Rho Resonance Parameters from Lattice QCD

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Overview

1 Introduction to Lattice QCD

2 Method

3 Results

4 Conclusion
Introduction to Lattice QCD

- Most visible matter in the universe are made up of particles called hadrons.

The interaction between hadrons is dominated by the strong force.

Quantum Chromodynamics (QCD) is a theory to describe the strong interaction between quarks and gluons which make up hadrons.

\[ \mathcal{L}_{QCD} = -\frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \sum_f \bar{\psi}_f \gamma^{\mu} [\partial_{\mu} - igA_\mu] \psi_f - \sum_f m_f \bar{\psi}_f \psi_f, \]  

Some techniques to work with QCD: Perturbation theory, Effective field theory, Lattice QCD and so on.
Introduction to Lattice QCD

For light hadron study, non-perturbative approach is needed. Lattice QCD is a non-perturbative approach to QCD. It formulates QCD in a discrete way.

Inputs:
- lattice geometry $N$
- lattice spacing $a$ set indirectly through the coupling constant $g$
- quark mass represented by pion mass $m_\pi$

For light hadron study, only light quarks $u$ and $d$ are important. $s$ quark introduces only small correction.

The role of Lattice QCD in resonance study is to extract the energy spectrum for two hadron states.
Why we study resonances from Lattice QCD?

- Lattice QCD offers us a way to study the resonances in terms of quark and gluon dynamics. It serves as a test of QCD for well determined resonance parameters.
- The techniques can be used to investigate systems where the experimental situation is less clear.
- Validate effective models used to describe hadron scattering.

How?

- We start from meson resonance because they have better signal-to-noise ratio.
- $\rho(770)$ resonance in $I = 1, J = 1$ $\pi - \pi$ scattering channel.
Symmetries on the lattice

On the lattice, the energy eigenstates $|n>$ of the system are computed in a given irrep of the lattice symmetry group.

$$\psi_n(R^{-1}x) = \psi_n(R^{-1}(x + nL)); \quad \langle \hat{O}_2(t)\hat{O}_1^\dagger(0) \rangle = \sum_n \langle 0|\hat{O}_2|n\rangle \langle n|\hat{O}_1|0\rangle e^{-tE_n} \quad (2)$$

Isospin, color and flavor symmetries are similar to the continuum.

Table: Irreducible representation in $SO(3)$, $O$ and $D_4$

<table>
<thead>
<tr>
<th>irep label</th>
<th>$SO(3)$</th>
<th>cubic box($O_h$)</th>
<th>elongated box($D_{4h}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim</td>
<td>$1, 3, \ldots, 2l + 1, \ldots \infty$</td>
<td>$1, 1, 2, 3, 3$</td>
<td>$1, 1, 2, 2, 2$</td>
</tr>
</tbody>
</table>

Table: Angular momentum mixing among the irreducible representations of the lattice group
Symmetries of the elongated box

\( \rho \) resonance is in \( I = 1, J^P = 1^- \) channel for pion-pion scattering. Elongated box method tunes the momentum of the scattering particles on the lattice \( \mathbf{p} \propto \left( \frac{2\pi}{\eta L} \right) \).

The \( \text{SO}(3) \) symmetry group reduce to discrete subgroup \( O_h \) or \( D_{4h} \)

<table>
<thead>
<tr>
<th>( J )</th>
<th>( O_h )</th>
<th>( D_{4h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( A_1^+ )</td>
<td>( A_1^+ )</td>
</tr>
<tr>
<td>1</td>
<td>( F_1^- )</td>
<td>( A_2^- \oplus E^- )</td>
</tr>
<tr>
<td>2</td>
<td>( E^+ \oplus F_2^+ )</td>
<td>( A_1^+ \oplus B_1^+ \oplus B_2^+ \oplus E^+ )</td>
</tr>
<tr>
<td>3</td>
<td>( A_2^- \oplus F_1^- \oplus F_2^- )</td>
<td>( A_2^- \oplus B_1^- \oplus B_2^- \oplus 2E^- )</td>
</tr>
<tr>
<td>4</td>
<td>( A_1^+ \oplus E^+ \oplus F_1^+ \oplus F_2^+ )</td>
<td>( 2A_1^+ \oplus A_2^+ \oplus B_1^+ \oplus B_2^+ \oplus 2E^+ )</td>
</tr>
</tbody>
</table>

For the p-wave (\( l = 1 \)) scattering channel, we only need to construct the interpolating fields in \( F_1^- \) in the \( O_h \) group, \( A_2^- \) representations in \( D_{4h} \) group because the energy contribution from angular momenta \( l \geq 3 \) is negligible.
Lüscher’s formula for elongated box [1]

Phase shift for $l = 1$, rest frame ($\mathbf{P} = 0$): 

\[
A_2^- : \cot \delta_1(k) = \mathcal{W}_{00} + \frac{2}{\sqrt{5}} \mathcal{W}_{20}
\]

(3)

\[
\mathcal{W}_{lm}(1, q^2, \eta) = \frac{Z_{lm}(1, q^2, \eta)}{\eta \pi^{3/2} q^{l+1}}; \quad q = \frac{kL}{2\pi}; \quad \eta = \frac{N_{el}}{N} : \text{elongation factor}
\]

(4)

(5)

Zeta function

\[
Z_{lm}(s; q^2, \eta) = \sum_{\tilde{n}} \mathcal{W}_{lm}(\tilde{n})(n^2 - q^2)^{-s}; \quad n \in m
\]

(6)

Total energy

\[
E = 2\sqrt{m^2 + k^2}; \quad k = \sqrt{\left(\frac{E}{2}\right)^2 - m^2}
\]

(7)

Lüscher’s formula for boost frame

In order to obtain new kinematic region, we boost the resonance along the elongated direction.

\[ A_2^{-} : \cot \delta_1(k) = \mathcal{W}_{00} + \frac{2}{\sqrt{5}} \mathcal{W}_{20} \]  

\[ \mathcal{W}_{lm}(1, q^2, \eta) = \frac{Z_{lm}^P(1, q^2, \eta)}{\gamma \eta \pi^{\frac{3}{2}} q^{l+1}}; \quad \eta = \frac{N_{el}}{N} : \text{elongation factor}; \quad \gamma : \text{boost factor}; \]  

\[ Z_{lm}^P(s; q^2, \eta) = \sum \mathcal{V}_{lm}(\tilde{n})(\tilde{n}^2 - q^2)^{-s}; \quad n \in \frac{1}{\gamma}(m + \hat{P}) \]
Interpolating field construction for $\rho$ resonance

Four $q\bar{q}$ operators and two scattering operators $\pi\pi$ in $A_2^-$ sector.

$$\rho^J(t_f) = \bar{u}(t_f)\Gamma_{t_f}A_{t_f}(p)d(t_f); \quad \rho^{J\dagger}(t_i) = \bar{d}(t_i)\Gamma_{t_i}^\dagger A_{t_i}^\dagger(p)u(t_i) \quad (12)$$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\Gamma_{t_f}$</th>
<th>$A_{t_f}$</th>
<th>$\Gamma_{t_i}^\dagger$</th>
<th>$A_{t_i}^\dagger$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\gamma_i$</td>
<td>$e^{ip}$</td>
<td>$-\gamma_i$</td>
<td>$e^{-ip}$</td>
</tr>
<tr>
<td>2</td>
<td>$\gamma_4\gamma_i$</td>
<td>$e^{ip}$</td>
<td>$\gamma_4\gamma_i$</td>
<td>$e^{-ip}$</td>
</tr>
<tr>
<td>3</td>
<td>$\gamma_i$</td>
<td>$\nabla_j e^{ip}\nabla_j$</td>
<td>$\gamma_i$</td>
<td>$\nabla_j e^{-ip}\nabla_j$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{2}$</td>
<td>${e^{ip}, \nabla_i}$</td>
<td>$-\frac{1}{2}$</td>
<td>${e^{-ip}, \nabla_i}$</td>
</tr>
</tbody>
</table>

$$\pi\pi_{P,\Lambda,\mu} = \sum_{p_1^*, p_2^*} C(P, \Lambda, \mu; p_1; p_2)\pi(p_1)\pi(p_2), \quad (13)$$

$$\pi\pi_{100}(p_1, p_2, t) = \frac{1}{\sqrt{2}}[\pi^+(p_1)\pi^-(p_2) - \pi^+(p_2)\pi^-(p_1)]; \quad p_1 = (1, 0, 0) \quad p_2 = (-1, 0, 0)$$

$$\pi\pi_{110} = \frac{1}{2}(\pi\pi(110) + \pi\pi(101) + \pi\pi(1 - 10) + \pi\pi(10 - 1))$$
6 × 6 correlation matrix

\[
C = \begin{pmatrix}
C_{\rho \rho} & C_{\rho \pi} & C_{\rho \pi} & C_{\rho \pi} & C_{\rho \pi} & C_{\rho \pi} \\
C_{\pi \rho} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} \\
C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} \\
C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} \\
C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} \\
C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} & C_{\pi \pi} \\
\end{pmatrix}.
\] (14)

The correlation functions: \( \bar{u}(t_i) \rightarrow u(t_f) \)

\[
C_{\rho \rho} = -\left\langle \Gamma_I^{J_f, (p, t_f)} \right\rangle = -\left\langle \text{Tr}[M^{-1}(t_i, t_f) \Gamma_I^{J_f} e^{ipM^{-1}(t_f, t_i) \Gamma_I^{J_f \dagger} e^{-ip}}] \right\rangle.
\] (15)

\[
C_{\pi \pi} = -\left\langle \Gamma_I^{J_f \dagger, (-p, t_i)} \right\rangle = -\left\langle \Gamma_I^{J_f \dagger} e^{-ip} \right\rangle.
\] (17)

\[
C_{\rho \pi} = \left\langle \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}\right\rangle - \left\langle \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}\right\rangle \quad \text{P} = 0 \quad \left\langle \begin{array}{c}
\begin{array}{c}
\end{array}\right\rangle.
\] (16)

\[
P = 0 \left\langle \begin{array}{c}
\begin{array}{c}
\end{array}\right\rangle - 2 \left\langle \begin{array}{c}
\begin{array}{c}
\end{array}\right\rangle + \left\langle \begin{array}{c}
\begin{array}{c}
\end{array} 
\end{array}\right\rangle.
\] (18)
Laplacian Heaviside smearing [2]

To estimate all-to-all propagators:

The 3-dimensional gauge-covariant Laplacian operator

$$\tilde{\Delta}^{ab}(x, y; U) = \sum_{k=1}^{3} \left\{ \tilde{U}_k^{ab}(x) \delta(y, x + \hat{k}) + \tilde{U}_k^{ba}(y)^* \delta(y, x - \hat{k}) - 2 \delta(x, y) \delta^{ab} \right\}. \quad (19)$$

$$S_\Lambda(t) = \sum_{\lambda(t)}^\Lambda |\lambda(t)\rangle \langle \lambda(t)|; \quad \tilde{u}(t) = S(t)u(t) = \sum_{\lambda_t} |\lambda_t\rangle \langle \lambda_t| u(t). \quad (20)$$

Energy spectrum

We implement the calculation in 3 ensembles ($\eta = 1.0, 1.25, 2.0$) at $m_\pi \approx 310$ MeV and 3 ensembles ($\eta = 1.0, 1.17, 1.33$) at $m_\pi \approx 227$ MeV with nHYP-smeared clover fermions and two mass-degenerated quark flavor.

![Graph](image)

**Figure:** The lowest three energy states with their error bars for $\eta = 1.0, m_\pi = 310$ MeV ensemble

We extract energy $E$ by using double exponential $f(t) = we^{-Et} + (1 - w)e^{-E't}$ to do the $\chi^2$ fitting for each eigenvalues.
Expectation for energy states

Figure: The lowest 3 energy states prediction from $U\chi$PT model. When $\eta = 2.0$ the 3rd state is from operator $\pi\pi_{200}$ instead of $\pi\pi_{110}$
\[ \cot(\delta_1(E)) = \frac{M_R^2 - E^2}{E \Gamma_r(E)}, \quad \Gamma_r(E) \equiv \frac{g_{R12}^2}{6\pi} \frac{p^3}{E^2}. \]  
\[ \delta_1(E) = \arccot \frac{6\pi(M_R^2 - E^2)E}{g^2p^3} \]
Phaseshift fitted with $U\chi$PT model

**Figure:** $m_\pi \approx 315$ MeV and $m_\pi \approx 227$ MeV data fitted with $U\chi$PT model.

<table>
<thead>
<tr>
<th></th>
<th>$m_\pi$ (MeV)</th>
<th>$m_\rho$</th>
<th>$\Gamma_\rho$ (MeV)</th>
<th>$g$ (GeV$^{-1}$)</th>
<th>$\chi^2$/dof</th>
</tr>
</thead>
<tbody>
<tr>
<td>Breit Wigner</td>
<td>315</td>
<td>794.6(6)</td>
<td>37.0(2)</td>
<td>5.57(11)</td>
<td>2.16</td>
</tr>
<tr>
<td>$U\chi$PT</td>
<td>795.2(3)</td>
<td>36.1(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Breit Wigner</td>
<td>227</td>
<td>748.4(1.6)</td>
<td>71.0(8)</td>
<td>5.70(12)</td>
<td>1.46</td>
</tr>
<tr>
<td>$U\chi$PT</td>
<td>748.2(7)</td>
<td>77.0(5)</td>
<td></td>
<td></td>
<td>1.53</td>
</tr>
</tbody>
</table>
**$K\bar{K}$ channel contribution to $\rho$ resonance**

We study the $K\bar{K}$ effect using $U\chi$PT model with input parameters $m_\pi$, $f_\pi$, $f_K$ and low energy constant $\hat{h}_{1,2}$.

**Figure:** (Left) Chiral extrapolation of the phase shift to the physical mass (red band), obtained from the simultaneous fit to pion masses. The blue band: phaseshift with $K\bar{K}$. Open circles: experiment data [3]
$m_\rho$ and $g_{\rho\pi\pi}$ comparison

**Figure:** Comparison of different lattice calculation for the $\rho$ resonance mass (left) and width parameter $g_{\rho\pi\pi}$ (right). The errors included here are only stochastic. The band in the left plot indicates a $N_f = 2 + 1$ expectation from $U_\chi$PT model constrained by some older lattice QCD data and some other physical input [4].

The results of ETMC are taken from [5]. PACS result is from [6].
Conclusions

- We complete a precision study of $\rho$ resonance with LapH smearing method and obtain the resonance parameters at $m_\pi \approx 310$ MeV and $m_\pi \approx 227$ MeV.
- For precise energy spectrum results, both Breit Wigner form and $U_\chi$PT model work well in the resonance region. Modification to the BW is needed when applied to a wider energy region.
- The extrapolation of $m_\rho$ to physical pion mass is smaller than $m_\rho^{\text{phy}} = 775$ MeV in a $N_f = 2$ situation, we believe that this comes from the absence of strange quark and the $K\bar{K}$ channel which is supported by our $U_\chi$PT study.


Symmetries on the lattice

The $SO(3)$ symmetry group reduce to discrete subgroup $O_h$ or $D_{4h}$

**Table:** Resolution of $2J + 1$ spherical harmonics into the irreducible representations of $O_h$ and $D_{4h}$

<table>
<thead>
<tr>
<th>$J$</th>
<th>$O_h$</th>
<th>$D_{4h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$A_1^+$</td>
<td>$A_1^+$</td>
</tr>
<tr>
<td>1</td>
<td>$F_1^-$</td>
<td>$A_2^- \oplus E^-$</td>
</tr>
<tr>
<td>2</td>
<td>$E^+ \oplus F_2^+$</td>
<td>$A_1^+ \oplus B_1^+ \oplus B_2^+ \oplus E^+$</td>
</tr>
<tr>
<td>3</td>
<td>$A_2^- \oplus F_1^- \oplus F_2^-$</td>
<td>$A_2^- \oplus B_1^- \oplus B_2^- \oplus 2E^-$</td>
</tr>
<tr>
<td>4</td>
<td>$A_1^+ \oplus E^+ \oplus F_1^+ \oplus F_2^+$</td>
<td>$2A_1^+ \oplus A_2^+ \oplus B_1^+ \oplus B_2^+ \oplus 2E^+$</td>
</tr>
</tbody>
</table>

Assume that the energy contribution from angular momenta $l \geq 3$ is negligible. For example, if we study the p-wave ($l = 1$) scattering channel, we should construct the interpolating field in $F_1^-$ in the $O_h$ group, $A_2^-$ and $E^-$ representations in $D_{4h}$ group.
Variational method is used to extract energy of the excited states. Construct correlation matrix in the interpolator basis

\[
C(t)_{ij} = \langle O_i(t) O_j^\dagger(0) \rangle; \ i, j = 1, 2, ..., \text{number of operators}
\]  

(24)

The eigenvalues of the correlation matrix are

\[
\lambda^{(n)}(t, t_0) \propto e^{-E_n t} (1 + O(e^{-\Delta E_n t})), \ n = 1, 2, ..., \text{number of operators}
\]  

(25)

where \(\Delta E_n = E_{\text{Number of operators} + 1} - E_n\).

Larger energy gap makes the high lying energy decay faster and effective mass plateau appear in an earlier time slice.
Appendix-B: LapH smearing

Benefit from LapH smearing:

- Keep low frequency mode up to \( \Lambda \) cutoff to compute the all to all propagators, \( u(x) \rightarrow u(y) \). The number of propagators \( M^{-1}(t_f, t_i) \) need to be computed reduce from \( 6.34 \times 10^{13} \) in position space to \( 3.7 \times 10^8 \) in momentum space for the \( 24^348 \) ensemble.

- The effective mass reach a plateau in an earlier time slice.

![Diagram](image.png)

**Figure:** pion effective mass with (red) and without LapH smearing (blue)
Appendix-C: Fitting phase-shift

Figure: $\chi^2$ fitting for the phase shift data to Breit Wigner form

$$\chi^2 = \Delta^T \text{COV}^{-1} \Delta$$ (26)

where

$$\Delta_i = \sqrt{s_{i\text{curve}}} - \sqrt{s_{i\text{data}}}$$ (27)
Appendix-D: Experiment data [7]

\[ \Gamma_{BW}(E) = \frac{g^2}{6\pi} \frac{p^3}{E^2} \]
\[ \Gamma_{CF}(E) = \frac{g^2}{6\pi} \frac{p^3}{E^2} \frac{1 + (p_R R)^2}{1 + (p R)^2} \]
\[ \Gamma_{GA}(E) = \frac{g^2}{6\pi} \frac{p^3}{E^2} \frac{e^{-p^2/6\beta^2}}{e^{-p_R^2/6\beta^2}} \]

**Figure:** $\pi \pi$ phase shift below $K \bar{K}$ threshold in experiment

K-matrix method

\[ T^{-1} = V^{-1} - G = \frac{-3(f^2 - 8l_1 m^2 + 4l_2 W^2)}{2p^2} - \text{Re}G(W) + \frac{ip}{8\pi W} \]  

(28)

For K-matrix method the \( \text{Re}G(W) = 0 \).

\[ T = \frac{-8\pi W}{p \cot \delta p - ip} \]  

(29)

\[ l_1 = \frac{1}{8\pi^2} \left( - \frac{1}{2} \frac{m^2}{g^2_{\rho\pi\pi}} + f^2 \right) \]  

(30)

\[ l_2 = - \frac{1}{8g^2_{\rho\pi\pi}} \]  

(31)