Nucleon Structure and PDFs on Euclidean Lattice

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Biography

- Born in Yichang City, China, on Sept. 20,1986
- 2005-2009, Bachelor's degree, Central China Normal University, Wuhan, China
- 2009-now, Ph.D. , Peking University, Beijing, China, Supervisor: Prof. Xiangdong Ji
- Research Interests: hadron multi-dimensional structure, spin structure and PDF(its extension).

  Also worked on atom collisions in Liquid Xenon Dark Matter detector

- # Publications

1. Cédric Lorcé, Barbara Pasquini, Xiaonu Xiong, Feng Yuan
The quark orbital angular momentum from Wigner distributions and light-cone wave functions
Phys. Rev. D 85, 114006 (2012)

2. Xiangdong Ji, Xiaonu Xiong, Feng Yuan
Proton Spin Structure from Measurable Parton Distributions
Phys. Rev. Lett. 109, 152005 (2012)

3. Xiangdong Ji, Xiaonu Xiong, Feng Yuan
Transverse Polarization of the Nucleon in Parton Picture
Phys.Lett. B 717, 214-218 (2012)

4. Xiangdong Ji, Xiaonu Xiong, Feng Yuan
Probing Parton Orbital Angular Momentum in Longitudinally Polarized Nucleon
Phys. Rev. D 88, 014041 (2013)

5. Wei Mu, Xiaonu Xiong, Xiangdong Ji
Scintillation Efficiency for Low-Energy Nuclear Recoils in Liquid-Xenon Dark Matter Detectors
arXiv:1306.0170v2 [nucl-ex] (2013)

6. Xiaonu Xiong, Xiangdong Ji, Jianhui Zhang, Yong Zhao
One-Loop Matching for Parton Distributions: Non-Singlet Case
arXiv:1310.7471 [hep-ph] (2013)

Parton Physics

- Wigner Distribution
- Light-Cone Quark Model
- Spin Structure
- New PDF on Euclidean Lattice

PART I

Parton Physics Introduction

Nucleon Structure

- Hadron properties ⟺ parton d.o.f
- Encoding the non-perturbative information of hadron(QCD)
- Essential for revealing the structure of hadron
- Setup:

  Light-Cone Coordinates: $P^{\pm} = \frac{1}{\sqrt{2}}(p^0 \pm p3)$

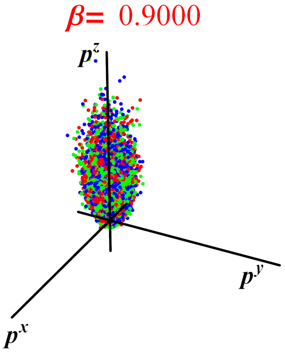
  Light-Cone gauge: $A^+ = 0$

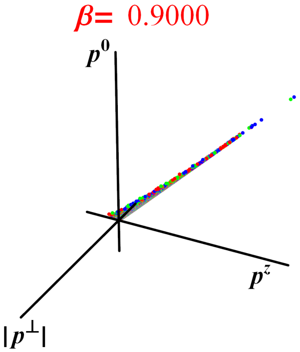
  Infinite Momentum Frame: $P^+ \to \infty$

- A Cartoon of boost to IMF

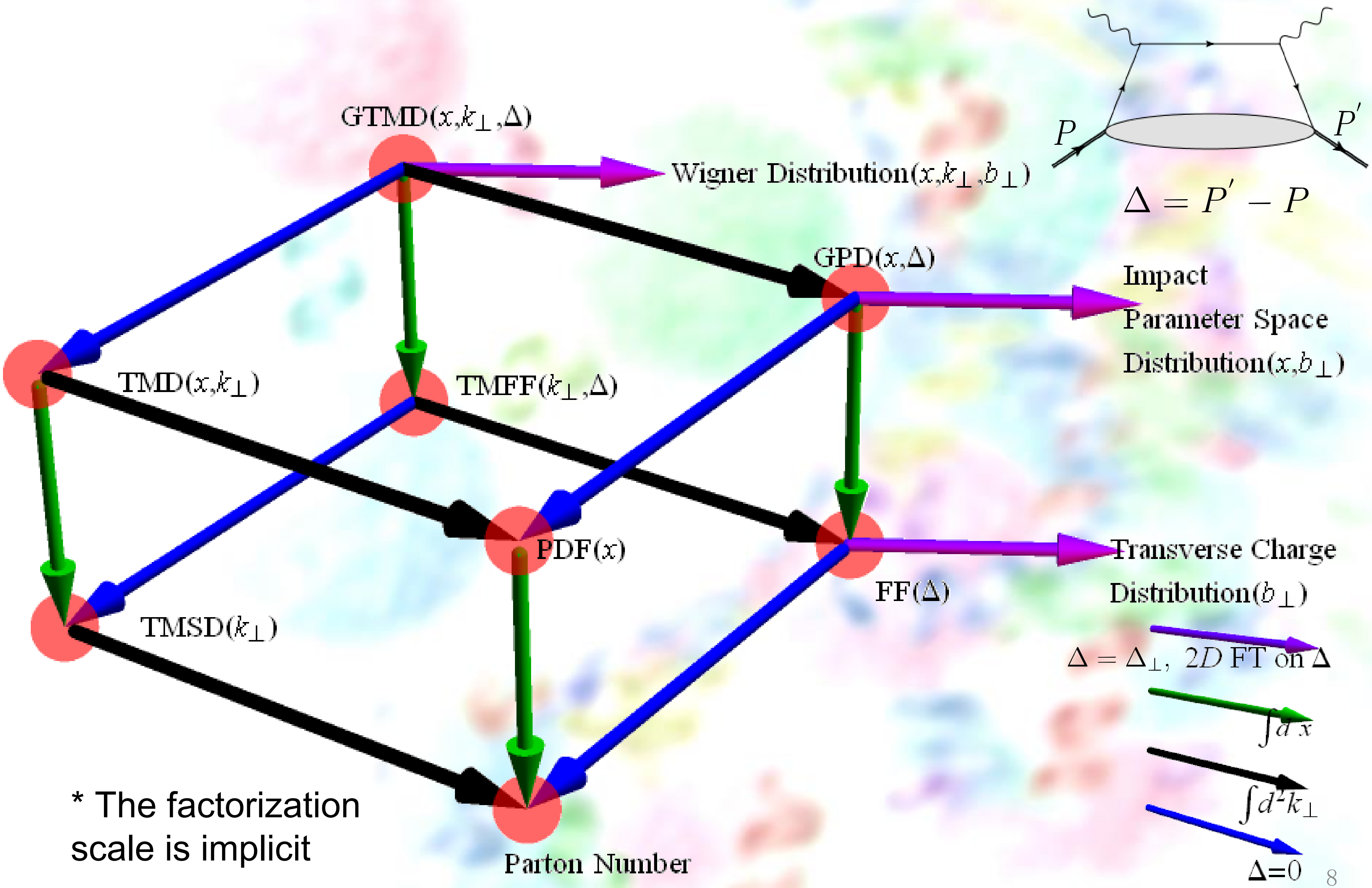
  Under large boost: $P^z, P^+ \to \infty$

  Nucleon $\to$ branch of collinear partons





PDF and its extension



* The factorization scale is implicit

Wigner Distribution

- Quantum phase-space distributions

  Provide the most complete information
- Not measurable — Uncertain Principle
- Not positive definite , no probability interpretation (projs. may have prob. int.)
- For any dynamic operator

$$\langle \hat{\mathcal{O}} \rangle = \int \mathrm{d}^n p \int \mathrm{d}^n r \, \hat{\mathcal{O}}(r, p) W(r, p)$$

- Definition in Parton Physics

$$W^q(x,\vec{k},\vec{b}_\perp)=\int\frac{d\eta^- d^2\vec{\eta}_\perp}{(2\pi)^3}e^{ik\eta}\left\langle P\left|\bar{\Psi}\left(\vec{b}_\perp-\tfrac{\eta}{2}\right)\gamma^+\Psi\left(\vec{b}_\perp+\tfrac{\eta}{2}\right)\right|P\right\rangle$$

$$=\int\frac{d^2\Delta_\perp}{(2\pi)^2}e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp}\int\frac{d\eta^- d^2\vec{\eta}_\perp}{(2\pi)^3}e^{ik\eta}\left\langle P+\tfrac{\vec{\Delta}_\perp}{2}\left|\bar{\Psi}\left(-\tfrac{\eta}{2}\right)\gamma^+\Psi\left(\tfrac{\eta}{2}\right)\right|P-\tfrac{\vec{\Delta}_\perp}{2}\right\rangle$$

$$W^g(x,\vec{k},\vec{b}_\perp)=\frac{1}{x}\int\frac{d\eta^- d^2\vec{\eta}_\perp}{(2\pi)^3}e^{ik\eta}\left\langle P\left|\mathbf{F}^{+i}\left(\vec{b}_\perp-\tfrac{\eta}{2}\right)\mathbf{F}^{+i}\left(\vec{b}_\perp+\tfrac{\eta}{2}\right)\right|P\right\rangle$$

$$=\frac{1}{x}\int\frac{d^2\Delta_\perp}{(2\pi)^2}e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp}\int\frac{d\eta^- d^2\vec{\eta}_\perp}{(2\pi)^3}e^{ik\eta}\left\langle P+\tfrac{\vec{\Delta}_\perp}{2}\left|\mathbf{F}^{+i}\left(-\tfrac{\eta}{2}\right)\mathbf{F}^{+i}\left(\tfrac{\eta}{2}\right)\right|P-\tfrac{\vec{\Delta}_\perp}{2}\right\rangle$$

GTMD

Ψ, F *Gauge invariant fields, contains gauge link*

- Quark OAM Distribution from Wigner Dis.

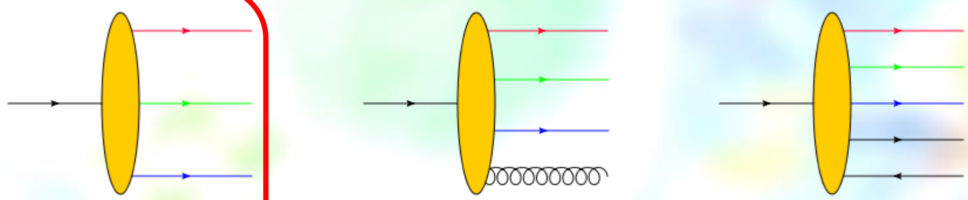
$$\boldsymbol{l}(x)=\int d^2\boldsymbol{k}_\perp d^2\boldsymbol{b}_\perp\,\boldsymbol{b}_\perp\times\boldsymbol{k}_\perp W(x,\boldsymbol{k}_\perp,\boldsymbol{b}_\perp)$$

PART II

Wigner Distribution and Quark OAM in Light-Cone Constituent Quark Model

Light-Cone Constituent Quark Model

- Light-Cone Wave Functions (LCWFs)
  the wave function of nucleon Fock States



$$|P\rangle = \Psi_{3q}\,|qqq\rangle + \Psi_{3qg}\,|qqqg\rangle + \Psi_{3q\bar{q}q}\,|qqq\bar{q}q\rangle \cdots$$

- Low-energy scale: D.O.F = valence quark only

$$\begin{aligned}|P\uparrow\rangle =&\Psi^{(l^z=-1)}\,|q_\uparrow q_\uparrow q_\uparrow\rangle + \Psi^{(l^z=0)}\,|q_\uparrow q_\uparrow q_\downarrow\rangle \\ &+ \Psi^{(l^z=1)}\,|q_\uparrow q_\downarrow q_\downarrow\rangle + \Psi^{(l^z=2)}\,|q_\downarrow q_\downarrow q_\downarrow\rangle\end{aligned}$$

Light-Cone Wave Functions

- LCWFs $\Psi^{l^z} = \Psi(\{x_1, x_2, x_3\}, \{k_1^\perp, k_2^\perp, k_3^\perp\})$

  momentum fraction: $x_i = \frac{p_i^+}{P^+}$

  relative transverse momentum: $k_i^\perp = p_i^\perp - x_i P^\perp$

  $\sum x_i = 1,\ \sum k_i^\perp = 0^\perp$

  eg. $\psi^{(l^z=0)} = \int \frac{[x]_3\,[^2k]_3}{\sqrt{x_1 x_2 x_3}} \left(\psi^{(1)}(1,2,3) + i\epsilon^{\alpha\beta} k_{1\alpha} k_{2\beta} \psi^{(2)}(1,2,3)\right)$

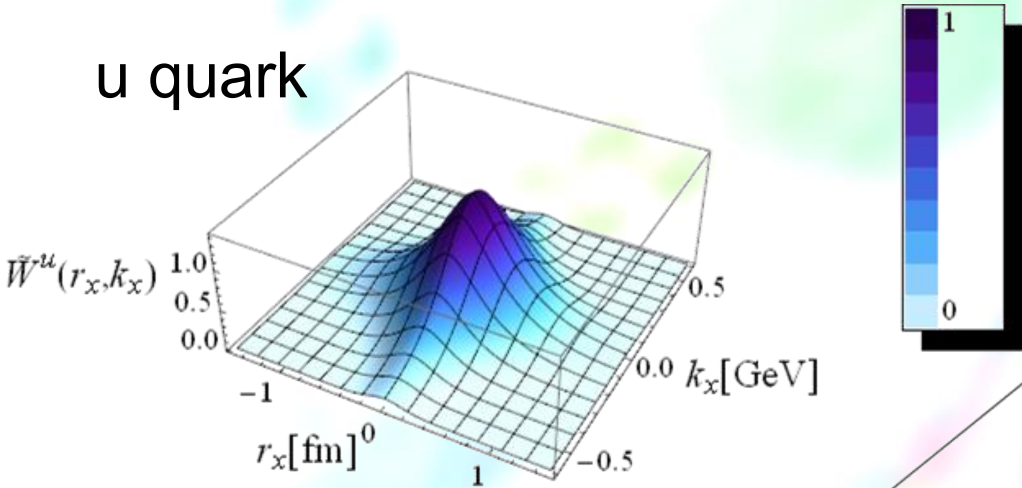
$$\psi^{(1)}(1,2,3) = \widetilde{\psi}(\{x_i, \boldsymbol{k}_{i\perp}\}) \frac{1}{\sqrt{3}} \prod_i \frac{1}{\sqrt{N(x_i, \boldsymbol{k}_{i\perp})}} (-a_1 a_2 a_3 + (a_3 + 2a_1) \boldsymbol{k}_{1\perp} \cdot \boldsymbol{k}_{2\perp} + 2a_1 \boldsymbol{k}_{2\perp}^2)$$

$$\psi^{(2)}(1,2,3) = \widetilde{\psi}(\{x_i, \boldsymbol{k}_{i\perp}\}) \frac{1}{\sqrt{3}} \prod_i \frac{1}{\sqrt{N(x_i, \boldsymbol{k}_{i\perp})}} (2a_1 + a_3)$$

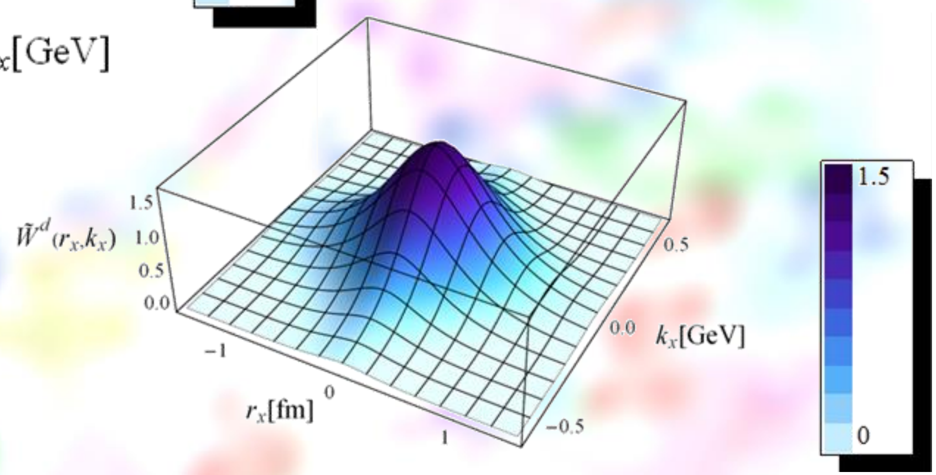
- Projected Wigner distributions

$$\tilde{W}^q(r_x, k_x) = \int \mathrm{d}x \int \mathrm{d}r_y \int \mathrm{d}k_y\, W^q(\boldsymbol{r}, \boldsymbol{k}, x)$$

u quark

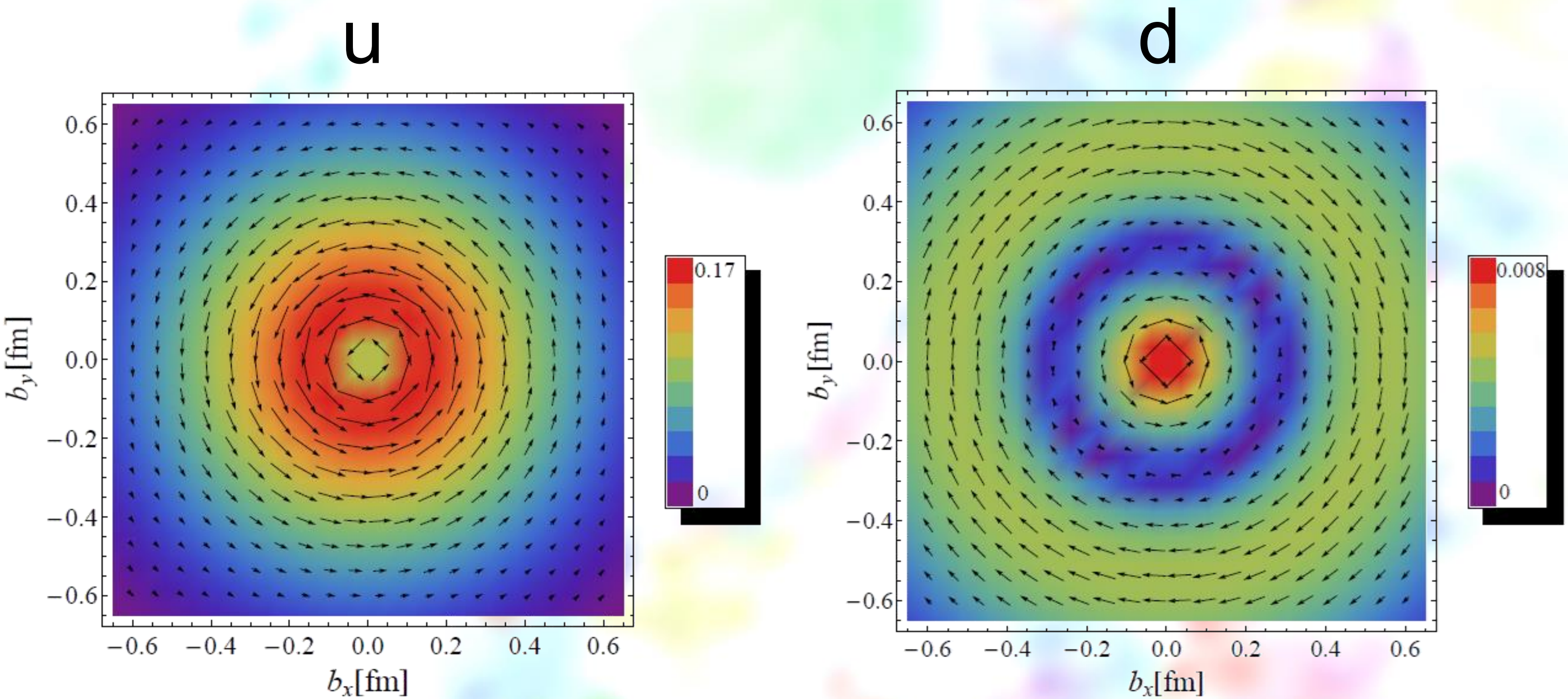


d quark



Quark $\langle \boldsymbol{k}_\perp \rangle$ Distribution

- $$\langle \boldsymbol{k}_\perp \rangle (\boldsymbol{b}_\perp) = \int \mathrm{d}^2 \boldsymbol{k}_\perp \int \mathrm{d}x \, \boldsymbol{k}_\perp W^q(\boldsymbol{b}, \boldsymbol{k}, x)$$



C. Lorce, B. Pasquini, X. Xiong, F. Yuan, PRD, 2012

Quark OAM in LCCQM

- $$l_z^q = \int \mathrm{d}^2\boldsymbol{r}_\perp \int \mathrm{d}^2\boldsymbol{k}_\perp \int \mathrm{d}x\, \boldsymbol{r}_\perp \times \boldsymbol{k}_\perp W^q(\boldsymbol{r}_\perp, \boldsymbol{k}_\perp, x)$$

	$l_z = 0$	$l_z = 1$	$l_z = -1$	$l_z = 2$	Total
ℓ_z^u	0.013	0.139	−0.046	0.025	0.131
ℓ_z^d	−0.013	0.087	−0.090	0.011	−0.005
ℓ_z	0	0.226	−0.136	0.036	0.126
ρ_{l_z}	0.620	0.226	0.136	0.018	1

- Cant be compared with high energy experiments and lattice, needs scale evolution

C. Lorce, B. Pasquini, X. Xiong, F. Yuan, PRD, 2012

PART III

Partonic Nucleon Spin Structure

Patronic Nucleon Spin Structure

- Transversely Polarized Nucleon

  Transverse Polarization Sum Rule

X. Ji, X. Xiong, F. Yuan, PRL, PLB, 2012

- Longitudinally Polarized Nucleon

  Longitudinally Helicity Decomposition

X. Ji, X. Xiong, F. Yuan, PRL2012, PRD,2013

Transverse Polarization

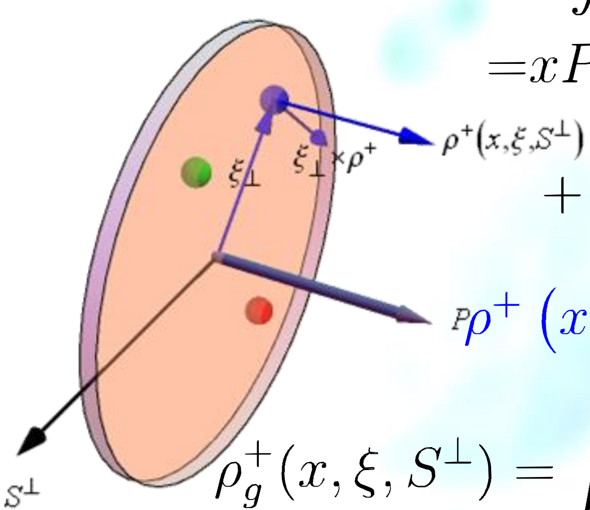
- Longitudinal Momentum Distribution

$$\rho_q^+(x,\xi,S^\perp) = x\int \frac{d\lambda}{4\pi}\langle PS^\perp|\bar{\psi}(-\tfrac{\lambda n}{2},\xi)\gamma^+\psi(\tfrac{\lambda n}{2},\xi)|PS^\perp\rangle$$

$$= xP^+H(x,0,0)$$

$$+\frac{1}{2}xP^+\left[H(x,0,0)+E(x,0,0)\right]\lim_{\Delta_\perp\to 0}\frac{S^{\perp\prime}}{M^2}\partial_\xi^\perp e^{i\xi_\perp\Delta_\perp}$$

non-zero under integral with $\xi_\perp$



$\rho^+(x,\xi,S^\perp)$ is already a Wigner distribution

$$\rho_g^+(x,\xi,S^\perp) = \int \frac{d\lambda}{4\pi}\langle PS^\perp|F^{+i}(-\tfrac{\lambda n}{2},\xi)F^{+i}(\tfrac{\lambda n}{2},\xi)|PS^\perp\rangle$$

$$= xP^+H(x,0,0)$$

$$+\frac{1}{2}xP^+\left[H(x,0,0)+E(x,0,0)\right]\lim_{\Delta_\perp\to 0}\frac{S^{\perp\prime}}{M^2}\partial_\xi^\perp e^{i\xi_\perp\Delta_\perp}$$

- # Transverse Polarization Sum Rule

Construct from Pauli–Lubanski vector

$V^{\perp'} = \epsilon^{-+\perp\alpha} V_\alpha$

$$W^{\perp}_{q/g}(x)|_{T^{++}} = \frac{M_N^2}{2P^+(2\pi)^2\delta^{(2)}(0)} \int d^2\xi \, \xi^{\perp'} \rho^{+}_{q/g}(x,\xi,S^{\perp})$$

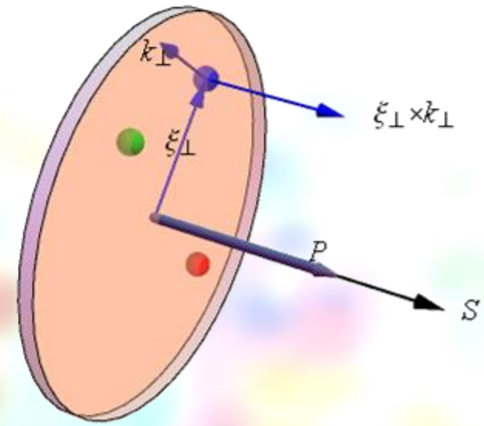
$$= S^{\perp}\frac{x}{4}\left[H_q(x,0,0) + E_q(x,0,0)\right]$$

$$W^{\perp}_{q/g}(x)|_{T^{+\perp}} = S^{\perp}\frac{x}{4}\left[H_q(x,0,0) + E_q(x,0,0)\right]$$

Transverse AM distribution

$$S^{\perp}_{q/g}(x) = \frac{x}{2}\left[H_{q/g}(x,0,0) + E_{q/g}(x,0,0)\right]$$

Helicity

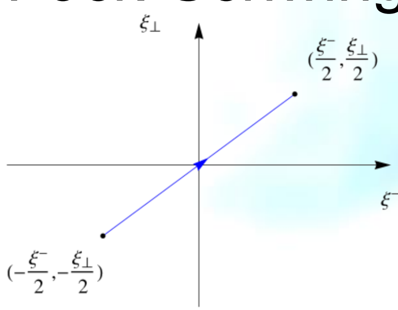


- Quark Wigner Distribution

$$W^q(x,\vec{k},\vec{b}_\perp)=\int\frac{d^2\Delta_\perp}{(2\pi)^2}e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp}\int\frac{d\eta^- d^2\vec{\xi}_\perp}{(2\pi)^3}e^{ik\xi}\left\langle P+\frac{\vec{\Delta}_\perp}{2}\right|\bar{\psi}\left(-\frac{\xi}{2}\right)\gamma^+\mathcal{L}\left[-\frac{\xi}{2},\frac{\xi}{2}\right]\psi\left(\frac{\xi}{2}\right)\left|P-\frac{\vec{\Delta}_\perp}{2}\right\rangle$$

- Gauge link choice

Fock-Schwinger gauge link:



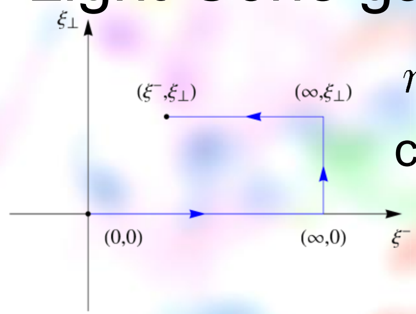
$$\xi\cdot A(\xi)=0$$

gauge inv. OAM

$$\frac{\left\langle P,S\left|\int d^3\vec{r}\,\overline{\psi}(\vec{r})\,\gamma^+\left(\vec{r}_\perp\times i\vec{D}_\perp\right)\psi(\vec{r})\right|P,S\right\rangle}{2}$$

$$=\int dx\int d^2\vec{b}_\perp d^2\vec{k}_\perp\left(\vec{b}_\perp\times\vec{k}_\perp\right)W_{FS}\left(x,\vec{k}_\perp,\vec{b}_\perp\right)$$

Light-Cone gauge link:



$$n\cdot A(\xi)=0$$

canonical OAM

$$\frac{\left\langle P,S\left|\int d^3\vec{r}\,\overline{\psi}(\vec{r})\,\gamma^+\left(\vec{r}_\perp\times i\vec{\partial}_\perp\right)\psi(\vec{r})\right|P,S\right\rangle}{2}$$

$$=\int dx\int d^2\vec{b}_\perp d^2\vec{k}_\perp\left(\vec{b}_\perp\times\vec{k}_\perp\right)W_{LC}\left(x,\vec{k}_\perp,\vec{b}_\perp\right)$$

- Gauge Inv. OAM

  ~ twist-2 + twist-3 GPD, measurable

  moments reduce to local operator

$$\frac{1}{n}\sum_i \overline{\psi}(0)\gamma^+ \left(iD^+\right)^i \left(\vec{r}_\perp \times i\vec{D}_\perp\right) \left(iD^+\right)^{n-1-i} \psi(0)$$

  lattice calculable

3-particle correlation, twist-3 GPD

- Canonical OAM

  ~ twist-2 + twist-3 GPD

  can be made gauge inv. through Gauge Invariant Extension (GIE), then measurable

$$i\tilde{\partial}^\perp = iD^\perp + \int^{\xi^-} d\eta^- L_{[\xi^-,\eta^-]} g F^{+\perp}\left(\eta^-, \xi_\perp\right) L_{[\eta^-,\xi^-]}$$

  but non-local

Twist-3 GPDs

- D-type

$$\int \frac{d\lambda}{2\pi} \int \frac{d\mu}{2\pi} e^{i\lambda(x-y)} e^{i\mu y} \langle P', S \left| \bar{\psi}(0)\gamma^+ iD^\perp(\mu n)\psi(\lambda n) \right| P, S \rangle$$
$$= \frac{i\epsilon^{\perp\alpha}}{2} \Delta_\alpha H_D^{q(3)}(x,y,\eta,t) \bar{U}(P')\gamma^+\gamma_5 U(P) + \cdots$$

- F-type

$$\int \frac{d\lambda}{2\pi} \int \frac{d\mu}{2\pi P^+} e^{i\lambda(x-y)} e^{i\mu y} \langle P', S \left| \bar{\psi}(0)\gamma^+ gF^{+\perp}(\mu n)\psi(\lambda n) \right| P, S \rangle$$
$$= \frac{\epsilon^{\perp\alpha}}{2} \Delta_\alpha H_F^{q(3)}(x,y,\eta,t) \bar{U}(P')\gamma^+\gamma_5 U(P) + \cdots$$

- Canonical

$$\int \frac{d\lambda}{2\pi} \int \frac{d\mu}{2\pi} e^{i\lambda(x-y)} e^{i\mu y} \left\langle P', S \left| \bar{\psi}(0)\gamma^+ i\tilde{\partial}^\perp(\mu n)\psi(\lambda n) \right| P, S \right\rangle$$
$$= \frac{i\epsilon^{\perp\alpha}}{2} \Delta_\alpha \tilde{H}_q^{(3)}(x,y,\eta,t) \bar{U}(P')\gamma^+\gamma_5 U(P) + \cdots$$

- Relation to OAM distributions

$$\text{Gauge Invariant } L_q^n = \int dx \int dy \, \frac{1}{n} \sum_{k=0}^{n-1} x^{n-1-k} (x-y)^k H_D^{q(3)}(x,y,0,0)$$

$$\text{GIE of Canonical } l_q(x) = \tilde{H}_q^{(3)}(x,0,0)$$

$$\text{potential term } l_{q,\mathrm{pot}}^n = -\int dx \int dy \, \frac{1}{n} \sum_{k=0}^{n-1} x^{n-1-k} (x-y)^k \, \mathrm{P.V.} \frac{1}{y} H_F^{q(3)}(x,y,0,0)$$

$$L_{q/g}(x) = l_{q/g}(x) + l_{q/g,\mathrm{pot}}(x)$$

⇕

$$H_D^{q/g(3)}(x,y,0,0) = -\mathrm{P.V.} \tfrac{1}{y} H_F^{q,g(3)} + \delta(y) \tilde{H}_{q,g}^{(3)}(x,0,0)$$

PART IV

New Parton Distributions on Euclidean Lattice

Light-Cone PDF

- Operator Definition $\Gamma = \gamma^+,\ \gamma^+\gamma^5,\ i\sigma^{+\perp}\gamma_5$

$$q(x) = \int \frac{d\xi^-}{2\pi} e^{-ixp^+\xi^-} \left\langle PS \left| \bar{\psi}(-\frac{\xi^-}{2})\Gamma\mathcal{L}[-\frac{\xi^-}{2};\frac{\xi^-}{2}]\psi(\frac{\xi^-}{2}) \right| PS \right\rangle$$

involving time component $\xi^0 \to i\xi^0_E$

measurable but can't be calculated on lattice

- Moments

$$q^n = \int dx\, x^{n-1} q(x) = \frac{1}{(p^+)^n} \left\langle PS \left| \bar{\psi}(0) \left(i \overleftrightarrow{D}^+ \right)^{n-1} \Gamma\psi(0) \right| PS \right\rangle$$

reduce to matrix elements of local operator

hard to simulate high order derivative on lattice

Quasi PDF

- Operator Definition $\tilde{\Gamma} = \gamma^z,\ \gamma^z\gamma^5,\ i\sigma^{z\perp}\gamma_5$

$$\tilde{q}(x) = \int \frac{dz}{2\pi} e^{-ixp^z z} \left\langle PS \left| \bar{\psi}(-\tfrac{z}{2})\tilde{\Gamma}\mathcal{L}[-\tfrac{z}{2};\tfrac{z}{2}]\psi(\tfrac{z}{2}) \right| PS \right\rangle$$

pure spatial correlation

directly calculated on lattice, no prob. int..

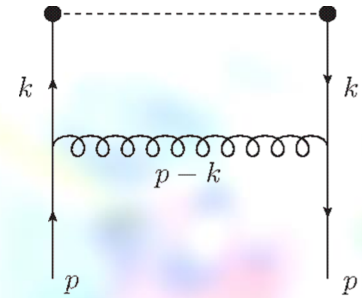
- Moments

$$\tilde{q}^n = \int dx\, x^{n-1}\tilde{q}(x) = \frac{1}{(p^z)^n} \left\langle PS \left| \bar{\psi}(0)\left(i\overleftrightarrow{D}^z\right)^{n-1}\tilde{\Gamma}\psi(0) \right| PS \right\rangle$$

recover L.C. moments when boost to IMF

with higher twist correction

x Regions



- For quark PDF in quark
  quark, gluon propagators $\frac{i(\not{k}-m)}{k^2-m^2+i\epsilon}, \quad \frac{D_{\mu\nu}(p-k)}{(p-k)^2+i\epsilon}$

- L.C k^- poles $k^- = \frac{\boldsymbol{k}_\perp^2+m^2-i\epsilon}{2xp^+}$ and $k^- = p^- + \frac{-(\boldsymbol{p}_\perp-\boldsymbol{k}_\perp)^2+i\epsilon}{2(1-x)}$

  $0 < x < 1 \implies \int dk^-(\cdots) \neq 0$

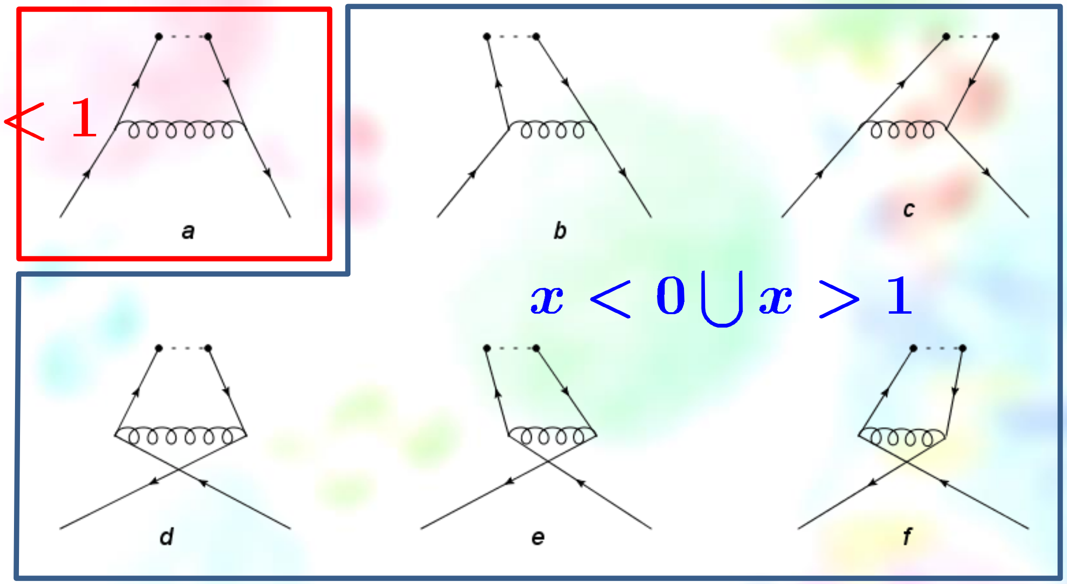
  large boost, no parton can move backward

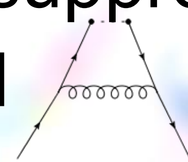
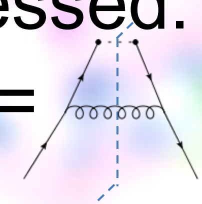
- Quasi. k^0 poles $k^0 = \pm\sqrt{\boldsymbol{k}^2+m^2} \mp i\epsilon$ and $k^0 = p^0 \pm \sqrt{(\boldsymbol{p}-\boldsymbol{k})^2} \mp i\epsilon$

  $x \in \mathbb{R} \implies \int dk^-(\cdots) \neq 0$

  large but finite P^z, parton can move backward

- $\int dk^0 \implies$ Time ordered diagrams



- L. C. : (b)~(f) are $1/p^+$ suppressed.
  only (a) survived  =  in $p^+ \to \infty$

- Quasi: (a)~(f) all contribute in finite p^z

Light-Cone Dis. VS Quasi Dis.

	LC dis.	Quasi dis.
operator definition	$\bar{\psi}(\xi^-)\mathcal{L}\left[\xi^-,0\right]\gamma^+\psi(0)$	$\bar{\psi}(\xi^z)\mathcal{L}\left[\xi^z,0\right]\gamma^z\psi(0)$
On Lattice	imaginary time, only calc. moments	spatial correlation directly calculable
moments	$\bar{\psi}(0)\left(i\overleftrightarrow{D}^+\right)^n\gamma^+\psi(0)$ $\sim(P^+)^n$	$\bar{\psi}(0)\left(i\overleftrightarrow{D}^z\right)^n\gamma^z\psi(0)$ $\sim(P^z)^n$
nucleon momentum	$p^+\to\infty$	p^z finite
momentum fraction	$0<x<1$	$-\infty<x<\infty$
Accessibility	experiments: DIS, D-Y	direct lattice calc.

Matching Condition
(Factorization theorem)

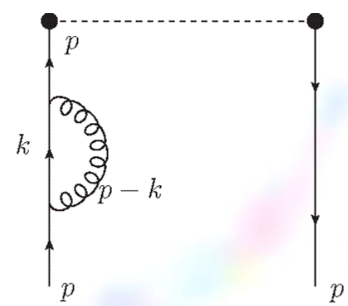
One-loop matching on PDFs

- Quark non-singlet case

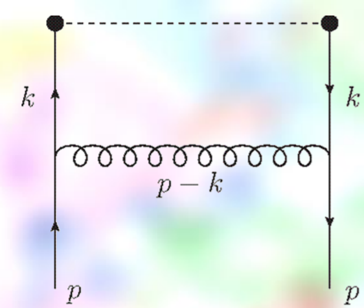
$$\tilde{u}(x) - \tilde{d}(x) = \int \frac{dy}{y} Z\left(\frac{x}{y}\right) \left[u(y) - d(y)\right]$$

- $Z^{(1)}$ from the wave function renormalization and the “vertex” correction

$$Z^{(1)}_{A^z=0} = 1 +$$



$$+$$



- Transverse cut-off regularization scheme

$$|\boldsymbol{k}_\perp| \leq \mu$$

X. Ji, X. Xiong, J,H. Zhang, Y. Zhao, 2013

- Matching condition $\Longleftrightarrow$ Factorization Theorem

$$\tilde{q} = Z \otimes q$$

$\tilde{q}$: no parton, probability interpretation

q: has parton, probability interpretation

rather than ~~$q = \tilde{Z} \otimes \tilde{q}$~~

- E.g. Collinear factorization $\sigma = \mathcal{H} \otimes q$

q: has parton, probability interpretation

rather than ~~$q = \mathcal{H}' \otimes \sigma$~~

- Matching Factor $Z\left(\xi\right)=\delta\left(\xi-1\right)+Z^{(1)}\left(\xi,\frac{P^z}{\mu}\right)$

quasi: $\tilde{q}\left(x\right)=\left(1+\delta\tilde{Z}_F^{(1)}\right)\delta\left(1-x\right)+\tilde{q}^{(1)}\left(x\right)$

L. C. : $\tilde{q}\left(x\right)=\left(1+\delta\tilde{Z}_F^{(1)}\right)\delta\left(1-x\right)+\tilde{q}^{(1)}\left(x\right)$

Self-energy correction ⟵ ⟶ vertex Correction

- Matching Condition

$$\tilde{q}(x)=\int_0^1\frac{dy}{y}Z\left(\frac{x}{y}\right)q(y)$$

$$\left(1+\delta\tilde{Z}_F\right)\delta\left(1-x\right)+\tilde{q}^{(1)}(x)$$

$$=\int_0^1\frac{dy}{y}\left[\delta\left(\frac{x}{y}-1\right)+Z^{(1)}\left(\frac{x}{y},\frac{P^z}{\mu}\right)\right]\left[\left(1+\delta Z_F\right)\delta\left(1-y\right)+q^{(1)}(y)\right]$$

- At $(\alpha_s)^0$:

$$\boldsymbol{\delta(1-x) = \int_0^1 \frac{dy}{y}\,\delta\left(\frac{x}{y}-1\right)\delta(1-y) = \delta(1-x)}$$

$$\boldsymbol{Z^{(0)}\left(x, \frac{P^z}{\mu}\right) = \delta(1-x)}$$

- At $(\alpha_s)^1$:

$$\begin{aligned}
&\delta\tilde{Z}_F\delta(1-x) + \tilde{q}^{(1)}(x) \\
=&\int_0^1 \frac{dy}{y}\,\delta\left(\frac{x}{y}-1\right)\left[\delta Z_F\delta(1-y) + q^{(1)}(y)\right] + \int_0^1 \frac{dy}{y}\,Z^{(1)}\left(\frac{x}{y}, \frac{P^z}{\mu}\right)\delta(1-y) \\
=&\delta Z_F\delta(1-x) + q^{(1)}(x) + Z^{(1)}\left(x, \frac{P^z}{\mu}\right)
\end{aligned}$$

$$\boldsymbol{Z^{(1)}\left(x, \frac{P^z}{\mu}\right) = \tilde{q}^{(1)}(x) - q^{(1)}(x) + \left(\delta\tilde{Z}_F - \delta Z_F\right)\delta(1-x)}$$

Matching Factor Results

- Unpolarized PDF:

Finite p^z

$$\tilde{q}^{(1)}(x,\mu,P^z) = \frac{\alpha_S C_F}{2\pi}\begin{cases} \frac{1+x^2}{1-x}\ln\frac{x}{x-1} + 1 + \frac{\mu}{(1-x)^2 P^z}\ , & x > 1\ , \\ \frac{1+x^2}{1-x}\ln\frac{(P^z)^2}{m^2} + \frac{1+x^2}{1-x}\ln\frac{4x}{1-x} - \frac{4x}{1-x} + 1 + \frac{\mu}{(1-x)^2 P^z}\ , & 0 < x < 1\ , \\ \frac{1+x^2}{1-x}\ln\frac{x-1}{x} - 1 + \frac{\mu}{(1-x)^2 P^z}\ , & x < 0\ , \end{cases}$$

IMF Limit

$$\tilde{q}^{(1)}(x) = \frac{\alpha_S C_F}{2\pi}\begin{cases} 0\ , & x > 1 \text{ or } x < 0\ , \\ \frac{1+x^2}{1-x}\ln\frac{\mu^2}{m^2} - \frac{1+x^2}{1-x}\ln(1-x)^2 - \frac{2x}{1-x}\ , & 0 < x < 1\ . \end{cases}$$

Mathcing Factor

$$Z^{(1)} = \frac{\alpha_S C_F}{2\pi}\begin{cases} \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{\xi}{\xi-1} + 1 + \frac{1}{(1-\xi)^2}\frac{\mu}{P^z}\ , & \xi > 1\ , \\ \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{(P^z)^2}{\mu^2} + \left(\frac{1+\xi^2}{1-\xi}\right)\ln[4\xi(1-\xi)] - \frac{2\xi}{1-\xi} + 1 + \frac{1}{(1-\xi)^2}\frac{\mu}{pz}\ , & 0 < \xi < 1\ , \\ \left(\frac{1+xi^2}{1-\xi}\right)\ln\frac{\xi-1}{\xi} - 1 + \frac{1}{(1-\xi)^2}\frac{\mu}{P^z}, & y < 0\ . \end{cases}$$

$$+\left(\tilde{Z}_F^{(1)} - Z_F^{(1)}\right)\delta(1-x)$$

Single, Double Pole

- Single Pole (S. P.) and Double Pole (D. P.) $(1-\xi)^{-1},\ (1-\xi)^{-2}$ originate form gluon propagator in axial gauge
- S. P. regularized by plus prescription

$$\int_0^1 d\xi\ \frac{f(\xi)}{(1-\xi)_+} = \int_0^1 d\xi\ \frac{f(\xi)-f(1)}{1-\xi}$$

- D. P. is associated with linear divergent term $\frac{\mu}{(1-\xi)^2 P^z}$ which disappear in Dim. Reg. and large P^z limit Reduce to S. P after including Z_F and regularized by P.V.

- Helicity distribution

Finite p^z

$$\Delta\tilde{q}^{(1)}(x) = \frac{\alpha_S C_F}{2\pi}\begin{cases} \frac{1+x^2}{1-x}\ln\frac{x}{x-1} + 1 + \frac{\mu}{(1-x)^2 P^z}\ , & x > 1\ , \\ \frac{1+x^2}{1-x}\ln\frac{(P^z)^2}{m^2} + \frac{1+x^2}{1-x}\ln\frac{4x}{1-x} - \frac{4}{1-x} + 2x + 3 + \frac{\mu}{(1-x)^2 P^z}\ , & 0 < x < 1\ , \\ \frac{1+x^2}{1-x}\ln\frac{x-1}{x} - 1 + \frac{\mu}{(1-x)^2 P^z}\ , & x < 0\ . \end{cases}$$

IMF Limit

$$\Delta\tilde{q}^{(1)}(x) = \frac{\alpha_S C_F}{2\pi}\begin{cases} 0\ , & x > 1 \text{ or } x < 0\ , \\ \frac{1+x^2}{1-x}\ln\frac{\mu^2}{m^2} - \frac{1+x^2}{1-x}\ln(1-x)^2 - \frac{2}{1-x} + 2x\ , & 0 < x < 1\ . \end{cases}$$

Mathcing Factor

$$\Delta Z^{(1)} = \frac{\alpha_S C_F}{2\pi}\begin{cases} \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{\xi}{\xi-1} + 1 + \frac{1}{(1-\xi)^2}\frac{\mu}{P^z}\ , & \xi > 1\ , \\ \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{(P^z)^2}{\mu^2} + \left(\frac{1+\xi^2}{1-\xi}\right)\ln\left[4\xi(1-\xi)\right] - \frac{2}{1-\xi} + 3 + \frac{1}{(1-\xi)^2}\frac{\mu}{pz}\ , & 0 < \xi < 1\ , \\ \left(\frac{1+xi^2}{1-\xi}\right)\ln\frac{\xi-1}{\xi} - 1 + \frac{1}{(1-\xi)^2}\frac{\mu}{P^z}, & y < 0\ . \end{cases}$$

$$+\left(\tilde{Z}_F^{(1)} - Z_F^{(1)}\right)\delta(1-x)$$

- Transversity distribution:

Finite p^z

$$\delta\tilde{q}^{(1)}(x) = \frac{\alpha_S C_F}{2\pi}\begin{cases} \frac{2x}{1-x}\ln\frac{x}{x-1} + \frac{\mu}{(1-x)^2 P^z}\ , & x>1\ , \\ \frac{2x}{1-x}\ln\frac{(P^z)^2}{m^2} + \frac{2x}{1-x}\ln\frac{4x}{1-x} - \frac{4x}{1-x} + \frac{\mu}{(1-x)^2 P^z}\ , & 0<x<1\ , \\ \frac{2x}{1-x}\ln\frac{x-1}{x} + \frac{\mu}{(1-x)^2 P^z}\ , & x<0\ , \end{cases}$$

IMF Limit

$$\delta q^{(1)}(x) = \frac{\alpha_S C_F}{2\pi}\begin{cases} 0\ , & x>1 \text{ or } x<0\ , \\ \frac{2x}{1-x}\ln\frac{\mu^2}{m^2} - \frac{2x}{1-x}\ln(1-x)^2 - \frac{2x}{1-x}\ , & 0<x<1\ . \end{cases}$$

Mathcing Factor

$$\delta Z^{(1)} = \frac{\alpha_S C_F}{2\pi}\begin{cases} \left(\frac{2\xi}{1-\xi}\right)\ln\frac{\xi}{\xi-1} + \frac{1}{(1-\xi)^2}\frac{\mu}{P^z}\ , & \xi>1\ , \\ \left(\frac{2\xi}{1-\xi}\right)\ln\frac{(P^z)^2}{\mu^2} + \left(\frac{2\xi}{1-\xi}\right)\ln[4\xi(1-\xi)] - \frac{2\xi}{1-\xi} + \frac{1}{(1-\xi)^2}\frac{\mu}{pz}\ , & 0<\xi<1\ , \\ \left(\frac{2\xi}{1-\xi}\right)\ln\frac{\xi-1}{\xi} + \frac{1}{(1-\xi)^2}\frac{\mu}{P^z}, & y<0\ . \end{cases}$$

$$+\left(\tilde{Z}_F^{(1)} - Z_F^{(1)}\right)\delta(1-x)$$

- The quasi dis. captures all the collinear behavior of the LC dis.
- Also working on quasi-TMDs, quasi-GPDs…

PART V

TMDs and GPDs on Euclidean lattice (under working)

Quasi TMDs

- The $k_\perp$ unintegrated dis.
- Has the same collinear behavior as the L.C.

Definitions

$$\tilde{q}\left(x,k_\perp\right)=\int\frac{dzd^2\vec{r}_\perp}{4\pi}e^{i\left(xP^zz+\vec{k}_\perp\vec{r}_\perp\right)}\left\langle P\left|\bar{\psi}\left(\vec{r}_\perp,z\right)\mathcal{L}^\dagger\left[\infty;\left(\vec{r}_\perp,z\right)\right]\right.\right.$$
$$\left.\left.\tilde{\Gamma}\mathcal{L}^\dagger\left[\infty;(0)\right]\psi\left(0\right)\right|P\right\rangle$$

Unpolarized, finite p^z

$$
\begin{aligned}
q^z(x,k_\perp) = \frac{C_F\alpha_s}{2p_z^2\pi}\Bigg(&\frac{1}{(-1+x)^2}\,\frac{p_z^5(-1+x)^4 + k_\perp^2 p^z m^2 x - p_z^3\left(-p^0\sqrt{k_\perp^2+p_z^2(-1+x)^2}(-1+x)^3 + k_\perp^2 x\right)}{\left(\sqrt{(p_z^2+m^2)\left(k_\perp^2+p_z^2(-1+x)^2\right)}+p_z^2(-1+x)\right)^2\sqrt{k_\perp^2+p_z^2(-1+x)^2}} \\
&+\frac{1}{x-1}\,\frac{p^z\left(-2m^6 - k_\perp^4 p_0^2 + p_z^4x^2\left(-1+x^2\right)\left(p_z^2x+p^0\sqrt{k_\perp^2+m^2+p_z^2x^2}\right) - 2m^4\left(-p_z^2(-3+x)x+p^0\sqrt{k_\perp^2+m^2+p_z^2x^2}\right)\right)}{\left(k_\perp^2+m^2+p_z^2x^2\right)^{3/2}\left(m^2+p_z^2x+p^0\sqrt{k_\perp^2+m^2+p_z^2x^2}\right)^2} \\
&+\frac{1}{x-1}\,\frac{p^z\left(p_z^2m^2\left(2p_z^2x^2\left(-2+x^2\right)+\left(1-6x+3x^2\right)p^0\sqrt{k_\perp^2+m^2+p_z^2x^2}\right)+k_\perp^2\left(-3m^4-m^2\left(p_z^2\left(1+4x-x^2\right)+2p^0\sqrt{k_\perp^2+m^2+p_z^2x^2}\right)\right)\right)}{\left(k_\perp^2+m^2+p_z^2x^2\right)^{3/2}\left(m^2+p_z^2x+\sqrt{(p_z^2+m^2)\left(k_\perp^2+m^2+p_z^2x^2\right)}\right)^2} \\
&+\frac{1}{x-1}\,\frac{p^z(-1+x)\left(+k_\perp^2\left(+p_z^2\left(p_z^2(-2+x)x^2+(-1+x)^2p^0\sqrt{k_\perp^2+m^2+p_z^2x^2}\right)\right)\right)}{\left(k_\perp^2+m^2+p_z^2x^2\right)^{3/2}\left(m^2+p_z^2x+p^0\sqrt{k_\perp^2+m^2+p_z^2x^2}\right)^2}\Bigg)
\end{aligned}
$$

Unpolarized IMF limit

$$
q^z_{IMF} = \begin{cases} \dfrac{C_F\alpha_s}{\pi}\,\dfrac{\left(1+x^2\right)k_\perp^2+(1-x)^4m^2}{(1-x)\left[k_\perp^2+(1-x)^2m^2\right]^2} & 0<x<1 \\ 0 & \text{other} \end{cases}
$$

Helicity, finite p^z

$$q^{z,5}=-\frac{C_F\alpha_s}{\pi p_z^2}\left(\frac{1}{(-1+x)^2}\frac{k_\perp^2 p^0\left(m^2+p_z^2x\right)-p_z^4(-1+x)^3\left(-p^0+\sqrt{k_\perp^2+p_z^2(-1+x)^2}+\sqrt{p_z^2+m^2}x\right)}{\left(p^0\sqrt{k_\perp^2+p_z^2(-1+x)^2}+p_z^2(-1+x)\right)^2\sqrt{k_\perp^2+p_z^2(-1+x)^2}}\right.$$
$$-\frac{1}{x-1}\frac{p_z^2\left(-k_\perp^4p^0+(-1+x)(m^2+p_z^2x^2)\left(2m^2\left(p^0+\sqrt{k_\perp^2+m^2+p_z^2x^2}\right)+p_z^2(1+x)\left(xp^0+\sqrt{k_\perp^2+m^2+p_z^2x^2}\right)\right)\right)}{(k_\perp^2+m^2+p_z^2x^2)^{3/2}\left(m^2+p_z^2x+p^0\sqrt{k_\perp^2+m^2+p_z^2x^2}\right)^2}$$
$$\left.-\frac{1}{x-1}\frac{p_z^2\left(k_\perp^2\left(m^2p^0(-2+x)+p_z^2\left(x^3p^0+\sqrt{k_\perp^2+m^2+p_z^2x^2}-2x\sqrt{k_\perp^2+m^2+p_z^2x^2}+x^2\left(-2p^0+\sqrt{k_\perp^2+m^2+p_z^2x^2}\right)\right)\right)\right)}{(k_\perp^2+m^2+p_z^2x^2)^{3/2}\left(m^2+p_z^2x+p^0\sqrt{k_\perp^2+m^2+p_z^2x^2}\right)^2}\right)$$

Helicity IMF limit

$$q_{IMF}^{z,5}=\begin{cases}\frac{C_F\alpha_s}{\pi}\frac{\left(1+x^2\right)k_\perp^2-(1-x)^4m^2}{(1-x)\left[k_\perp^2+(1-x)^2m^2\right]^2} & 0<x<1\\ 0 & \text{other}\end{cases}$$

Transveristy, finite p^z, μ

$$q^{z,\perp} = \frac{C_F\alpha_s}{2\pi p^z}\left(\frac{1}{(1-x)^2}\frac{x p_0^2 k_\perp^2}{\left(p^0\sqrt{k_\perp^2+{p_z}^2(-1+x)^2}+{p_z}^2(-1+x)\right)^2\sqrt{k_\perp^2+{p_z}^2(-1+x)^2}}\right.$$

$$+\frac{1}{1-x}\frac{p_0^2k_\perp^4+m^2\left(2m^4+2p_z^2x\left(p_z^2x+p^0\sqrt{k_\perp^2+m^2+{p_z}^2x^2}\right)+m^2\left(p_z^2(1+x)^2+2p^0\sqrt{k_\perp^2+m^2+{p_z}^2x^2}\right)\right)}{(k_\perp^2+m^2+p_z^2x^2)^{3/2}\left(m^2+p_z^2x+p^0\sqrt{k_\perp^2+m^2+{p_z}^2x^2}\right)^2}$$

$$\left.+\frac{1}{1-x}\frac{k_\perp^2\left(3m^4+p_z^4x^2+m^2\left(p_z^2\left(2+x+x^2\right)+2p^0\sqrt{k_\perp^2+m^2+p_z^2x^2}\right)\right)}{(k_\perp^2+m^2+{p_z}^2x^2)^{3/2}\left(m^2+{p_z}^2x+p^0\sqrt{k_\perp^2+m^2+{p_z}^2x^2}\right)^2}\right)$$

Transversity IMF limit

$$q_{IMF}^{z,\perp} = \begin{cases} \frac{C_F\alpha_s}{\pi}\frac{2xk_\perp^2 S^\perp}{(1-x)\left[k_\perp^2+(1-x)^2m^2\right]^2} & 0<x<1 \\ 0 & \text{other} \end{cases}$$

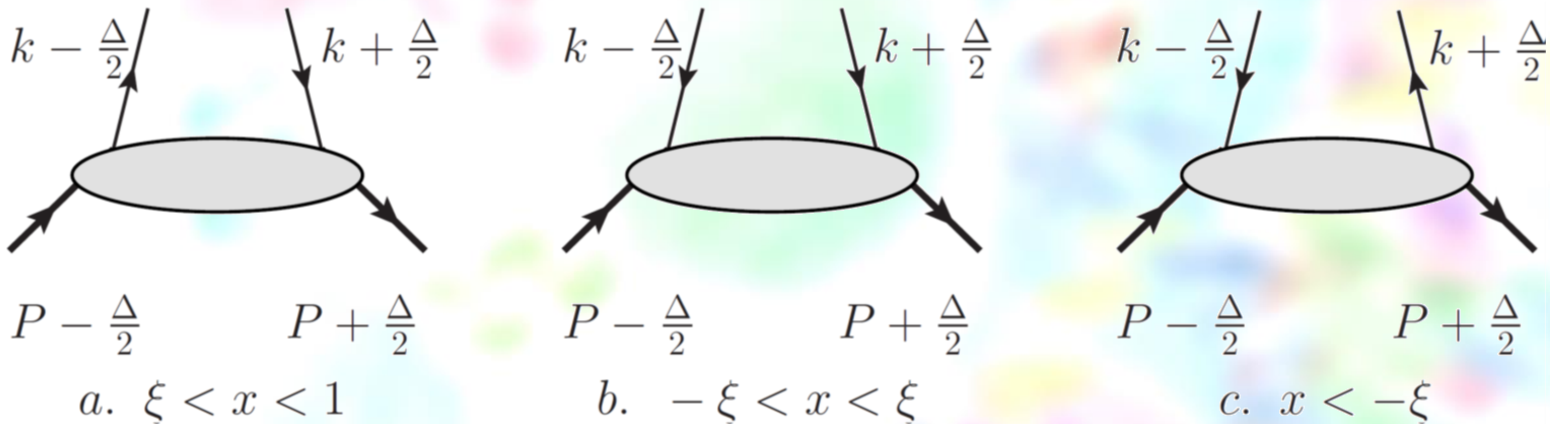
Matching on GPDs

- Tree level only has $H(x,\xi,\Delta^2)$
  Unpolarized quark GPD

$$\int \frac{d\xi^-}{2\pi} e^{-ixp^z z} \left\langle p+\frac{\Delta}{2}, S \middle| \bar{\psi}(-\frac{z}{2})\gamma^z \mathcal{L}[-\frac{z}{2};\frac{z}{2}]\psi(\frac{z}{2}) \middle| p-\frac{\Delta}{2}, S \right\rangle$$
$$= H(x,\xi,\Delta^2)\bar{U}(p+\frac{\Delta}{2})\gamma^z U(p-\frac{\Delta}{2})$$
$$+ E(x,\xi,\Delta^2)\bar{U}(p+\frac{\Delta}{2})\frac{i\sigma^{z\rho}\Delta_\rho}{2m}U(p-\frac{\Delta}{2})$$

- Momentum fraction: $x = k^z/p^z$
- Skewness: $\xi = \Delta^z/2p^z$

$x,\ \xi$ Regions



- a. take out a quark then insert back
- b. take out a quark anti-quark pair
- c. take out a anti-quark then insert back

- $k^-,\ k^0$ poles in $\frac{i(\not{k}\pm\not{\Delta}/2-m)}{(k\pm\Delta/2)^2-m^2+i\epsilon},\quad \frac{D_{\mu\nu}(p-k)}{(p-k)^2+i\epsilon}$

L. C. : $k^- = \mp\frac{\Delta^-}{2} + \frac{(\boldsymbol{k}_\perp-\boldsymbol{\Delta}_\perp/2)^2+m^2-i\epsilon}{2P^+(x\pm\xi)}$ and $k^- = p^- + \frac{-(\boldsymbol{p}_\perp-\boldsymbol{k}_\perp)^2+i\epsilon}{2(1-x)}$

$$x < 1 \implies \int dk^-(\cdots) \neq 0$$

Quasi: $k^0 = -\frac{\Delta^0}{2} \pm \sqrt{(\boldsymbol{k}+\boldsymbol{\Delta}/2)^2+m^2} \mp i\epsilon$

$$k^0 = \frac{\Delta^0}{2} \pm \sqrt{(\boldsymbol{k}-\boldsymbol{\Delta}/2)^2+m^2} \mp i\epsilon$$

and $k^0 = p^0 \pm \sqrt{(\boldsymbol{p}-\boldsymbol{k})^2} \mp i\epsilon$

$$x \in \mathbb{R} \implies \int dk^0(\cdots) \neq 0$$

- Complexity : involve mother parton’s transverse momentum, transverse cut reg. scheme

Further More…

- Continuous limit and lattice lagrangian matching
- Quasi pion distribution amplitude, Wigner distribution, LCWF, higher-twist distributions…

Thanks！

Backup Slices

Spin, Polarization and Helicity

- Transverse AM doesn't commute with the longitudinal boost

$$[J_\perp, K_z] \neq 0 \quad \Longrightarrow \quad \text{Frame dependent}$$

- Pauli-Lubanski Vector $W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma}$

$$W^\perp = \epsilon^{-+\perp\sigma}\left(P^+ J^{-\sigma} - P^- J^{+\sigma}\right)$$

Boost ($J^{-\sigma}$) AM ($J^{+\sigma}$)

Frame independent

- NR

  Spin Operator $\hat{\vec{S}} = \frac{1}{2}\vec{\sigma}$

  Polarized along $\vec{n}$: eigen state of $\vec{n} \cdot \hat{\vec{S}}$

- SR

  Polarized along n^μ ($n^2 = -1, \; n \cdot P = 0$):

  eigen state of $-n \cdot W/M$

  (1) rest frame $-W_i/M = J_i$

  (2) trans. polarized $n^\mu = (0, \vec{n}_\perp, 0)$

  not trans. spin

  $-n \cdot W/M = \gamma^0 J_\perp \neq J_\perp \Longrightarrow$ trans. polarization

  (3) long. polarized $n^\mu = \left(0, \mathbf{0}_\perp, \vec{P}/|\vec{P}|\right)$

  $-n \cdot W/M = \vec{J} \cdot \vec{P}/|\vec{P}| \Longrightarrow$ helicity

Long. Mom. Dis.

- Operator definition

$$\lim_{\Delta_\perp\to 0} x\int\frac{\mathrm{d}\lambda}{4\pi}\, e^{i\lambda x}\langle P+\tfrac{\Delta_\perp}{2},S^\perp|\bar\psi\big(-\tfrac{\lambda n}{2},\xi_\perp\big)\,\gamma^+\psi\big(\tfrac{\lambda n}{2},\xi_\perp\big)|P+\tfrac{\Delta_\perp}{2},S^\perp\rangle$$

- Transition on $\mathcal{O}\left(\xi_\perp\right)=e^{i\hat{P}_\perp\xi_\perp}\mathcal{O}(0)e^{-i\hat{P}_\perp\xi_\perp}$

$$\lim_{\Delta_\perp\to 0} x\int\frac{\mathrm{d}\lambda}{4\pi}\, e^{i\lambda x}\langle P+\tfrac{\Delta_\perp}{2},S^\perp|\bar\psi\big(-\tfrac{\lambda n}{2}\big)\,\gamma^+\psi\big(\tfrac{\lambda n}{2}\big)|P+\tfrac{\Delta_\perp}{2},S^\perp\rangle e^{i\Delta_\perp\xi_\perp}$$

- Expand in $\Delta_\perp$

$$\lim_{\Delta_\perp\to 0} x\int\frac{\mathrm{d}\lambda}{4\pi}\, e^{i\lambda x}\langle P,S^\perp|\bar\psi(-\tfrac{\lambda n}{2})\gamma^+\psi(\tfrac{\lambda n}{2})|P,S^\perp\rangle\left(1+i\Delta\xi_\perp\right)$$

$$+\lim_{\Delta_\perp\to 0}\frac{\partial}{\partial\Delta_\perp}\left\{x\int\frac{\mathrm{d}\lambda}{4\pi}e^{i\lambda x}\left\langle P+\tfrac{\Delta_\perp}{2},S^\perp|\cdots|\,P+\tfrac{\Delta_\perp}{2},S^\perp\right\rangle\right\}_{\Delta_\perp=0}\Delta_\perp e^{i\Delta_\perp\xi_\perp}$$

- Apply GPD definition, Gordon Identity

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \left\langle P', S \left| \bar{\psi}\left(-\tfrac{\lambda n}{2}\right) \gamma^{+} \psi\left(\tfrac{\lambda}{2}\right) \right| P, S \right\rangle$$

$$= \overline{U}_S(P')\gamma^{+} U_S(P) H(x,\xi,t) + \overline{U}_S(P') \frac{i\sigma^{+\rho}\Delta_\rho}{2M_N} U_S(P) E(x,\xi,t)$$

- only keeps the linear term in $\frac{\partial}{\partial \boldsymbol{\Delta}_\perp}\{\cdots\}_{\boldsymbol{\Delta}_\perp=0}$ gives

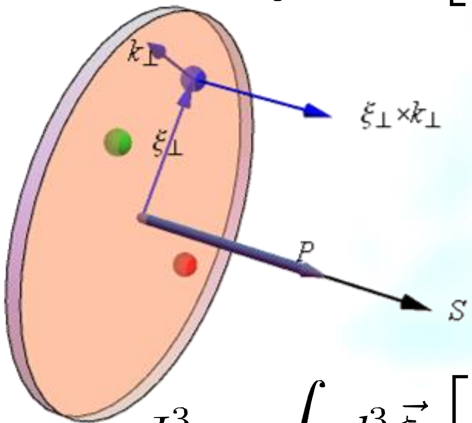
$$\begin{aligned}\rho_q^{+}(x,\xi,S^{\perp}) =& x\int \frac{d\lambda}{4\pi}\langle PS^{\perp}|\bar{\psi}(-\tfrac{\lambda n}{2},\xi)\gamma^{+}\psi(\tfrac{\lambda n}{2},\xi)|PS^{\perp}\rangle \\ =& xP^{+}H(x,0,0) \\ &+\frac{1}{2}xP^{+}\left[H(x,0,0)+E(x,0,0)\right]\lim_{\Delta_\perp\to 0}\frac{S^{\perp\prime}}{M^2}\partial_\xi^{\perp} e^{i\xi_\perp \Delta_\perp}\end{aligned}$$

Longitudinal Polarization

- Longitudinal AM of quark and gluon

$$J_q^3 = \int d^3\vec{\xi}\left[\bar{\psi}\gamma^+\left(\frac{\Sigma^3}{2}\right)\psi + \bar{\psi}\gamma^+\left(\vec{\xi}\times i\vec{\partial}\right)^3\psi - \bar{\psi}\gamma^+\left(\vec{\xi}\times g\vec{A}\right)^3\psi\right]$$

$\frac{\Sigma_q}{2}$, l_q, l_{pot}, L_q



$$J_g^3 = \int d^3\vec{\xi}\left[\epsilon_{\alpha\beta}F^{+\alpha}A^\beta + F^{+i}\left(\vec{\xi}\times\vec{\partial}\right)^3 A_i + \bar{\psi}\gamma^+\left(\vec{\xi}\times g\vec{A}\right)^3\psi\right]$$

Quark OAM Distributions

- Canonical

$$l_q(x) = \frac{\epsilon_\perp^{ij}}{2\pi P^+} \int d^2\xi^\perp \int d\lambda e^i \lambda x \langle PS|\bar{\psi}(0)\gamma^+ \xi^i i\partial_\perp^j \psi(\xi)|PS\rangle$$

$$= \epsilon_\perp^{ij} \frac{i\partial}{\partial_{\perp i}}|_{\Delta=0} \left[\int \frac{d\xi^-}{2\pi} e^{ixP^+\xi^-} \langle P'S|\bar{\psi}(0)\gamma^+ i\partial_\perp^j \psi(\xi)|PS\rangle \right]$$

- Gauge Invariant

$$L_q(x) = \frac{\epsilon_\perp^{ij}}{2\pi P^+} \int d^2\xi^\perp \int d\lambda e^{i\lambda x} \langle PS|\bar{\psi}(0)\gamma^+ \xi^i iD_\perp^j \psi(\xi)|PS\rangle$$

$$= \epsilon_\perp^{ij} \frac{i\partial}{\partial_\perp^i}|_{\Delta=0} \left[\int \frac{d\xi^-}{2\pi} e^{ixP^+\xi^-} \langle P'S|\bar{\psi}(0)\gamma^+ iD_\perp^j \psi(\xi)|PS\rangle \right]$$

- Quark potential AM

Defined through it's moments

$$l^n_{q,\mathrm{pot}} = \frac{-\epsilon_\perp^{\alpha\beta}}{(P^+)^n} \frac{i\partial}{\partial\Delta_{\perp\alpha}}\Big|_{\Delta_\perp=0} \left[\left\langle P'S\left|\bar{\psi}(0)\gamma^+\frac{1}{n}\sum_{k=0}^{n-1}(iD^+)^{n-1-k}gA_\perp^\beta(0)(iD^+)^k\psi(0)\right|PS\right\rangle\right]$$

Inverse M. T.

- Relation between quark AM distribution

$$L_q(x) = l_q(x) + l_{q,\mathrm{pot}}(x)$$

- Analogous for the gluon case

eg. gluon potential AM

$$l^n_{g,\mathrm{pot}} = \frac{-\epsilon_\perp^{\alpha\beta}}{4\pi(P^+)^n} \frac{i\partial}{\partial\Delta_{\perp\alpha}}\Big|_{\Delta_\perp=0} \left[\left\langle P'S\left|\frac{1}{n}\sum_{k=0}^{n-1}F^{+i}(0)(iD^+)^{n-1-k}gA_\perp^\beta(0)(iD^+)^kA^i(0)\right|PS\right\rangle\right]$$

OAM Distribution and GPDs

- Twist-3 GPDs, D-type

quark:

$$\int \frac{d\lambda}{2\pi} \int \frac{d\mu}{2\pi} e^{i\lambda(x-y)} e^{i\mu y} \left\langle P', S \left| \bar{\psi}(0)\gamma^+ iD^\perp(\mu n)\psi(\lambda n) \right| P, S \right\rangle$$

$$= \frac{i\epsilon^{\perp\alpha}}{2} \Delta_\alpha H_D^{q(3)}(x, y, \eta, t) \bar{U}(P')\gamma^+\gamma_5 U(P) + \cdots$$

gluon

$$\int \frac{d\lambda}{2\pi P^+} \int \frac{d\mu}{2\pi} e^{i\lambda(x-y)} e^{i\mu y} \left\langle P', S \left| F^{+i}(0) iD^\perp(\mu n) F^{+i}(\lambda n) \right| P, S \right\rangle$$

$$= \frac{i\epsilon^{\perp\alpha}}{4} \Delta_\alpha H_D^{g(3)}(x, y, \eta, t) \bar{U}(P')\gamma^+\gamma_5 U(P) + \cdots$$

- Twist-3 GPDs, F-type

quark:

$$\int \frac{d\lambda}{2\pi}\int \frac{d\mu}{2\pi P^+} e^{i\lambda(x-y)} e^{i\mu y} \left\langle P', S \left| \bar{\psi}(0)\gamma^+ g F^{+\perp}(\mu n)\psi(\lambda n) \right| P, S \right\rangle$$

$$= \frac{\epsilon^{\perp\alpha}}{2}\Delta_\alpha H_F^{q(3)}(x,y,\eta,t)\bar{U}(P')\gamma^+\gamma_5 U(P) + \cdots$$

gluon:

$$\int \frac{d\lambda}{2\pi P^+}\int \frac{d\mu}{2\pi P^+} e^{i\lambda(x-y)} e^{i\mu y} \left\langle P', S \left| F^{+i}(0) g F^{+\perp}(\mu n) F^{+i}(\lambda n) \right| P, S \right\rangle$$

$$= \frac{\epsilon^{\perp\alpha}}{4}\Delta_\alpha H_F^{g(3)}(x,y,\eta,t)\bar{U}(P')\gamma^+\gamma_5 U(P) + \cdots$$

- Canonical (GIE)

quark:

$$\int \frac{d\lambda}{2\pi} \int \frac{d\mu}{2\pi} e^{i\lambda(x-y)} e^{i\mu y} \left\langle P', S \left| \bar{\psi}(0)\gamma^+ i\tilde{\partial}^\perp(\mu n)\psi(\lambda n) \right| P, S \right\rangle$$

$$= \frac{i\epsilon^{\perp\alpha}}{2} \Delta_\alpha \tilde{H}_q^{(3)}(x, y, \eta, t) \bar{U}(P')\gamma^+\gamma_5 U(P) + \cdots$$

gluon:

$$\int \frac{d\lambda}{2\pi P^+} \int \frac{d\mu}{2\pi} e^{i\lambda(x-y)} e^{i\mu y} \left\langle P', S \left| F^{+i}(0) i\tilde{\partial}^\perp(\mu n) F^{+i}(\lambda n) \right| P, S \right\rangle$$

$$= \frac{i\epsilon^{\perp\alpha}}{4} \Delta_\alpha \tilde{H}_g^{(3)}(x, y, \eta, t) \bar{U}(P')\gamma^+\gamma_5 U(P) + \cdots$$

- OAM distribution and GPDs are related in the forward limit:

quark:

$$L_q^n = \int dx \int dy \, \frac{1}{n} \sum_{k=0}^{n-1} x^{n-1-k} (x-y)^k H_D^{q(3)}(x,y,0,0)$$

$$l_q(x) = \tilde{H}_q^{(3)}(x,0,0)$$

$$l_{q,\mathrm{pot}}^n = - \int dx \int dy \, \frac{1}{n} \sum_{k=0}^{n-1} x^{n-1-k} (x-y)^k P\frac{1}{y} H_F^{q(3)}(x,y,0,0)$$

gluon:

$$L_g^n = \int dx \int dy \, \frac{1}{n} \sum_{k=0}^{n-1} x^{n-1-k} (x-y)^{k-1} H_D^{g(3)}(x,y,0,0)$$

$$l_g(x) = \tilde{H}_g^{(3)}(x,0,0)$$

$$l_{g,\mathrm{pot}}^n = - \int dx \int dy \, \frac{1}{n} \sum_{k=0}^{n-1} x^{n-1-k} (x-y)^{k-1} P\frac{1}{y} H_F^{g(3)}(x,y,0,0)$$

- Relations between twist-3 GPD and OAM distribution

$$H_D^{q/g(3)}(x,y,0,0) = -P\frac{1}{y}H_F^{q,g(3)} + \delta(y)\tilde{H}_{q,g}^{(3)}(x,0,0)$$

$$\Updownarrow$$

$$L_{q/g}(x) = l_{q/g}(x) + l_{q/g,\mathrm{pot}}(x)$$

verified by taking moments

- Longitudinal helicity sum rule

  Angular momentum distribution

$$J_q(x) = \tfrac{1}{2}\Delta\Sigma(x) + l_q(x) + l_{q,\mathrm{pot}}(x)$$

$$J_g(x) = \Delta g(x) + l_g(x) - l_{q,\mathrm{pot}}(x)$$

$$\begin{aligned} J(x) &= J_q(x) + J_g(x) \\ &= \frac{1}{2}\Delta\Sigma(x) + l_q(x) + (x) + l_g(x) \end{aligned}$$

$$\frac{1}{2} = \int dx \left[\frac{1}{2}\Delta\Sigma(x) + l_g(x) + \Delta g(x) + l_g(x)\right]$$

Gauge Invariant Extension

- fixed-gauge result gauge-invariantly extrapolated to any other gauge

eg. gluon spin is not gauge invariant

$$S_g^3 = \int d^3\vec{r} \, \left(\vec{E}_\perp \times \vec{B}_\perp\right)^3$$

gluon helicity operator is gauge invariant

$$S_g^{inv.} = \frac{i}{2} \int \frac{dx}{xP^+} \int d^3\xi e^{ix\xi^- P^+} F^{+\rho}(\xi^-) \mathcal{L}[\xi^-, 0] \tilde{F}^{+}{}_{\rho}(0)$$

and $S_g^{inv.} |_{A^+=0} = S_g^3$

$S_g^{inv.}$ is the GIE of gluon spin

GIE: non-local, no simple Lorentz Transformation
Hard to compute and measure, scale mixing

Impact Parameter Space

[Burkardt, 2002]

- Origin: $|P^+, \boldsymbol{R}_\perp = 0_\perp\rangle = N\int \frac{d^2\boldsymbol{P}_\perp}{(2\pi)^2}|P^+, \boldsymbol{P}_\perp\rangle$

  $\boldsymbol{R}_\perp = \frac{1}{P^+}\int dr^- d^2\boldsymbol{r}_\perp\, \boldsymbol{r}_\perp T^{++}$

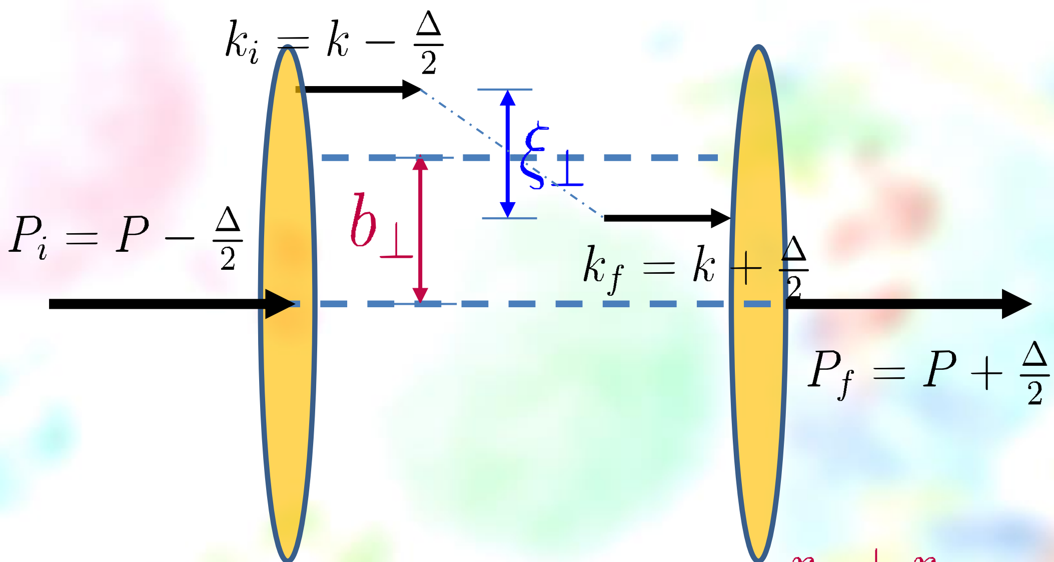
  in parton language $\boldsymbol{R}_\perp = \sum_i x_i \boldsymbol{r}_\perp$ (like CMS)

- Define impact parameter dependent distribution

$$q(x,\boldsymbol{b}_\perp) = |N|^2\int\frac{d\xi^-}{2\pi}e^{ixP^+\xi^-}\left\langle P^+, \boldsymbol{R}_\perp = \boldsymbol{0}\right|\bar{\psi}\left(-\tfrac{\xi^-}{2}, \boldsymbol{b}_\perp\right)\gamma^+\psi\left(\tfrac{\xi^-}{2}, \boldsymbol{b}_\perp\right)\left|P^+, \boldsymbol{R}_\perp = \boldsymbol{0}\right\rangle$$

$$=|N|^2\int\frac{d\xi^-}{2\pi}e^{ixP^+\xi^-}\int\frac{d^2\boldsymbol{P}_\perp}{(2\pi)^2}\int\frac{d^2\boldsymbol{P}'_\perp}{(2\pi)^2}\left\langle P^+,\boldsymbol{P}'_\perp\right|e^{-i\hat{\boldsymbol{P}}'_\perp\boldsymbol{b}_\perp}\bar{\psi}\left(-\tfrac{\xi^-}{2},\boldsymbol{0}_\perp\right)\gamma^+\psi\left(\tfrac{\xi^-}{2},\boldsymbol{0}_\perp\right)e^{i\hat{\boldsymbol{P}}_\perp\boldsymbol{b}_\perp}\left|P^+,\boldsymbol{P}_\perp\right\rangle$$

$$=\int\frac{d^2\boldsymbol{\Delta}_\perp}{2\pi}e^{-i\boldsymbol{\Delta}_\perp\boldsymbol{b}_\perp}\left(|N|^2\int\frac{d^2\bar{\boldsymbol{P}}_\perp}{2\pi}\right)\int\frac{d\xi^-}{2\pi}e^{ixP^+\xi^-}\left\langle P^+,\boldsymbol{P}'_\perp\right|\bar{\psi}\left(-\tfrac{\xi^-}{2},\boldsymbol{0}_\perp\right)\gamma^+\psi\left(\tfrac{\xi^-}{2},\boldsymbol{0}_\perp\right)\left|P^+,\boldsymbol{P}_\perp\right\rangle$$



$$\xi = r_f - r_i \qquad b = \frac{r_f + r_i}{2}$$

$$\Delta = k_f - k_i \qquad k = \frac{k_f + k_i}{2}$$

conjugated variables

$$k_f \cdot r_i - k_f \cdot r_i = \Delta \cdot b - k \cdot \xi$$

GPD: $\int d^2k_\perp \mathrm{GTMD}\left(x, k_\perp, \Delta\right)$ ⇨ $\xi_\perp = 0, \Delta \neq 0$

TMD: $\int d^2k_\perp \mathrm{Wigner}\left(x, k_\perp, b_\perp\right)$ ⇨ $\xi_\perp \neq 0, \Delta = 0$

Trans. Coordinate and OAM

- $$\vec{l} = \sum_n \vec{r}_n \times \vec{p}_n = \sum_n \left(\vec{r}_n - \vec{R}\right) \times \left(\vec{p}_n - x_n \vec{P}\right) + \vec{R} \times \sum_n \left(\vec{p}_n - x_n \vec{P}\right) + \sum_n x_n \vec{r}_n \times \vec{P} = \sum_n \vec{r}_i^{\,\mathrm{rel}} \times \vec{p}_i^{\,\mathrm{rel}} + \vec{R} \times \vec{P}$$

where $\vec{R} = \sum_n x_n \vec{r}_n, \ \sum_n x_n = 1, \ \sum_n \vec{p}_n = \vec{P}$

Gauge Invariant Extension

- GIE of $i\partial_\perp^\alpha$ and $A_\perp^\alpha$

$$i\tilde{\partial}_\perp^\alpha = iD_\perp^\alpha + \int^{\xi^-} d\eta^-\, L_{[\xi^-,\eta^-]} g F^{+\alpha}(\eta^-,\xi_\perp) L_{[\eta^-,\xi^-]}$$

$$\tilde{A}_\perp^\alpha = \int^{\xi^-} d\eta^-\, L_{[\xi^-,\eta^-]} F^{+\alpha}(\eta^-,\xi_\perp) L_{[\eta^-,\xi^-]}$$

$l(x)$ and $l_{\rm pot}(x)$ are made gauge invariant through GIE

GIE reduce to normal $i\partial_\perp^\alpha$, $A_\perp^\alpha$ in light-cone gauge,

also $l(x)$, $l_{\rm pot}(x)$ and their GIE coincide

LCWF

- There are only two parameters in the model
  1. quark mass m
  2. confinement parameter β, enters in the S – wave orbital wave function

Operator Definition of XPDs

- PDF

$$\int \frac{d\xi^-}{2\pi}\, e^{ixp^+\xi^-} \left\langle PS \left| \bar{\psi}(-\frac{\xi^-}{2})\gamma^+ \mathcal{L}[-\frac{\xi^-}{2};\frac{\xi^-}{2}]\psi(\frac{\xi^-}{2}) \right| PS \right\rangle$$

- TMD

$$\int \frac{d\xi d^2k_\perp}{(2\pi)^2}\, e^{i\left(xp^+\xi^- - k^\perp\cdot\xi^\perp\right)} \left\langle PS \left| \bar{\psi}(-\frac{\xi^-}{2},-\frac{\xi^\perp}{2})\gamma^+ \mathcal{L}[-\frac{\xi}{2};\frac{\xi}{2}]\psi(\frac{\xi^-}{2},\frac{\xi^\perp}{2}) \right| PS \right\rangle$$

- GPD

$$\int \frac{d\xi^-}{2\pi}\, e^{ixp^+\xi^-} \left\langle P'S \left| \bar{\psi}(-\frac{\xi^-}{2})\gamma^+ \mathcal{L}[-\frac{\xi^-}{2};\frac{\xi^-}{2}]\psi(\frac{\xi^-}{2}) \right| PS \right\rangle$$

- GTMD

$$\int \frac{d\xi d^2k_\perp}{(2\pi)^2}\, e^{i\left(xp^+\xi^- - k^\perp\cdot\xi^\perp\right)} \left\langle P'S \left| \bar{\psi}(-\frac{\xi^-}{2},-\frac{\xi^\perp}{2})\gamma^+ \mathcal{L}[-\frac{\xi}{2};\frac{\xi}{2}]\psi(\frac{\xi^-}{2},\frac{\xi^\perp}{2}) \right| PS \right\rangle$$

Meilin Transformation

- Melin Moments

$$f^n = \int dx\, x^{n-1} f(x)$$

- Application

  Convert convolution to product (*eg.* Evolution of PDF)

$$\int_0^1 dx\, x^{n-1} \int_x^1 \frac{dy}{y} f(y) g\left(\frac{x}{y}\right) = \boldsymbol{f^n g^n}$$

- Analytical Inverse Transformation

  On a complex plane

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dn\, x^{-n} f^n$$

- Numerical Inverse Transformation

e.g. 1. Least square approximation:

assuming $g(x) = \sum C_k x^k$,minimize $\mathcal{M}(C_k) = \int dx\,[f(x) - g(x)]^2$

$\delta\mathcal{M}(C_k) = 0 = \sum_k \frac{\partial \mathcal{M}^k}{\partial C_k}\delta C_k \rightarrow \frac{\partial \mathcal{M}}{\partial C_k} = 0$ solving C_k

e.g. 2. Fixing Parameterization

Assuming $f(x) \approx g(x, p_1, p_2, \cdots p_n)$, $f^n = g^n(\{p_i\})$, solving $\{p_i\}$

- A distribution function is equivalent to its moments ⟺ *distribution could be defined through it's moments*

δZ_F in axial gauge

- In $n \cdot A = 0$ gauge

$$\Sigma(p) = A(p^2, n\cdot p)\not{p} + \frac{B(p^2, n\cdot p)}{2n\cdot p}\not{n}$$

$$\delta Z_F = A + B$$

- Transverse cut-off breaks Lorentz Symmetry

$$\Sigma(p) = A(p^2, n\cdot p)\not{p} + \frac{B(p^2, n\cdot p)}{2n\cdot p}\not{n} + Cn\cdot p\not{n}$$

$$\delta Z_F = A + B - C = n^\mu \bar{u}(p)\frac{\partial\Sigma(p)}{\partial p^\mu}u(p)$$

S. P. and D. P.

- S. P. regularized by plus-prescription

$$\int_0^1 dx\, \frac{f(x)}{(1-x)_+} = \int_0^1 dx\, \frac{f(x)-f(1)}{1-x}$$

- D. P. reduce to S. P., then regularized by Principle Value prescription

Ward Identity

$$Z_F = q$$

$$\tilde{q}(x) - q(x) = \int_{-1}^{1} \frac{dy}{|y|} Z^{(1)}\left(\frac{x}{y}\right) - \int_{-\infty}^{\infty} d\xi\, Z^{(1)}(\xi)\, q(x)$$

$$= \int_{-\infty}^{\infty} dy \left[Z^{(1)}\left(\frac{x}{y}\right) \frac{q(y)}{|y|} - Z^{(1)}\left(\frac{y}{x}\right) \frac{q(x)}{|x|} \right]$$

- Near $\xi = 1 \implies y = x + x\delta,\ \delta \to 0$

$$Z^{(1)}_{D.P.}\left(\frac{x}{y}\right)\frac{q(y)}{|y|} - Z^{(1)}_{D.P.}\left(\frac{y}{x}\right)\frac{q(x)}{|x|}$$
$$\simeq \frac{1}{\delta^2}\left[\frac{q(x)+q'(x)\delta x)}{|x|\,(1+\delta)} - \frac{q(x)}{|x|}\right]$$
$$\simeq \frac{xq'(x)-q(x)}{\delta\,|x|}$$

The divergence is like $(1-x)^{-1}$ and odd in δ

$$\mathrm{PV}\int_{1-\epsilon}^{1+\epsilon} dx\ \frac{1}{(1-x)} = 0$$

We don't know for higher order in α_s

P.V. prescription on D.P.

$$Z(\xi) \sim P\,\frac{1}{(1-\xi)^2} = \lim_{\epsilon\to 0}\frac{1}{2}\left[\frac{1}{(1-\xi+i\epsilon)^2}+\frac{1}{(1-\xi-i\epsilon)^2}\right]$$

near $\xi = 1 \implies y = x + x\delta,\ \delta \to 0$

$$\begin{aligned}
&Z^{(1)}\!\left(\frac{x}{y}\right)\frac{q(y)}{|y|} - Z^{(1)}\!\left(\frac{y}{x}\right)\frac{q(x)}{|x|}\\
\simeq& Z^{(1)}(1-\delta)\frac{q(x+\delta x)}{|x+\delta x|} - Z^{(1)}(1+\delta)\frac{q(x)}{|x|}\\
\simeq& \frac{1}{2}\left[\frac{1}{(-\delta+i\epsilon)^2}+\frac{1}{(-\delta-i\epsilon)^2}\right]\frac{q(x)+q'(x)x\delta}{|x|\,(1+\delta)}\\
&-\frac{1}{2}\left[\frac{1}{(\delta+i\epsilon)^2}+\frac{1}{(\delta-i\epsilon)^2}\right]\frac{q(x)}{|x|}\\
\simeq& \frac{\delta\,[q(x)-xq'(x)]}{\epsilon^2\,|x|}
\end{aligned}$$

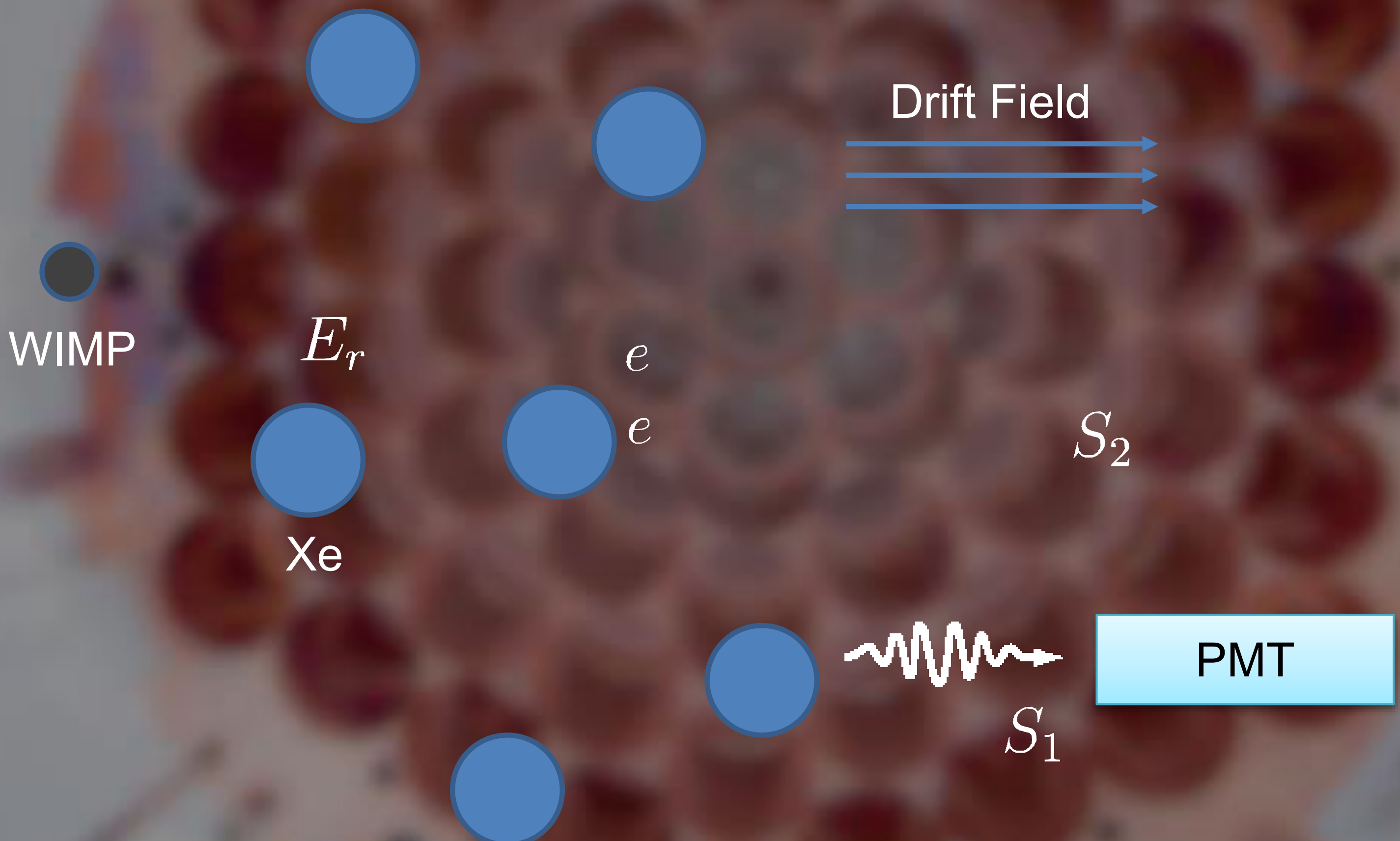
Liquid Xenon Detector

- Monte-Carlo simulation on Effective Scintillation Efficiency
- $\mathcal{L}_{eff}$ is the correspondence between signal detected and WIMP-Nuclei interacting energy

Effective Scintillation Efficiency

$\mathcal{L}_{eff}$

- Dark Matter Direct Detection with Lxe



- $\mathcal{L}_{eff}$ is used to reconstruct the WIMP-Xe interaction energy--

  lack of experiments an theoretical calculation

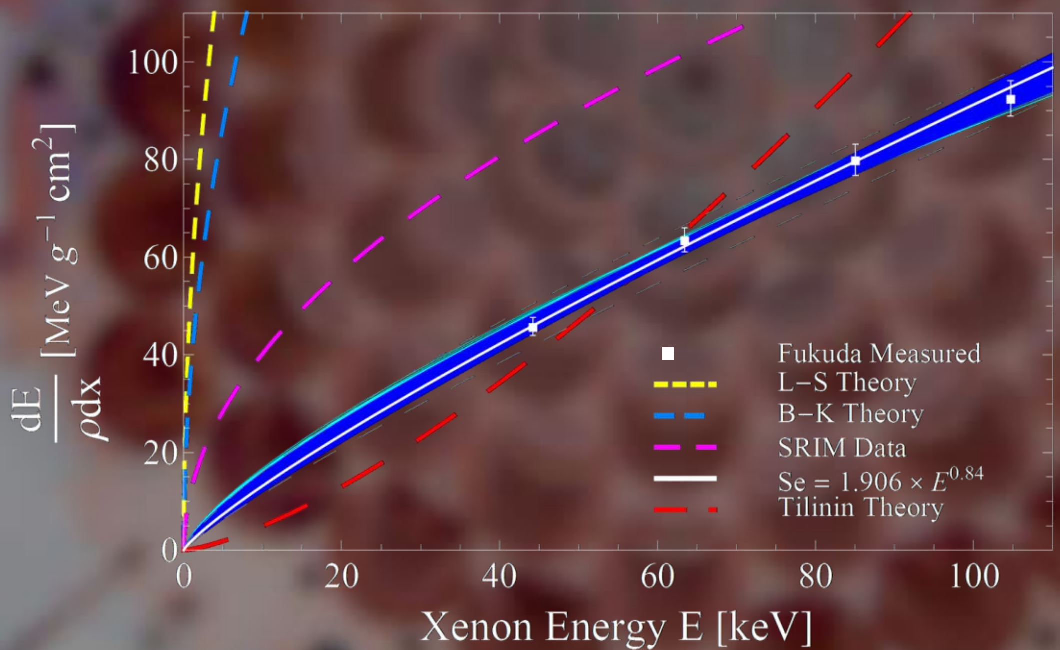
  in low energy region ($E_r \leq 30\text{keV}$)

- Binary collision theory
- Quenching Factors
- *Atom spatial distribution (isotopic not homogenous)*
- Monte-Carlo simulation on cascade

Stopping Power

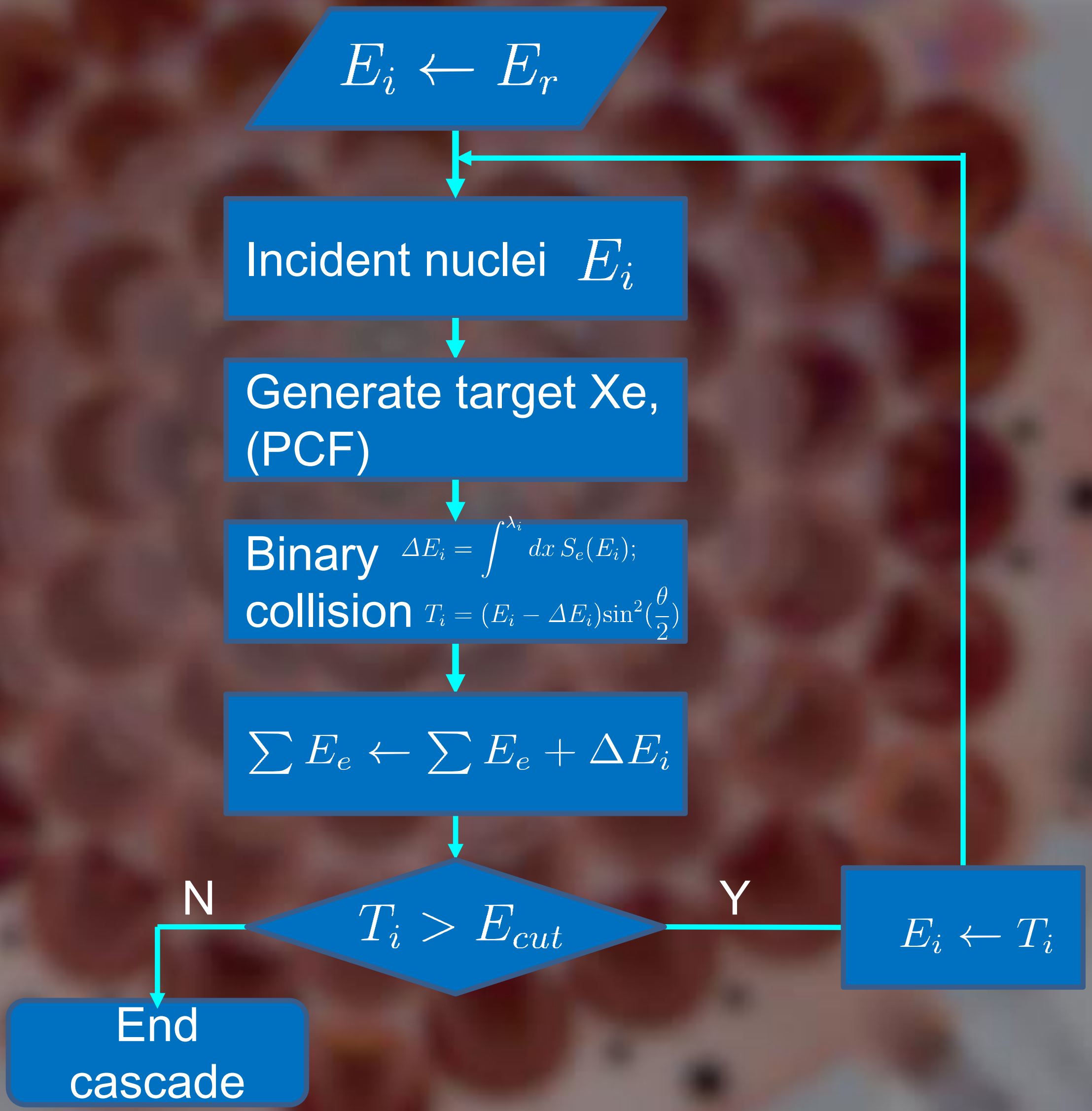
- Describe an energetic charged particle traveling inside a medium

$$S = -\frac{dE}{dx}$$



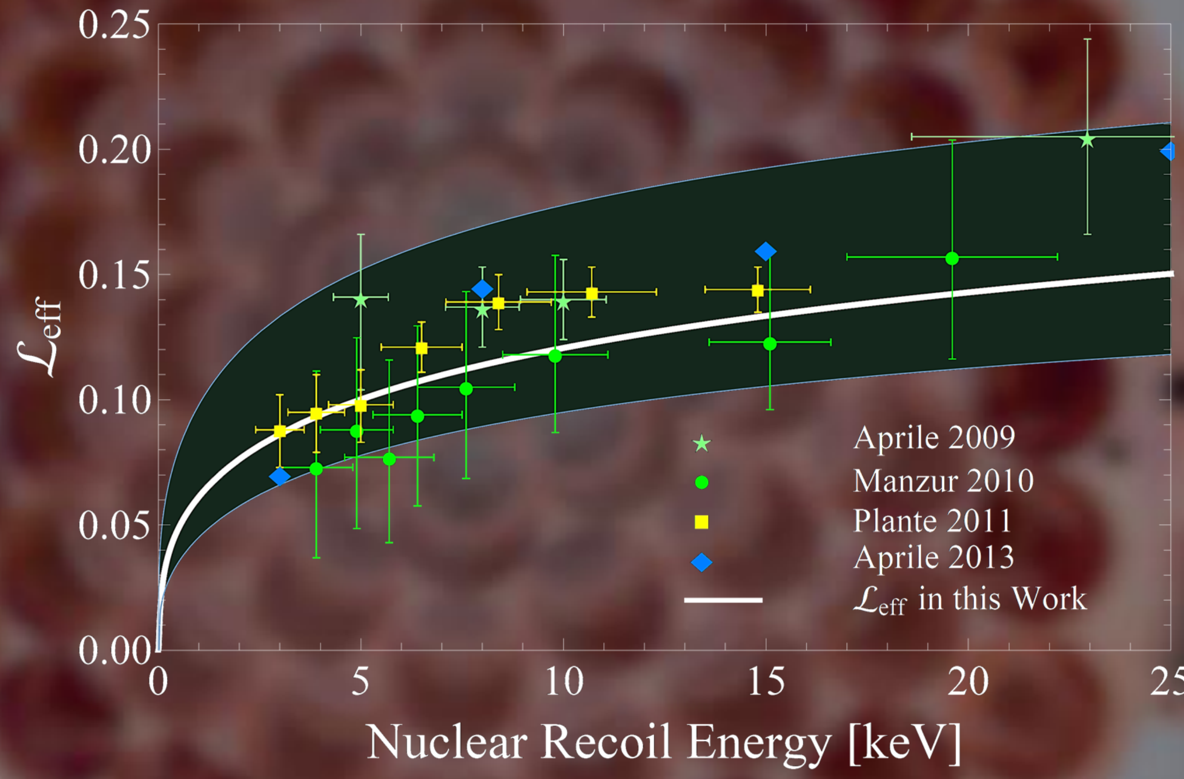
Stopping power (devided by density) for Xe in Xe

Simulation Algorithm



Results for $\mathcal{L}_{eff}$

- $\mathcal{L}_{eff}$

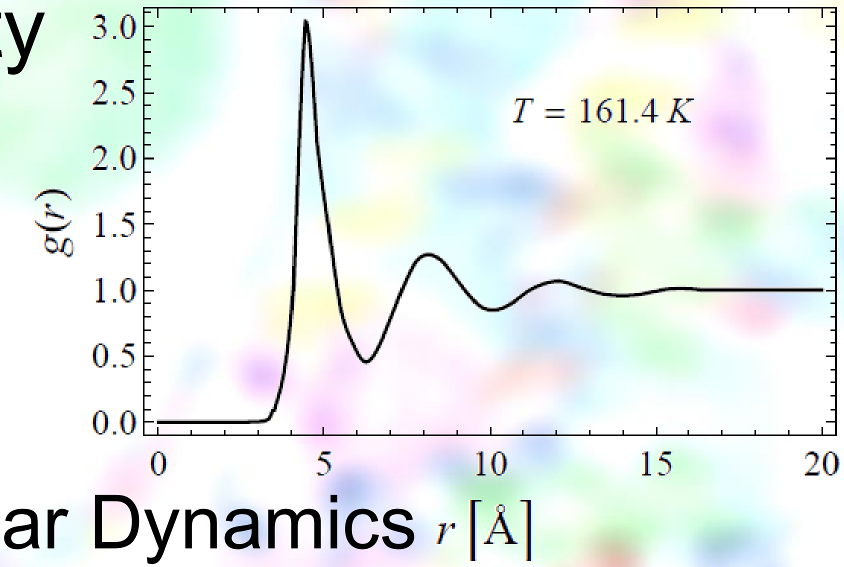


Atom Spatial Distribution

- Pair Correlation Function (PCF)

local/global density

$$\rho(\vec{r}) = g(\vec{r})\rho_0$$



Theoretically: Molecular Dynamics

Experimentally: Neutron Diffraction

Screened Potential

- The interaction between to nuclei dressed by the electrons
- General Form

$$U(r) = \frac{Z_1 Z_2 e^2}{r} \Phi\left(\frac{r}{a}\right)$$

Hartree-Fock Screening radius

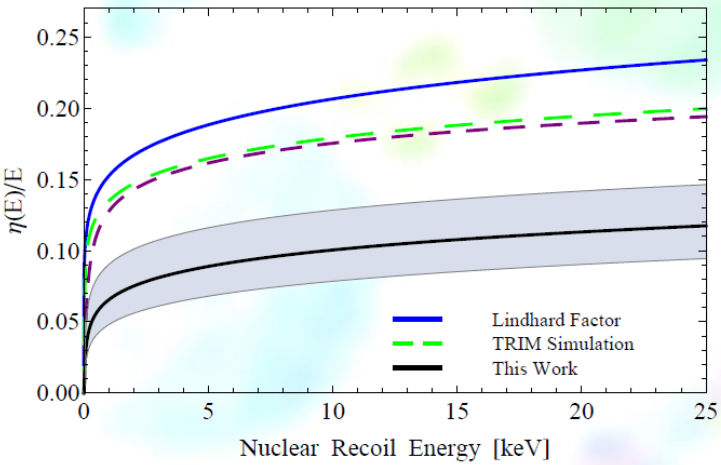
Hartree-Fock Screening Function

$$\Phi(x) = 0.1818e^{-3.2x} + 0.5099e^{-0.9423x} + 0.2802e^{-0.4028x} + 0.02817e^{-0.2016x}$$

Quenching Factor

- Nuclear Quenching Factor

$$q_{nc}(E_r) = \frac{\eta(E_r)}{E_r}$$



- Scintillation Quenching Factor

$$q_{sc}(E_r) = \frac{\eta_{sc}(E_r)}{E_r}$$