-pion states $\sim \bar{\psi}_q \gamma_5 \psi_q \cdot \bar{\psi}_q \gamma_5 \psi_q$

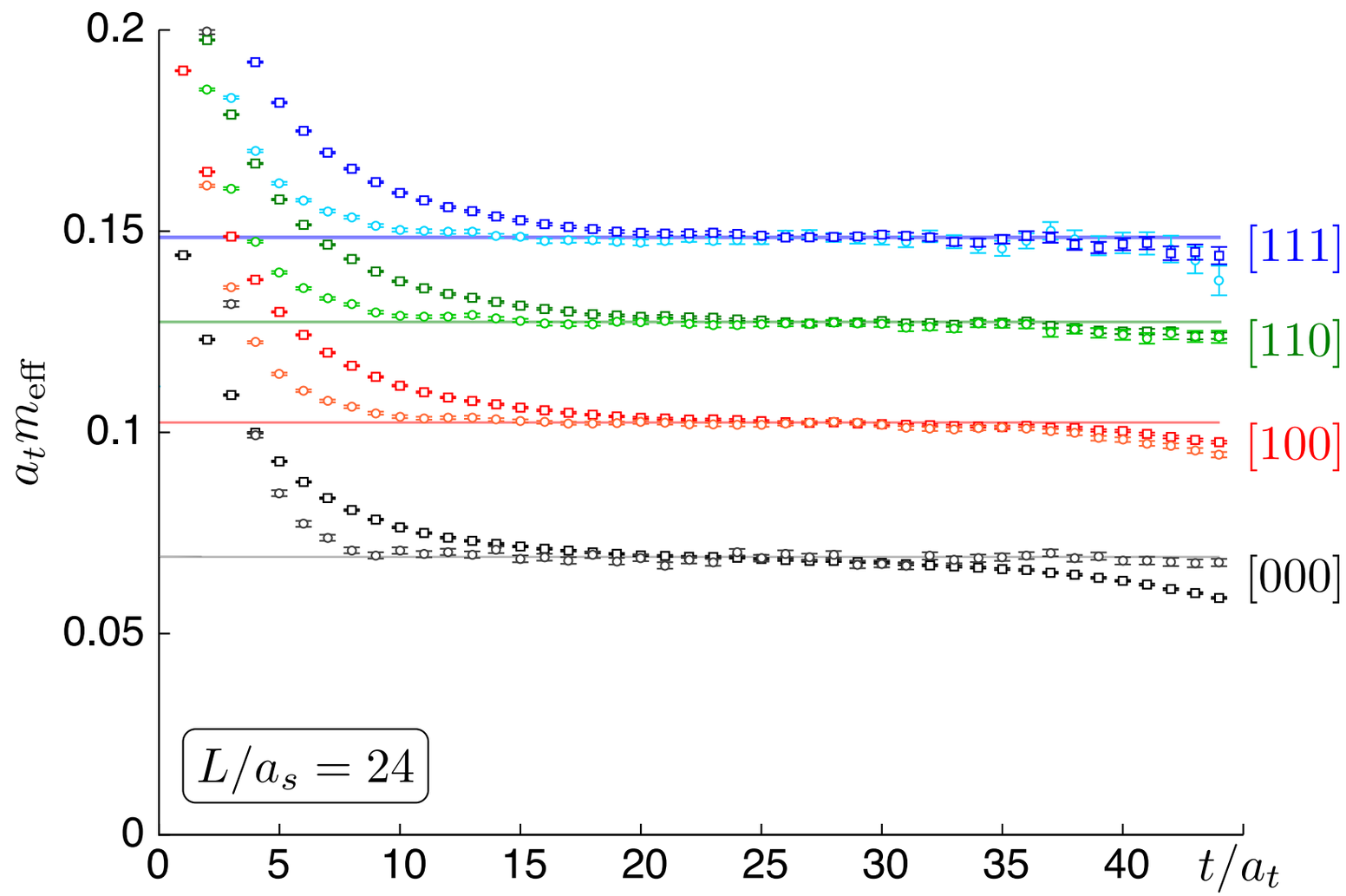
But it's better to make use of variational method solutions: $C(t)v^n = \lambda_n(t) C(t_0) v^n$

$$\lambda_n(t) \sim e^{-E_n(t-t_0)}$$

v^n represents the variationally-optimal pion,
so to create such a state we use:

$$\Omega_n^\dagger = \sum_i v_i^n \mathcal{O}_i^\dagger$$





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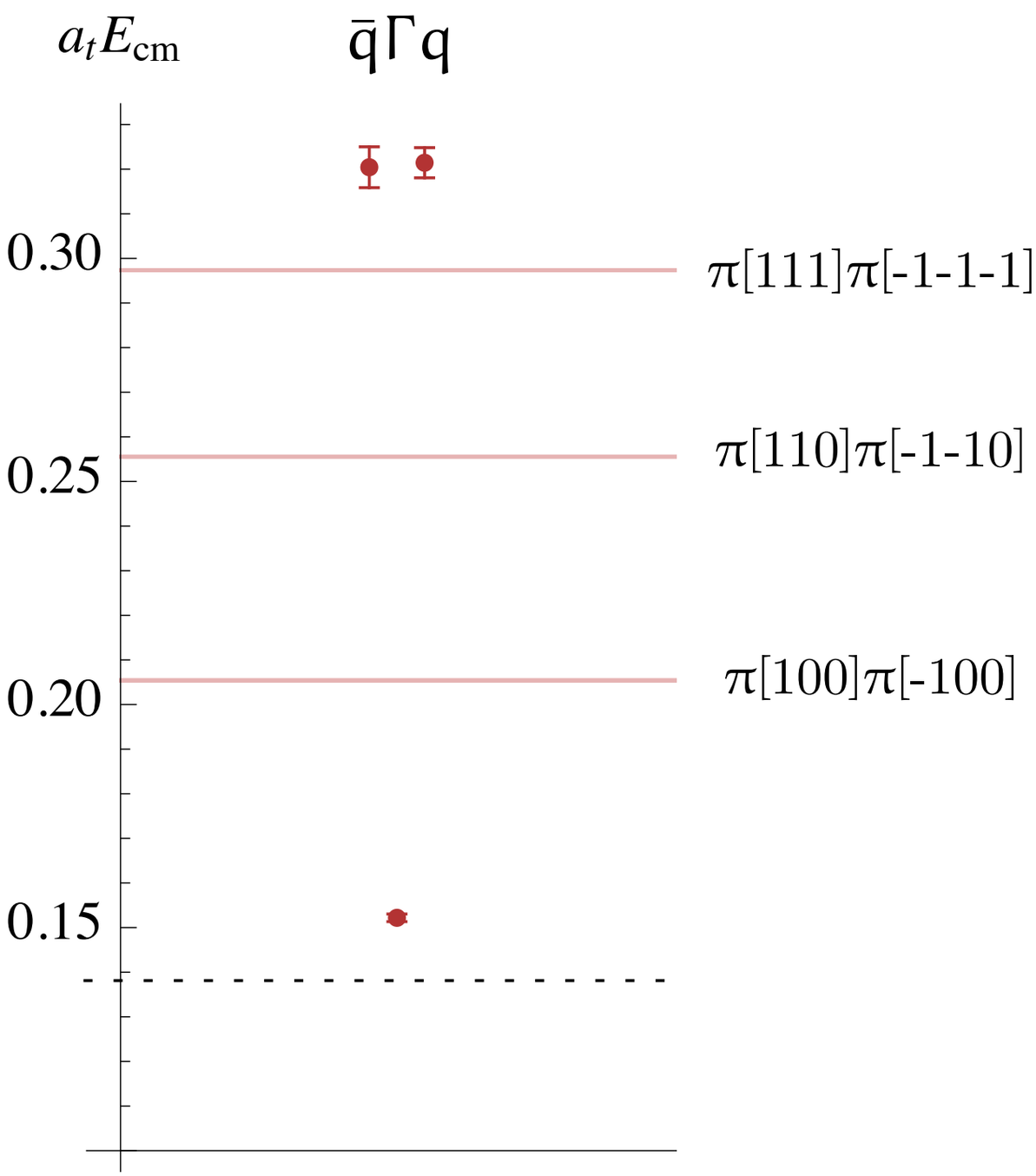
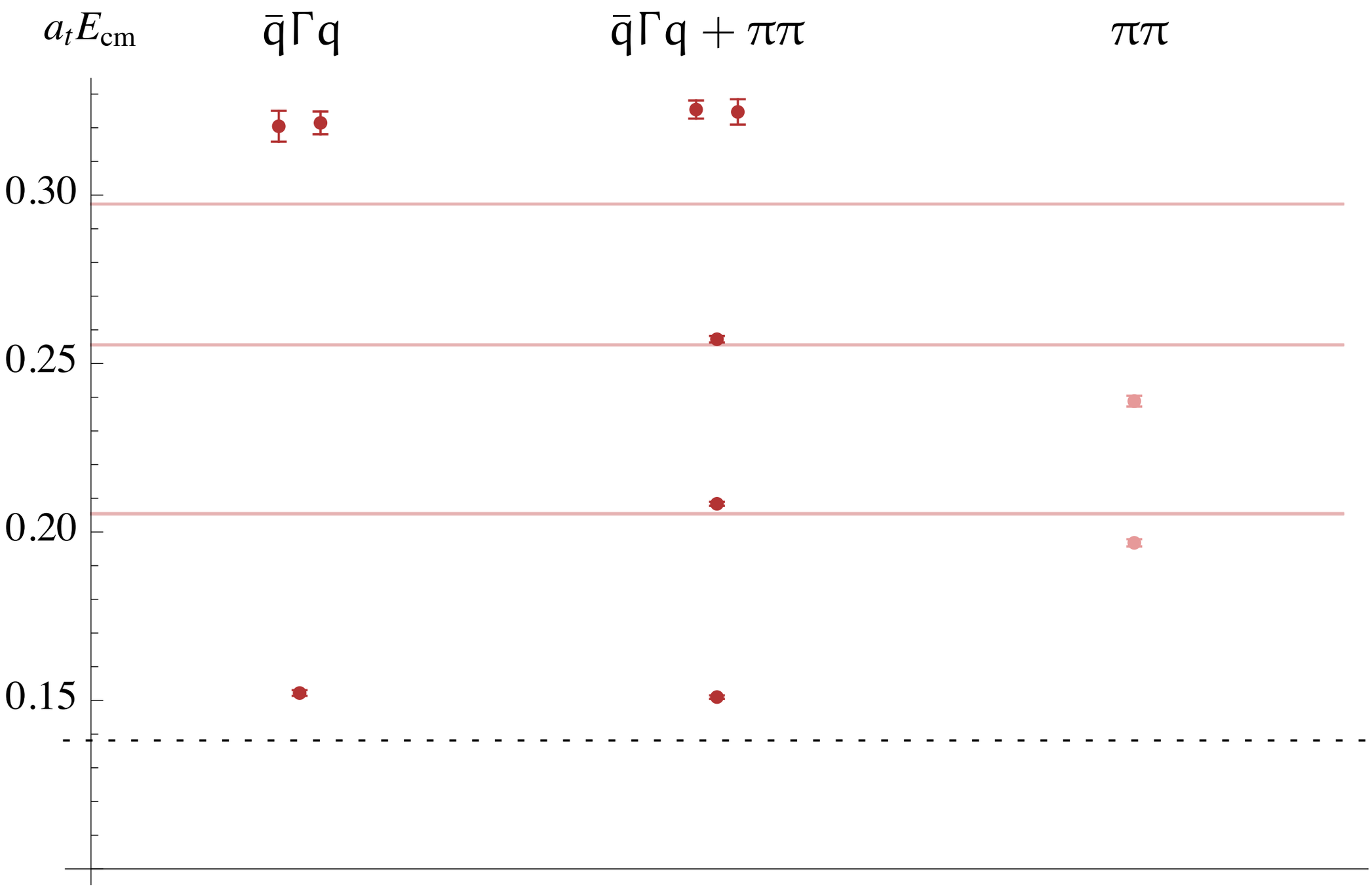


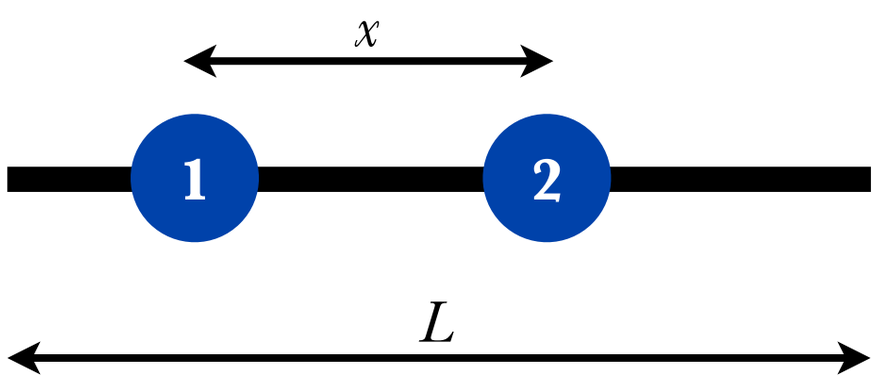
Meson-meson energy levels

T_1^- irrep

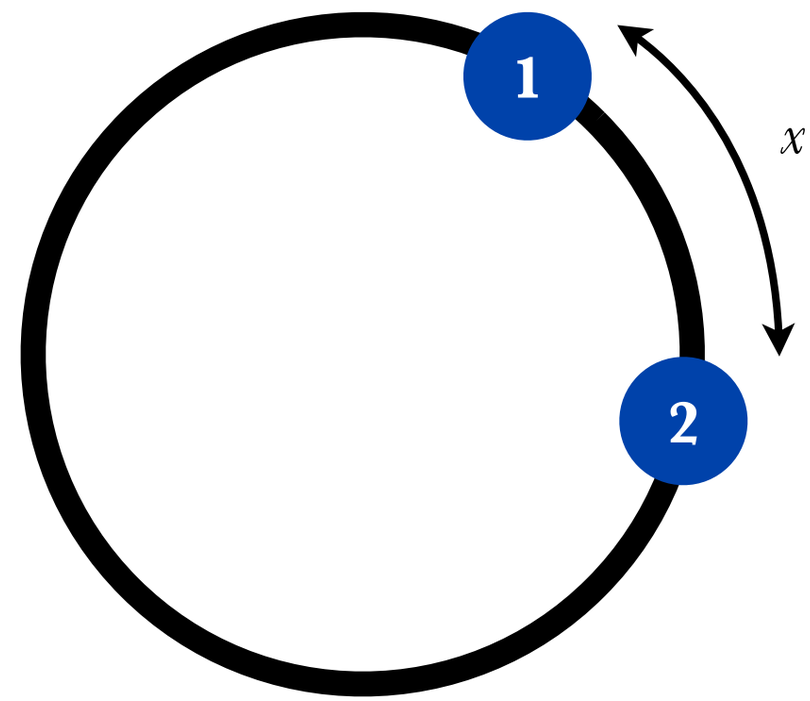


Two particles in a finite volume

Simple 1-d problem



Periodic B.C.'s



No interactions → total energy is just the sum

$$E = (\vec{p}_1^{\,2} + m_1^2)^{\frac{1}{2}} + (\vec{p}_2^{\,2} + m_2^2)^{\frac{1}{2}}$$

For a single particle:

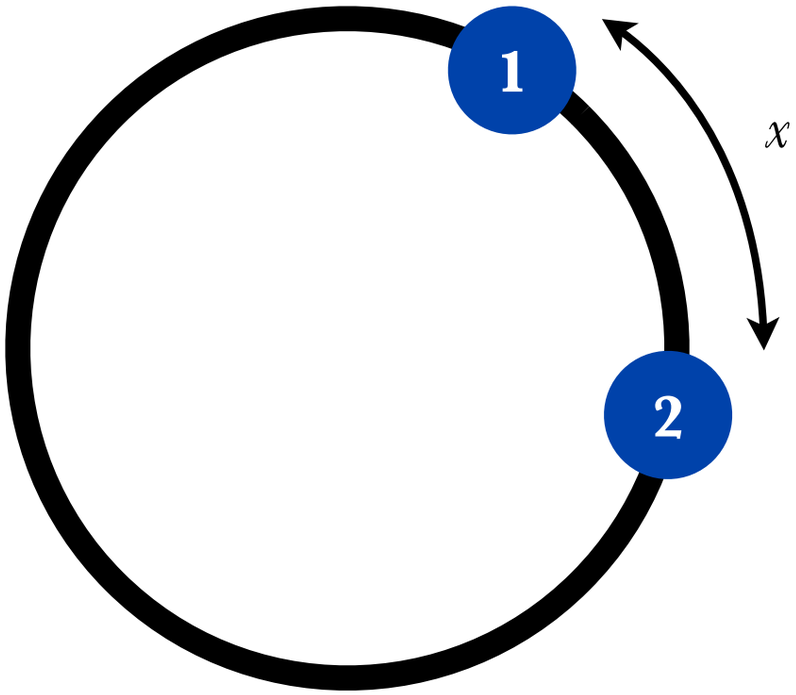
$$\vec{p}_i^{\,2} = \left(\frac{2\pi n}{L}\right)^2$$

Non-interacting energies in a finite volume are known from the single-particle analysis

If we measure the energies on the lattice and find a difference, this shift must be due to interactions.

Lüscher *et al*

Two particles in a finite volume



In simple QM: Interactions lead to phase shift δ on the wavefunction $\psi(x) \sim e^{\pm ipx}$

Periodic boundary conditions for interacting particles.

$$\psi(0) = \psi(L), \quad \left.\frac{\partial\psi}{\partial x}\right|_{x=0} = \left.\frac{\partial\psi}{\partial x}\right|_{x=L}$$

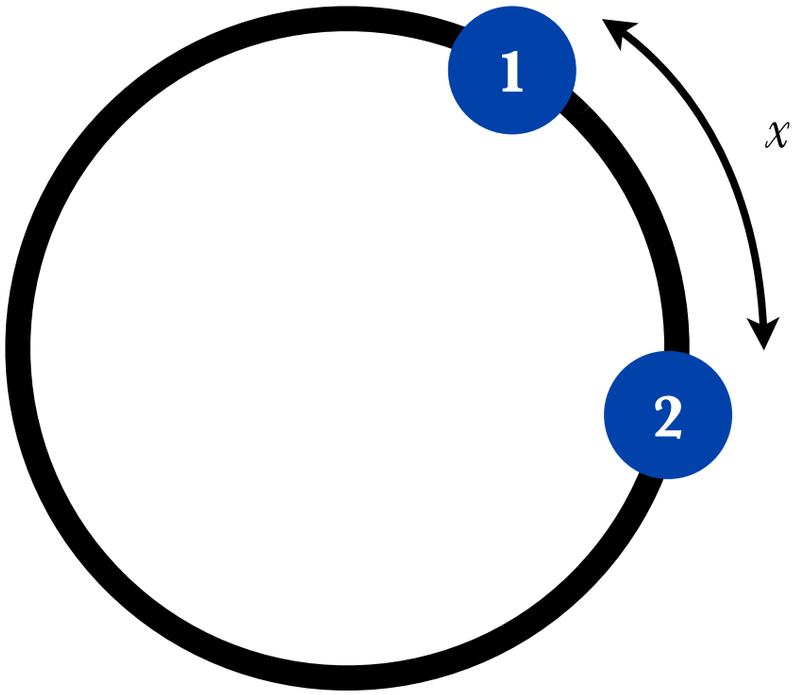
$$\sin\left(\frac{pL}{2} + \delta(p)\right) = 0$$

$$p = \frac{2\pi n}{L} - \frac{2}{L}\delta(p)$$

Discrete spectrum of allowed energies directly connected to the phase shift.

If we measure the energies on the lattice and find a difference, this shift must be due to interactions.

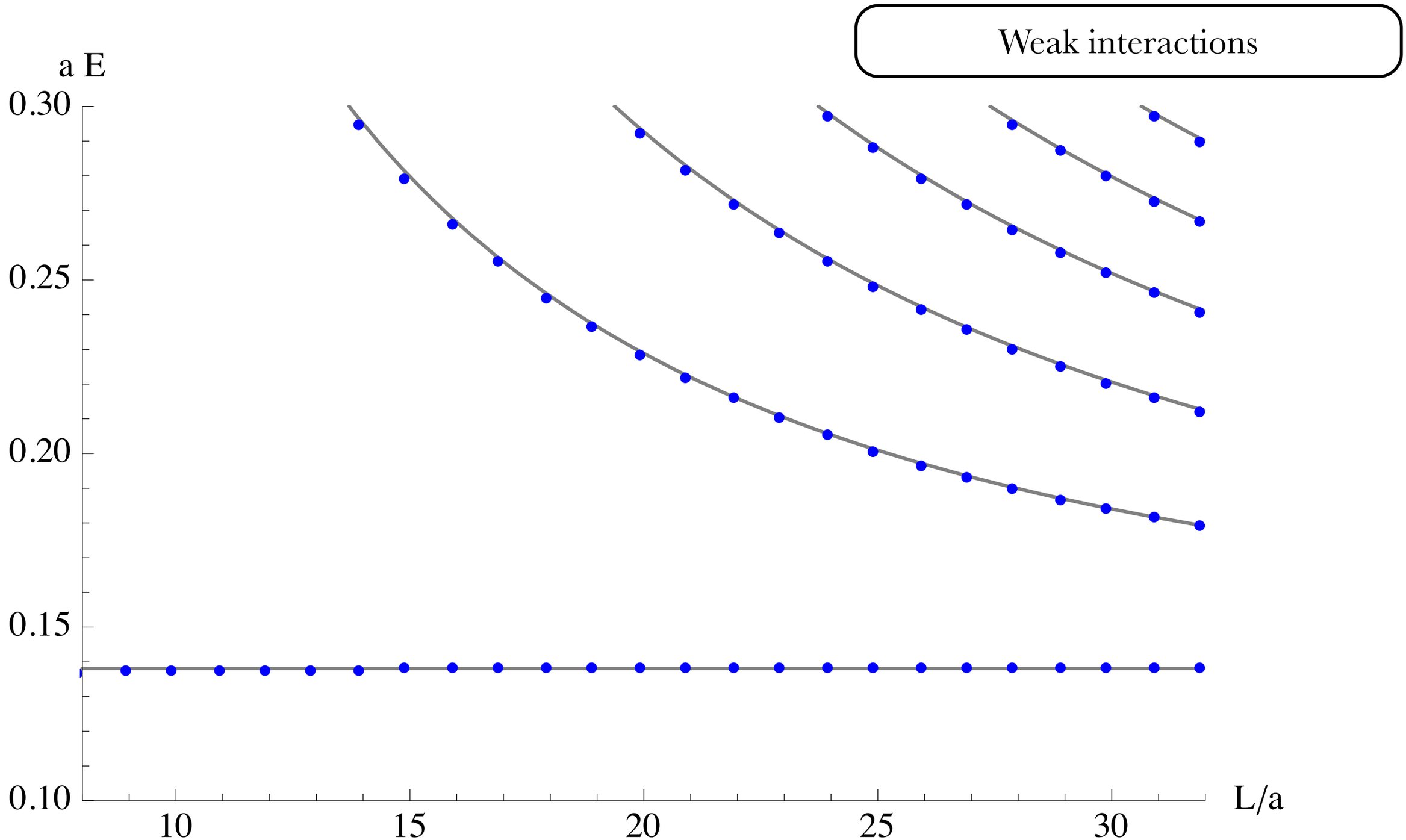
Two particles in a finite volume



In 3+1 dimensions, this leads to a simple relation between the finite volume energy and the S-wave scattering length:

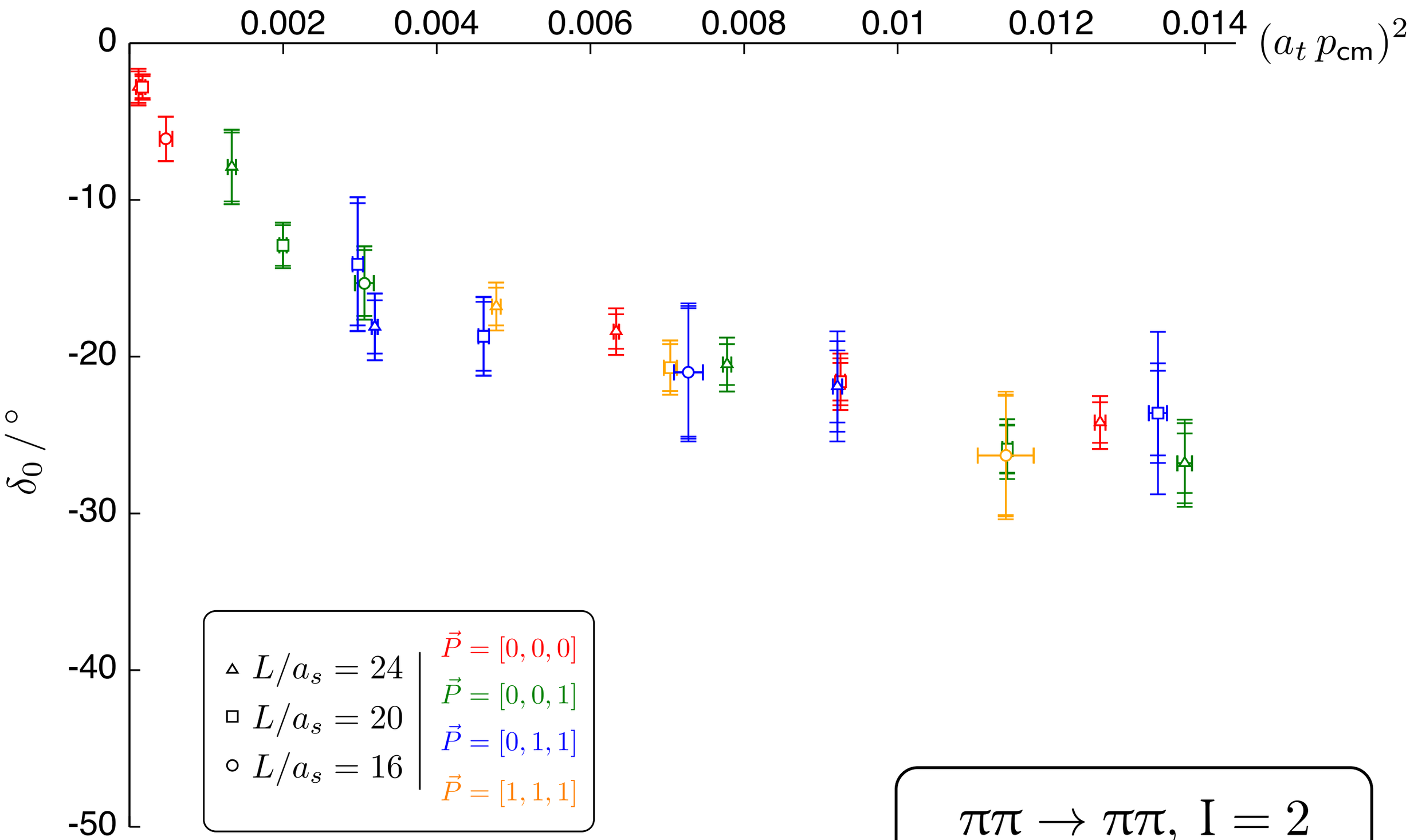
$$k \cot \delta = \frac{1}{a} + \frac{1}{2} r k^2 + \mathcal{O}(k^4)$$
$$= \frac{1}{\pi L} \sum_{\vec{n} \in \mathbb{Z}_3} \frac{1}{\left(|\vec{n}|^2 - \left(|\vec{k}| L/(2\pi) \right)^2 \right)}$$

Finite volume spectra



Small, +ve scattering length (weakly attractive)

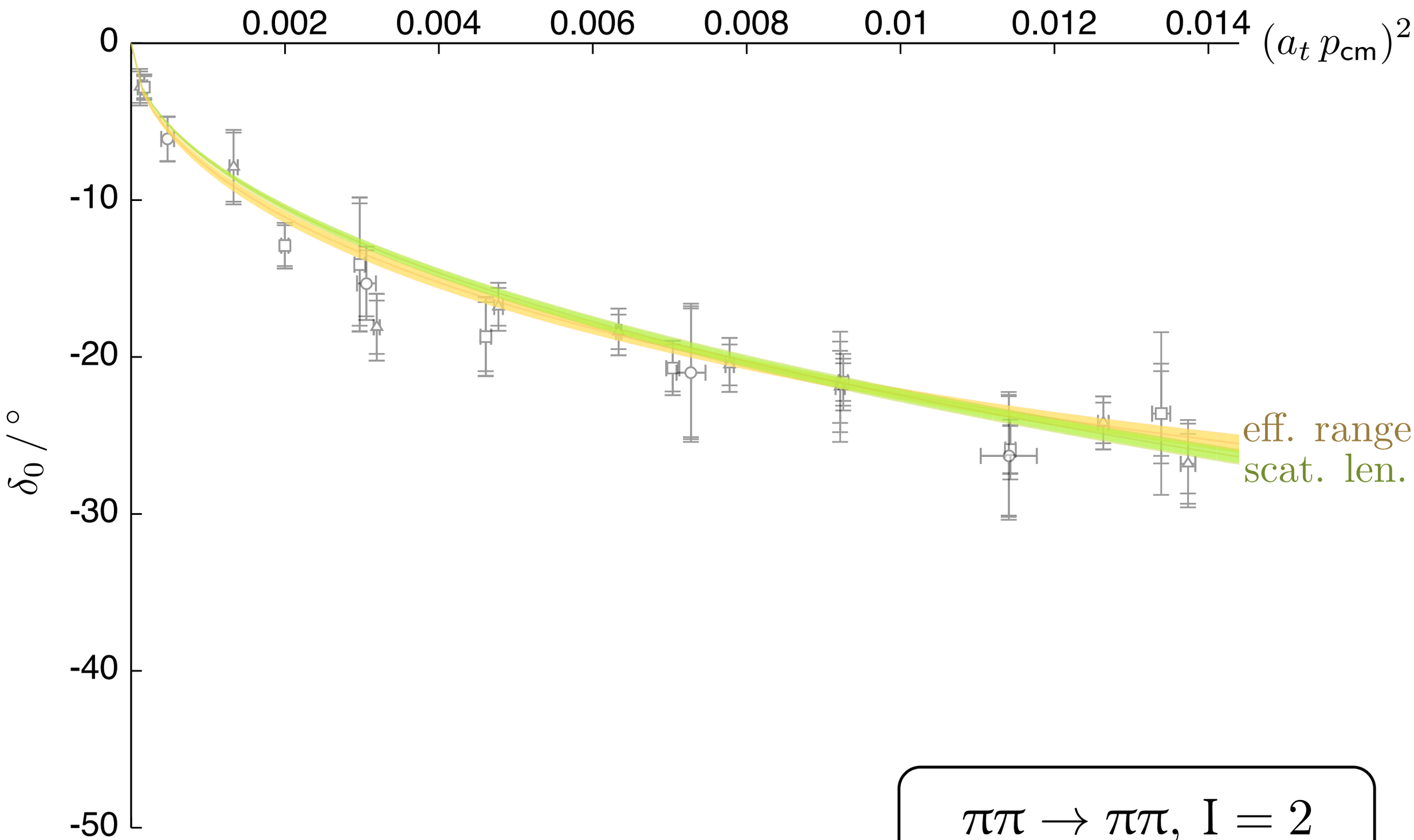
Weakly repulsive scattering from QCD



$$\pi\pi \to \pi\pi,\ \mathrm{I} = 2$$

Dudek, Edwards and Thomas

Weakly repulsive scattering from QCD

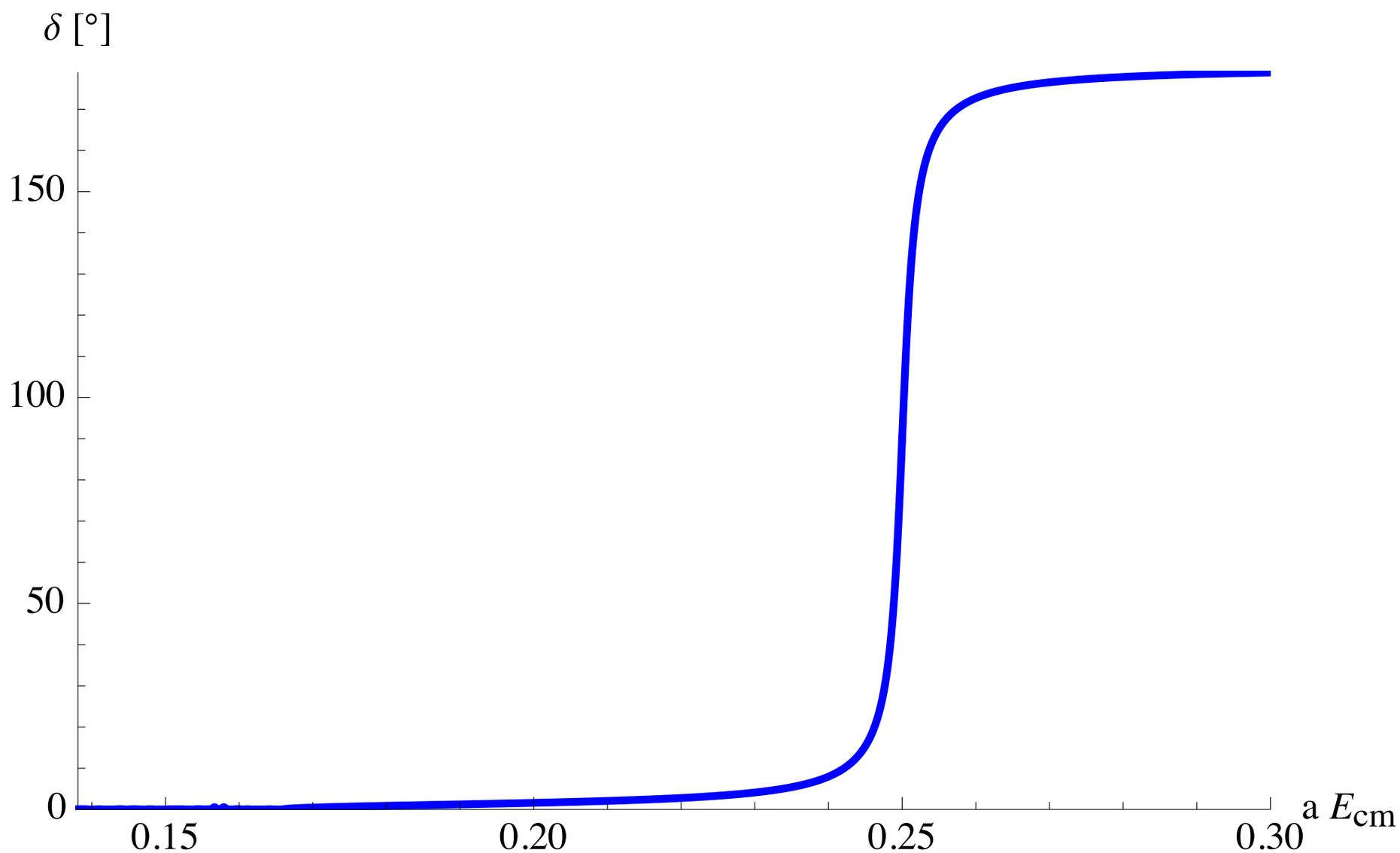


$$\pi\pi \to \pi\pi,\ \mathrm{I} = 2$$

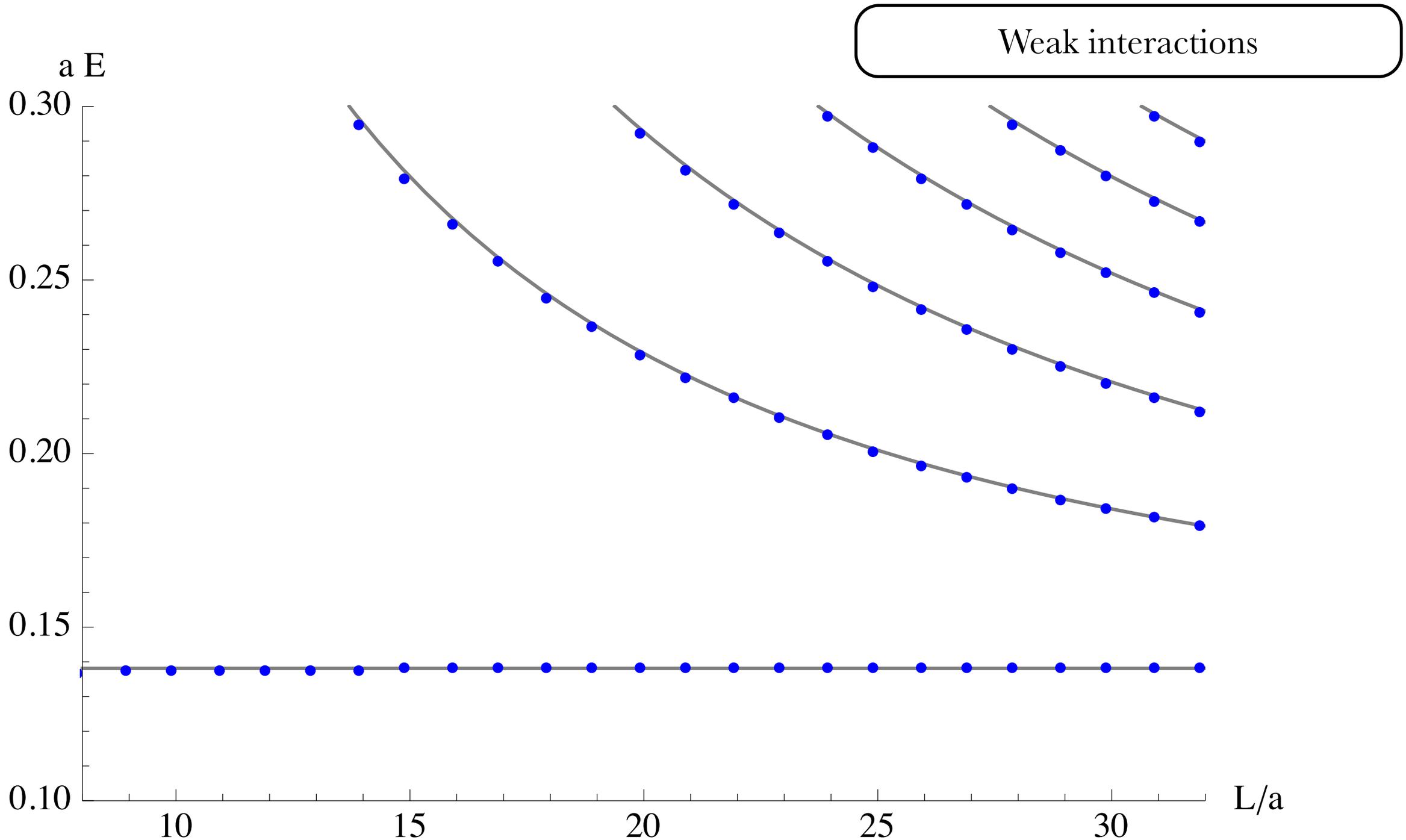
Resonances

$$t = \frac{1}{\rho(E^2)} \frac{E\Gamma(E^2)}{m_R^2 - E^2 - iE\Gamma(E^2)} \qquad \Gamma(s) = \frac{g_R^2}{6\pi} \frac{k_{cm}^3}{s}$$

$$k_{cm}^3 \cot \delta_1 = \frac{6\pi}{g^2} E \left(m^2 - E^2\right)$$



Finite volume spectra with a resonance



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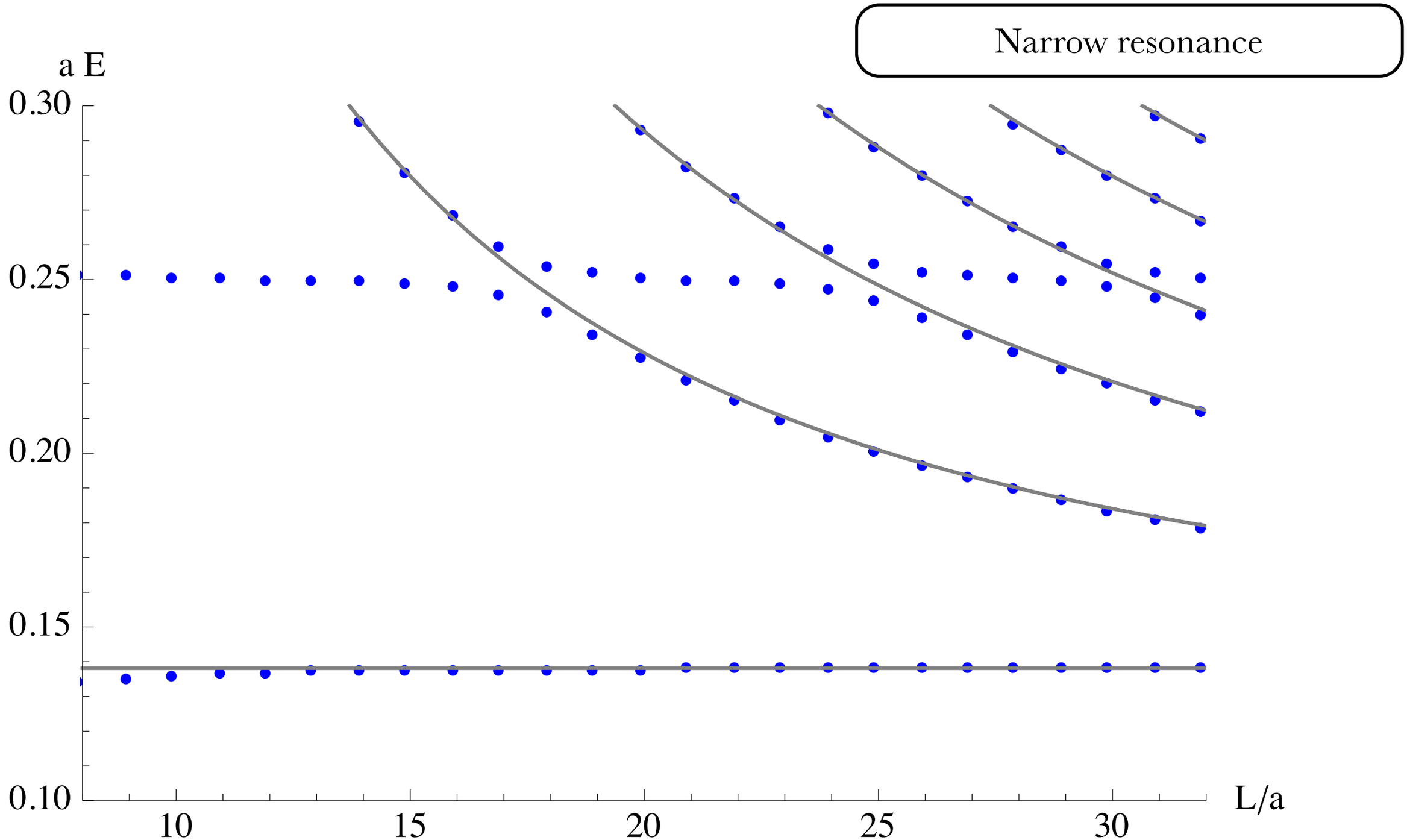

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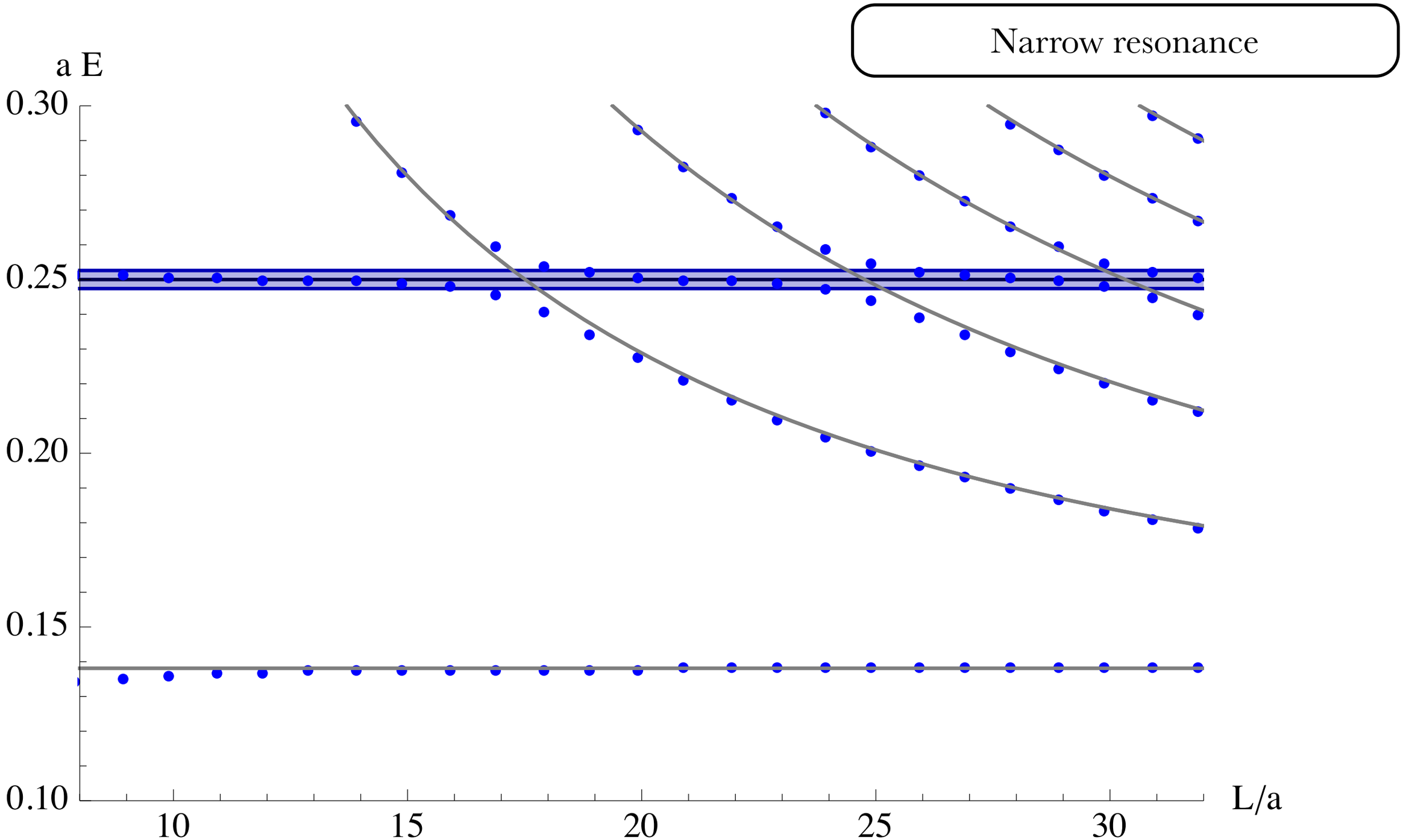


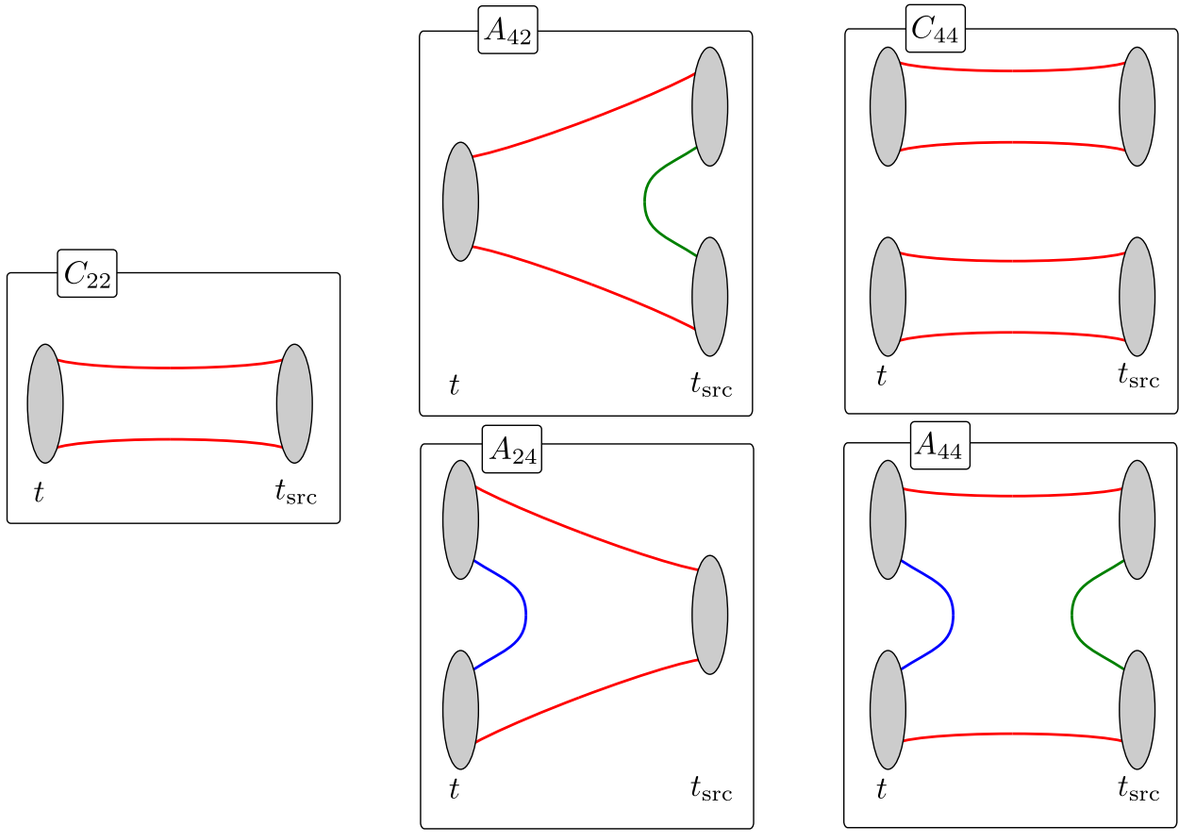
Extracting the ρ resonance

Several volumes: L=16, 20, 24.
Operators in several moving frames, upto n=(2,0,0).

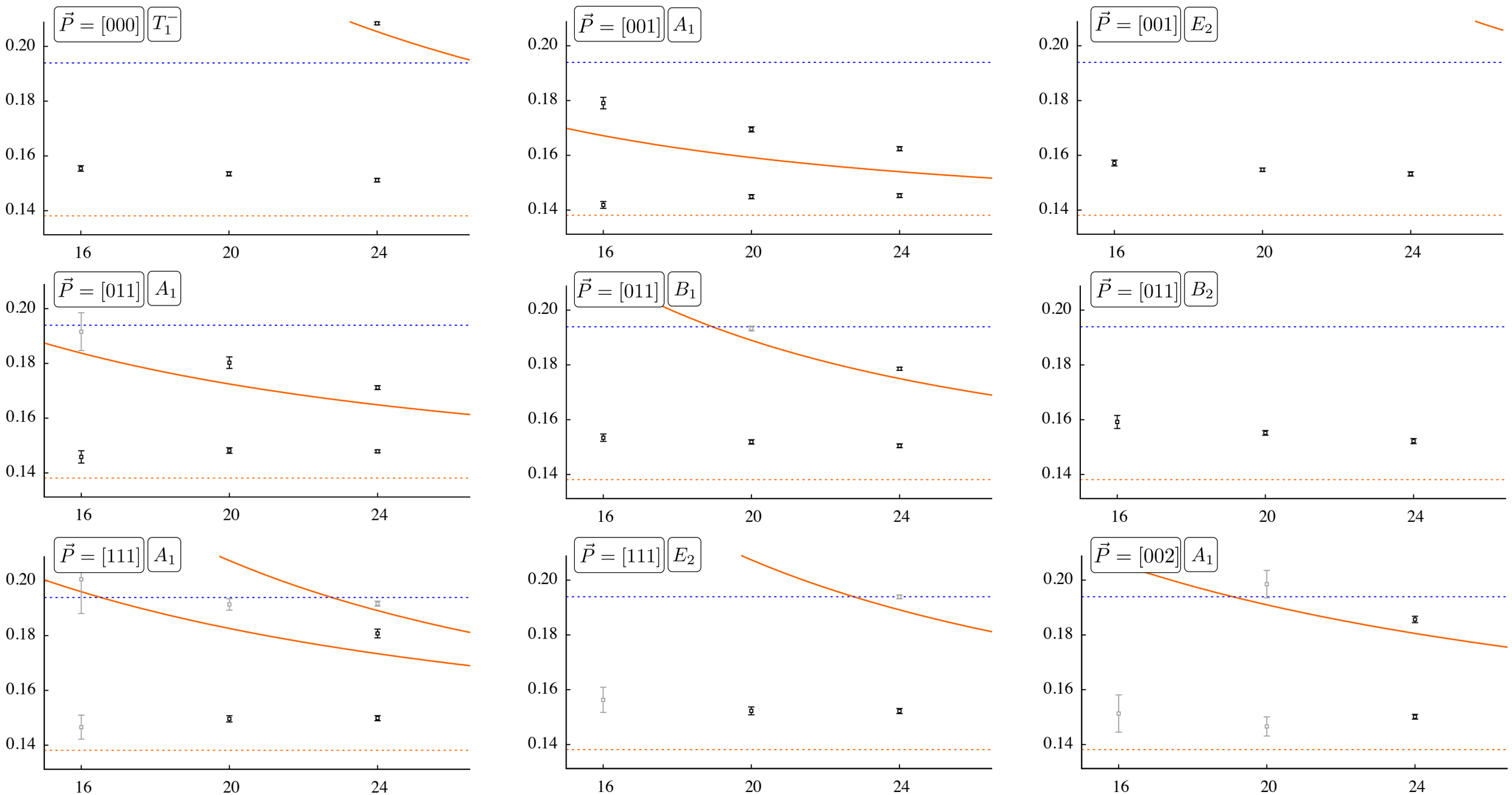
Anisotropic lattices:
temporal spacing 3.5 times finer for better energy resolution.

Combination of single particle and meson-meson operators.

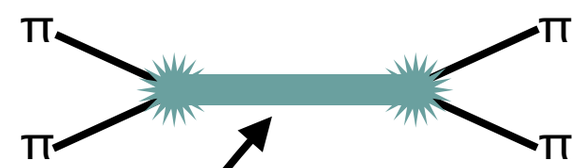
m_π=391 MeV



Finite volume spectra in I=1 J=1

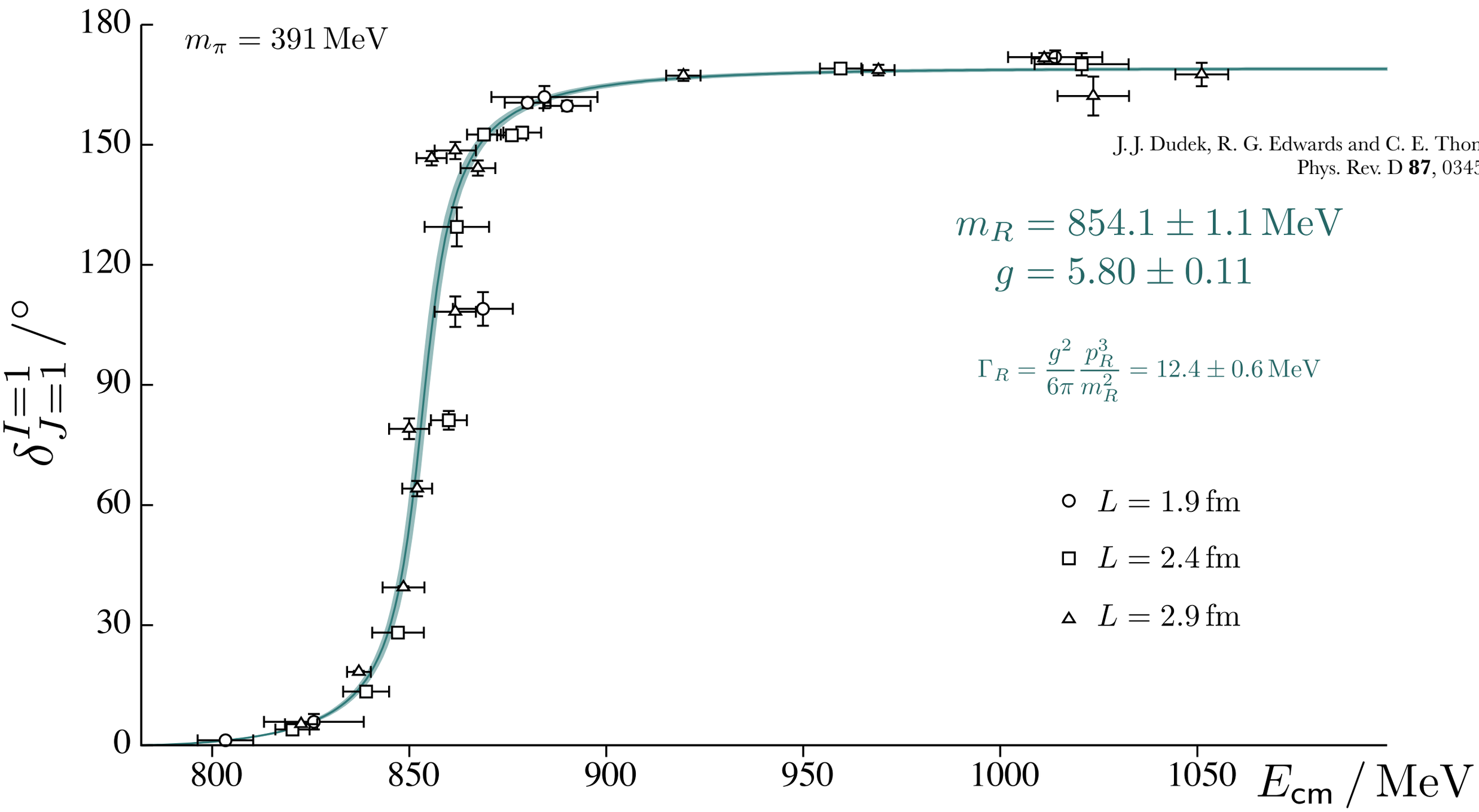


A resonance from QCD



$$\sim \frac{1}{\rho(s)} \frac{s^{\frac{1}{2}}\Gamma(s)}{m_R^2 - s - i s^{\frac{1}{2}}\Gamma(s)}$$

$$\Gamma(s) = \frac{g_R^2}{6\pi} \frac{k_{cm}^3}{E_{cm}^2}$$



J. J. Dudek, R. G. Edwards and C. E. Thomas
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Extensions

Coupled-channel scattering, eg: $\pi K \to \pi K, \pi K \to \eta K$
see Jo's talk next week.

Also $\pi\eta \to K\overline{K}$, $\pi\pi \to K\overline{K}$

Nucleons, eg: $N\pi \to N\pi$

Form factors, matrix elements, eg: $\gamma\pi \to \pi\pi$

$\gamma N \to N\pi$

Summary

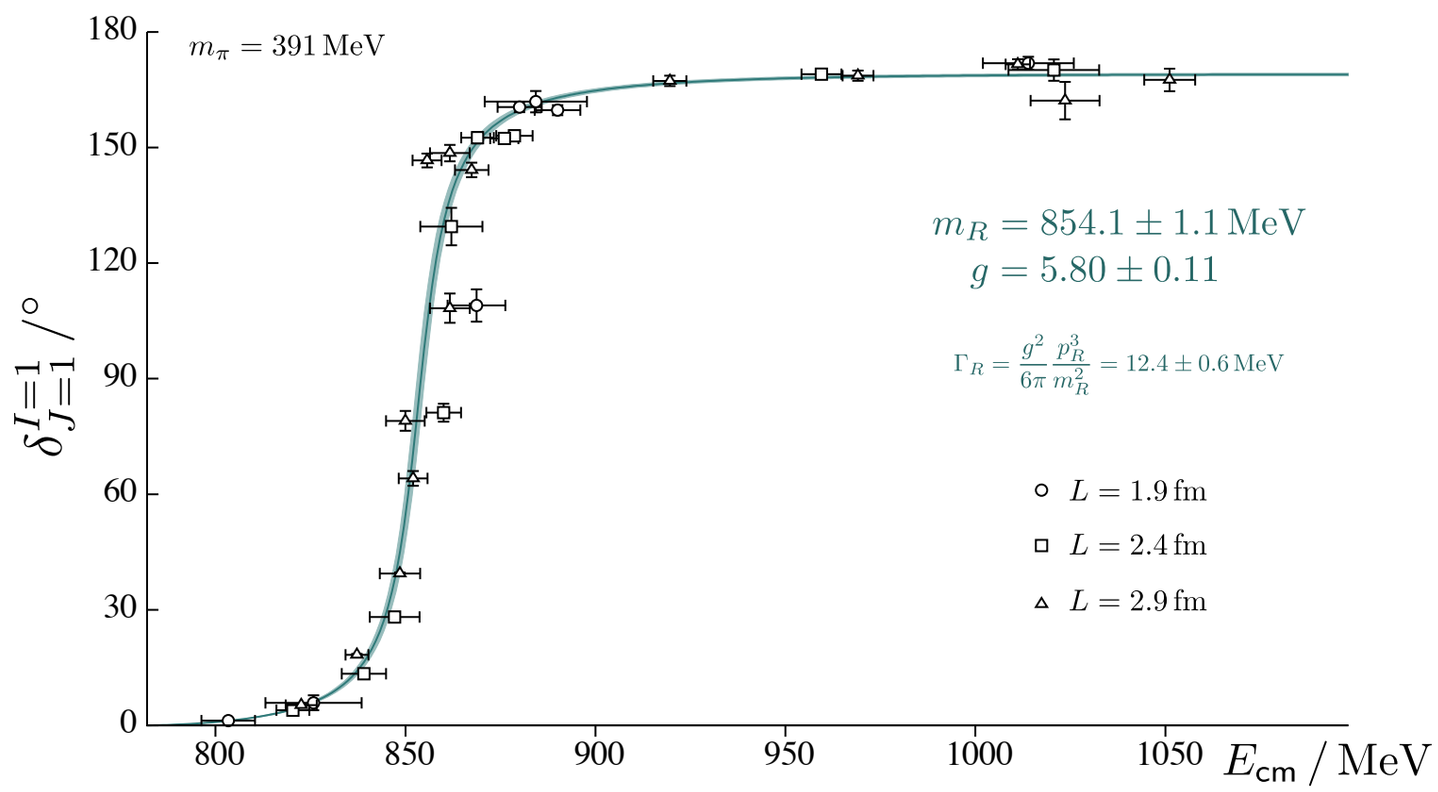
Using finite volume formalism of Lüscher and others, it is possible to translate finite volume energy levels into scattering amplitudes.

Scattering information, including resonances, can be obtained using lattice QCD.

A large basis of operators makes it possible, through the variational method, to obtain excited states with the same quantum numbers.

In order to extract energy levels that can mostly be attributed to meson-meson states we found it necessary to construct operators.

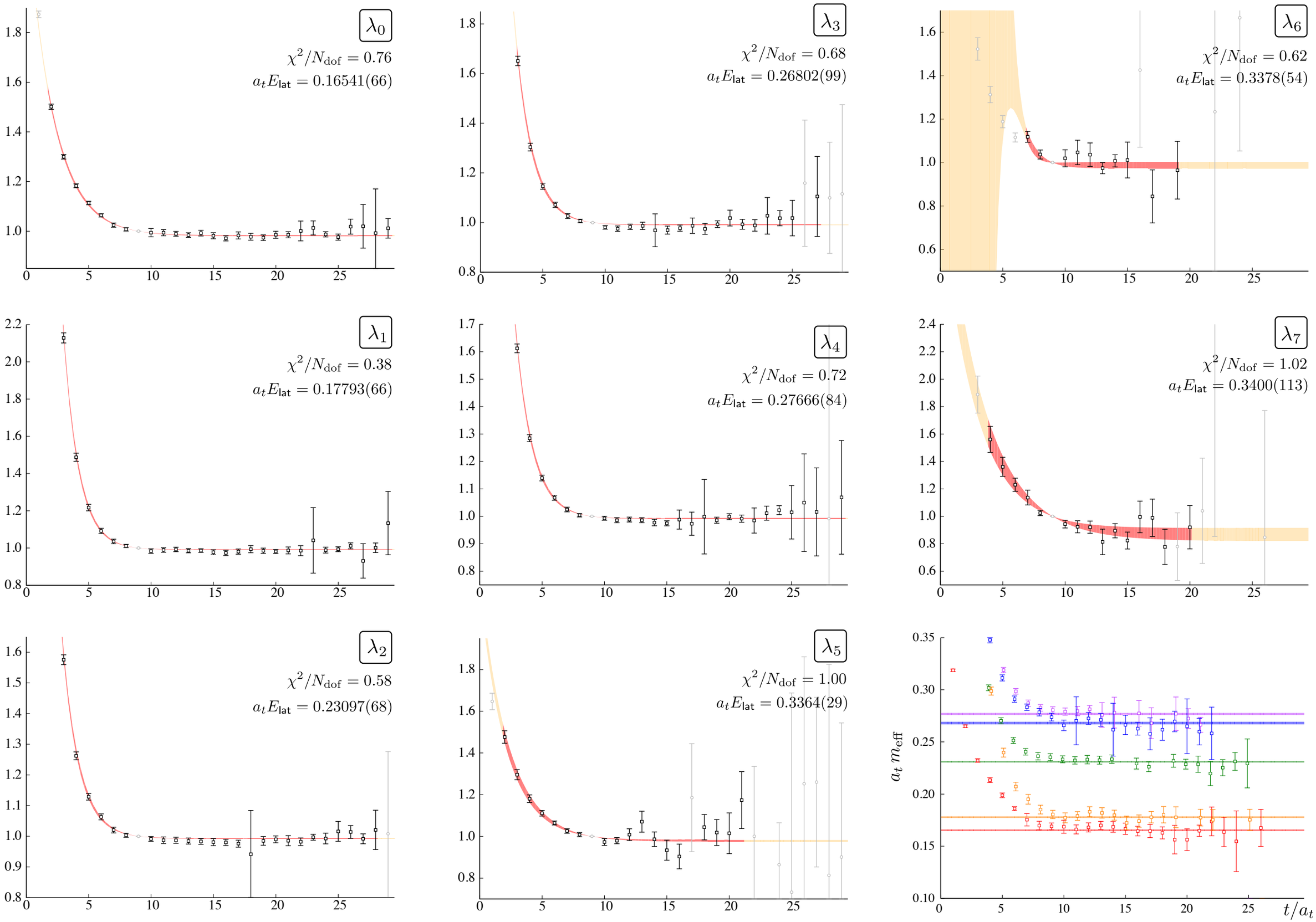
Many exciting opportunities for future extensions.



Backup slides

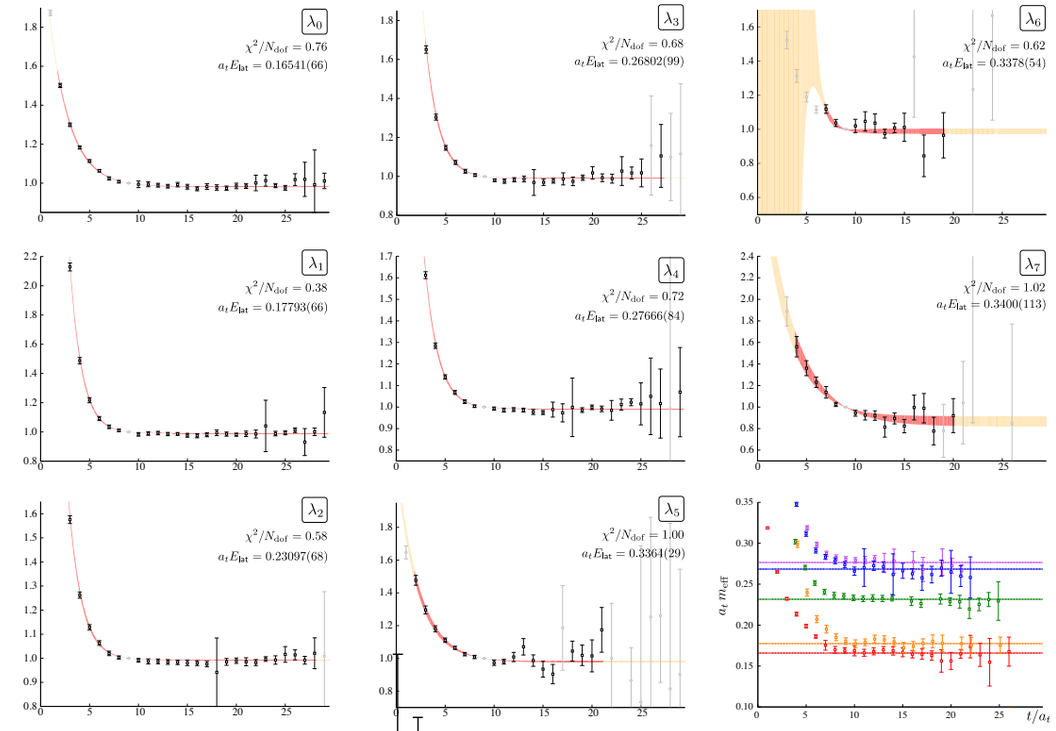
Principal correlators

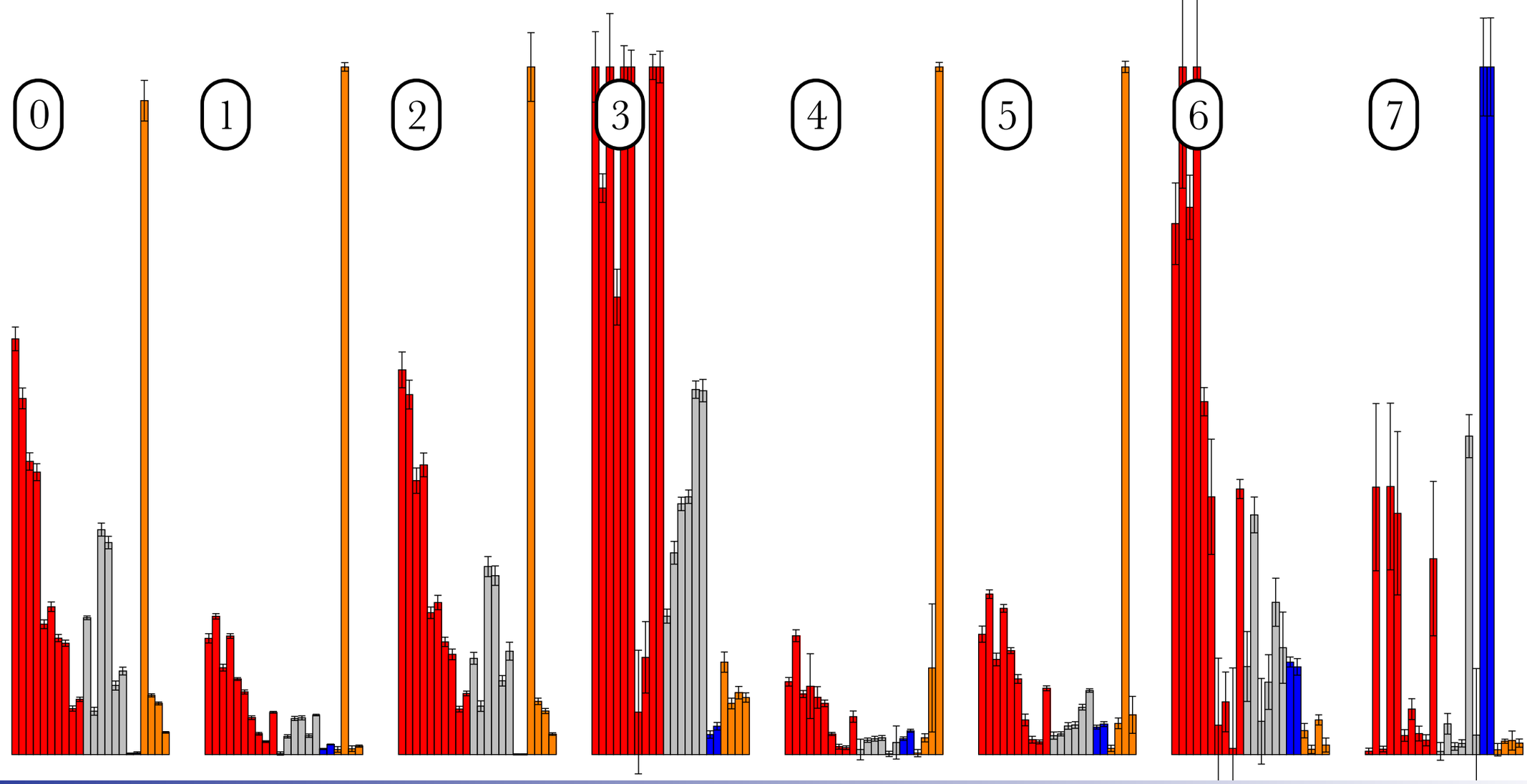
$$e^{E_{\mathbf{n}} t} \lambda_{\mathbf{n}}(t)$$

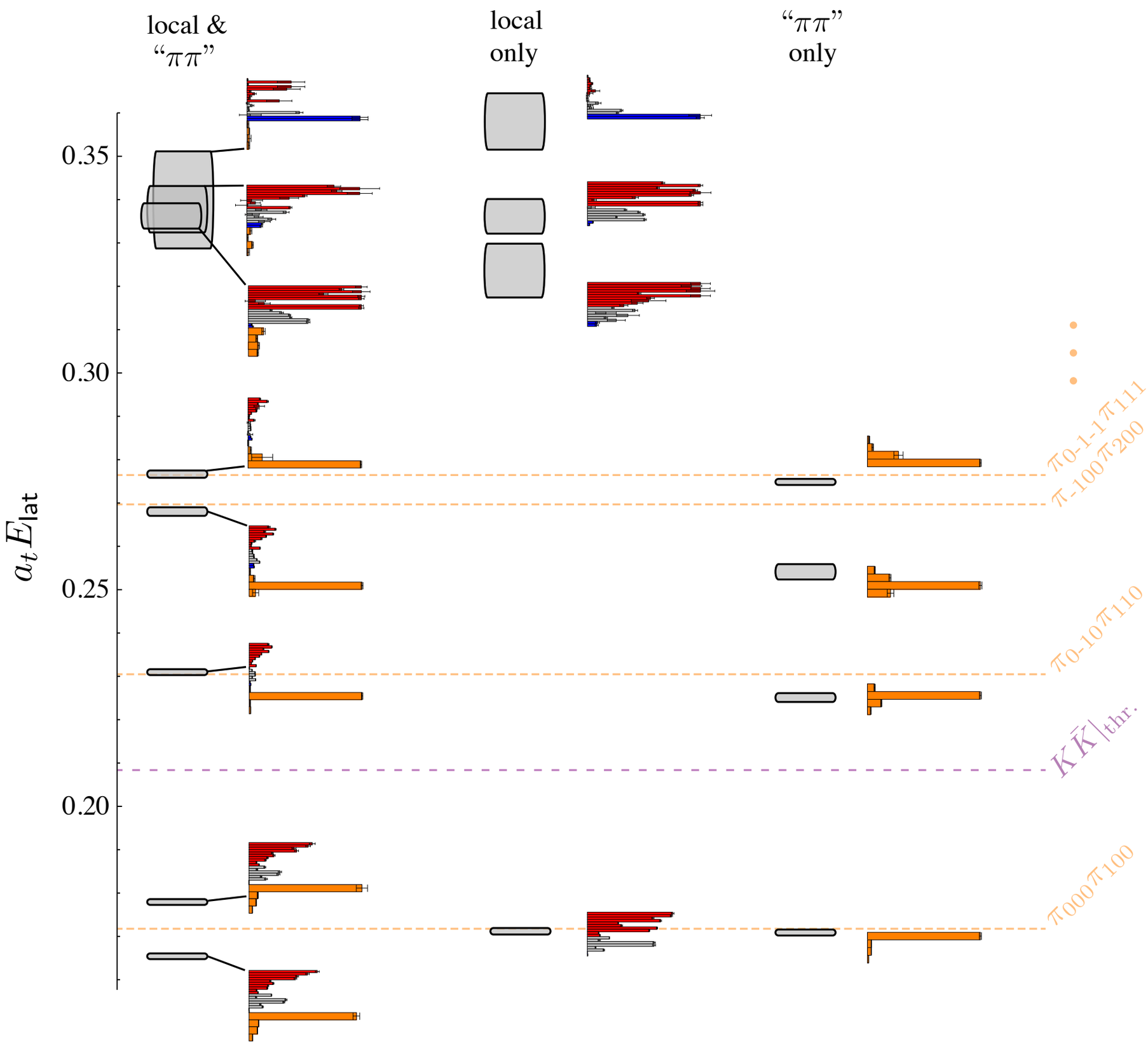


Relative operator overlaps

$$Z_i = \langle \mathbf{n} | \mathcal{O}_i | 0 \rangle$$






local &
"ππ"
local
only
"ππ"
only
0.35
0.30
0.25
0.20
$a_t E_{\mathrm{lat}}$
$\pi_{0\text{-}1\text{-}1}\pi_{111}$
$\pi_{\text{-}100}\pi_{200}$
$\pi_{0\text{-}10}\pi_{110}$
$K\bar{K}|_{\mathrm{thr.}}$
$\pi_{000}\pi_{100}$