Scattering amplitudes from lattice QCD

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Based on work in collaboration with J.J. Dudek, R.G. Edwards and C.E. Thomas.
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Two talks

Today:
Introductory stuff
The methods we are using
Elastic scattering

Next Week:
Jo Dudek
Resonances in coupled channel scattering from lattice QCD
Introduction

Lagrangian of QCD

\[ \mathcal{L}_{\text{QCD}} = \sum_q \bar{\psi}_q \left( i\mathcal{D} - m_q \right) \psi_q - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \]

\[ \mathcal{D} = \gamma^\mu (\partial_\mu - igA_\mu) \]

\[ \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g [A_\mu, A_\nu] \]

Coloured quark and gluon degrees of freedom.

- Excited state spectrum contains many interesting open questions.
- Interesting effects near thresholds: tetraquarks? meson-meson bound states? \( f_0(980) \) in \( \pi\pi \) scattering and new charmonium states, eg: \( Z(4430) \).
- Hybrid states, containing explicit gluonic degrees of freedom. Could be seen in new experiments, like Glue-X.

Spectrum of hadrons

Only colourless states, no asymptotic quarks and gluons.
Strong coupling

Lagrangian of QCD

\[ \mathcal{L}_{\text{QCD}} = \sum_q \bar{\psi}_q (i\slashed{D} - m_q) \psi_q - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \]

\[ \slashed{D} = \gamma^\mu (\partial_\mu - igA_\mu) \]

\[ \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g [A_\mu, A_\nu] \]

Many interesting consequences:
Confinement - no asymptotic quarks or gluons.
Dynamical chiral symmetry breaking.
Light physical pion ~ goldstone boson of the symmetry breaking.
Cannot use perturbation theory: Non-perturbative methods needed.

Several options including:
Models
Schwinger-Dyson+Bethe-Salpeter
Effective Field Theories
Lattice QCD
Path integrals

The starting point is the path integral:

\[ \langle x_b | e^{-iHT} | x_a \rangle = \int \mathcal{D}x(t) e^{iS[x(t)]} \]
Path integrals

To solve numerically, consider the discretised version

$$\langle \chi_b | e^{-iHT} | \chi_a \rangle = \Pi_{x_{t_i}} \int dx_{t_i} e^{iS(x_{t_i})}$$
Path integrals

Evaluate correlation functions from the path integral:

\[ \langle 0 | \mathcal{O}_i(t) \mathcal{O}_j^\dagger(0) | 0 \rangle = \frac{1}{Z_0} \int \mathcal{D} \bar{\Psi} \mathcal{D} \Psi \mathcal{D} A \, \mathcal{O}_i(t) \mathcal{O}_j^\dagger(0) \, e^{iS[\bar{\Psi}, \Psi, A]} \]

In order to deal with strong coupling: Solve the QCD path integral numerically.

Integrate over gauge field configurations: Infinitely many possibilities.

⇒ Store field values on a discrete set of points
Use a finite spacetime volume, $L^3 \times t$. ($L$ Roughly 2-3fm in these studies).

Use a finite number of points, with separation $a \sim 0.1$fm. ($L/a = 16, 20, 24$)

Quarks live on discrete points and the gluons live on the links between them.

Use periodic boundary conditions: Volume becomes a torus.
Use a finite spacetime volume, $L^3 \times t$. ($L$ Roughly 2-3fm in these studies).

Use a finite number of points, with separation $a \sim 0.1$fm ($L/a = 16, 20, 24$).

Change variables to Euclidean spacetime to simplify integration.

Quarks live on discrete points and the gluons live on the links between them.

Use periodic boundary conditions: Volume becomes a torus.
Correlation functions

Evaluate correlation functions from the path integral:

\[ C_{ij}(t) = \langle 0 | \mathcal{O}_i(t) \mathcal{O}_j^\dagger(0) | 0 \rangle \]

\[ = \frac{1}{Z_0} \int \mathcal{D} \bar{\psi} \mathcal{D} \psi \mathcal{D} A \ \mathcal{O}_i(t) \mathcal{O}_j^\dagger(0) \ e^{-S[\bar{\psi}, \psi, A]} \]

Leads to the ground state energy for large \( t \):

\[ C_{ij}(t) = \sum_n \frac{1}{2E_n} \langle 0 | \mathcal{O}_i | n \rangle \langle n | \mathcal{O}_j^\dagger | 0 \rangle \ e^{-E_n t} \]

\[ = \frac{Z_i^* Z_j}{2E_n} e^{-E_n t} \]

The symmetries of the operators dictate which states can be extracted:

\[ \bar{\psi} \Gamma \psi \]

\[ \Gamma \]

\[ \gamma_5 \sim \pi \quad J^P = 0^- \]

\[ \gamma_i \sim \rho \quad J^P = 1^- \]
Symmetry on the lattice

The lattice has a cubic symmetry.
It does not have the O(3) symmetry of continuous space.

Eg: 2D QM

Continuous rotational spatial symmetry

\[ e^{i\phi} \rightarrow e^{i\phi + i\alpha} \]  \[ e^{i\phi} \rightarrow e^{i\phi + in\pi/2} \]

Only symmetric at discrete angles
Symmetry on the lattice

Continuous rotational spatial symmetry

- \( e^{i\phi} \rightarrow e^{i\phi + i\alpha} \)
- \( e^{i\phi} \rightarrow e^{i\phi + in\pi/2} \)

Only symmetric at discrete angles

Cubic symmetry groups mix the continuum angular momentum:

<table>
<thead>
<tr>
<th>Irrep</th>
<th>( J^P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1^+ )</td>
<td>0(^+), 4(^+), ...</td>
</tr>
<tr>
<td>( T_1^- )</td>
<td>1(^-), 3(^-), ...</td>
</tr>
</tbody>
</table>

\( e^{i\phi} \rightarrow e^{i\phi + in\pi/2} \)
Operators with overall momentum

Because momentum is quantised, different energies can be accessed by considering operators with an overall momentum

\[ \vec{p} = \frac{2\pi}{\xi L} \vec{n} \]

\[ E_{\text{lat}}^2 = E_{\text{cm}}^2 + \left( \frac{2\pi}{\xi L} |\vec{n}| \right)^2 \]

Useful to consider systems with \( \vec{n} = (0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1), (2, 0, 0) \)

Overall zero momentum: \( \pi(0, 0, 0)\pi(0, 0, 0), \pi(1, 0, 0)\pi(-1, 0, 0), \ldots \)
One unit: \( \pi(1, 0, 0)\pi(0, 0, 0), \pi(1, 1, 0)\pi(-1, 0, 0), \ldots \)

Less symmetry: More mixing of angular momentum!
Extracting a spectrum

Getting the ground state is useful, but we want to extract the whole spectrum in a finite volume.

Fitting subleading exponentials doesn’t get very far:
With very precise data, sometimes a second state can be found.

A solution: The variational method.

\[ C_{ij}(t)\nu_j^n = \lambda_n(t)C_{ij}(t_0)\nu_j^n \]

If more than one operator overlaps onto the same state represented by some eigenvector \( \nu_i^n \) the generalised eigenvalue problem can be solved and then as many states as operators may be extracted.

\[ \lambda_n(t) \sim e^{-E_n(t-t_0)} \]

... a large basis of operators are needed
Operators and the variational method

\[ C(t)\psi^n = \lambda_n(t)C(t_0)\psi^n \]
\[ \lambda_n(t) \sim e^{-E_n(t-t_0)} \]

Use a large basis of operators

\[ \mathcal{O}_i = \bar{\psi} \Gamma \psi \]
\[ \mathcal{O}_i = \bar{\psi} \Gamma \overset{\downarrow}{D} \ldots \overset{\downarrow}{D} \psi \]

\[ \Gamma_i = \{ 1, \gamma_0, \gamma_5, \gamma_0\gamma_5, \gamma_i, \gamma_0\gamma_i, \gamma_5\gamma_i, [\gamma_i, \gamma_j] \} \]

Use the variational method with a large correlation matrix to obtain an optimal spectrum.
Operators and the variational method

\[ e^{E_1 t} \lambda_1(t) \]

\[ e^{E_2 t} \lambda_2(t) \]

\[ e^{E_3 t} \lambda_3(t) \]

\[ e^{E_4 t} \lambda_4(t) \]

\[ T_1^- \text{ irrep (} J^P = 1^- \text{) using } c\bar{c} \text{ operators} \]

\[ C(t) \nu^n = \lambda_n(t) C(t_0) \nu^n \]

\[ \lambda_n(t) \approx e^{-E_n(t-t_0)} \]

L. Liu et al.
Meson-meson energy levels

$T_1^-$ irrep: contains states with $J^P = 1^-, 3^-, 4^-, \ldots$

The spectrum should also contain multiparticle states: $\pi(\bar{p}_1)\pi(\bar{p}_2)$

No continuum of energies: allowed momentum is quantised

Two meson states do not appear to overlap well onto the “single-particle” operators we have used.
Meson-meson energy levels

We could construct something simple to overlap on to two-pion states \( \sim \bar{\psi} q \gamma_5 \psi q \cdot \bar{\psi} q \gamma_5 \psi q \)

But it’s better to make use of variational method solutions:

\[
C(t)\nu^n = \lambda_n(t)C(t_0)\nu^n \\
\lambda_n(t) \sim e^{-E_n(t-t_0)}
\]

\(\nu^n\) represents the variationally-optimal pion, so to create such a state we use:

\[
\Omega_n^\dagger = \sum_i \nu^n_i \Omega_i^\dagger
\]
Meson-meson energy levels

$T_1^-$ irrep: contains states with $J^P = 1^-, 3^-, 4^-, ...$

$a_t E_{\text{cm}} \quad \bar{q} \Gamma q$

- $0.30 \quad \pi[111] \pi[-1-1-1]$
- $0.25 \quad \pi[110] \pi[-1-10]$
- $0.20 \quad \pi[100] \pi[-100]$
- $0.15 \quad \vdots$

\[ \pi[100] \pi[-100] \]
Meson-meson energy levels

$T_1$-irrep

$a_t E_{cm}$  |  $\bar{q}\Gamma q$  |  $\bar{q}\Gamma q + \pi\pi$  |  $\pi\pi$
--- | --- | --- | ---
0.15 | | | |
0.20 | | | |
0.25 | | | |
0.30 | | | |
Two particles in a finite volume

Simple 1-d problem

No interactions $\rightarrow$ total energy is just the sum

$$E = (\vec{p}_1^2 + m_1^2)^{\frac{1}{2}} + (\vec{p}_2^2 + m_2^2)^{\frac{1}{2}}$$

For a single particle:

$$\vec{p}_i^2 = \left(\frac{2\pi n}{L}\right)^2$$

Non-interacting energies in a finite volume are known from the single-particle analysis

If we measure the energies on the lattice and find a difference, this shift must be due to interactions.

Lüscher et al
Two particles in a finite volume

In simple QM: Interactions lead to phase shift \( \delta \) on the wavefunction \( \psi(x) \sim e^{\pm ipx} \)

Periodic boundary conditions for interacting particles.

\[
\psi(0) = \psi(L), \quad \frac{\partial \psi}{\partial x} \bigg|_{x=0} = \frac{\partial \psi}{\partial x} \bigg|_{x=L}
\]

\[
\sin \left( \frac{pL}{2} + \delta(p) \right) = 0
\]

\[
p = \frac{2\pi n}{L} - \frac{2}{L} \delta(p)
\]

Discrete spectrum of allowed energies directly connected to the phase shift.

If we measure the energies on the lattice and find a difference, this shift must be due to interactions.
Two particles in a finite volume

In 3+1 dimensions, this leads to a simple relation between the finite volume energy and the S-wave scattering length:

\[ k \cot \delta = \frac{1}{a} + \frac{1}{2} r k^2 + \mathcal{O}(k^4) \]

\[ = \frac{1}{\pi L} \sum_{\vec{n} \in \mathbb{Z}_3} \frac{1}{\left( |\vec{n}|^2 - \left( |\vec{k}|L/(2\pi) \right)^2 \right)^2} \]
Finite volume spectra

Weak interactions

Small, +ve scattering length (weakly attractive)
Weakly repulsive scattering from QCD

Coupled-channel scattering from lattice QCD

\[ \pi\pi \rightarrow \pi\pi, \ I = 2 \]

\[ (a_t \rho_{cm})^2 \]

\[ \delta_0 \]°

\[ L/\alpha_s = 24 \]
\[ L/\alpha_s = 20 \]
\[ L/\alpha_s = 16 \]

\[ \bar{P} = [0, 0, 0] \]
\[ \bar{P} = [0, 0, 1] \]
\[ \bar{P} = [1, 1, 1] \]

Dudek, Edwards and Thomas
Weakly repulsive scattering from QCD

\[ \pi\pi \rightarrow \pi\pi, \ I = 2 \]
Resonances

\[ t = \frac{1}{\rho(E^2)} \frac{\Gamma(E^2)}{m_R^2 - E^2 - i\Gamma(E^2)} \quad \Gamma(s) = \frac{g_R^2}{6\pi} \frac{k_{cm}^3}{s} \]

\[ k_{cm}^3 \cot \delta_1 = \frac{6\pi}{g^2} E (m^2 - E^2) \]
Finite volume spectra with a resonance

Weak interactions
Finite volume spectra with a resonance

Narrow resonance

Coupled-channel scattering from lattice QCD

Monday, 20 October 14
Finite volume spectra with a resonance

Narrow resonance

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Coupled-channel scattering from lattice QCD
Monday, 20 October 14
Extracting the $\rho$ resonance

Several volumes: $L=16, 20, 24$.
Operators in several moving frames, upto $n=(2,0,0)$. 

Anisotropic lattices:
temporal spacing 3.5 times finer for better energy resolution.

Combination of single particle and meson-meson operators.

$m_{\pi}=391\text{ MeV}$
Finite volume spectra in I=1 J=1

\[ \bar{\rho} = [000] T_1 \]

\[ \bar{\rho} = [001] A_1 \]

\[ \bar{\rho} = [001] A_2 \]

\[ \bar{\rho} = [001] E_2 \]

\[ \bar{\rho} = [011] A_1 \]

\[ \bar{\rho} = [011] B_1 \]

\[ \bar{\rho} = [011] B_2 \]

\[ \bar{\rho} = [111] A_1 \]

\[ \bar{\rho} = [111] E_2 \]

\[ \bar{\rho} = [002] A_1 \]
A resonance from QCD

\[ \Gamma(s) = \frac{g_R^2 k_{cm}^3}{6\pi E_{cm}^2} \]

\[ m_R = 854.1 \pm 1.1 \text{ MeV} \]
\[ g = 5.80 \pm 0.11 \]
\[ \Gamma_R = \frac{g^2 p_R^3}{6\pi m_R^2} = 12.4 \pm 0.6 \text{ MeV} \]

- \( \circ \) \( L = 1.9 \text{ fm} \)
- \( \square \) \( L = 2.4 \text{ fm} \)
- \( \triangle \) \( L = 2.9 \text{ fm} \)
Extensions

Coupled-channel scattering, eg: \( \pi K \rightarrow \pi K, \pi K \rightarrow \eta K \)
see Jo’s talk next week.

Also \( \pi \eta \rightarrow K\bar{K}, \pi \pi \rightarrow K\bar{K} \)

Nucleons, eg: \( N\pi \rightarrow N\pi \)

Form factors, matrix elements, eg: \( \gamma\pi \rightarrow \pi\pi \)
\( \gamma N \rightarrow N\pi \)
Summary

Using finite volume formalism of Lüscher and others, it is possible to translate finite volume energy levels into scattering amplitudes.

Scattering information, including resonances, can be obtained using lattice QCD.

A large basis of operators makes it possible, through the variational method, to obtain excited states with the same quantum numbers.

In order to extract energy levels that can mostly be attributed to meson-meson states we found it necessary to construct operators.

Many exciting opportunities for future extensions.
Principal correlators

$e^{E_n t} \lambda_n(t)$

$\chi^2 / N_{dof} = 0.76$
$\alpha_t E_{lat} = 0.16541(66)$

$\chi^2 / N_{dof} = 0.68$
$\alpha_t E_{lat} = 0.26802(99)$

$\chi^2 / N_{dof} = 0.62$
$\alpha_t E_{lat} = 0.3378(54)$

$\chi^2 / N_{dof} = 0.38$
$\alpha_t E_{lat} = 0.17793(66)$

$\chi^2 / N_{dof} = 0.72$
$\alpha_t E_{lat} = 0.27666(84)$

$\chi^2 / N_{dof} = 1.02$
$\alpha_t E_{lat} = 0.3400(113)$

$\chi^2 / N_{dof} = 0.58$
$\alpha_t E_{lat} = 0.23097(88)$

$\chi^2 / N_{dof} = 1.00$
$\alpha_t E_{lat} = 0.3364(29)$

David Wilson

Coupled-channel scattering from lattice QCD

Monday, 20 October 14
Relative operator overlaps

\[ Z_i = \langle n | O_i | 0 \rangle \]
Coupled-channel scattering from lattice QCD