Effective actions for SU(3) gauge theories and mean-field solutions at finite density

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Except near $\mu = 0$, it is mostly conjecture. The location and nature of phase transitions in the QCD phase diagram involves non-perturbative physics, but our most reliable tool, lattice Monte Carlo simulations, doesn’t seem applicable at large finite densities.
In order to vary particle density, we need to introduce a chemical potential $\mu$ into the fermion action.

When the quark fields are integrated out, each flavor contributes a factor

$$\det(\mathcal{D} + m + \mu \gamma_0)$$

But

$$\gamma_5(\mathcal{D} + m + \mu \gamma_0)\gamma_5 = \mathcal{D}^\dagger + m - \mu \gamma_0 = (\mathcal{D} + m - \mu^* \gamma_0)^\dagger$$

Take the determinant of both sides

$$\det(\mathcal{D} + m + \mu \gamma_0) = \det^*(\mathcal{D} + m - \mu^* \gamma_0)$$

So the determinant factors are only real if $\mu = 0$, or purely imaginary, or if there are two flavors with $\mu$ of opposite sign (isospin chemical potential).
Using $\det(M) = \exp \text{Tr} \log(M)$, the fermion determinant becomes part of the action (the starting point for hybrid Monte Carlo), but for $\mu \neq 0$ the action is complex.

A toy 0-dimensional example:

$$Z(\lambda) = \int_{-\infty}^{\infty} dx \ e^{-x^2 + i\lambda x}$$

Straightforward importance sampling is impossible!
What to do?

There are several “direct” approaches to the sign problem in QCD, which are under development:

- **Reweighting + cumulant expansion** *(WHOT collaboration)*
  - Treat the imaginary part of the action as an observable, rather than as part of the Boltzman weight.

- **Stochastic quantization** *(Aarts, Seiler, Sexty, Stamatescu...)*
  - Complexify the field variables and apply the Langevin equation.

- **Lefschetz thimbles** *(Cristoforetti, Di Renzo, Mukherjee, Scorzato)*
  - Shift functional integration contours into the complex plane.

In this talk I will discuss an “indirect” approach: Map the gauge-matter theory onto a much simpler theory — a **Polyakov line action (or “SU(3) spin”) model** — where the sign problem has already been solved by a number of different techniques.
Effective Polyakov Line Action

Start with lattice gauge theory and integrate out all d.o.f. subject to the constraint that the Polyakov line holonomies are held fixed. In temporal gauge

\[ e^{S_P[U_x]} = \int DU_0(x, 0) DU_k D\phi \left\{ \prod_x \delta[U_x - U_0(x, 0)] \right\} e^{S_L} \]

At leading order in the strong coupling/hopping parameter expansion \( S_P \) has the form of an SU(3) spin model

\[ S_{spin} = J \sum_x \sum_{k=1}^3 \left( \text{Tr}[U_x] \text{Tr}[U_x^{\dagger}] + \text{c.c.} \right) \]

\[ + h \sum_x \left( e^{\mu/T} \text{Tr}[U_x] + e^{-\mu/T} \text{Tr}[U_x] \right) \]
The SU(3) spin model has been solved successfully, for a wide range of parameters $J, h, \mu$, in several different ways:

### Methods

1. flux representation *(Gattringer and Mercado)*
2. stochastic quantization *(Aarts and James)*
3. reweighting *(Fromm, Langelage, Lottini and Philipsen)*
4. mean field *(Splittorff and JG)*

Since these methods work for the simple SU(3) spin model $S_{\text{spin}}$, perhaps they also work for the more complicated effective action $S_P$.

*The problem is to find the effective action $S_P$, corresponding to lattice gauge theory at weaker couplings, finite $\mu$, and light quark masses.*
Avoid dynamical fermion simulations for now, work instead with an SU(3) gauge-Higgs model:

\[ S_L = \frac{\beta}{3} \sum_p \text{ReTr}[U(p)] + \frac{\kappa}{3} \sum_x \sum_{\mu=1}^4 \text{ReTr}\left[ \Omega^\dagger(x) U_\mu(x) \Omega(x + \hat{\mu}) \right] \]

If we can derive \( S_P \) at \( \mu = 0 \), then we also have \( S_P \) at \( \mu > 0 \) by the following identity:

\[ S_P^\mu[U_x, U_\mu^\dagger] = S_P^{\mu=0} \left[ e^{N_\mu x} U_x, e^{-N_\mu x} U_\mu^\dagger \right] \]

which is true to all orders in the strong coupling/hopping parameter expansion.
How to compute $S_P$ at $\mu = 0$?

- strong-coupling expansions \((Philipsen \ et \ al.)\)
- inverse Monte Carlo \((Heinzl \ et \ al.)\)
- relative weights \(this \ talk\)

And how do we know that we have derived $S_P$ correctly?

One test: compare Polyakov line correlators

$$G(R) = \frac{1}{N_c^2} \left\langle \text{Tr}[U_x]\text{Tr}[U_y^\dagger] \right\rangle, \quad R = |x - y|$$

computed for the effective action, and in the underlying lattice gauge theory.

Agreement has not been demonstrated in other approaches to deriving $S_P$ beyond $R = 2$ or 3 lattice spacings (see, e.g., Bergner et al., arXiv:1311.6745)
In previous papers we worked out $S_P$ for pure SU(2) gauge theory:

$$S_P = \sum_{x,y} P_x K(x - y) P_y$$

where

$$P_x = \frac{1}{2} \text{Tr} U_x$$

Here is the correlator comparison for

$$G(R) = \langle P_x P_y \rangle$$

The underlying lattice gauge theory is at $\beta = 2.2$ on a $24^3 \times 4$ lattice.
The Relative Weights Method

Let $S'_L$ be the lattice action in temporal gauge with $U_0(x,0)$ fixed to $U'_x$. It is not so easy to compute

$$\exp[S_P[U'_x]] = \int DU_k D\phi \ e^{S'_L}$$

directly. But the ratio (“relative weights”)

$$e^{\Delta S_P} = \frac{\exp[S_P[U'_x]]}{\exp[S_P[U'_{x''}]]}$$

is easily computed as an expectation value

$$\exp[\Delta S_P] = \frac{\int DU_k D\phi \ e^{S'_L}}{\int DU_k D\phi \ e^{S''_L}}$$

$$= \frac{\int DU_k D\phi \ \exp[S'_L - S''_L] e^{S'_L}}{\int DU_k D\phi \ e^{S''_L}}$$

$$= \left\langle \exp[S'_L - S''_L] \right\rangle''$$

where $\left\langle \ldots \right\rangle''$ means the VEV in the Boltzmann weight $\propto e^{S''_L}$.
Suppose $U_x(\lambda)$ is some path through configuration space parametrized by $\lambda$, and suppose $U'_x$ and $U''_x$ differ by a small change in that parameter, i.e.

$$U'_x = U_x(\lambda_0 + \frac{1}{2} \Delta \lambda), \quad U''_x = U_x(\lambda_0 - \frac{1}{2} \Delta \lambda)$$

Then the relative weights method gives us the derivative of the true effective action $S_P$ along the path:

$$\left( \frac{dS_P}{d\lambda} \right)_{\lambda=\lambda_0} \approx \frac{\Delta S}{\Delta \lambda}$$

The question is: which derivatives will help us to determine $S_P$ itself?
Fourier components of $P_x$

\[ P_x \equiv \frac{1}{N_c} \text{Tr} U_x = \sum_k a_k e^{i k \cdot x} \]

We first set a particular momentum mode $a_k$ to zero. Call the resulting configuration $\tilde{P}_x$. Then define ($f \approx 1$)

\[ P''_x = \left( \alpha - \frac{1}{2} \Delta \alpha \right) e^{i k \cdot x} + f \tilde{P}_x \]
\[ P'_x = \left( \alpha + \frac{1}{2} \Delta \alpha \right) e^{i k \cdot x} + f \tilde{P}_x \]

which uniquely determine (in SU(2) and SU(3)) the eigenvalues of the corresponding holonomies $U'_x, U''_x$. 
$S_P$ has a remnant local symmetry $U_x \rightarrow g_x U_x g_x^\dagger$, so the holonomies $U_x'$, $U_x''$ can be taken to be diagonal. We then compute

$$\frac{1}{L^3} \left( \frac{\partial S_P}{\partial a_{\alpha_k}^R} \right)_{a_k = \alpha}$$

by the relative weights simulation ($a_{\alpha_k}^R$ is the real part of $a_k$).

For a pure gauge theory, the part of $S_P$ bilinear in $P_x$ is constrained to have the form

$$S_P = \sum_{x y} P_x P_y^\dagger K(x - y)$$

Then, going over to Fourier modes

$$\frac{1}{\alpha} \frac{1}{L^3} \left( \frac{\partial S_P}{\partial a_{\alpha_k}^R} \right)_{a_k = \alpha} = 2 \tilde{K}(k)$$
SU(3) Pure Gauge Theory

We work on a $16^3 \times 6$ lattice volume; there is a deconfinement transition at $\beta = 5.89$, but we are interested in the confinement (or, with matter, the “confinement-like”) regime. Here are the relative weights results at $\beta = 5.7$:

1. Rotation invariance: data points only depend on $k$ through $k_L$

   \[ k_L = 2\sqrt{\sum_{i=1}^{3} \sin^2(k_i/2)} \]

2. Except at $k_L = 0$, the data points fall on one of two straight lines, with different slopes.

3. Deviation at $k_L = 0$ is handled by a long range cutoff in the kernel $K(x - y)$, which would otherwise be proportional to $\sqrt{-\nabla^2}$. 

\[ \beta = 5.7, \kappa = 0 \]
\[ \alpha \rightarrow 0 \]
\[ \alpha = 0.01 \]
\[ \alpha = 0.02 \]
\[ \alpha = 0.03 \]
\[ \alpha = 0.04 \]
To compute $K(x - y)$

1. Fit the $\tilde{K}(k_L)$ data to

$$\tilde{K}^{fit}(k_L) = \begin{cases} 
  c_1 - c_2 k_L & k_L \leq k_0 \\
  b_1 - b_2 k_L & k_L > k_0 
\end{cases}$$

2. Introduce a long-range cutoff $r_{max}$

$$K(x - y) = \begin{cases} 
  \frac{1}{L^3} \sum_k \tilde{K}^{fit}(k_L)e^{ik \cdot (x-y)} & |x - y| \leq r_{max} \\
  0 & |x - y| > r_{max} 
\end{cases}$$

3. Transform back to momentum space. Choose cutoff $r_{max}$ so that $\tilde{K}(0)$ matches the data point at $k_L = 0$. 
The red points are the Fourier transform of $K(x - y)$, which gives us the effective action $S_P$

\[ \beta = 5.7, \kappa = 0 \]

\[ 2K(k) \]
\[ \alpha \rightarrow 0 \]
\[ \alpha = 0.01 \]
\[ \alpha = 0.02 \]
\[ \alpha = 0.03 \]
\[ \alpha = 0.04 \]
Correlator comparisons at $\beta = 5.6, 5.7$

$$S_P = \sum_{xy} P_x P_y^\dagger K(x - y)$$

Simulate the effective theory in the usual way, and compare the Polyakov line correlators in the effective theory with the correlators in the underlying pure gauge theory.
Gauge-Higgs model: General remarks

In an SU(N) lattice gauge theory with matter in the fundamental representation, there is no absolute separation in coupling-constant space between a confining and a Higgs phase.

We are considering the SU(3) gauge-Higgs action

\[ S_L = \frac{\beta}{3} \sum_p \text{ReTr}[U(p)] + \frac{\kappa}{3} \sum_x \sum_{\mu=1}^{4} \text{ReTr}[\Omega^{\dagger}(x) U_\mu(x) \Omega(x + \hat{\mu})] \]

In our case, keeping \( \beta = 5.6 \) fixed and varying \( \kappa \), there is a rapid crossover from a “confinement-like” to a “Higgs-like” region at \( \kappa \approx 4.0 \).
This plot shows the Polyakov line correlator $G(R) = \langle P_x P_y \rangle$ vs. $R$ for the SU(3) gauge-Higgs model, computed by standard lattice Monte Carlo (+ Lüscher-Weisz noise reduction), at $\beta = 5.6$ and various $\kappa$. 
Introducing matter fields introduces a dependence on chemical potential in $S_P$:

$$S_P = \sum_S e^{s\mu/T} S_P^{(s)} [U_x, U_x^\dagger]$$

- Truncation is inevitable.
- But terms which are negligible at $\mu = 0$ can become significant at large enough $\mu$.
- The hope is to calculate enough of $S_P$ so that the approximation works in the region of interest in the $\mu - T$ plane.
- For now we will determine $S_P$ up to 2nd order in fugacity, and 2nd order in products of Polyakov lines.

The starting point is to include, in the center symmetry-breaking terms, $\text{Tr} U_x$, $\text{Tr} U_x^2$ (+ complex conjugates), and products of no more than two of these terms.
1. Write down all possible terms in $S_P$ involving $\text{Tr} U_x$, $\text{Tr} U_x^2$, $\text{Tr} U_x^\dagger$, $\text{Tr} U_x^{\dagger 2}$ and nonlocal products of any two of these terms.

2. Introduce a finite chemical potential via the transformation

$$U_x \rightarrow e^{N_t \mu} U_x, \quad U_x^\dagger \rightarrow e^{-N_t \mu} U_x^\dagger$$

3. Make use of the SU(3) identities

$$\text{Tr}[U_x^2] = 9P_x^2 - 6P_x^\dagger, \quad \text{Tr}[U_x^{\dagger 2}] = 9P_x^{\dagger 2} - 6P_x$$

4. Discard terms involving a product of three or more $P_x$'s.
We end up with the bilinear action

\[
S_P = \sum_{xy} P_x P^\dagger_y K(x - y) + \sum_{xy} (P_x P_y Q(x - y, \mu) + P^\dagger_x P^\dagger_y Q(x - y; -\mu)) \\
+ \sum_x \left\{ (d_1 e^{\mu/T} - d_2 e^{-2\mu/T}) P_x + (d_1 e^{-\mu/T} - d_2 e^{2\mu/T}) P^\dagger_x \right\}
\]

where

\[
Q(x - y; \mu) = Q^{(1)}(x - y)e^{-\mu/T} + Q^{(2)}(x - y)e^{2\mu/T} + Q^{(4)}(x - y)e^{-4\mu/T}
\]

The problem is to determine \( K(x - y), d_1, d_2, Q(x - y; \mu) \).
Use of the imaginary chemical potential $\mu/T = i\theta$

In terms of Fourier amplitudes

$$\frac{1}{L^3} S_P = \sum_k a_k a_k^* \tilde{K}(k_L) + a_0 \left( d_1 e^{i\theta} - d_2 e^{-2i\theta} \right) + a_0^* \left( d_1 e^{-i\theta} - d_2 e^{2i\theta} \right)$$

$$+ \sum_k \left( a_k a_{-k} \tilde{Q}(k_L, \theta) + a_k^* a_k^* \tilde{Q}(k_L, \theta) \right)$$

Then

$$\frac{1}{L^3} \left( \frac{\partial S_P}{\partial a_0^R} \right)_{a_0 = \alpha} = 2\tilde{K}(0)\alpha + 2d_1 \cos(\theta) - (2d_2 - 4\tilde{Q}(0)\alpha) \cos(2\theta)$$

Fit to

$$\frac{1}{L^3} \left( \frac{\partial S_P}{\partial a_0^R} \right)_{a_0^R = \alpha} = A(\alpha) + B(\alpha) \cos(\theta) - C(\alpha) \cos(2\theta)$$

Compare the data to the fit, and we find $d_1, d_2, \tilde{K}(0), \tilde{Q}(0)$. 
Gauge-Higgs theory at $\beta = 5.6$, $\kappa = 3.9$ on a $16^3 \times 6$ lattice. Calculate (lhs) and fit (rhs)

$$\frac{1}{L^3} \left( \frac{\partial S_P}{\partial a_0^R} \right)_{a_0^R = \alpha} = A(\alpha) + B(\alpha) \cos(\theta) - C(\alpha) \cos(2\theta)$$

at 15 values of $\theta$ and several $\alpha$ values:

We can then extract coefficients of center symmetry-breaking terms (in this case $d_1 = 0.0585$, $d_2 = 0.0115$), as well as $\tilde{K}(0)$ and $\tilde{Q}(0)$. 
For $k \neq 0$, the derivative wrt $a_k$ has terms proportional to $a_{-k}$. We set $a_{-k}$ to some constant real value $a_{-k} = \sigma$. Then

$$\frac{1}{L^3} \left( \frac{\partial S_P}{\partial a_k^R} \right)^{a_{-k}=\sigma}_{a_k=\alpha} = 2\tilde{K}(k_L)\alpha + 4\left( \tilde{Q}^{(1)}(k_L) \cos(\theta) \right. + \tilde{Q}^{(2)}(k_L) \cos(2\theta) + \tilde{Q}^{(4)}(k_L) \cos(4\theta) \left. \right) \sigma$$

First, setting $\sigma = 0$, we have

$$\tilde{K}(k_L) = \frac{1}{2L^3} \frac{1}{\alpha} \left( \frac{\partial S_P}{\partial a_k^R} \right)^{\alpha_{-k}=0}_{a_k=\alpha}$$

Then, at small but finite $\sigma$, we can determine the $\tilde{Q}^{(n)}(k_L)$ from the $\theta$-dependence of the data.
$\tilde{Q}(k_L, \mu)$ seems calculable, but the magnitude is small and the errorbars are large:

(a) $a_k$ derivative at smallest $k_L \neq 0$, vs. $\theta$

(b) $\tilde{K}(k_L)$ and (estimate of) $\tilde{Q}_1(k_L)$ vs. $k_L$

For now we will ignore the $Q(x - y; \mu)$ term in the action.
The underlying lattice gauge-Higgs theory is at $\beta = 5.6$, $\mu = 0$ and $\kappa = 3.6, 3.8, 3.9$ on a $16^3 \times 6$ lattice volume.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{plot.png}
\caption{$\kappa = 3.6$}
\end{figure}
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Mean Field solution at $\mu > 0$

- $S_P$ still has a sign problem.

- It can be addressed in various ways: flux representation, stochastic quantization, reweighting, and mean field.

- In general – mean field becomes more reliable the more spins are coupled to a given spin. But for $S_P$, many spins are coupled to any given spin, especially for light scalar masses, through the non-local kernel $K(x - y)$.

- Perhaps the mean field method is more reliable, when applied to $S_P$ at finite $\mu$, than one might expect.

Whether or not that is true, we have applied mean field to $S_P$, following the treatment in *Splittorff and JG, arXiv:1206.1159* for the SU(3) spin model.
Solution of $S_P$ for $\langle \text{Tr} U_x \rangle$, $\langle \text{Tr} U_x^\dagger \rangle$ and particle number/site $n$, for an underlying lattice gauge-Higgs theory at $\beta = 5.6$ and $\kappa = 3.8$, $16^3 \times 6$ lattice volume, varying $\mu$.

(a) $\langle \text{Tr} U_x \rangle$, $\langle \text{Tr} U_x^\dagger \rangle$

(b) particle number density
In a certain limit where the inverse mass (staggered) or hopping parameter (Wilson) is very small, and the chemical potential $\mu$ is large, the fermion determinant simplifies. In temporal gauge, the lattice action is

\[ e^{S_L} = \prod_x \det \left[ 1 + h e^{\mu/T} U_0(x, 0) \right]^p \det \left[ 1 + h e^{-\mu/T} U_0^*(x, 0) \right]^p e^{S_{\text{plaq}}} \]

where $S_{\text{plaq}}$ is the plaquette action, $p = 1$ for staggered fermions, $p = 2N_f$ for Wilson fermions. If we know the Polyakov line action for the pure gauge theory $S^p_{\text{pg}}$, then the Polyakov line action in this heavy quark limit is obtained immediately:

\[ e^{S_P} = \prod_x \det \left[ 1 + h e^{\mu/T} U_x \right]^p \det \left[ 1 + h e^{-\mu/T} U_x^* \right]^p e^{S^p_{\text{pg}}} \]

This action is also amenable to a mean field solution.
Here are some mean field results for staggered fermions at $\beta = 5.6$ and $h = 0.0001 \rightarrow m = 2.32/a$. Note the saturation in number density at large $\mu$. 

![Polyakov lines](image1)

![Number density](image2)
We have determined the effective Polyakov line actions $S_P$, up to terms bilinear in $P_x$, corresponding to SU(3) pure gauge theory, to SU(3) gauge-Higgs theory, and to SU(3) with heavy quarks, at finite chemical potential.

**Next Steps:**

1. Beyond bilinear: determine contributions to $S_P$ involving products of three or four Polyakov line variables $P_x$.

2. Beyond mean field: reweighting, stochastic quantization, flux representation...