Extracting scattering and resonance properties from the lattice

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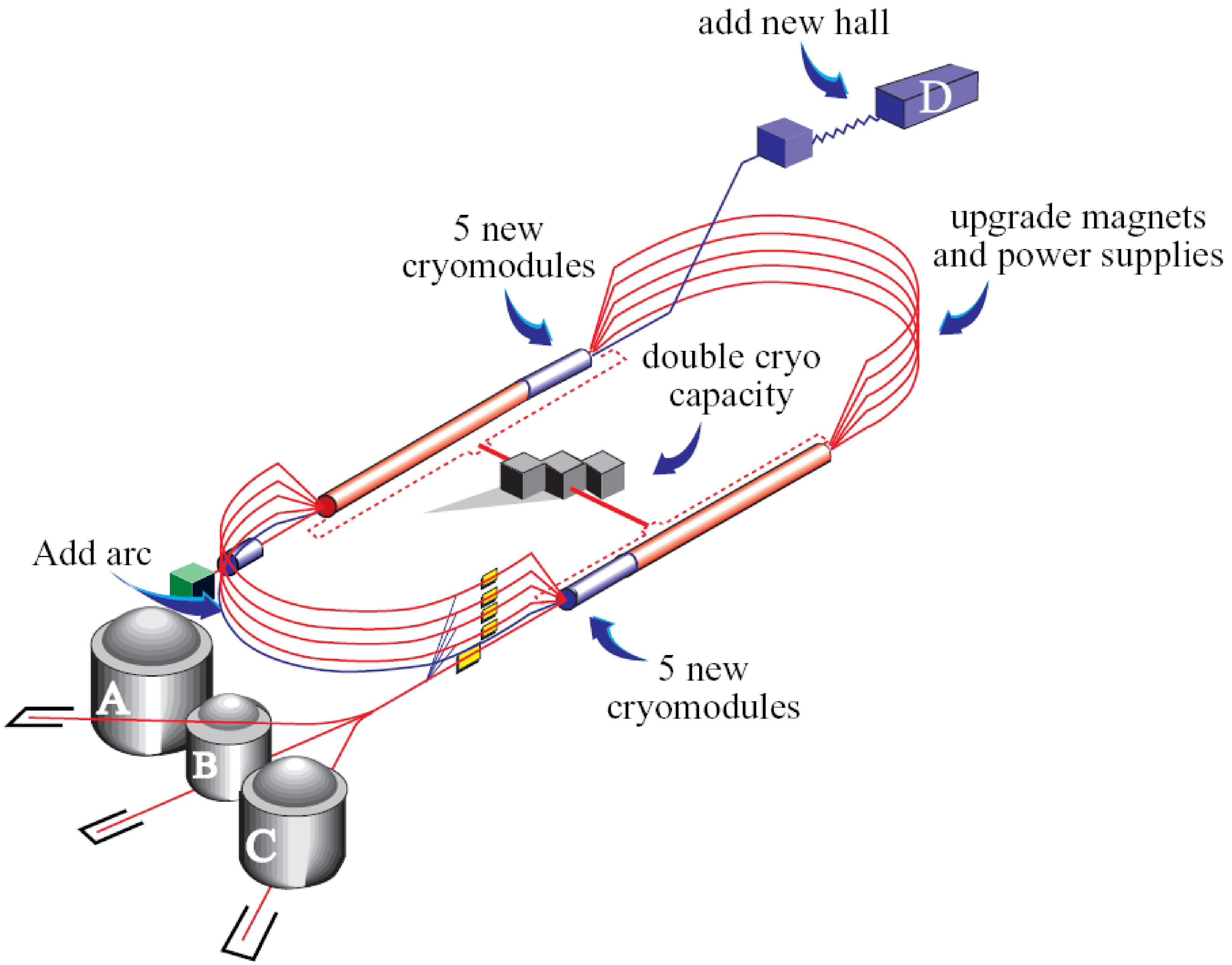
February 10th, 2016

Helmholtz-Institut Mainz

JLab Physics



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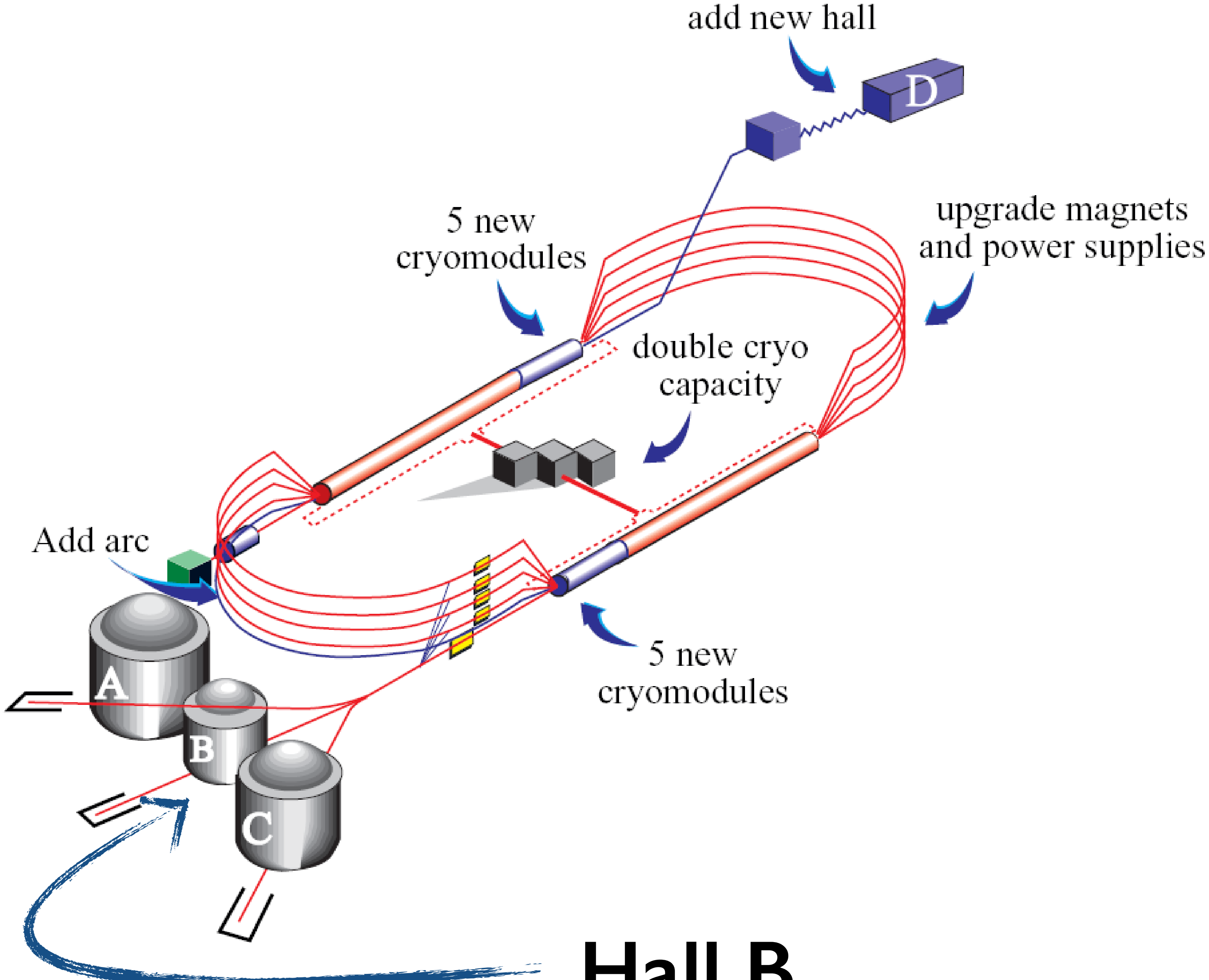

Hall B

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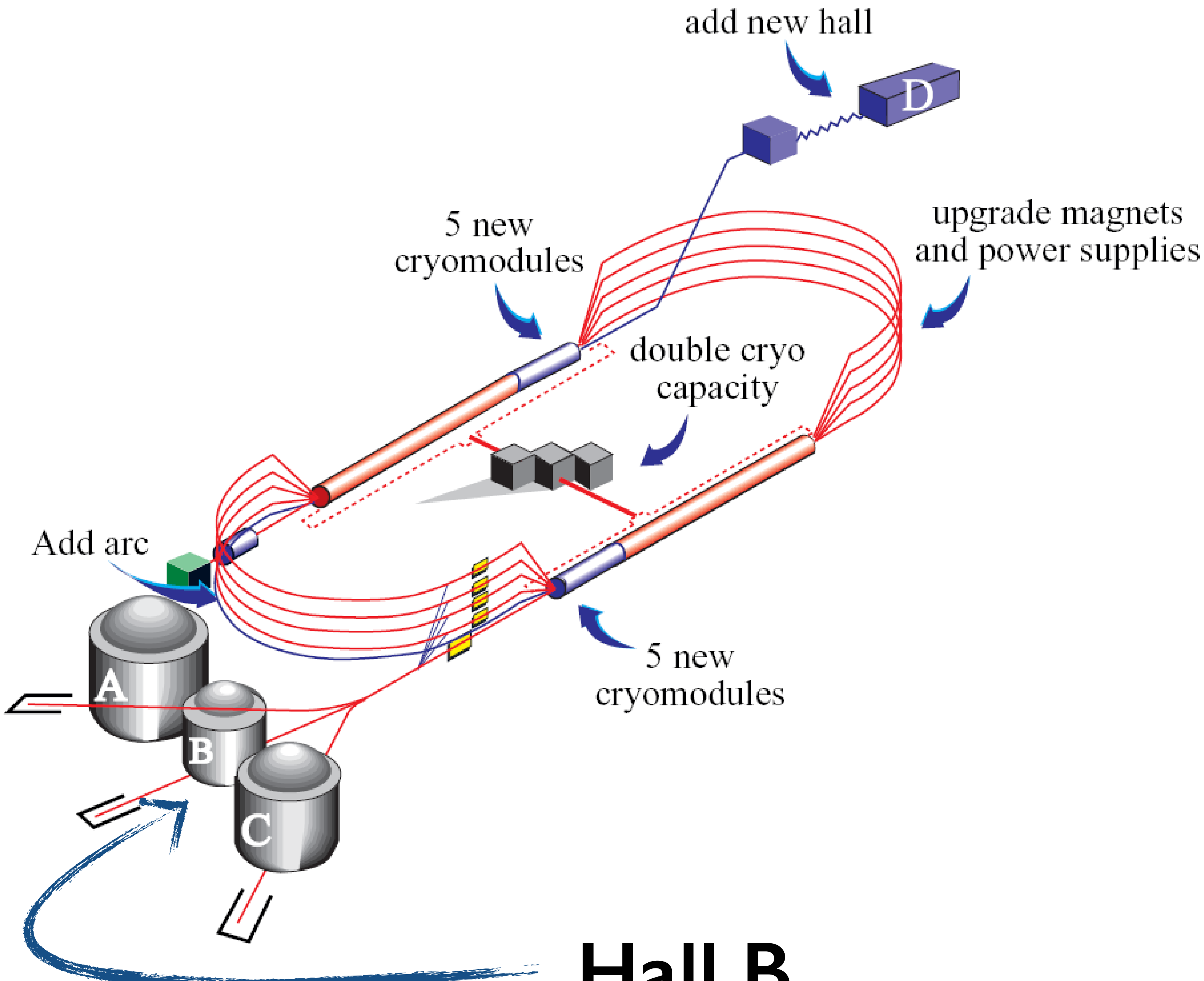

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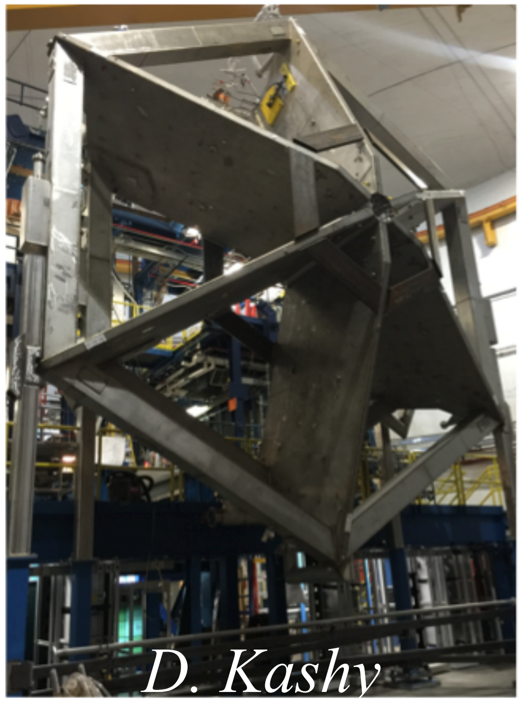

CLAS12 Torus Magnet complete

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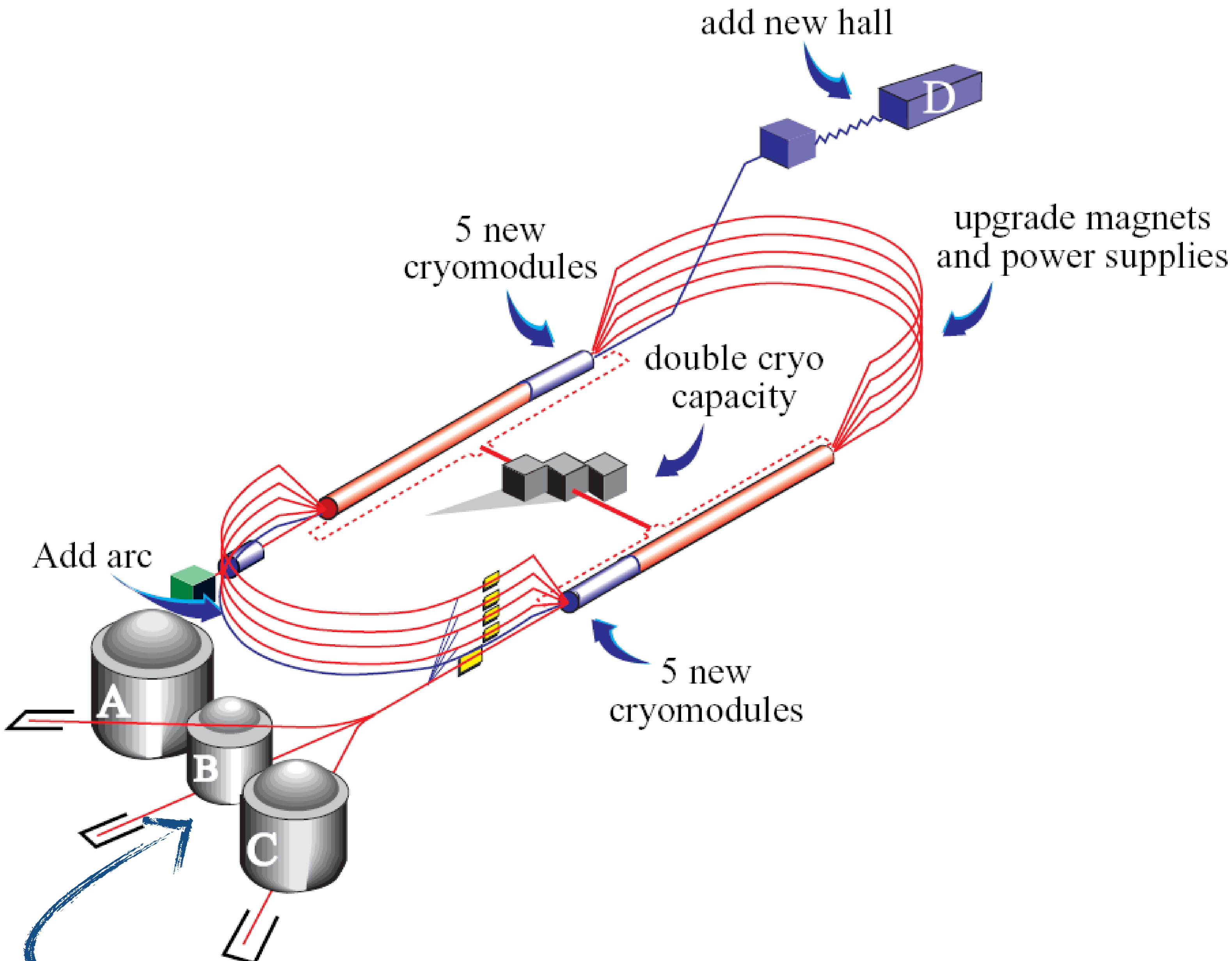

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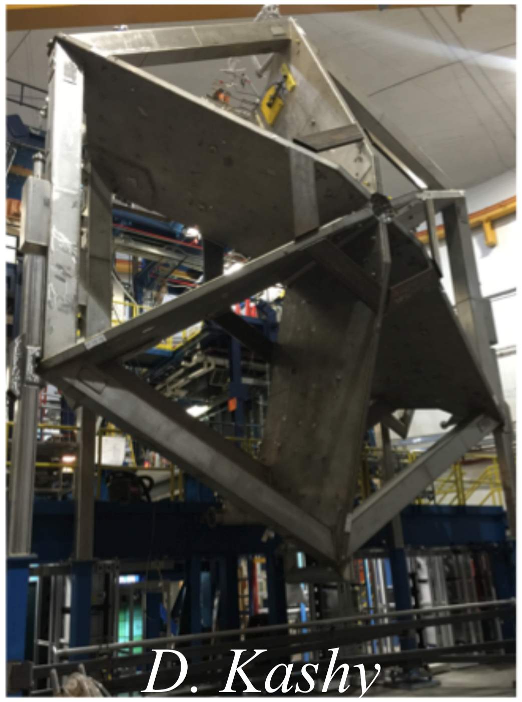

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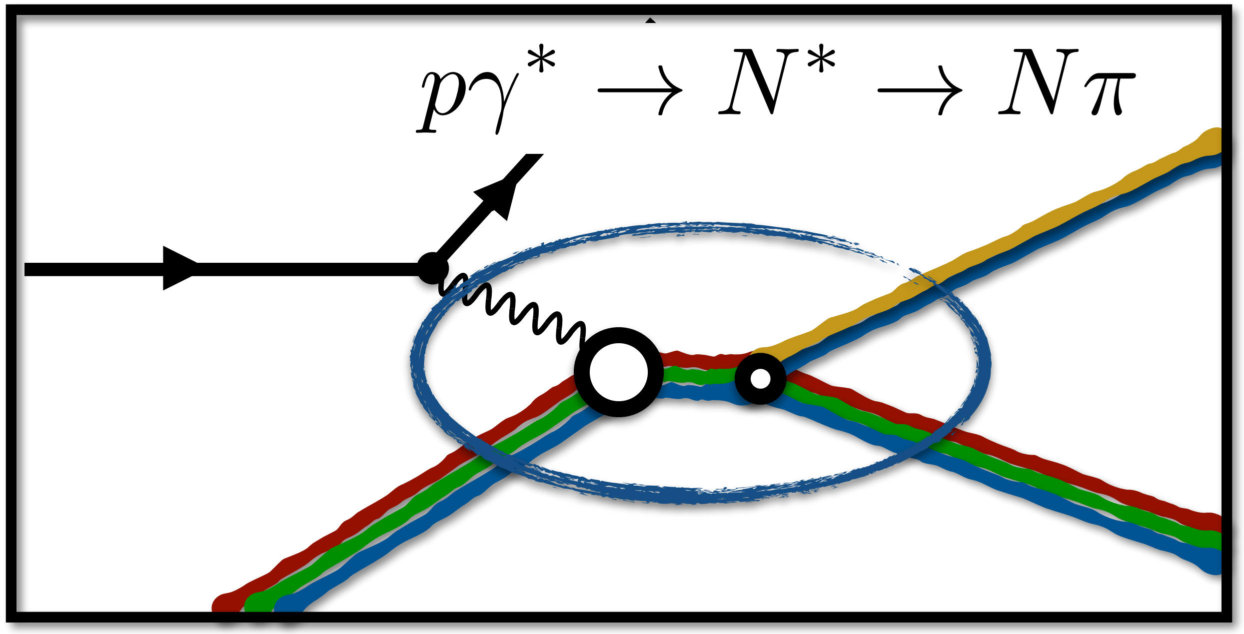

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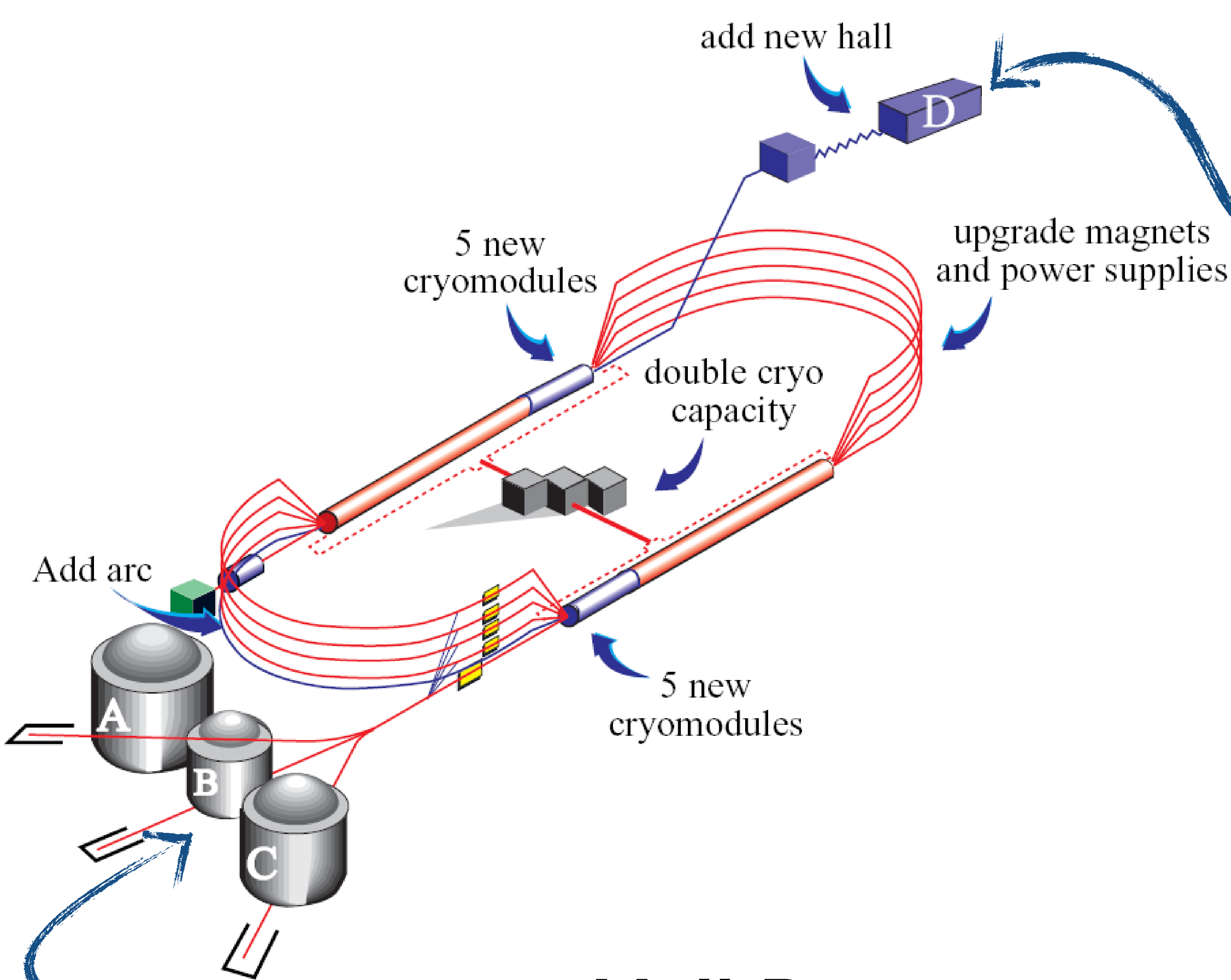
Hall D (GlueX)



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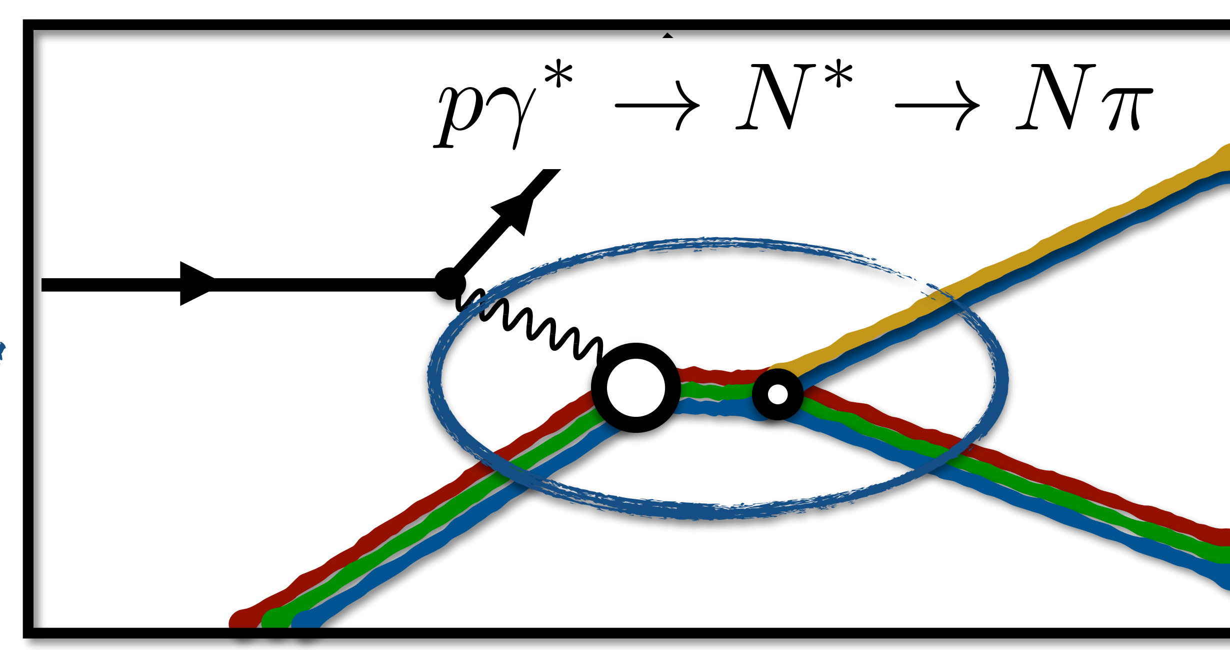

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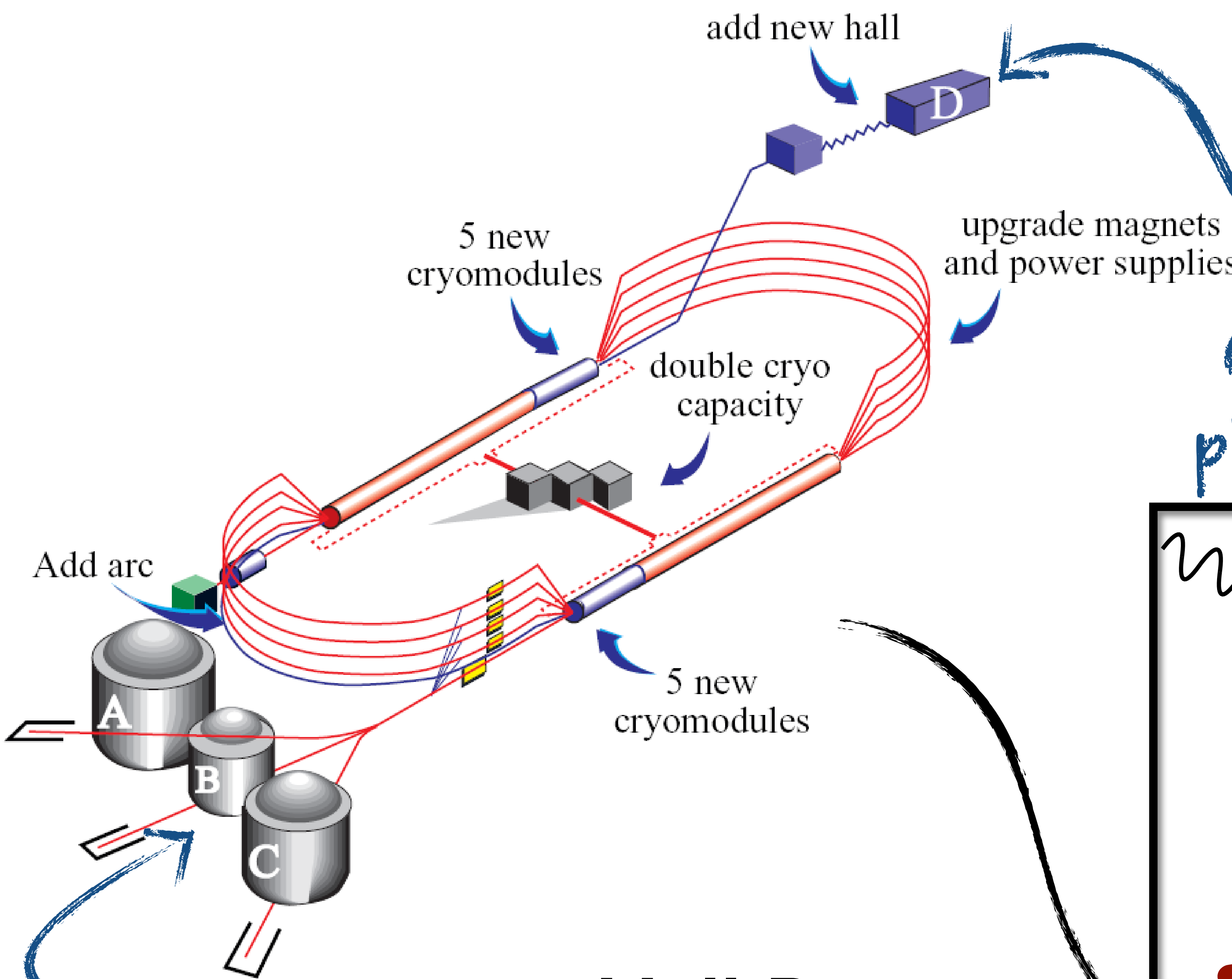

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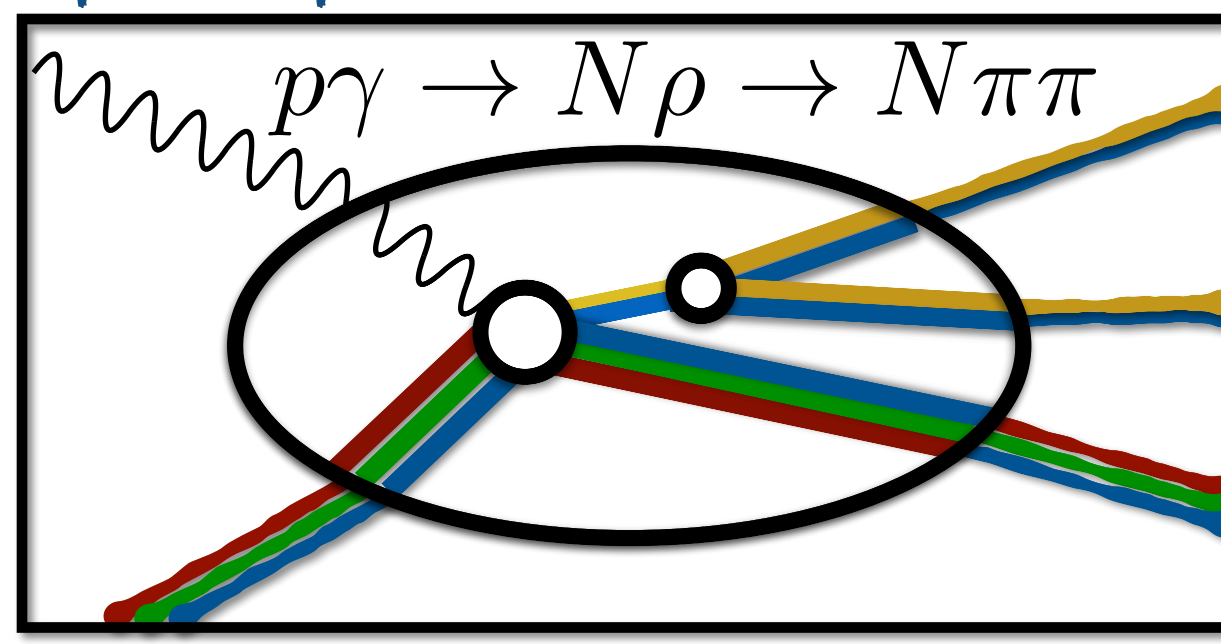

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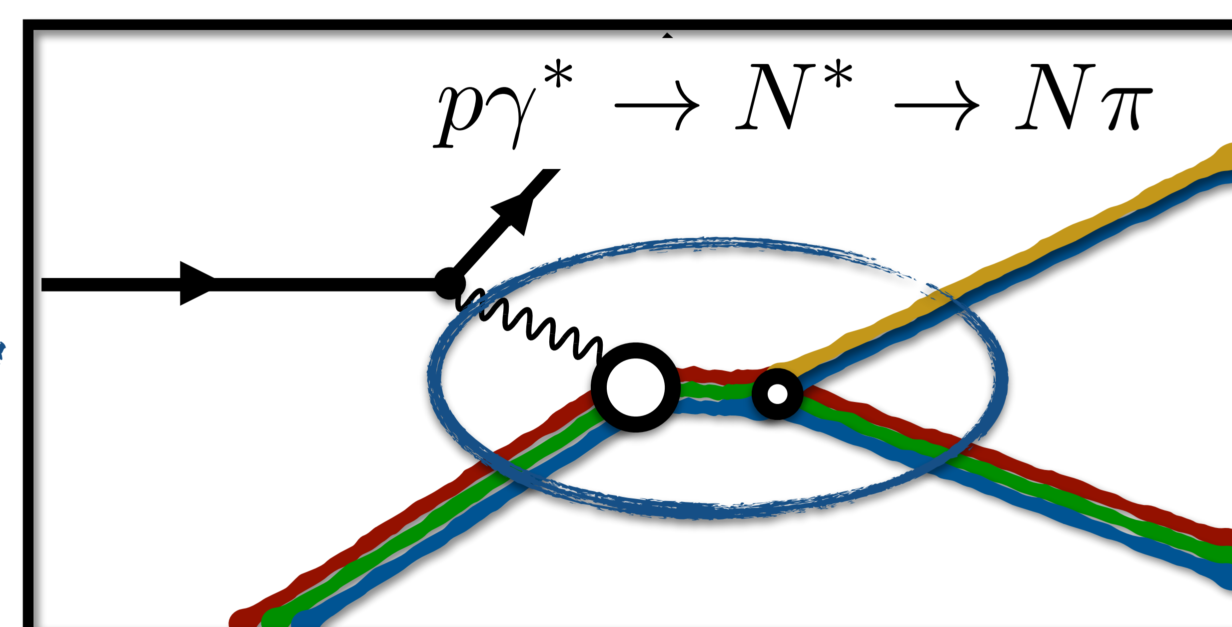

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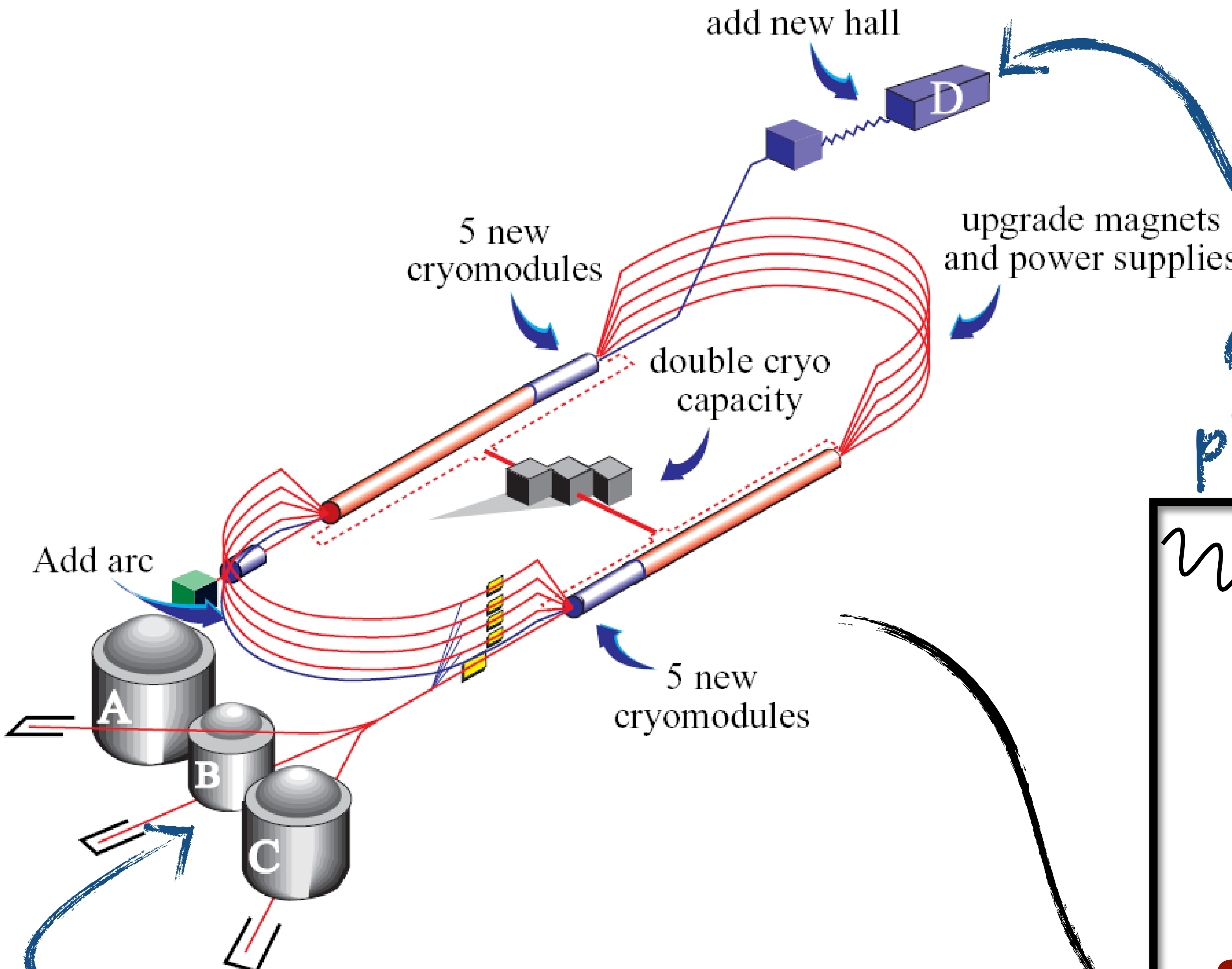

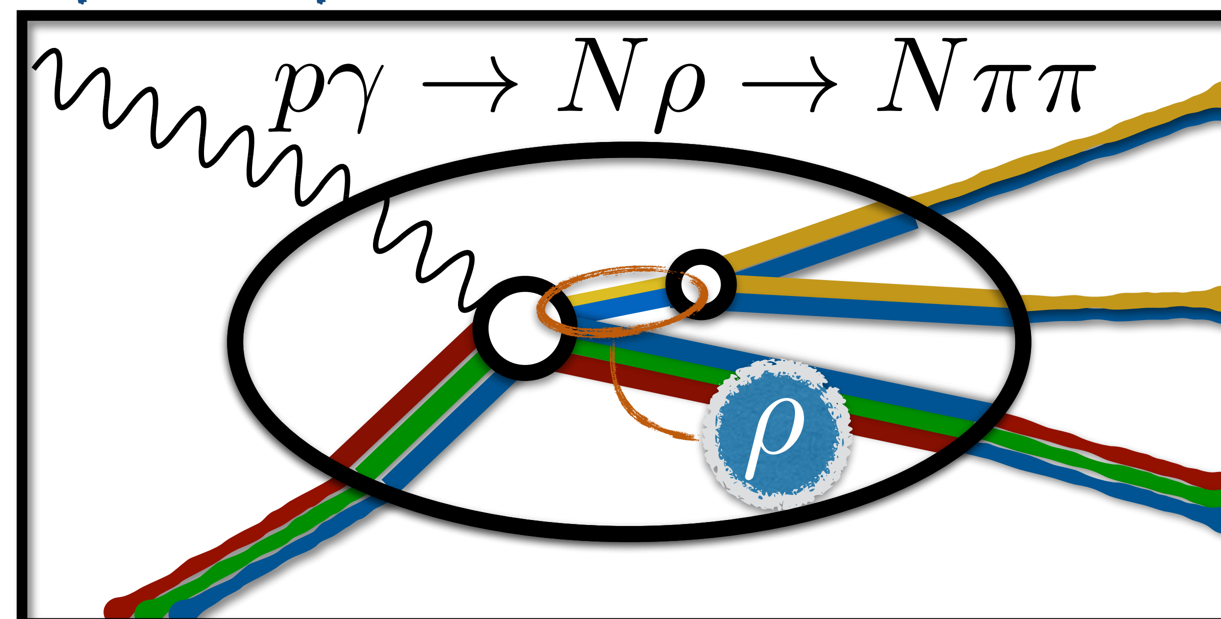

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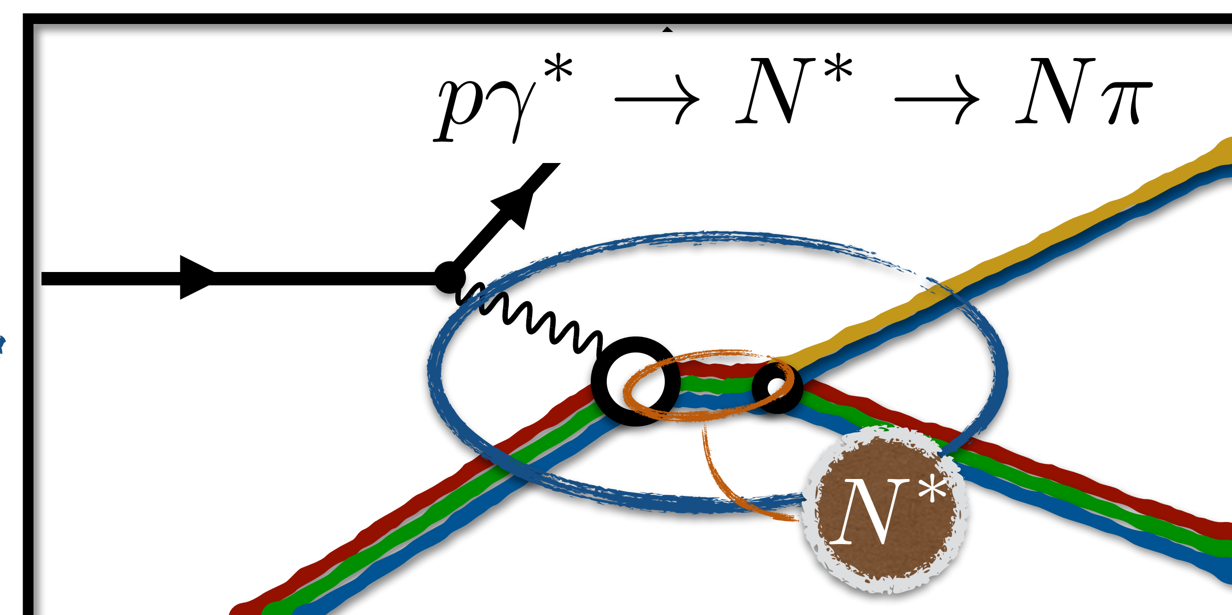
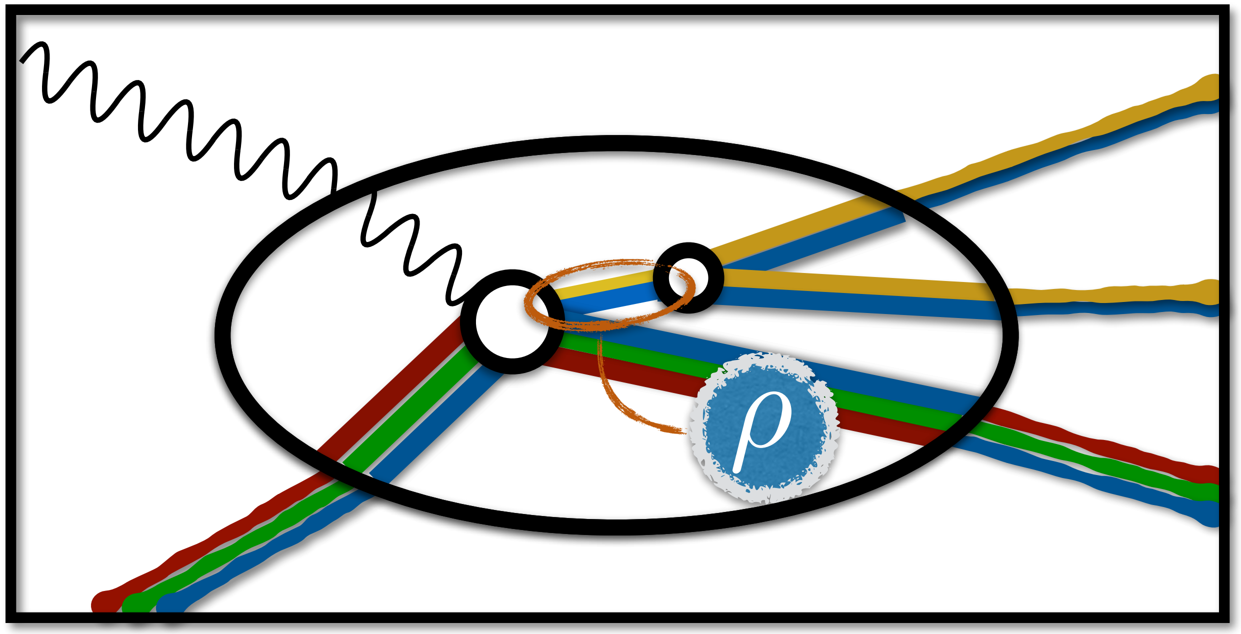
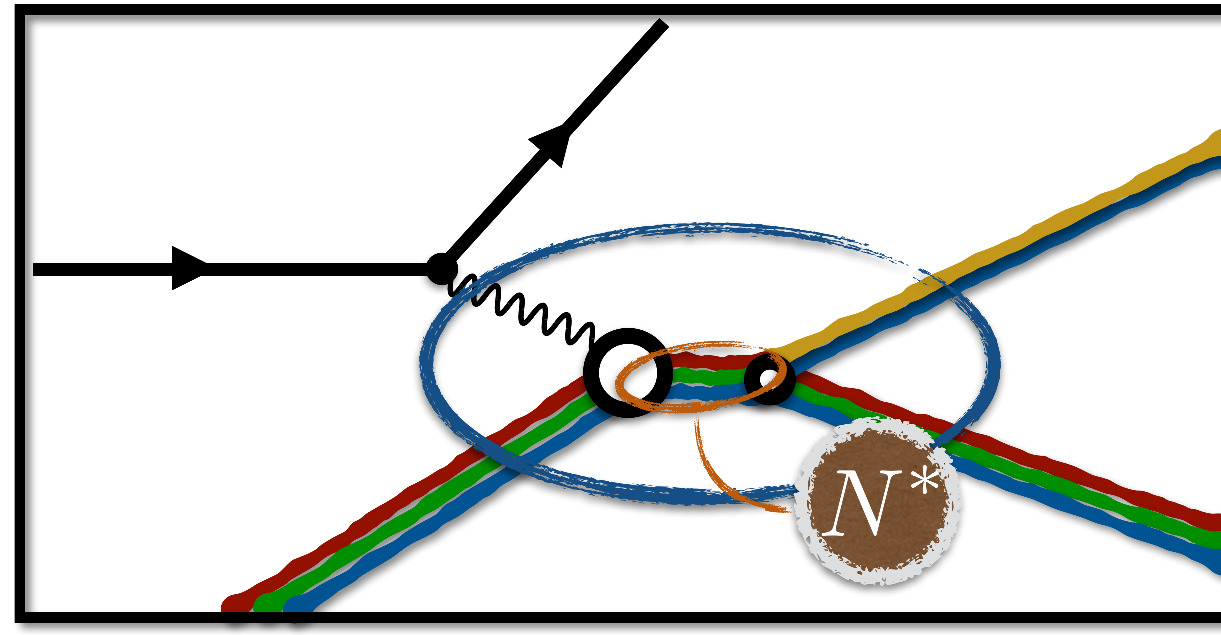




$$p\gamma \to N\rho \to N\pi\pi$$



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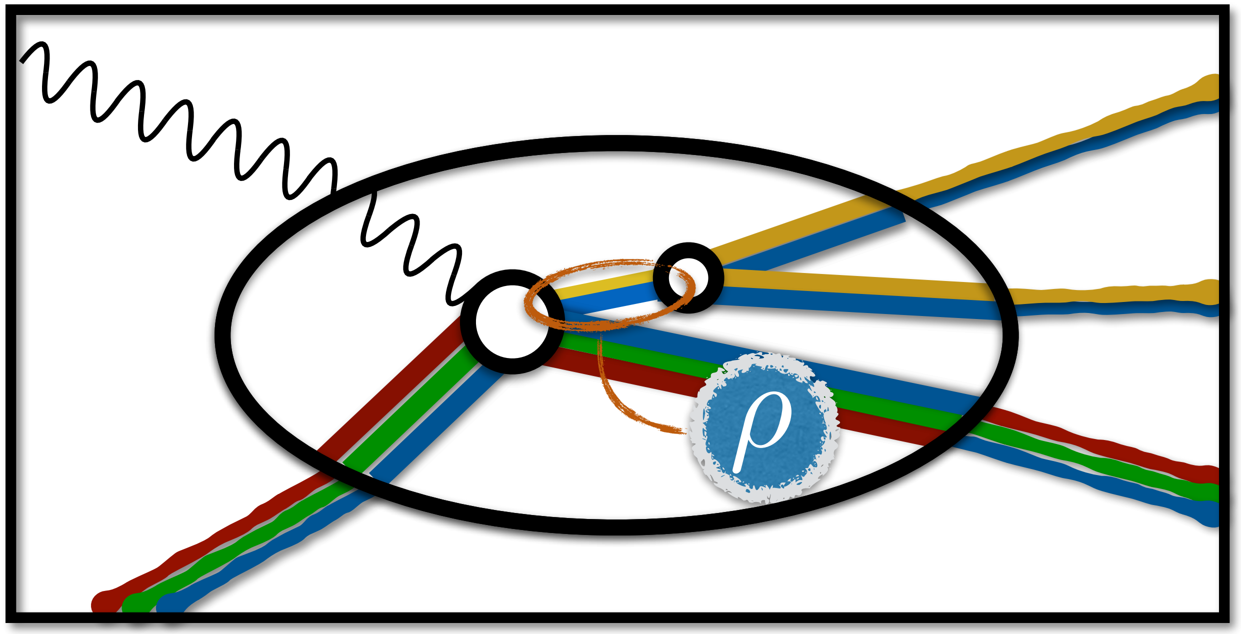

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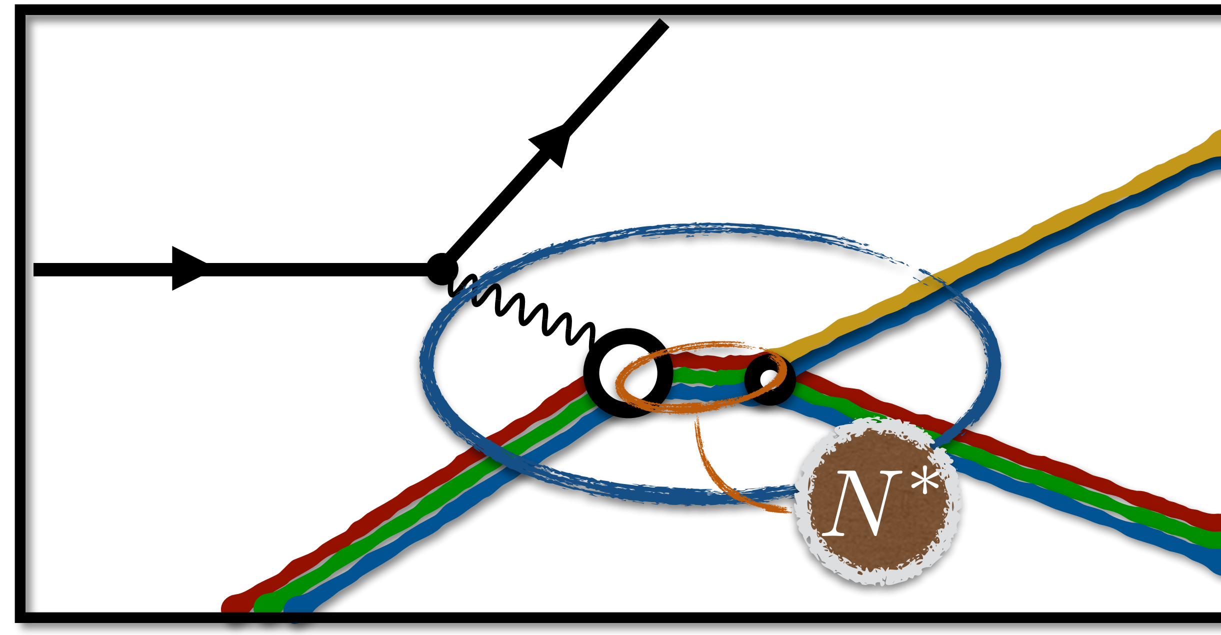

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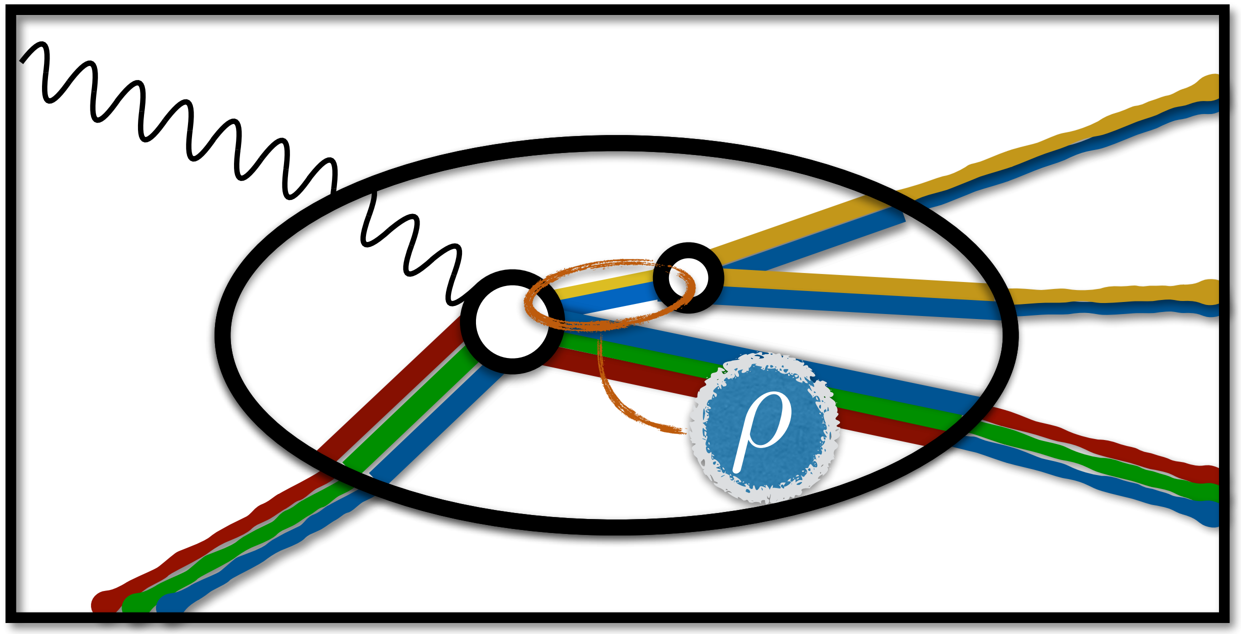
Outgoing hadrons are used to reconstruct resonance properties.

It is thus highly valuable to predict these transition amplitudes from the underlying theory of QCD



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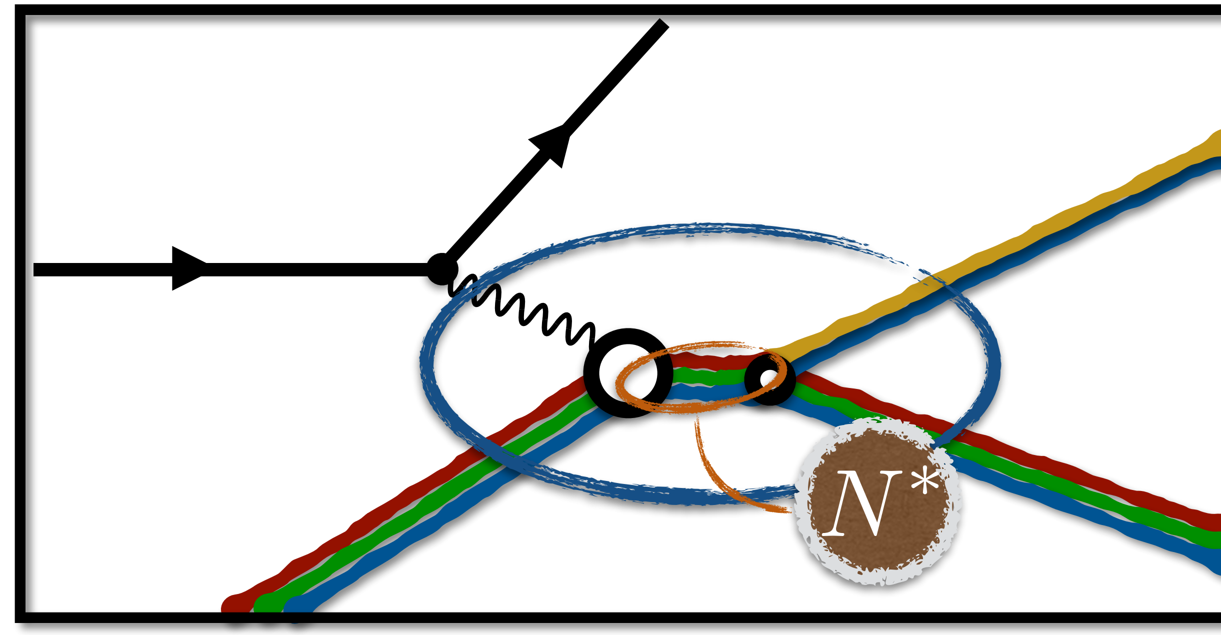

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Combining accurate, model-independent predictions with experiment will lead to a deeper understanding of QCD's rich resonance structure

What can we extract from the lattice?

We are trying to evaluate a difficult integral numerically

$$\textbf{observable} = \int \mathcal{D}\phi \; e^{iS} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

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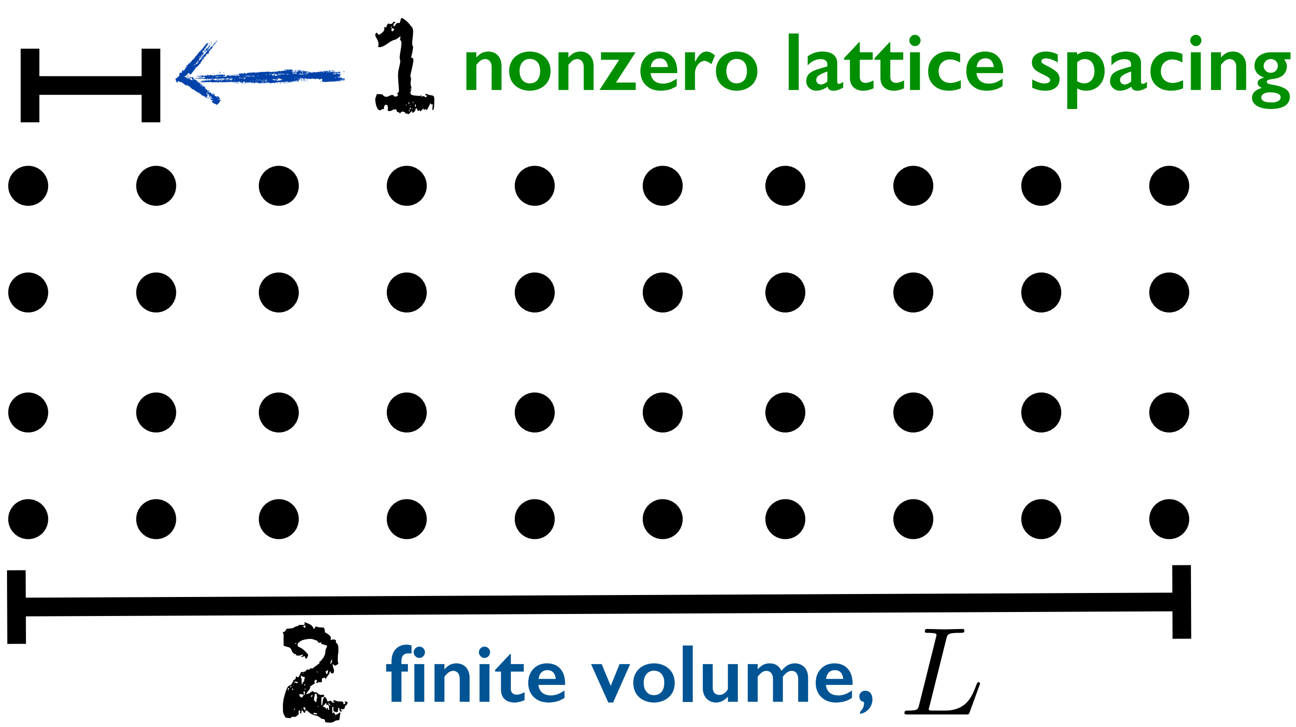

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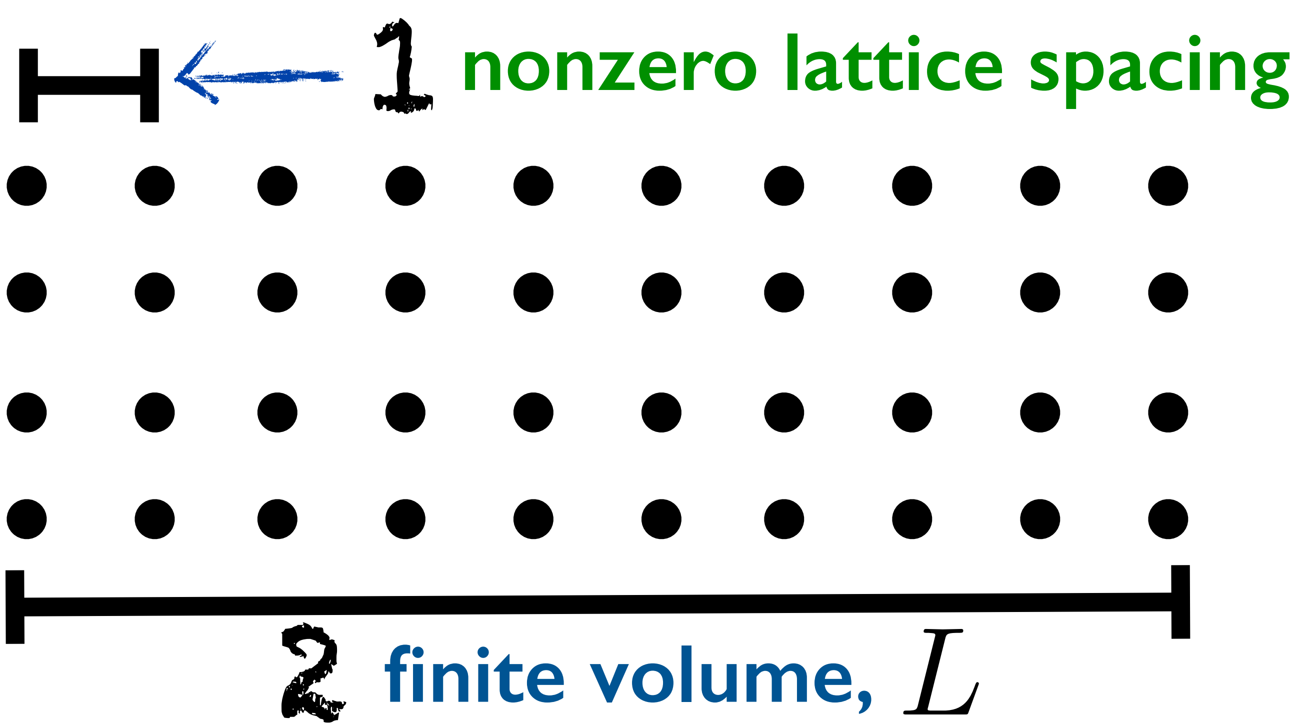

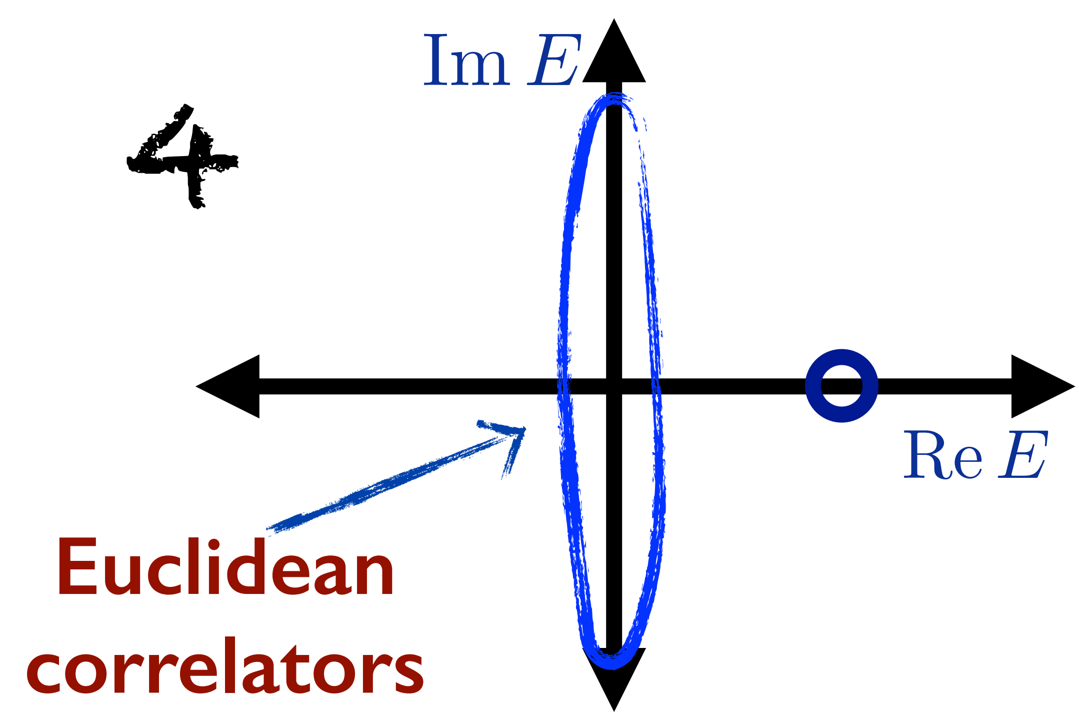

3 **Unphysical pion masses** $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

But calculations at the physical pion mass do now exist

What can we extract from the lattice?

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$$\left(\begin{array}{c}\textbf{observable?}\\ \text{discretized, finite volume,}\\ \text{Euclidean, heavy pions}\end{array}\right) = \int \prod_i^N d\phi_i \; e^{-S} \left[\begin{array}{c}\textbf{interpolator}\\ \textbf{for observable}\end{array}\right]$$

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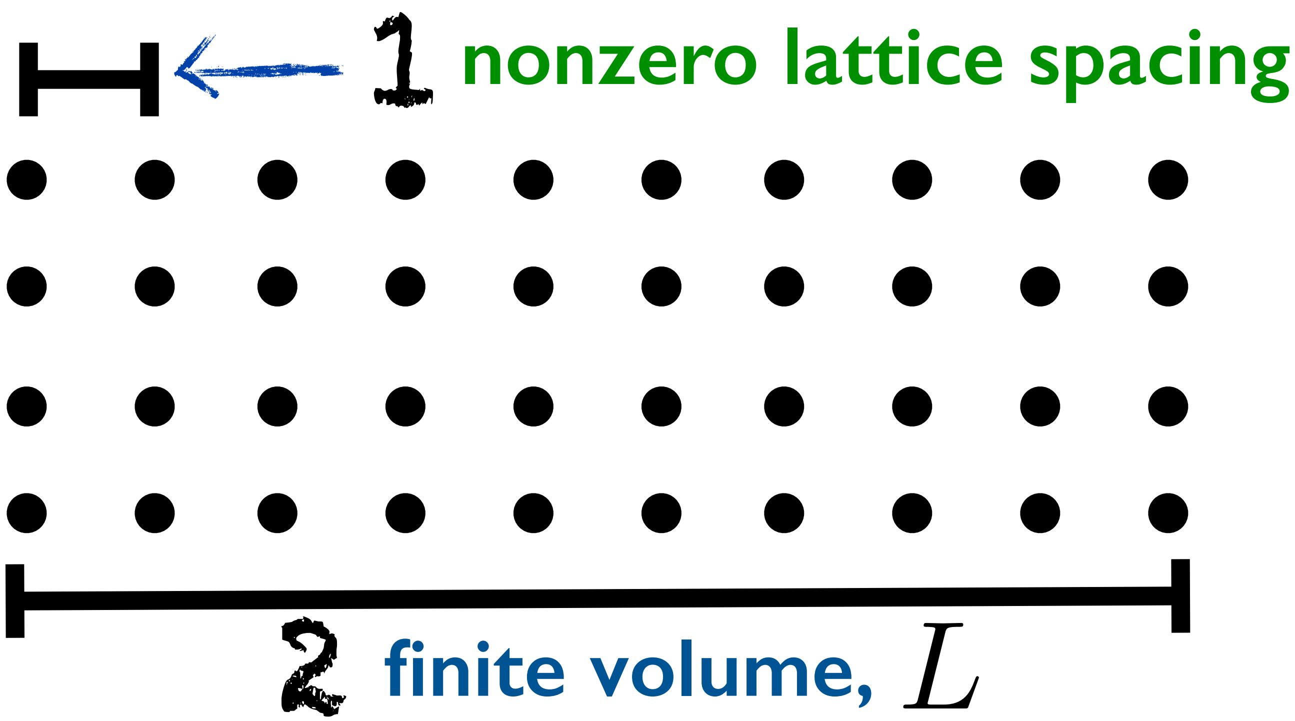

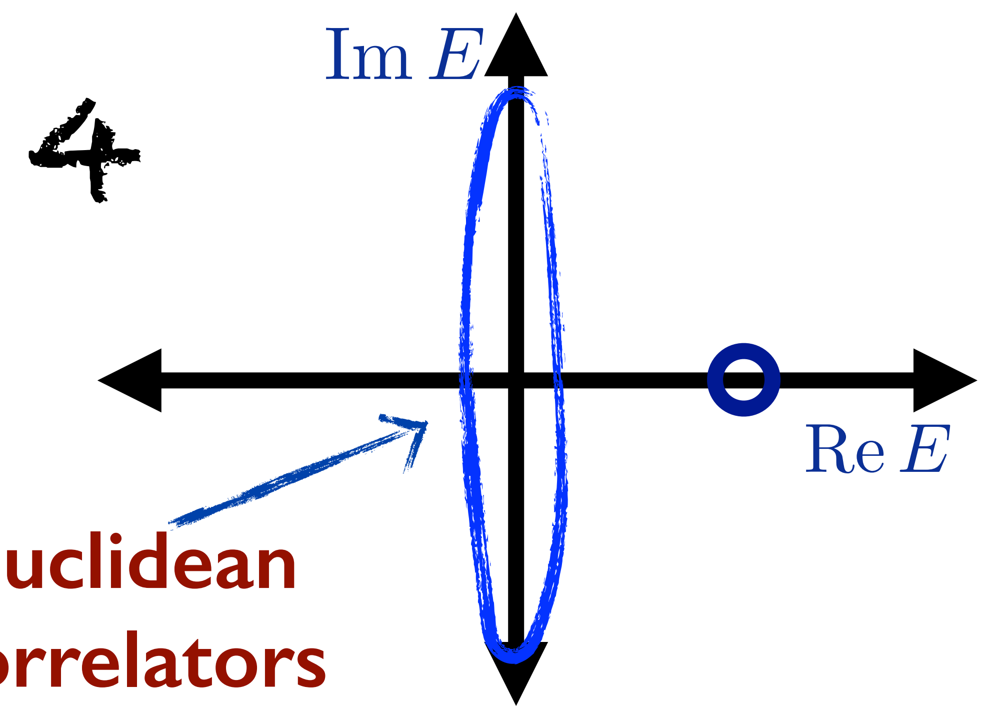

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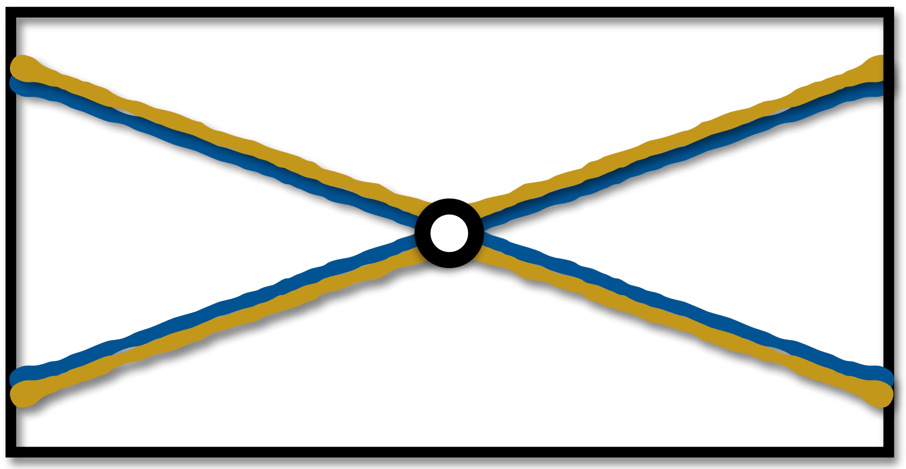
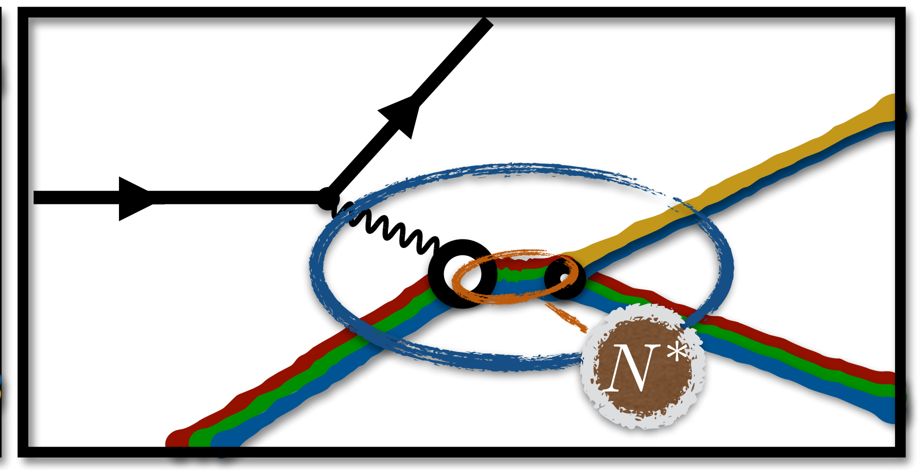


$\langle \pi\pi | \pi\pi \rangle$



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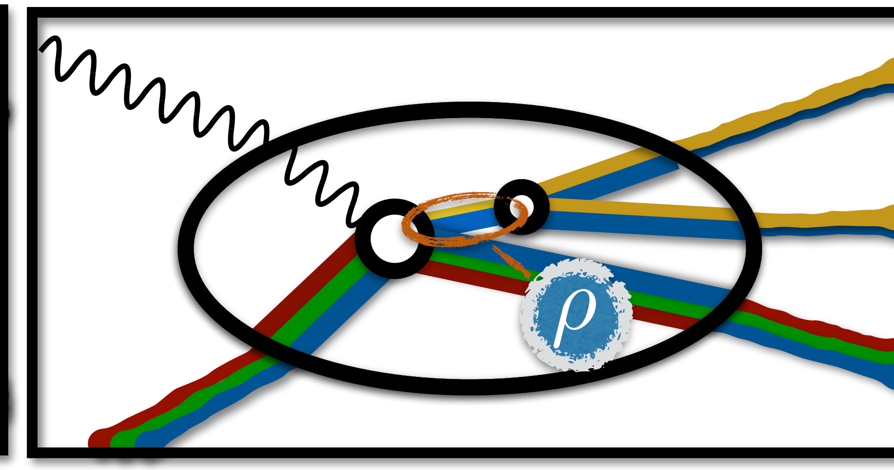

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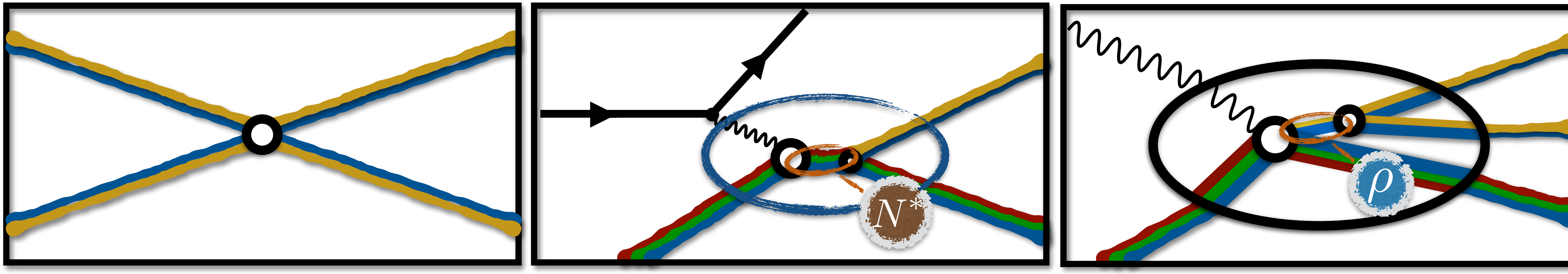

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multi-particle in- and outstates

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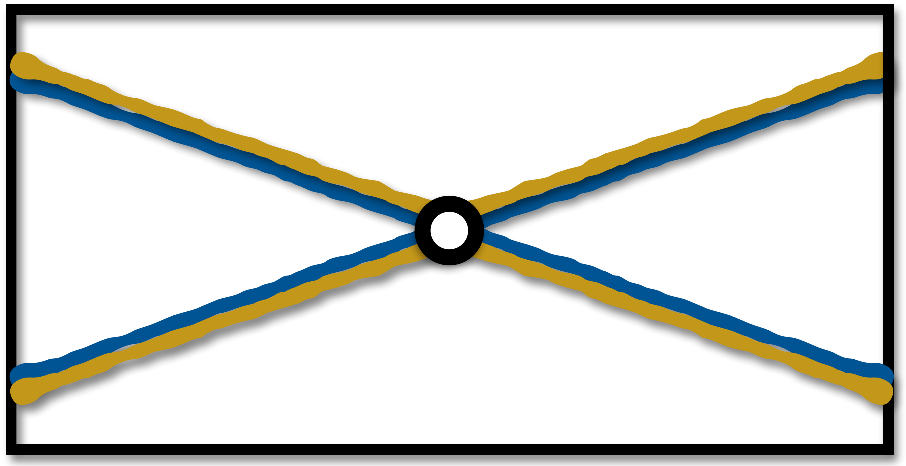

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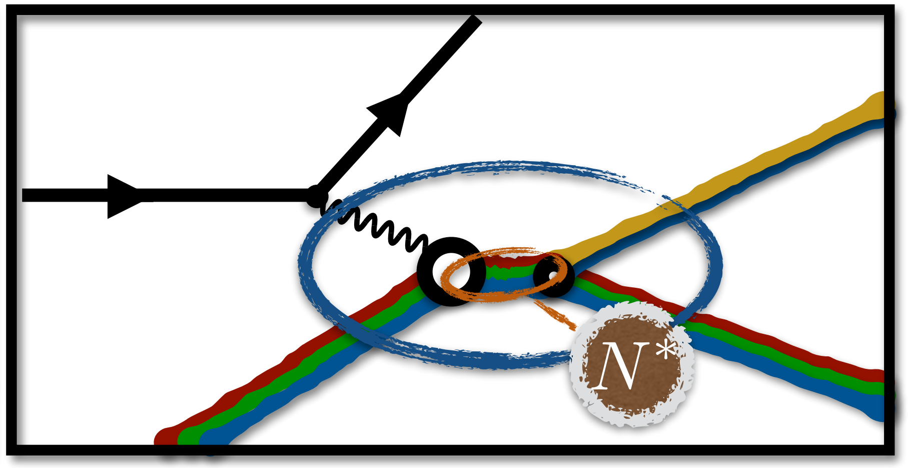

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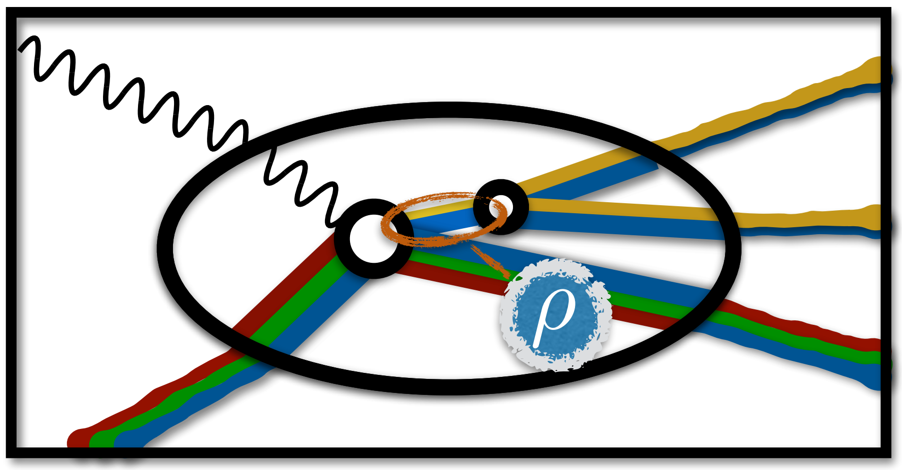

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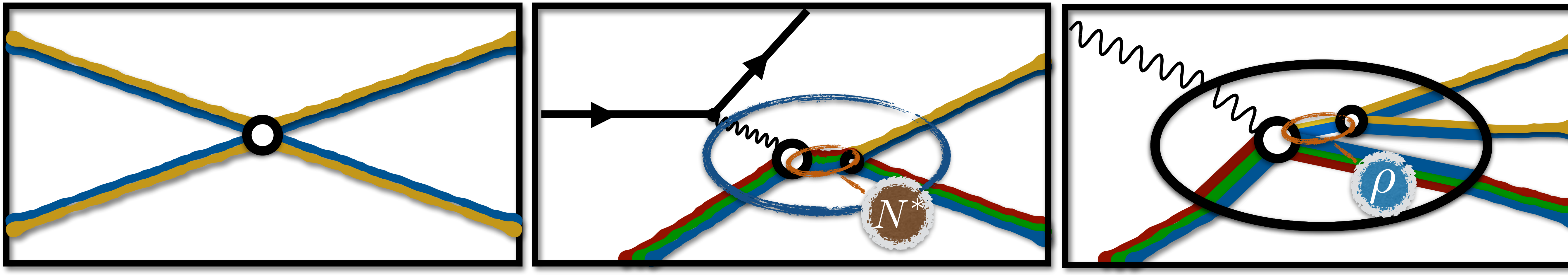

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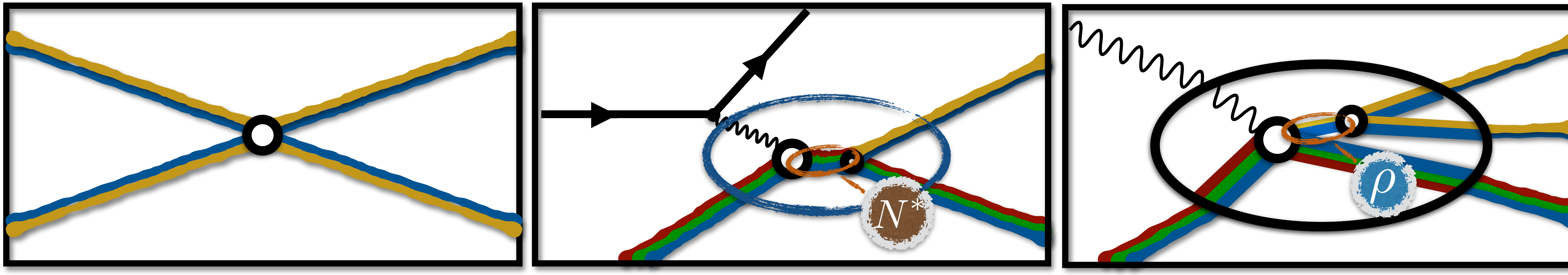

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Requires Minkowski momenta and infinite volume

What can we extract from the lattice?

Instead we can only access

$$H_{\text{QCD}}|n, L\rangle = |n, L\rangle E_n(L) \qquad \langle n, L, \text{“}N\pi\pi\text{”}|\mathcal{J}_\mu(x)|\text{“}N\text{”}, L\rangle$$

finite-volume energies and matrix elements

labels in quotes indicate quantum numbers

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How can we determine

$\langle \pi\pi, \text{out}|\pi\pi, \text{in}\rangle$ and $\langle N\pi\pi, \text{out}|\mathcal{J}_\mu(x)|N\rangle$

from

$E_n(L)$ and $\langle n, L, \text{“}N\pi\pi\text{”}|\mathcal{J}_\mu(x)|\text{“}N\text{”}, L\rangle$?

It is possible to derive relations between finite- and infinite-volume physics

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Two-particle scattering

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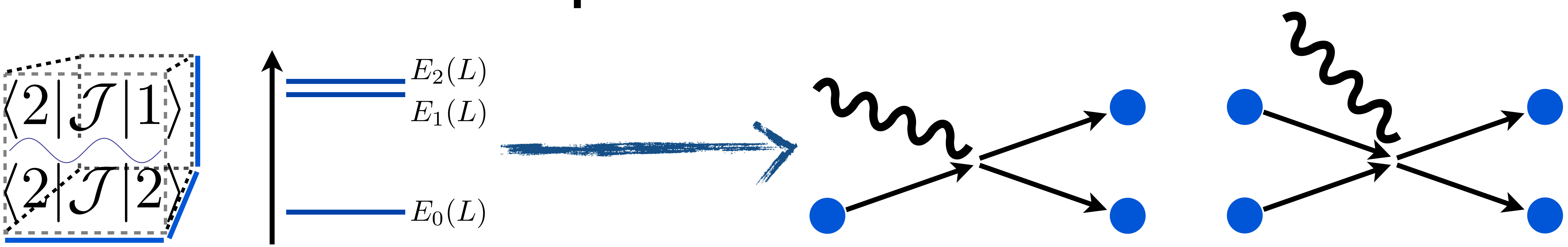
Two-particle scattering

Photo- and electroproduction



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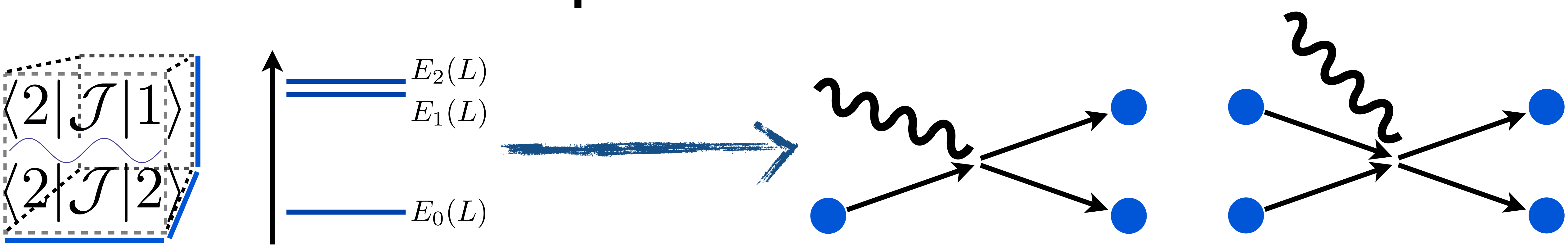
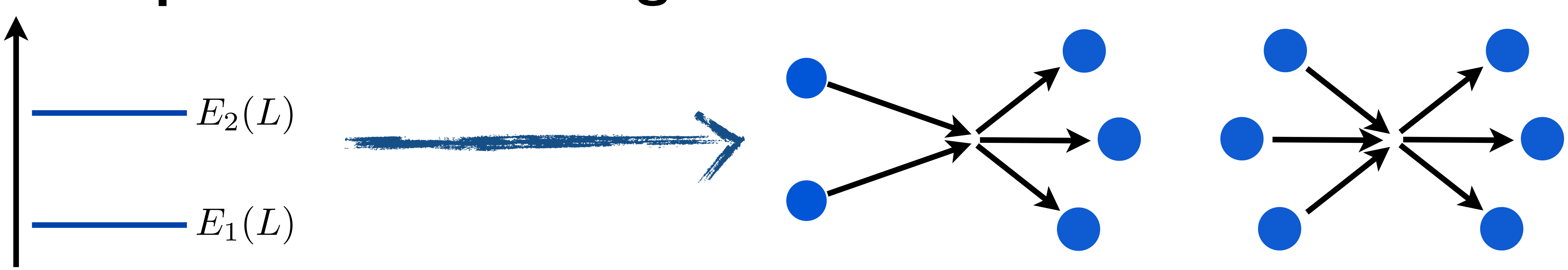


Three-particle scattering



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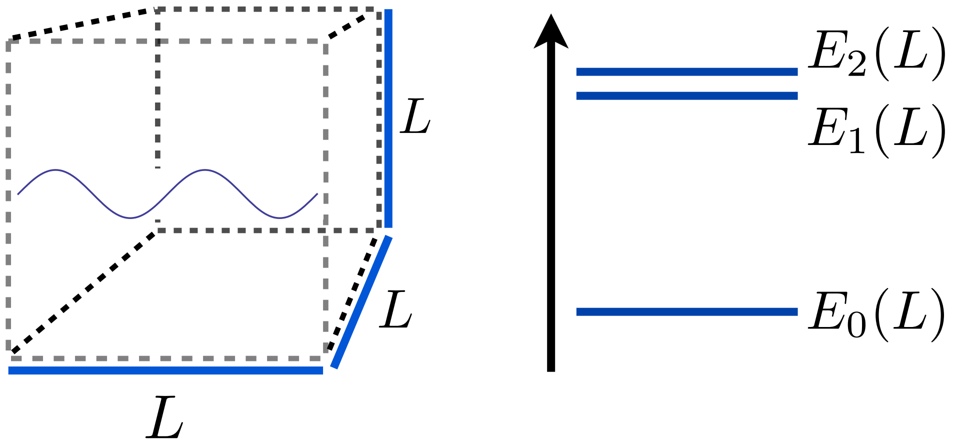

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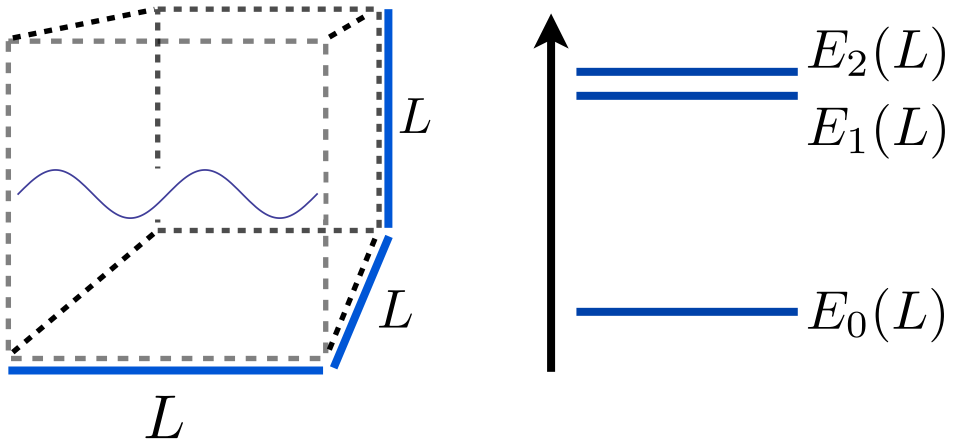


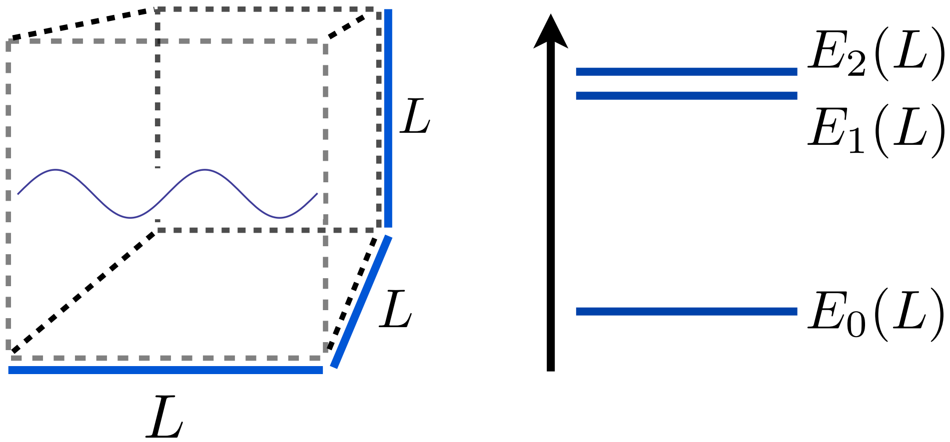
cubic, spatial volume (extent L)

periodic boundary conditions

$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

time direction **infinite**

Finite volume



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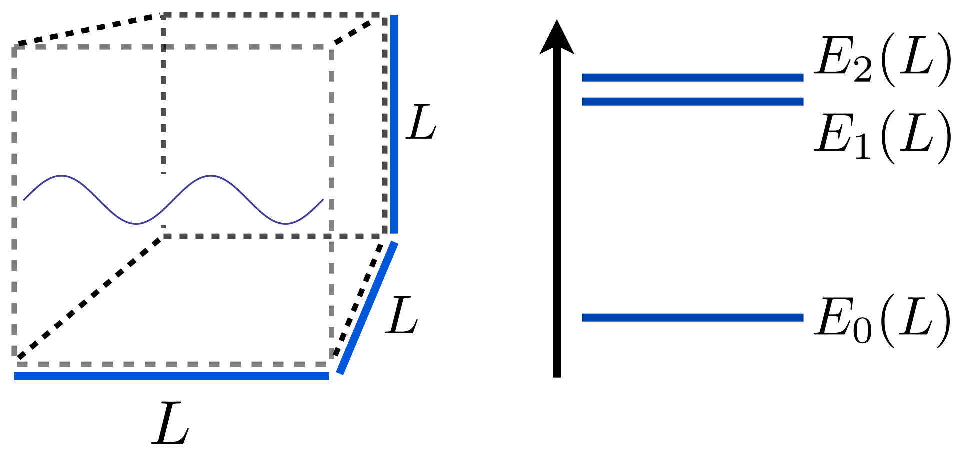

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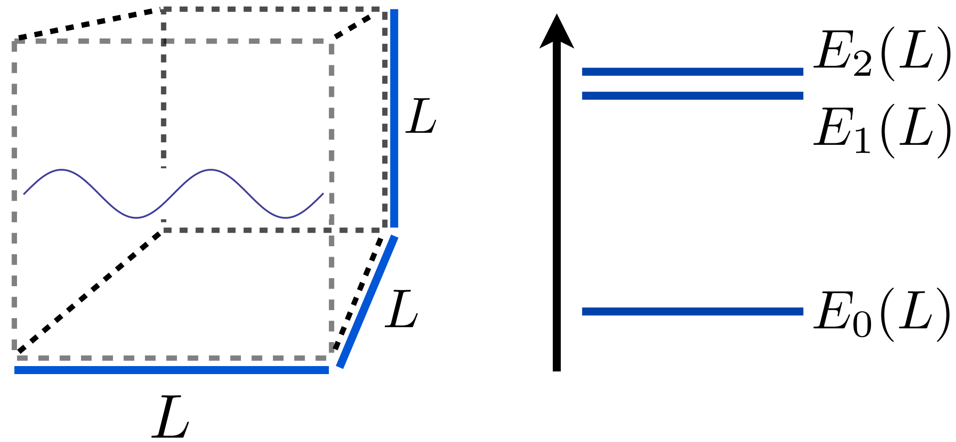
Finite volume quantum field theory



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generic relativistic QFT

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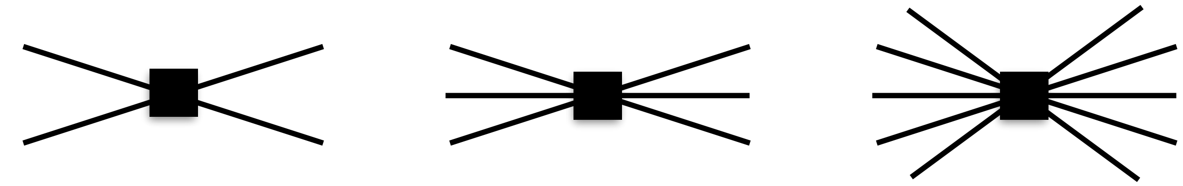

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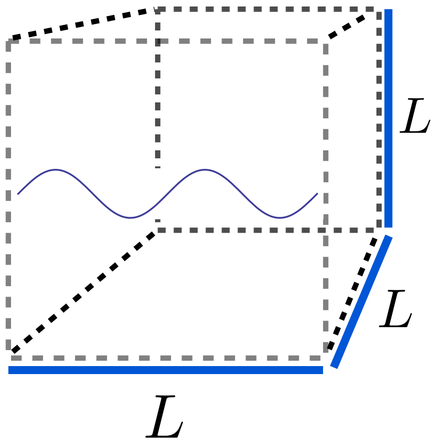

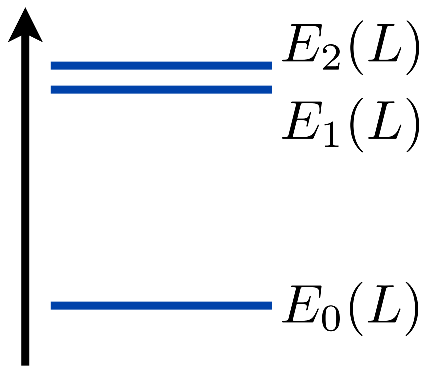

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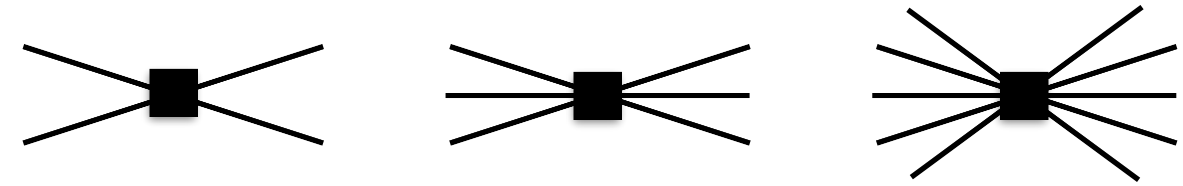

2. no power-counting scheme

Not possible to directly calculate scattering observables to all orders

But it is possible to derive *general, all-orders relations* to finite-volume quantities

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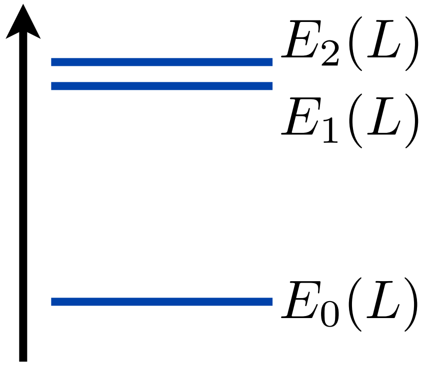
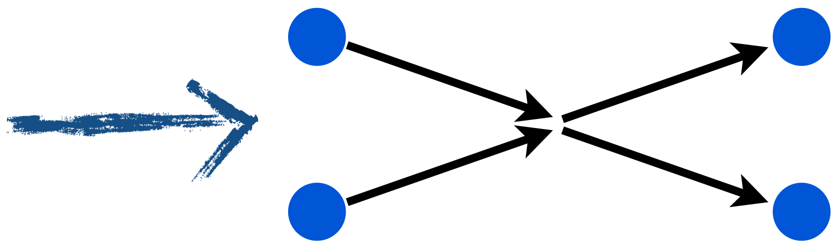
Work in continuum field theory throughout

Two-to-two scattering

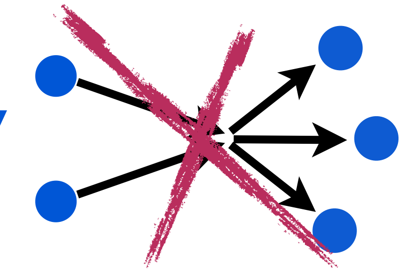




For now assume…

identical scalars, mass m

$\mathbb{Z}_2$ **symmetry**



Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys*. B727, 218-243 (2005)

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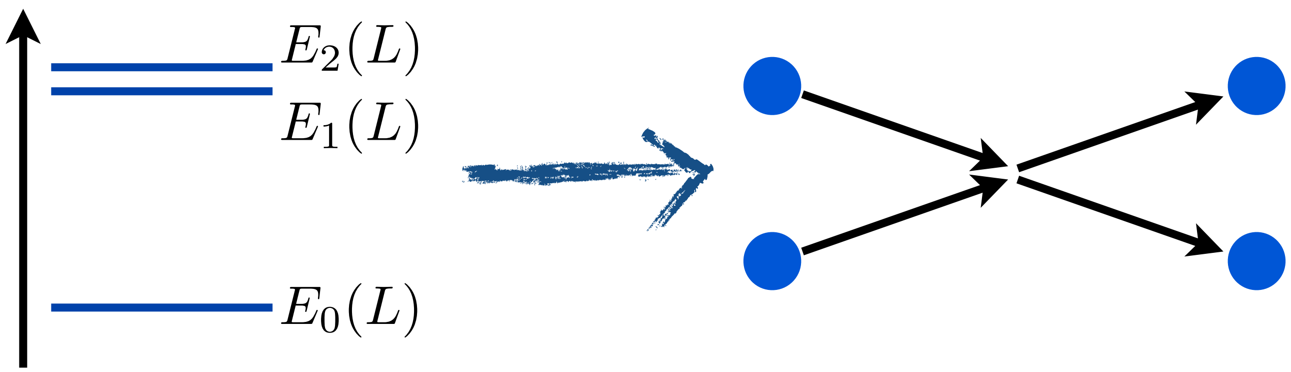

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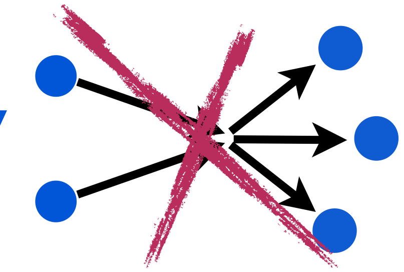
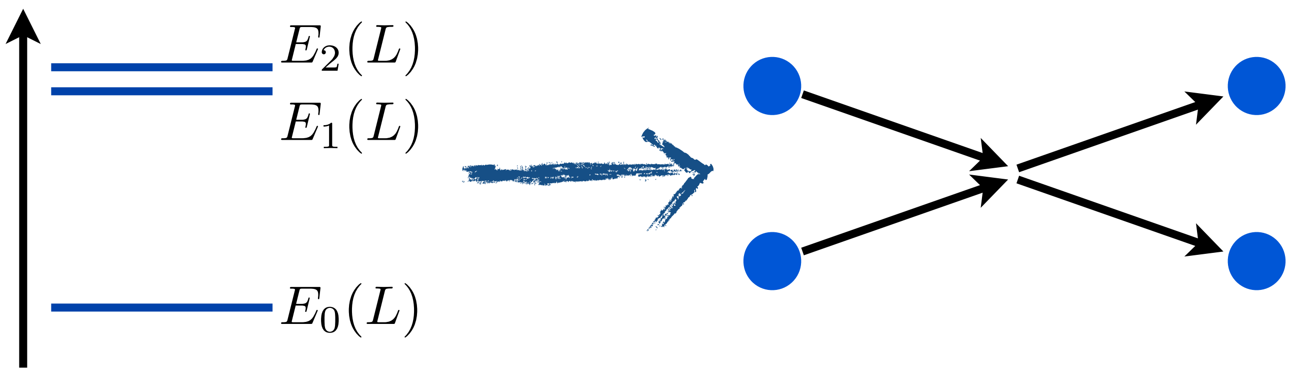


$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

two-particle interpolator

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys*. B727, 218-243 (2005)

Two-to-two scattering



For now assume…

identical scalars, mass m

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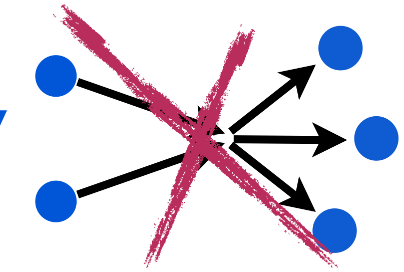

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Euclidean convention

two-particle interpolator

$$P = (P_4, \vec{P}) = (P_4, 2\pi\vec{n}/L)$$

but allow P_4 to be real or imaginary

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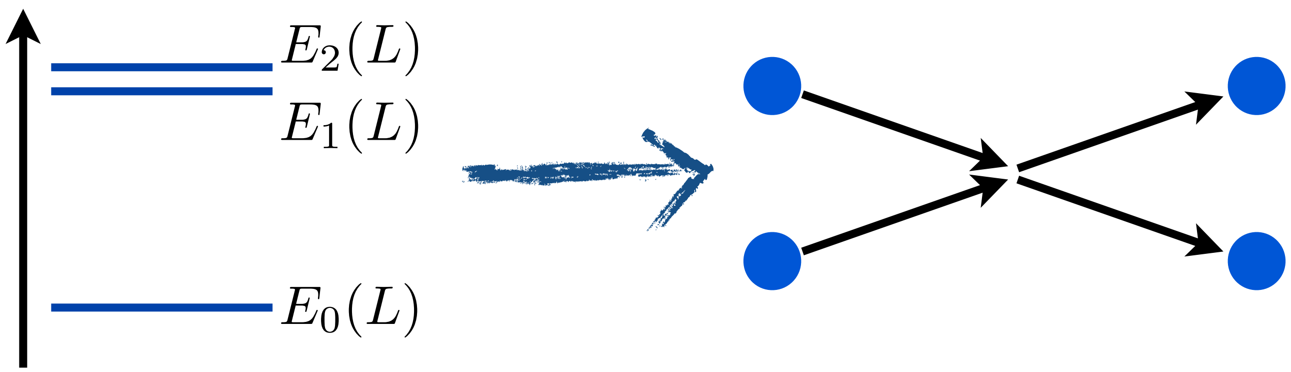

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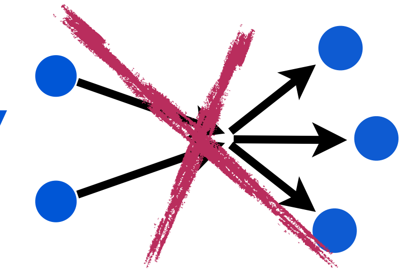

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Euclidean convention **two-particle interpolator**

$$P = (P_4, \vec{P}) = (P_4, 2\pi\vec{n}/L)$$

but allow P_4 to be real or imaginary

CM frame energy is then $E^{*2} = -P_4^2 - \vec{P}^2$

Require $E^* < 4m$ to isolate two-to-two scattering

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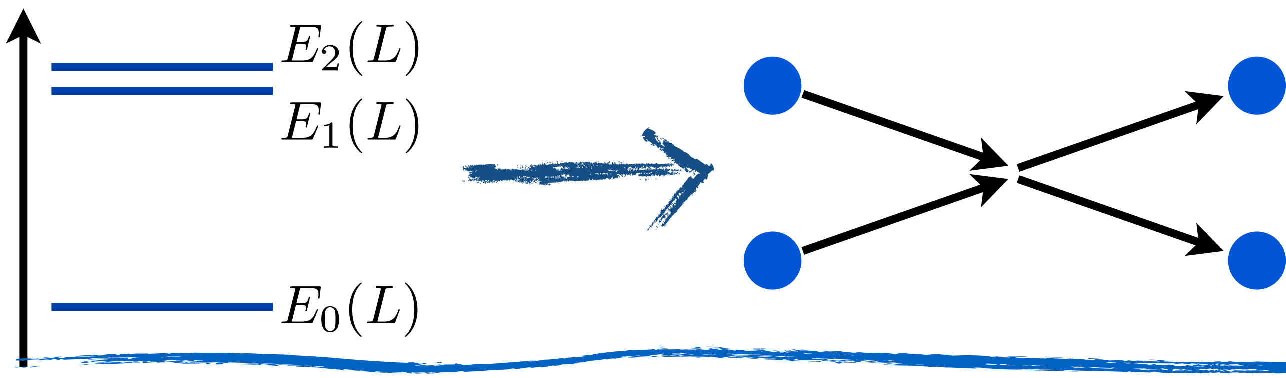
Two-to-two scattering



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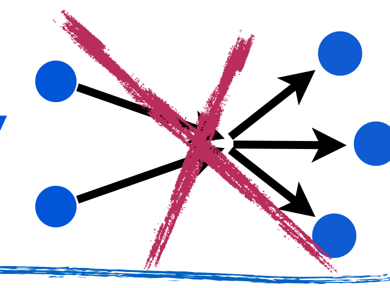
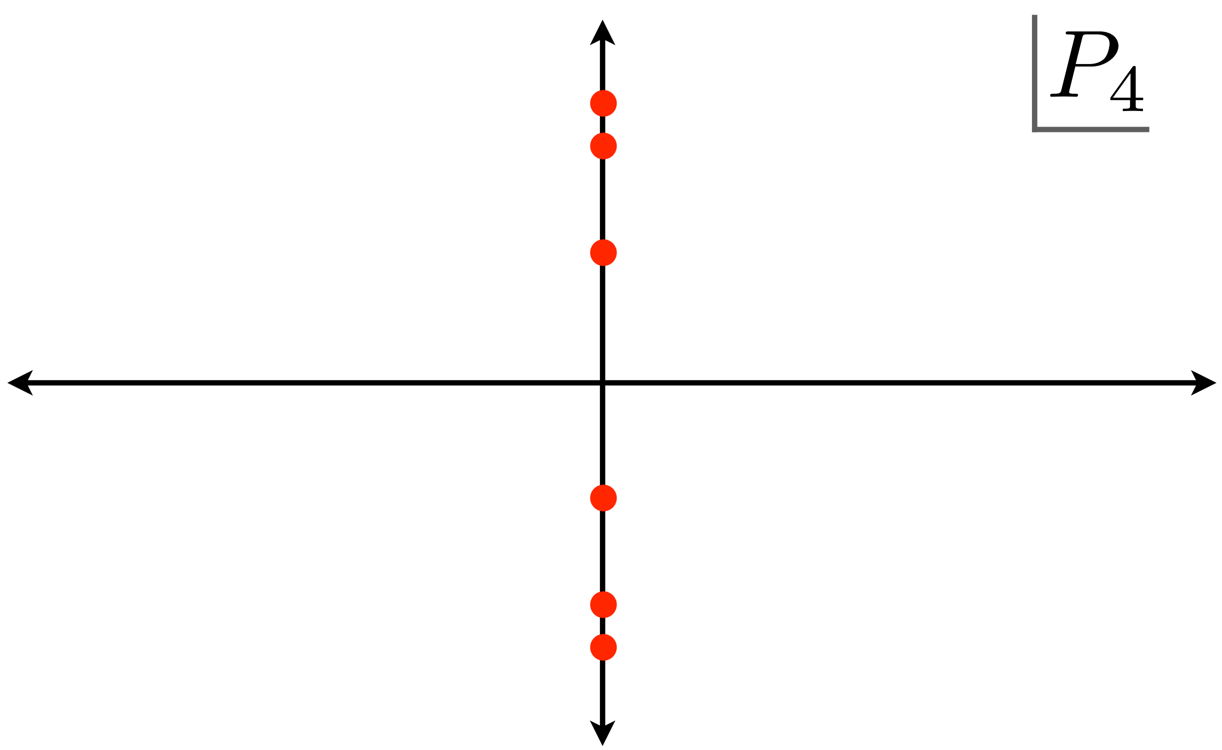
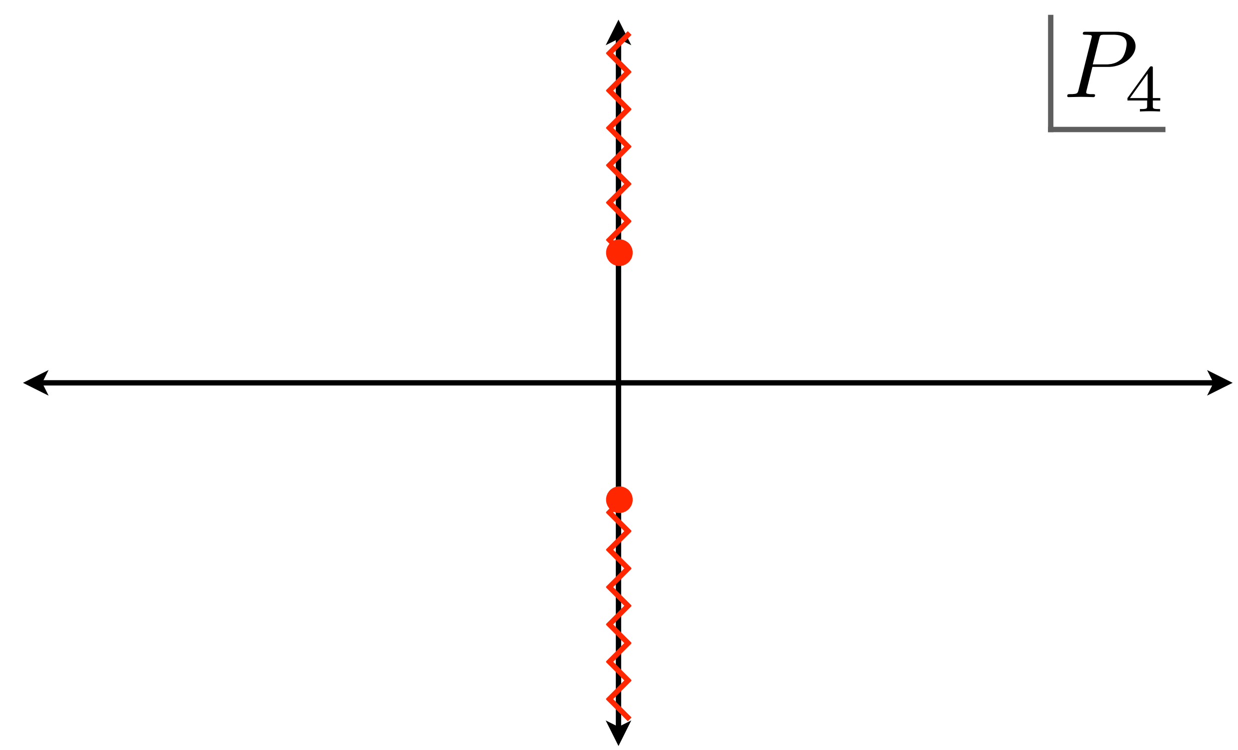


$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

At fixed L, $\vec{P}$, **poles in** C_L **give finite-volume spectrum**

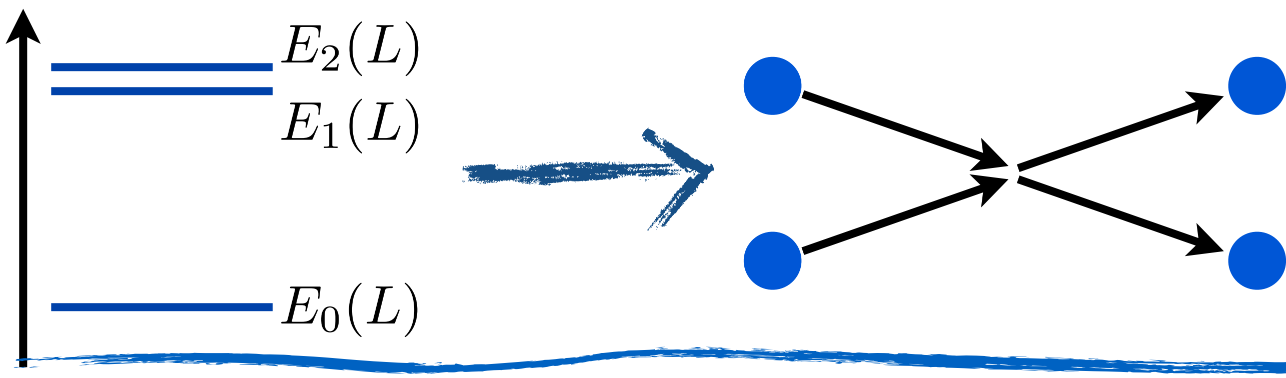


C_L **analytic structure**



C_∞ **analytic structure**

Two-to-two scattering



For now assume…

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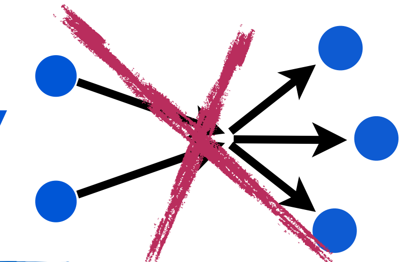

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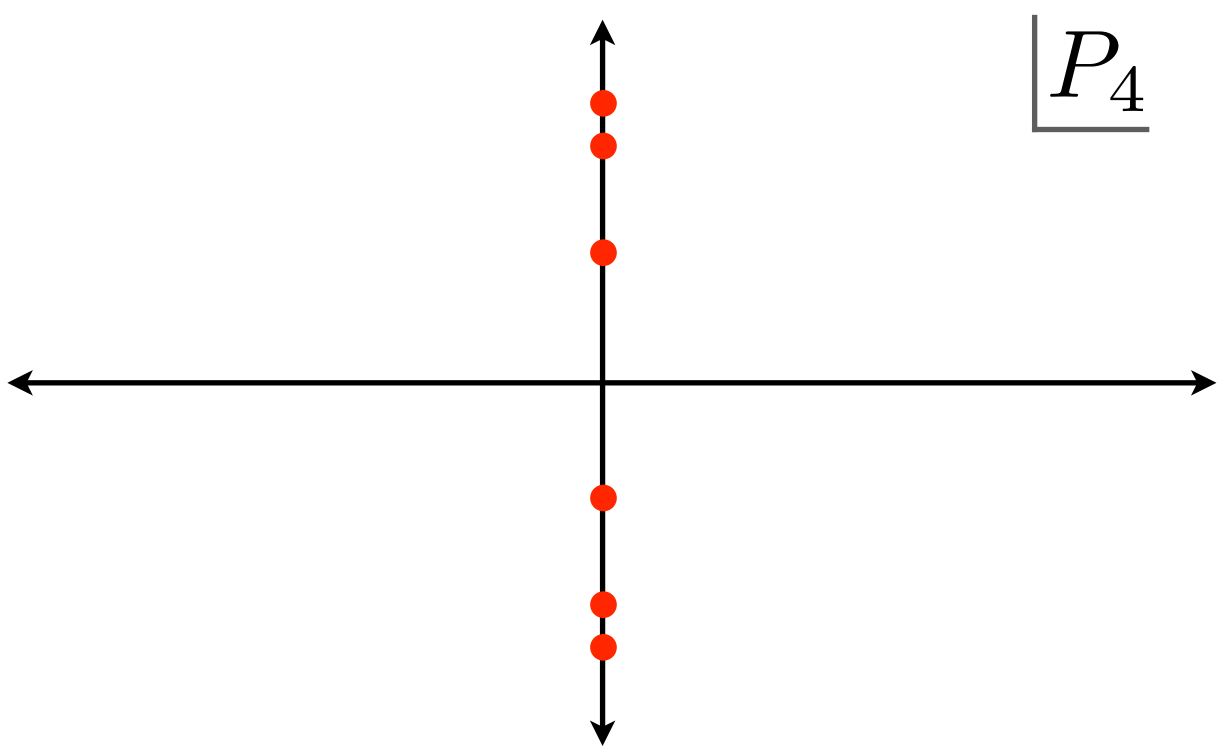
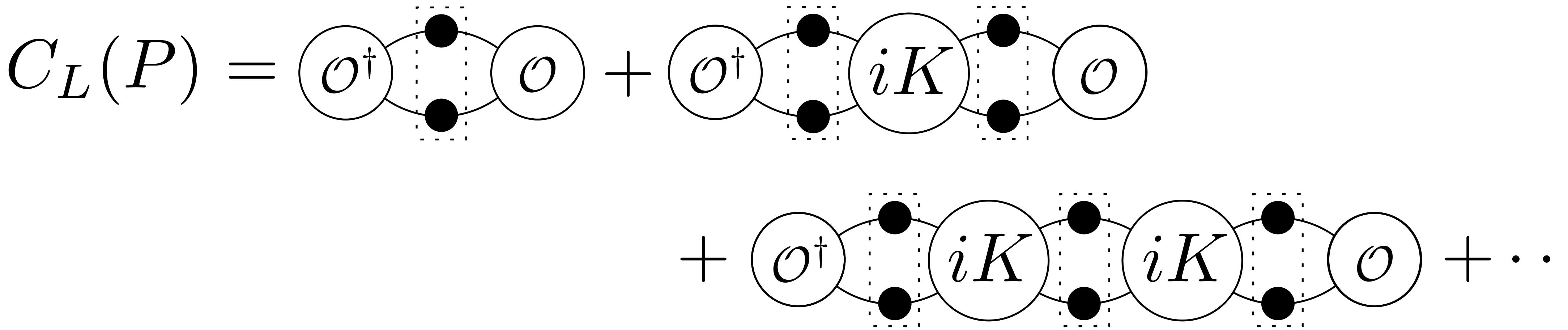


C_L **analytic structure**

Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.



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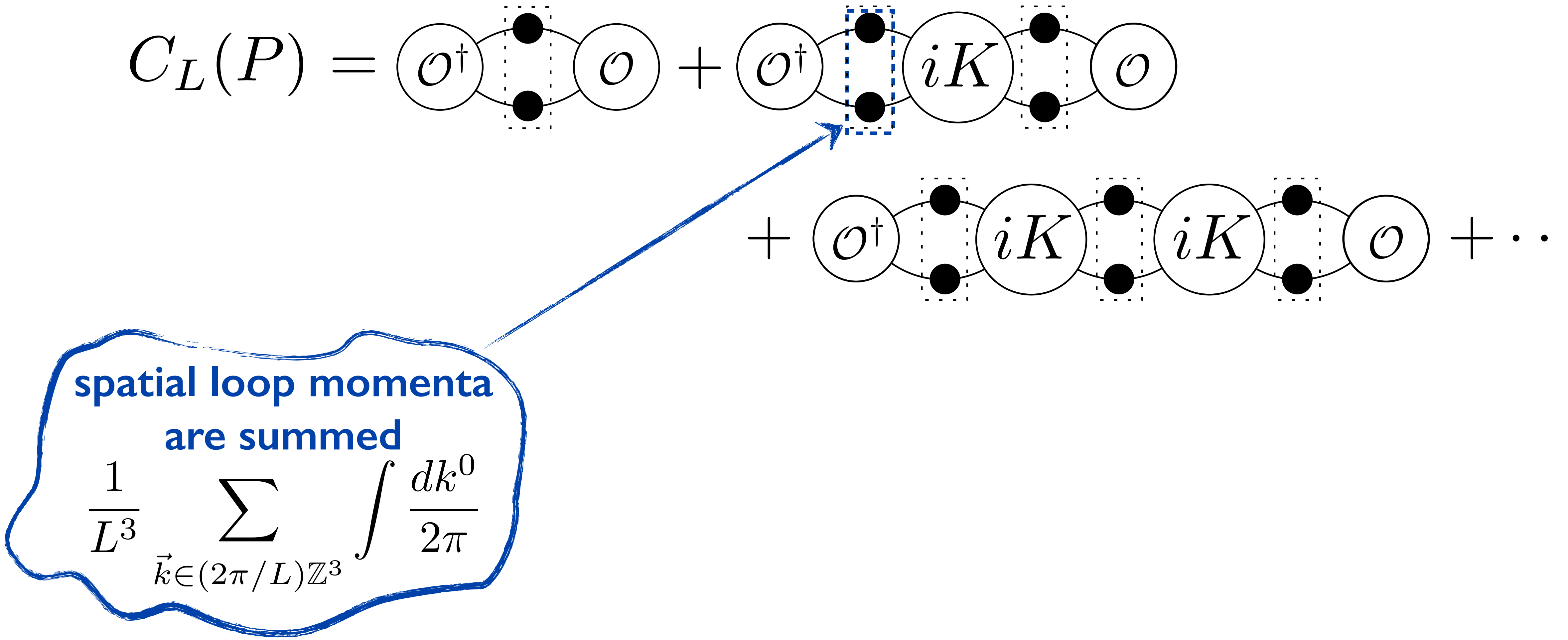

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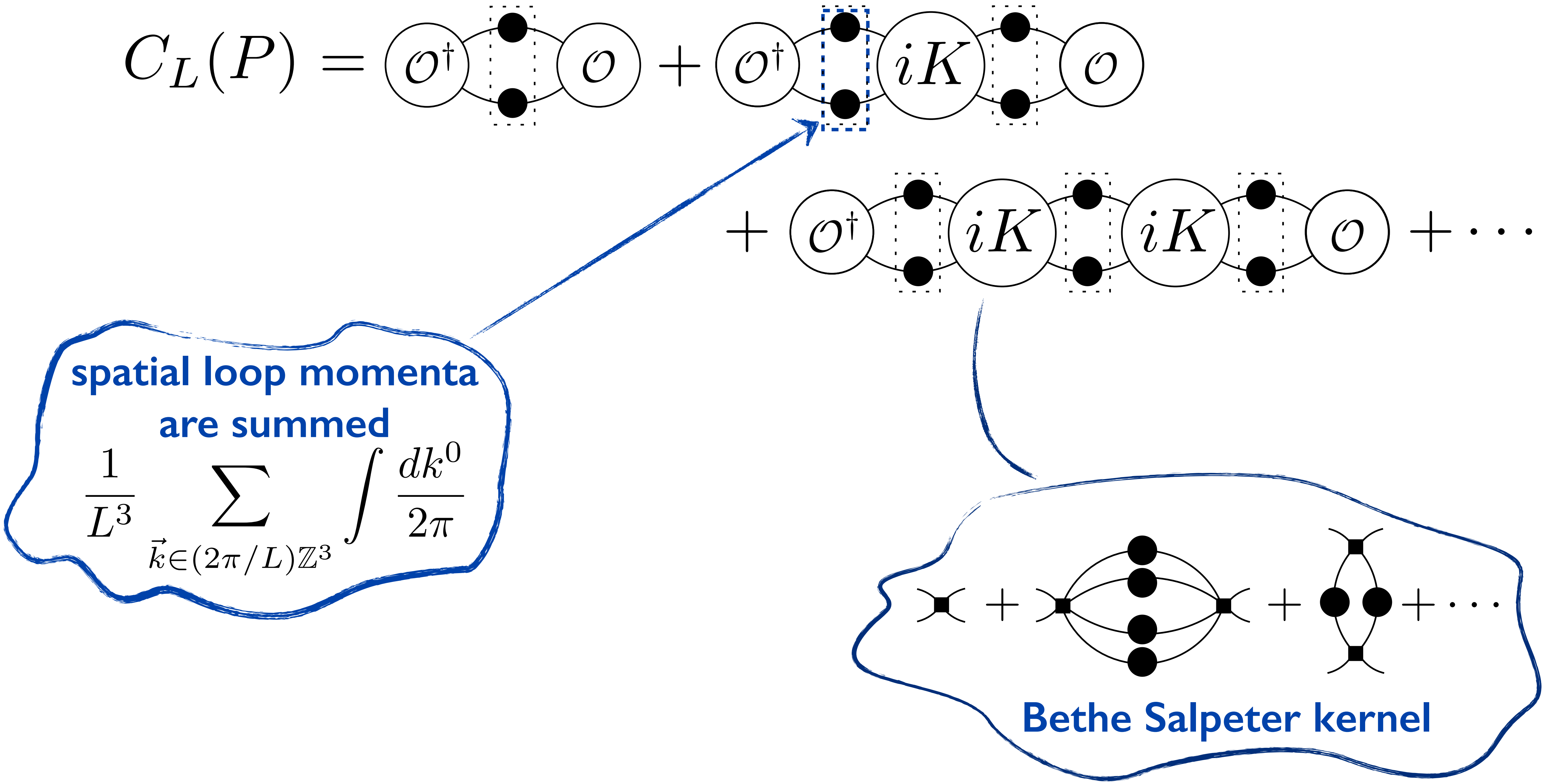

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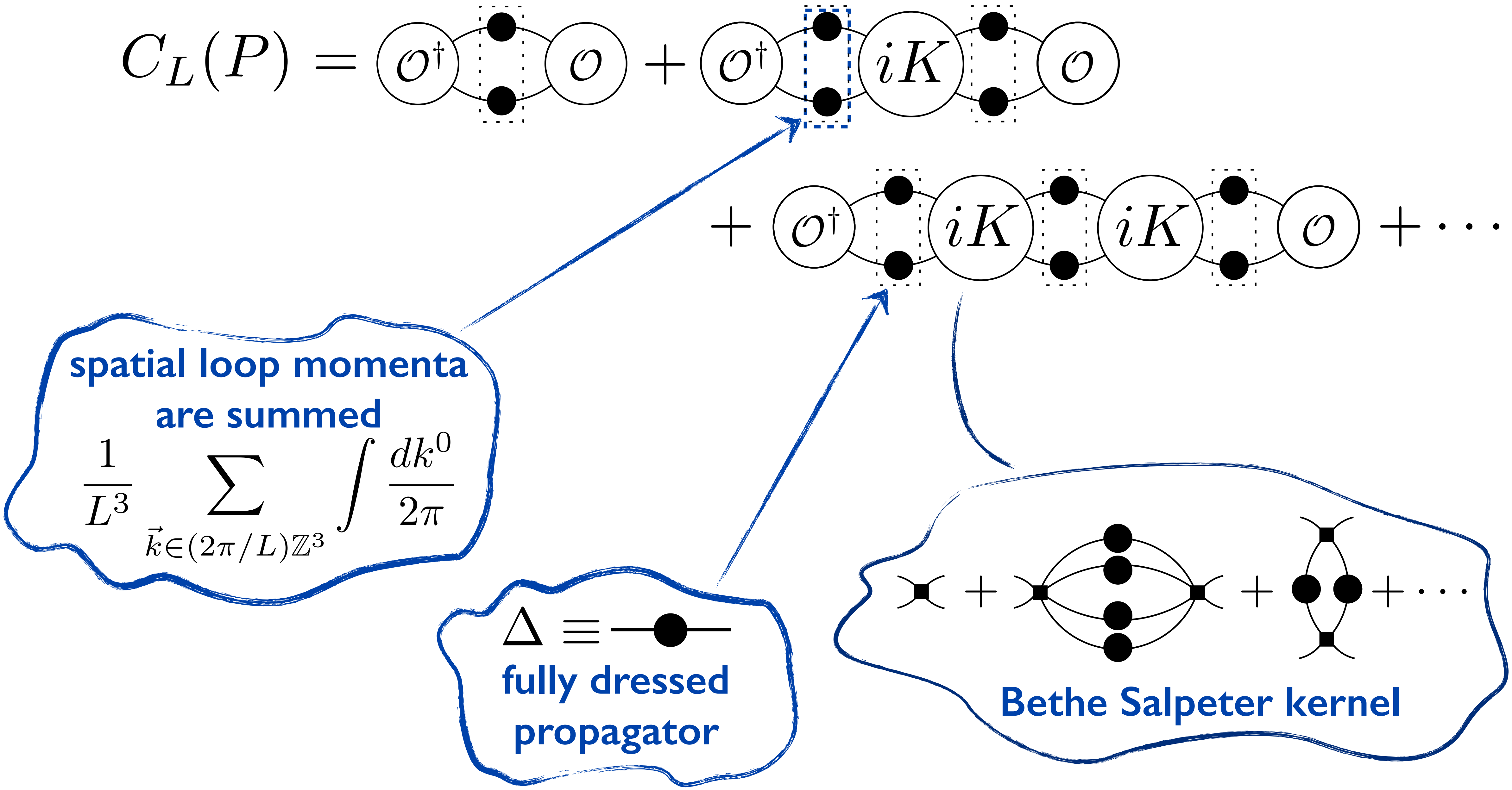

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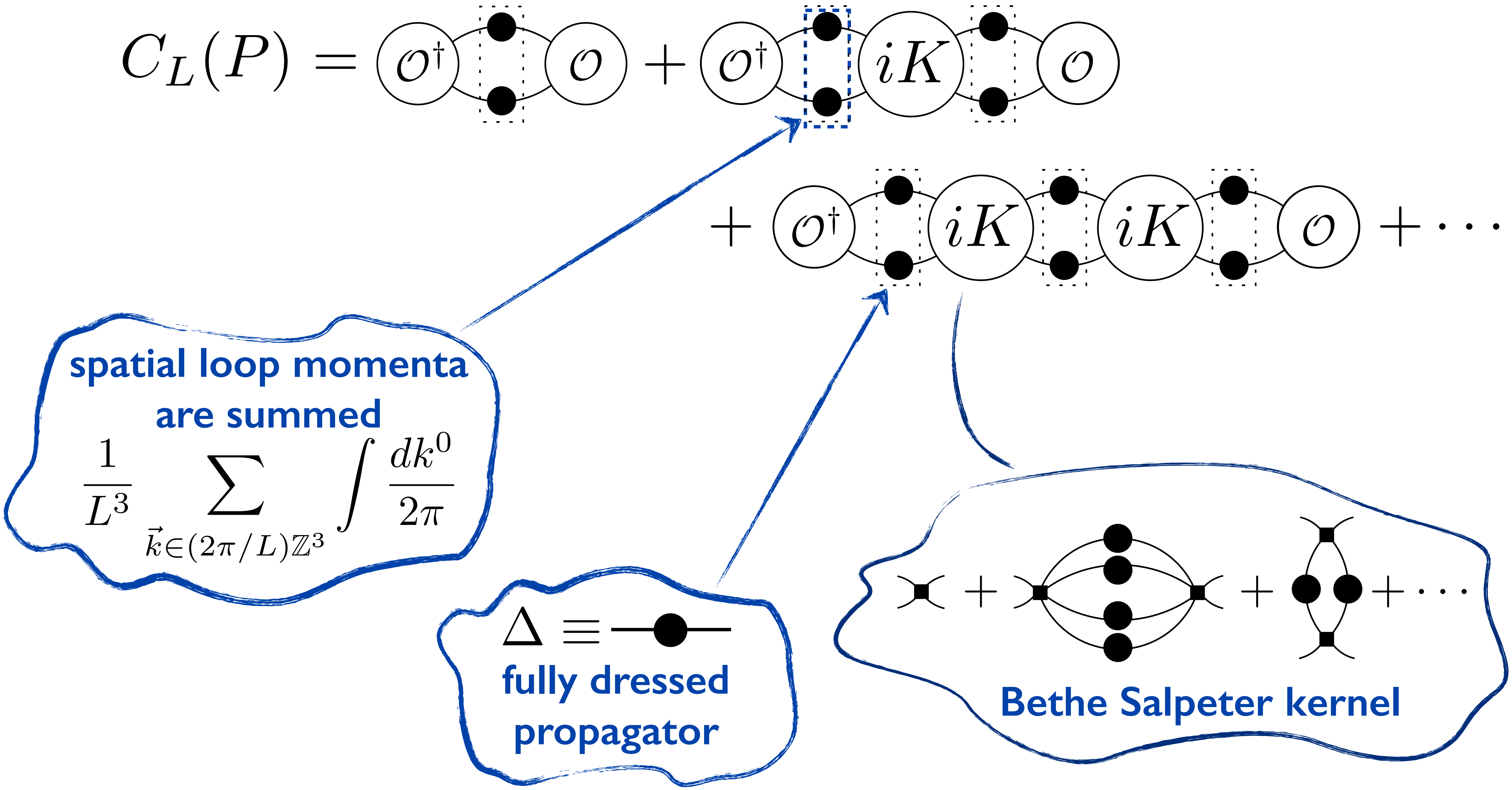

If $E^* < 4m$ **then**
$$K_L = K_\infty + \mathcal{O}(e^{-mL})$$
$$\Delta_L = \Delta_\infty + \mathcal{O}(e^{-mL})$$

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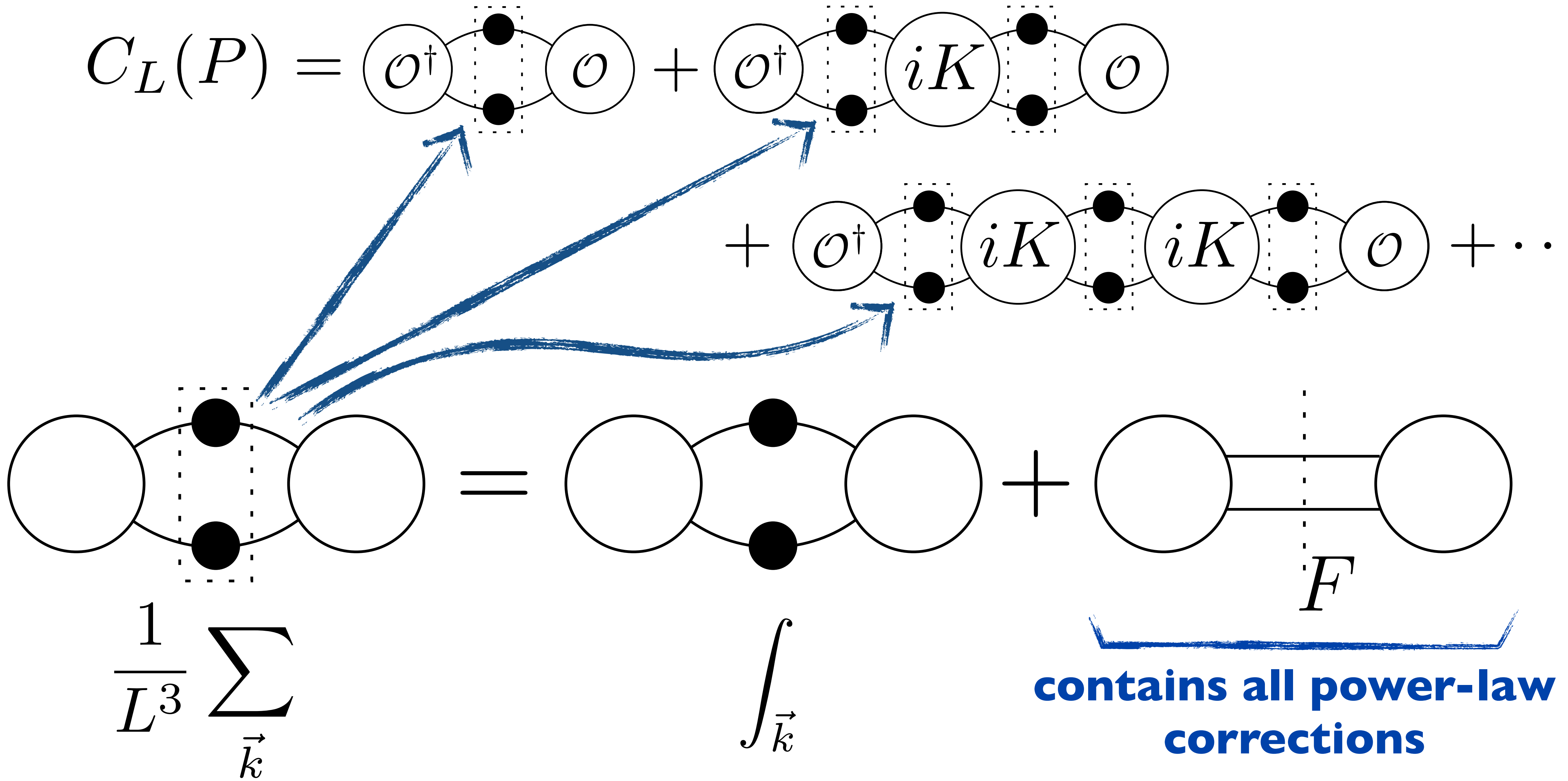

Now we introduce an important identity.

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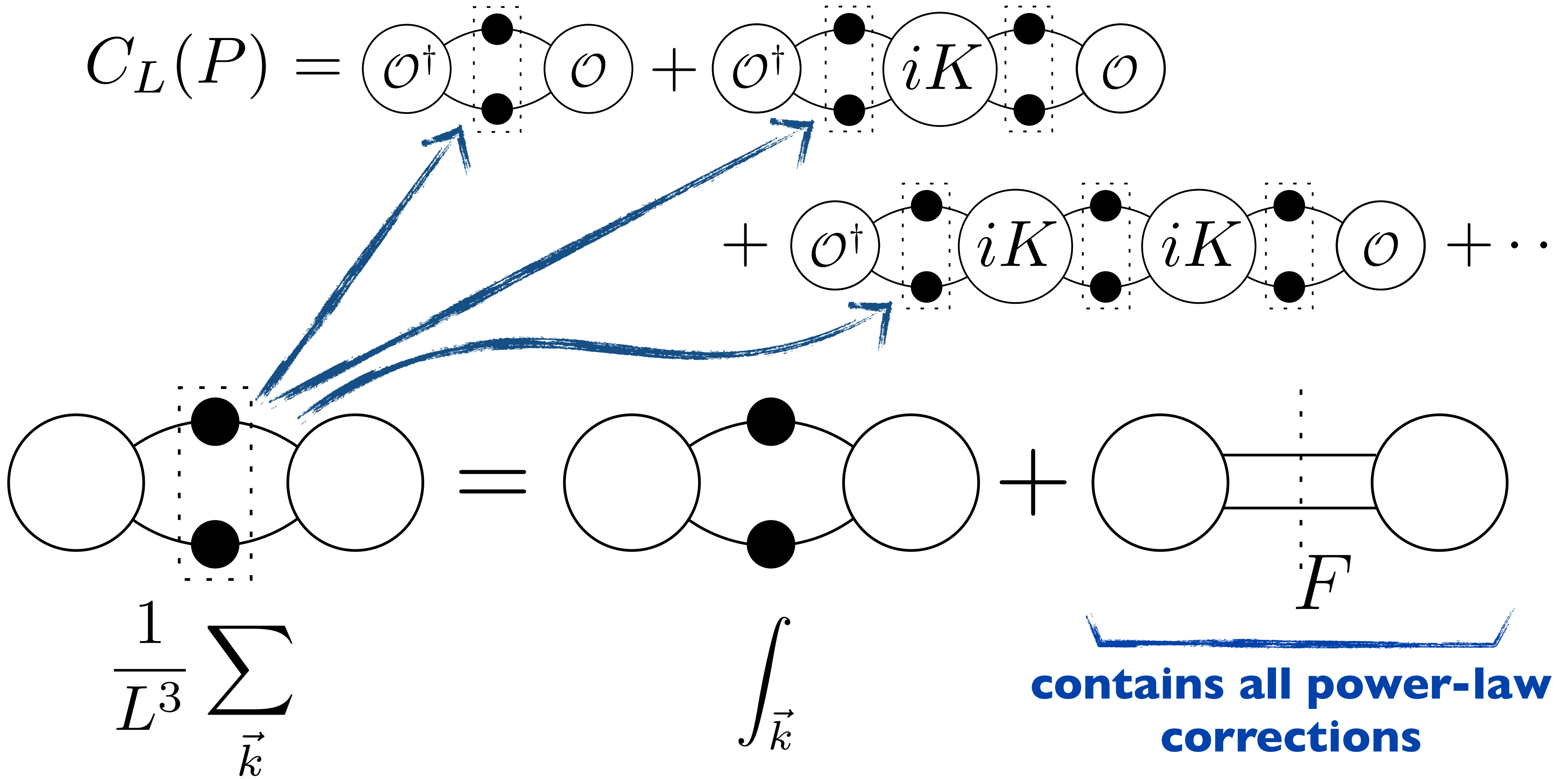
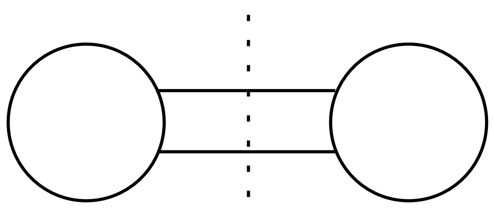


Now we introduce an important identity.

In  all four-momenta are projected **on shell.**

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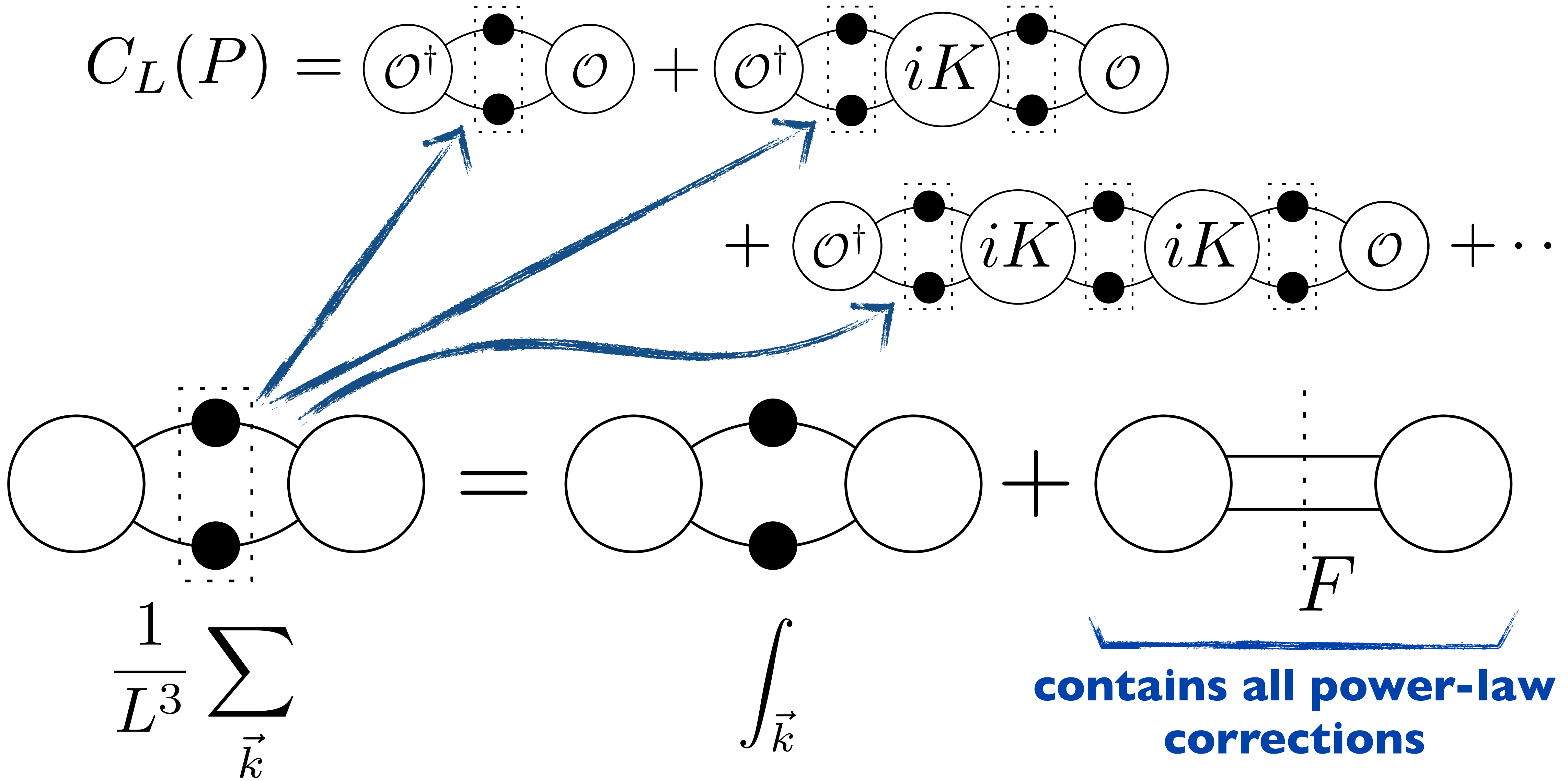

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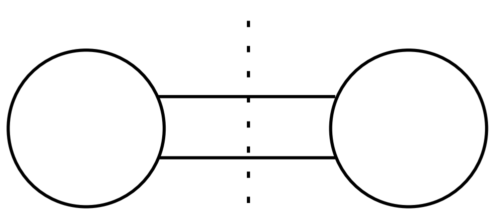
In  **all four-momenta are projected on shell.**

Physical, propagating states give dominate finite-volume effects.

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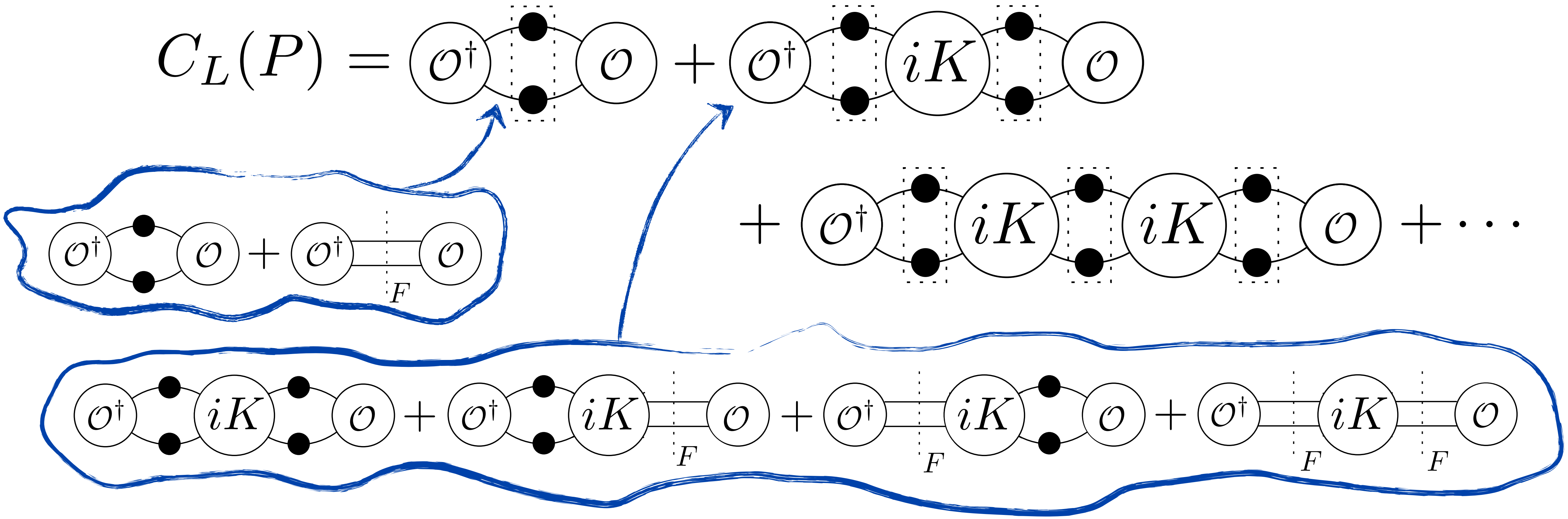
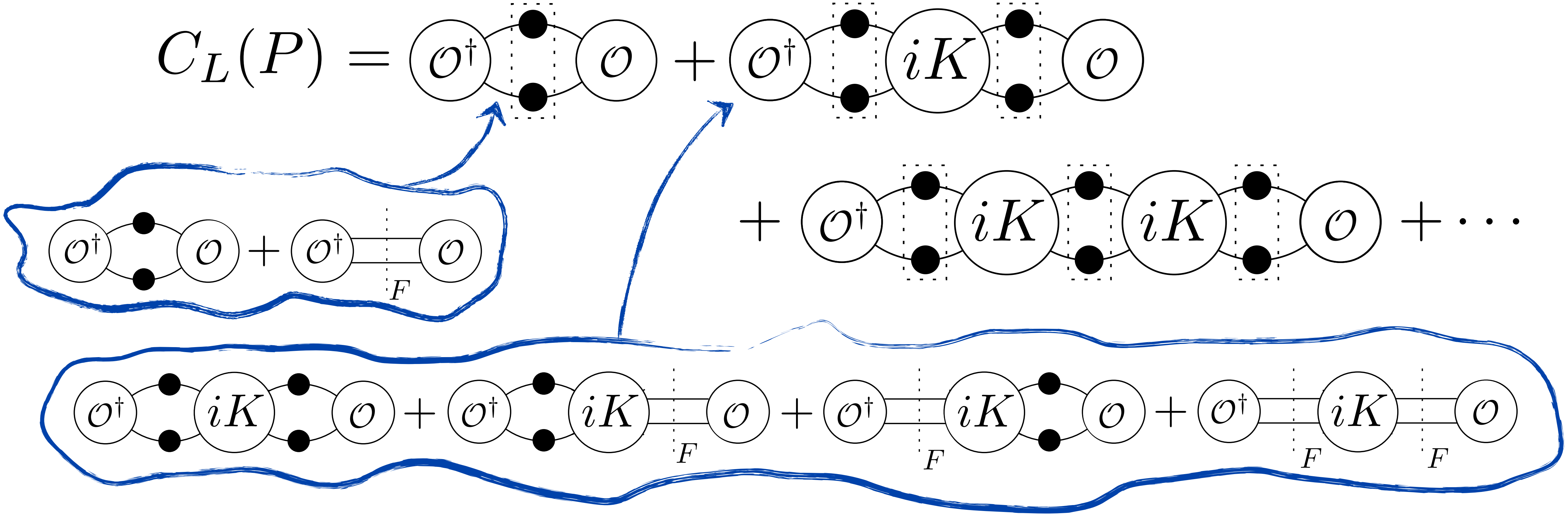
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)



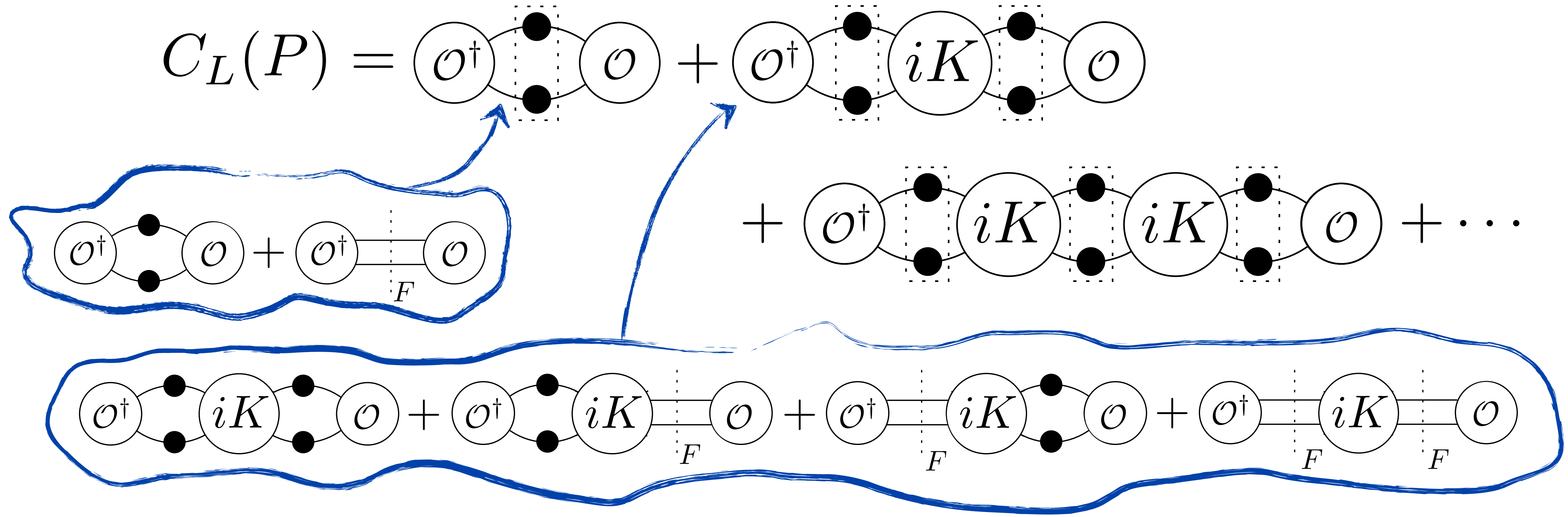
$C_L(P) =$
$\mathcal{O}^\dagger$
$\mathcal{O}$
iK
F
$+ \cdots$



Now regroup by number of Fs

zero Fs

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) +$$



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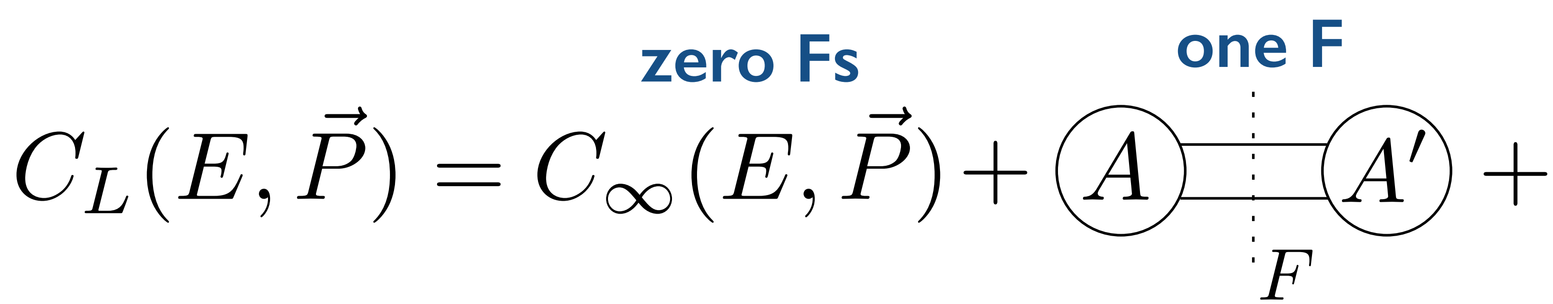

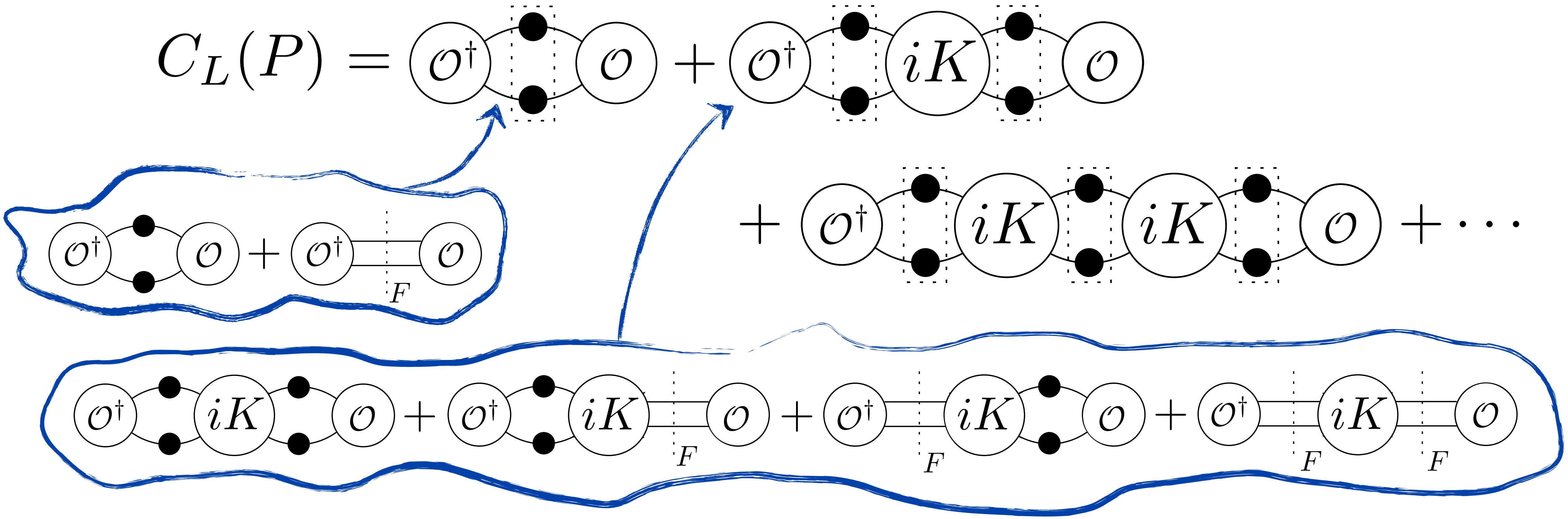

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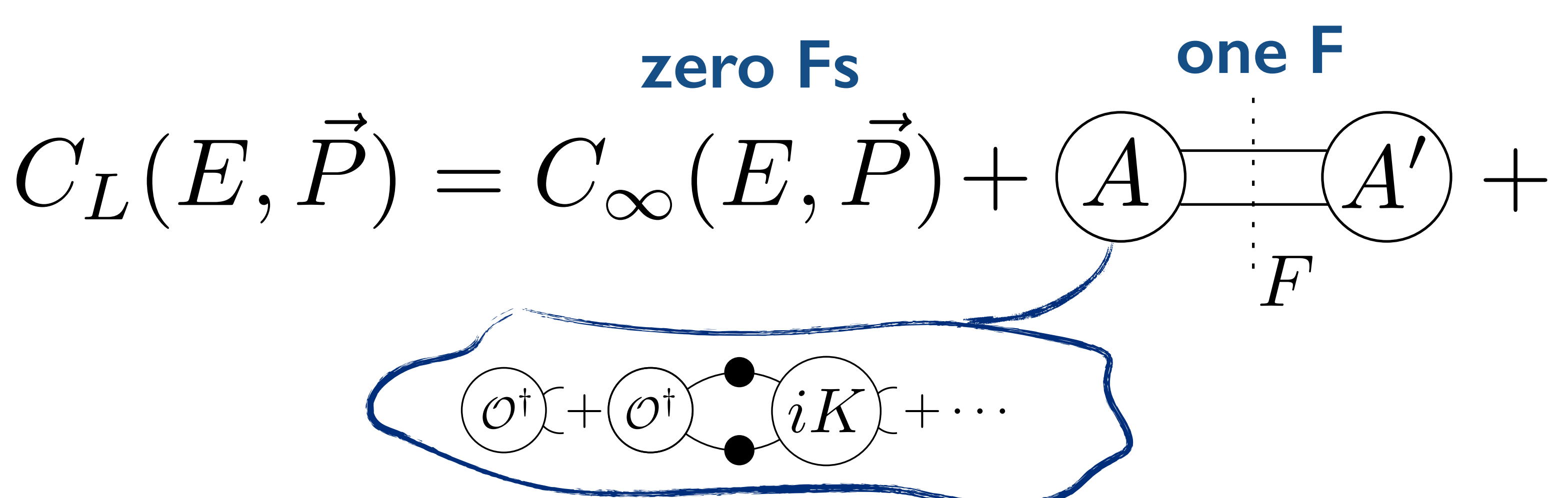
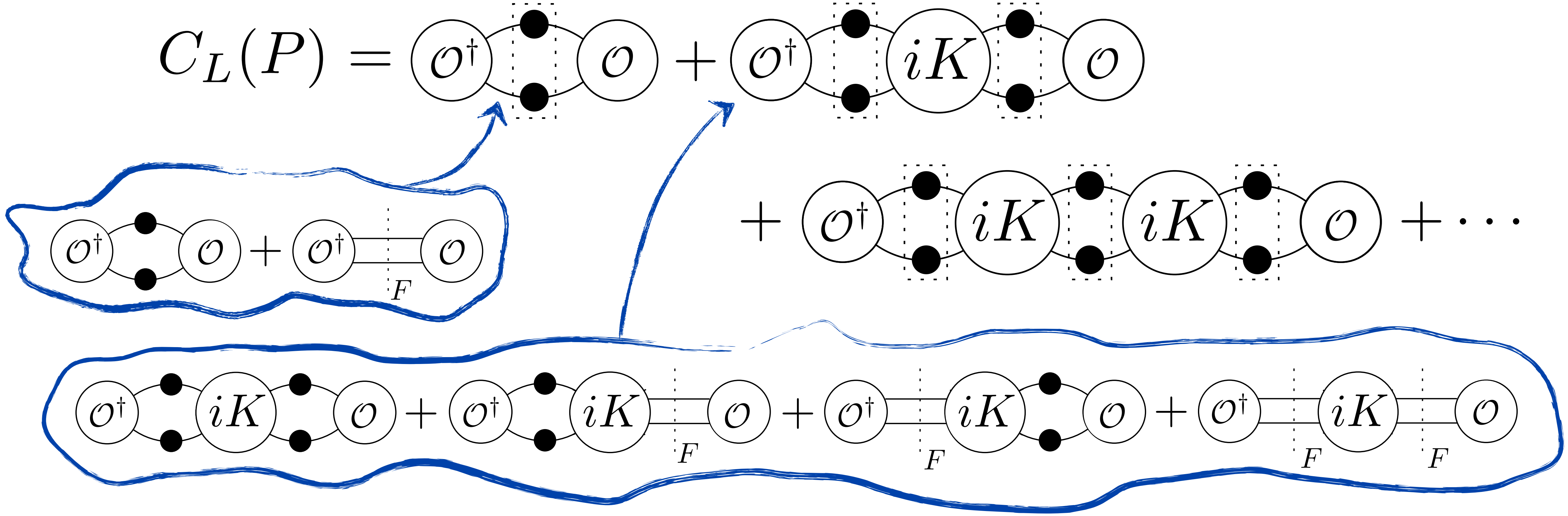


$$= \langle \pi\pi, \mathrm{out}|\mathcal{O}^\dagger|0\rangle$$



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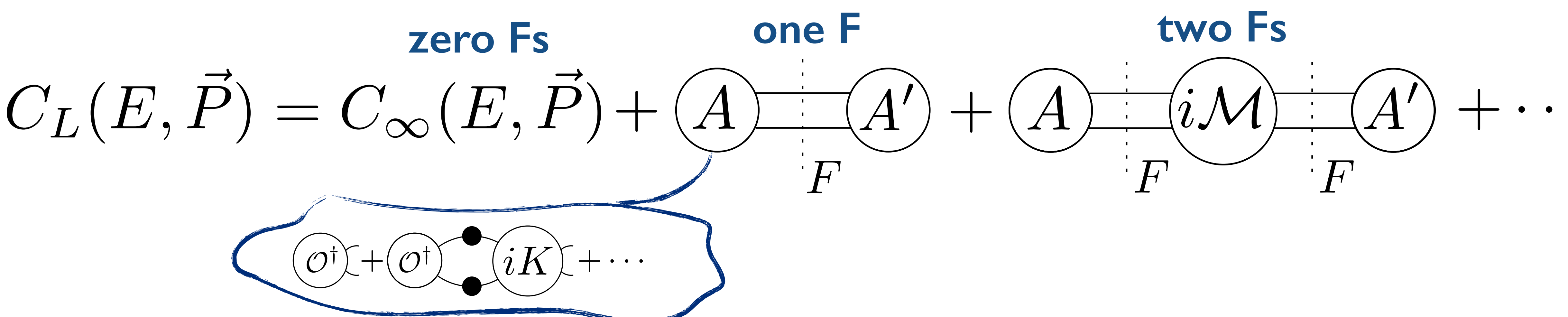

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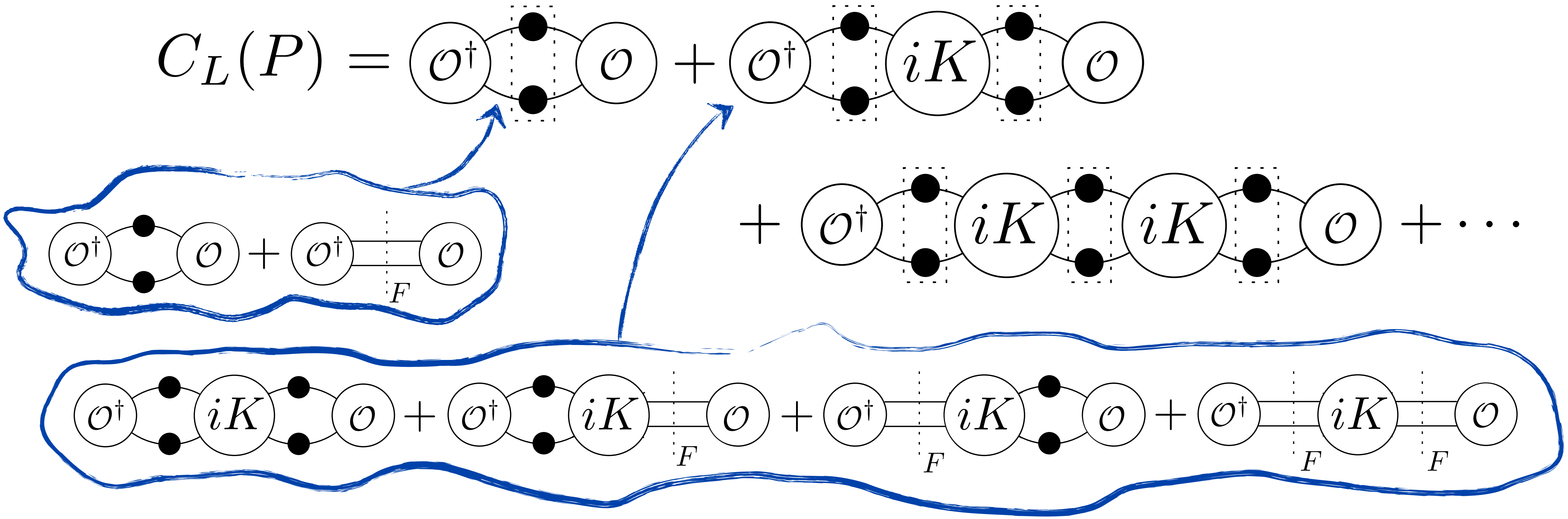

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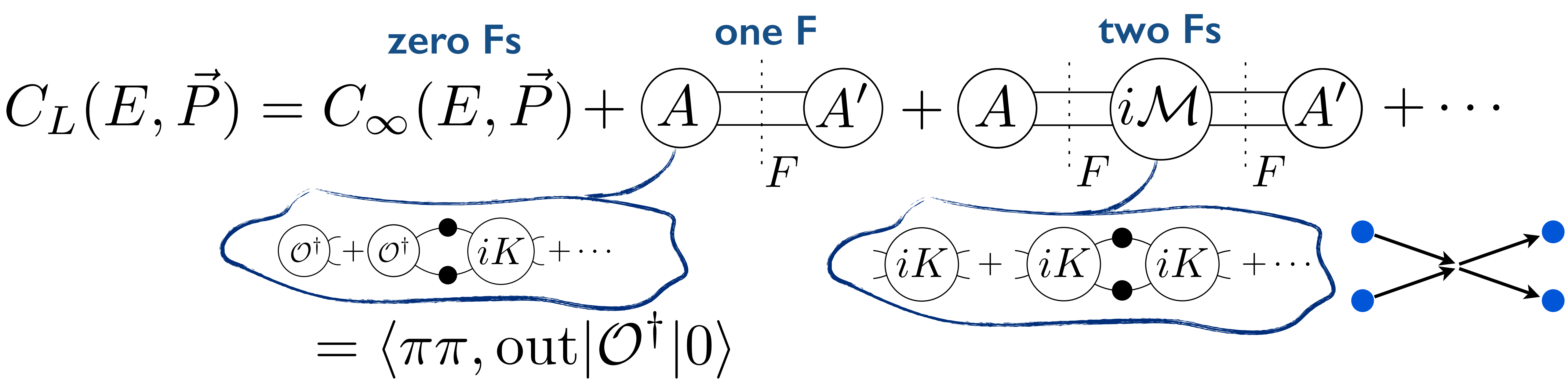

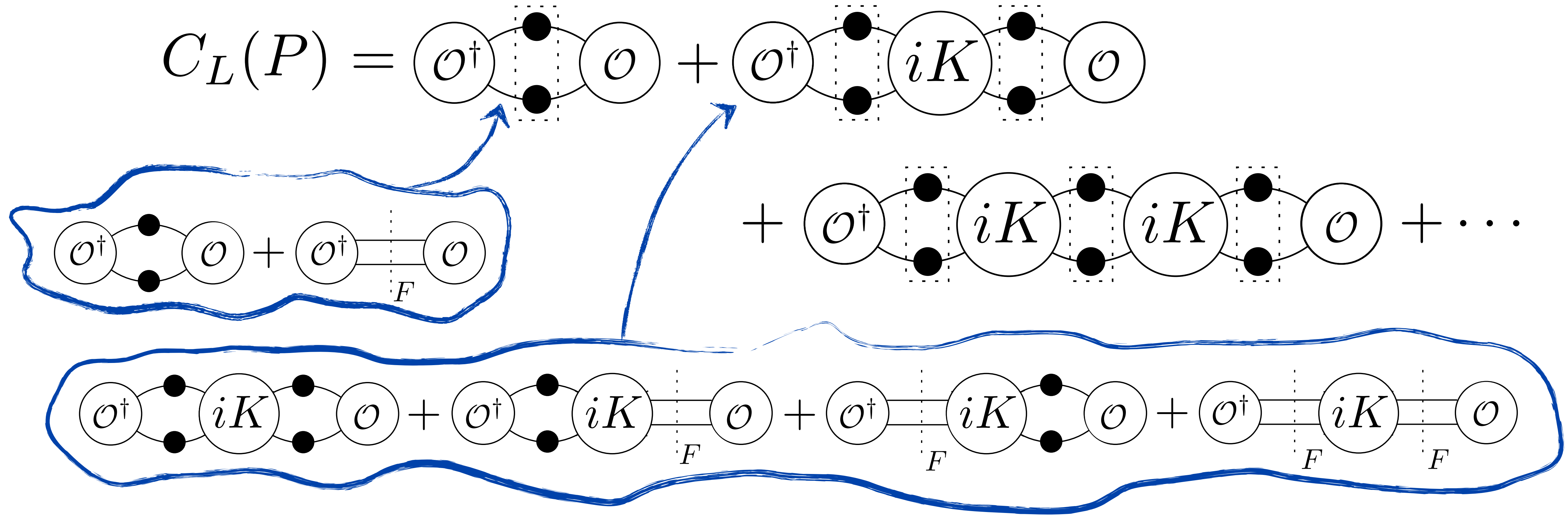

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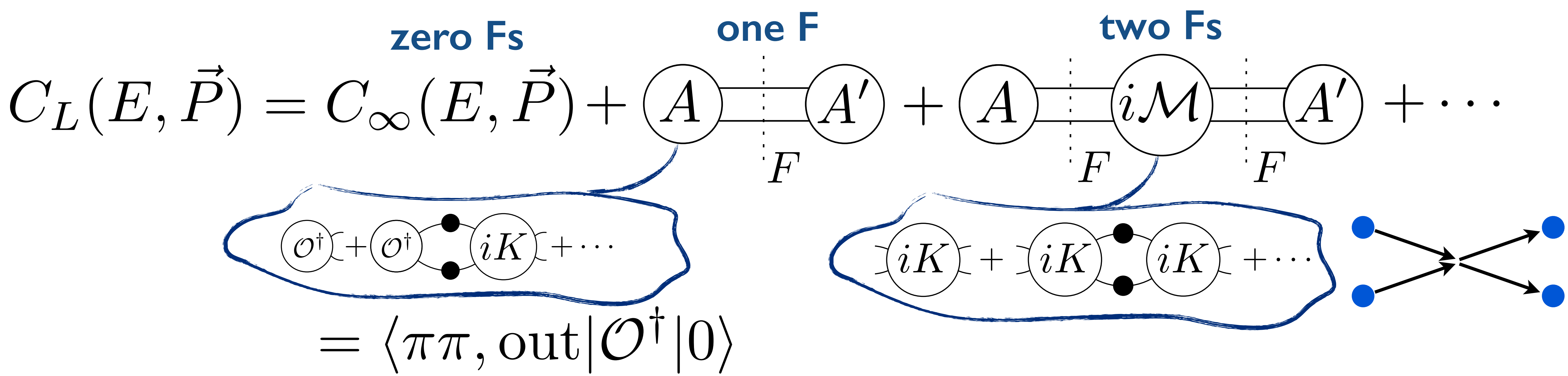


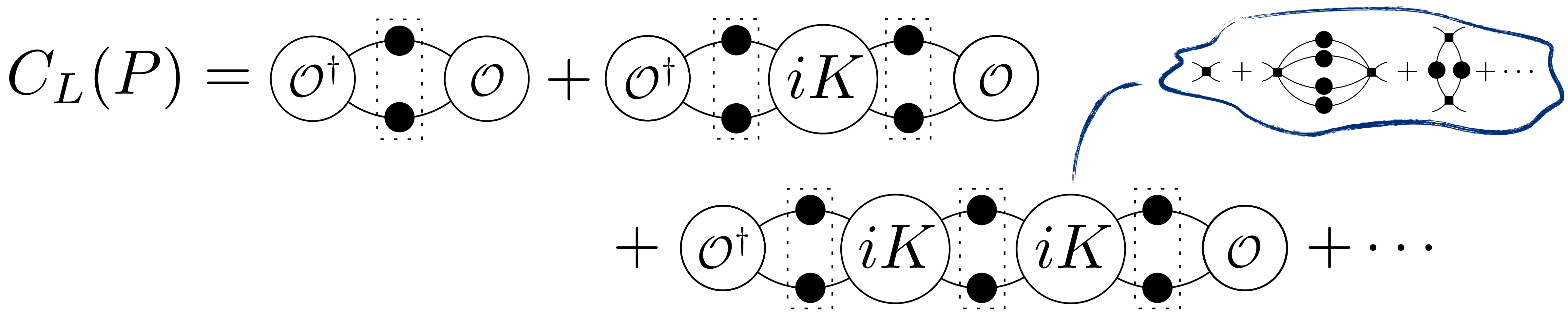
When we factorize diagrams and group infinite-volume parts…

physical observables emerge!

Review...

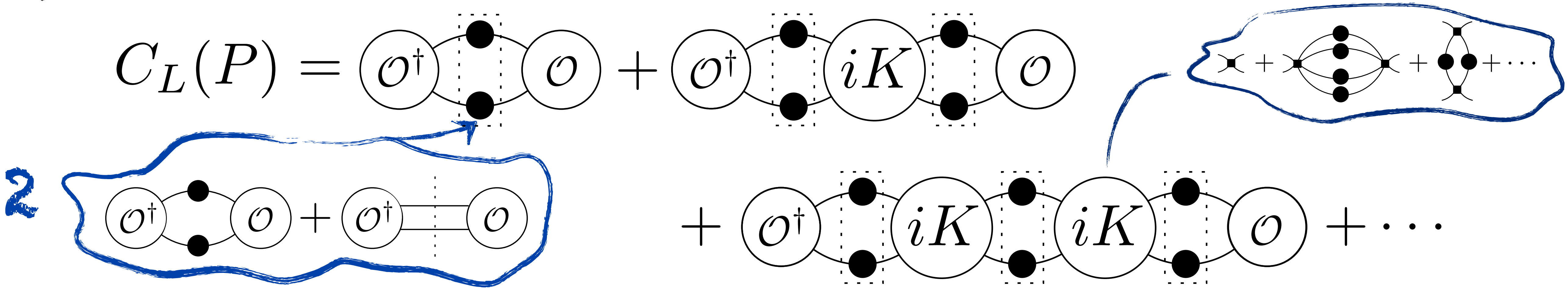
1

Review...

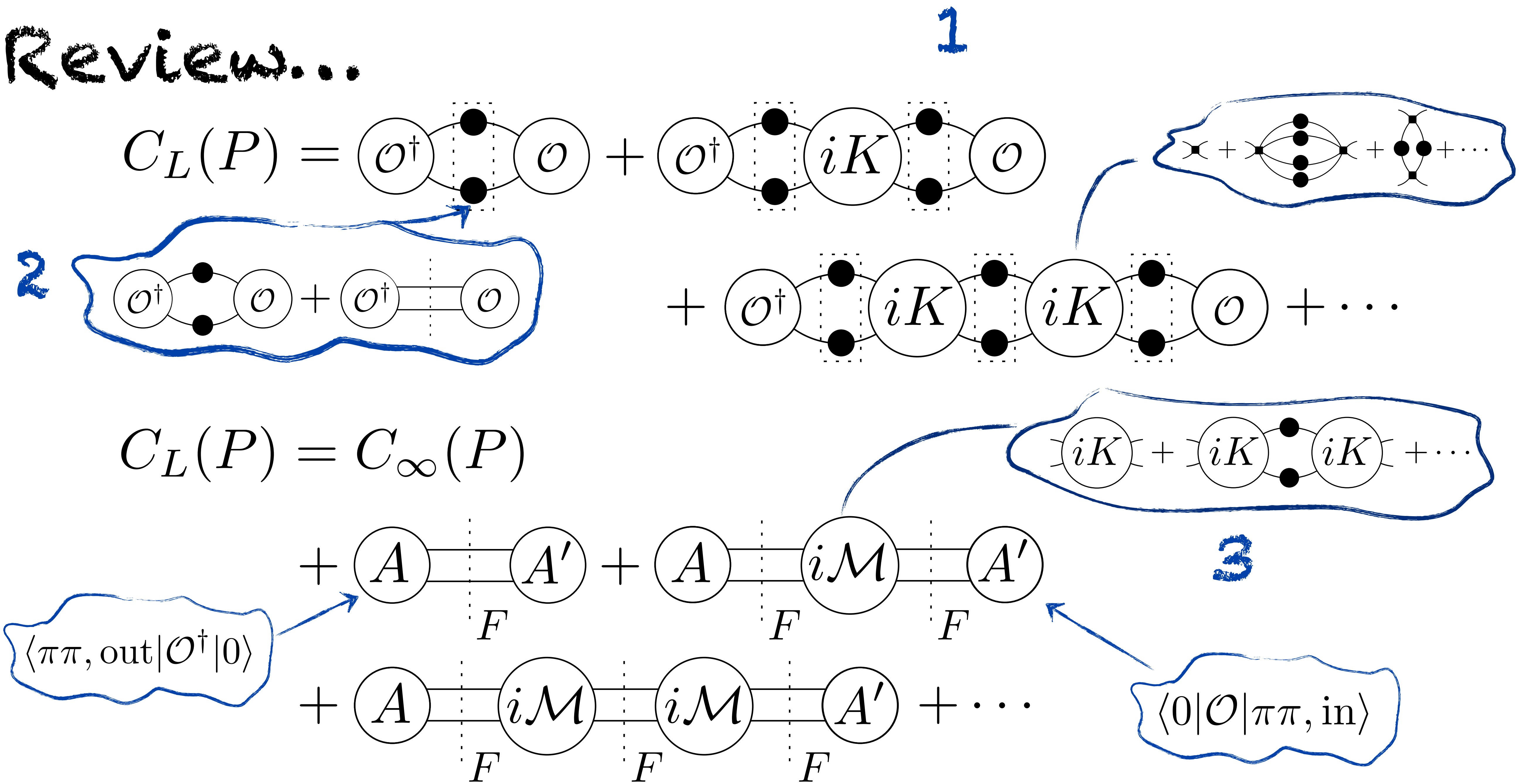


Review...

1

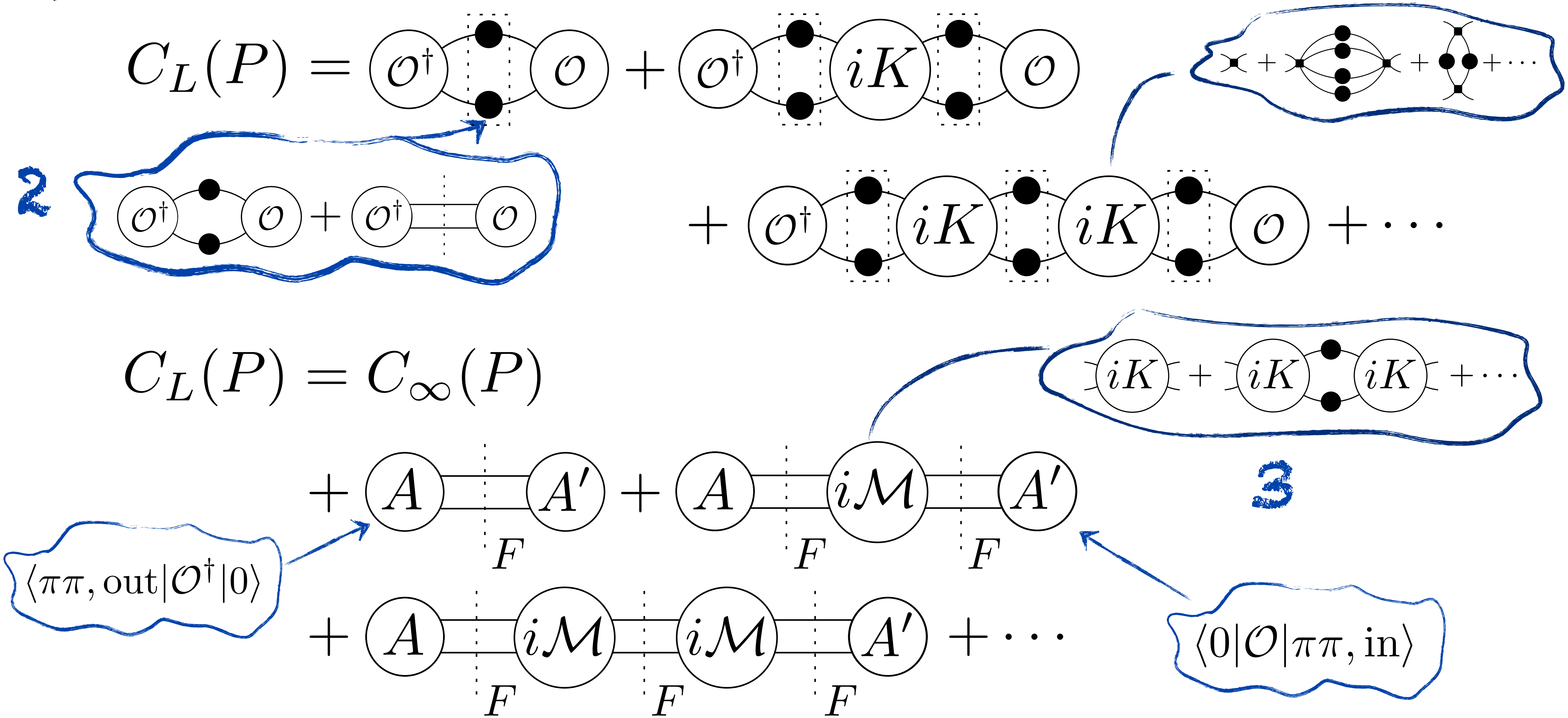


Review...


1
$C_L(P) =$
$\mathcal{O}^\dagger$ $\mathcal{O}$ + $\mathcal{O}^\dagger$ iK $\mathcal{O}$
+ $\mathcal{O}^\dagger$ iK iK $\mathcal{O}$ + ⋯
2
$\mathcal{O}^\dagger$ $\mathcal{O}$ + $\mathcal{O}^\dagger$ $\mathcal{O}$
$C_L(P) = C_\infty(P)$
+ A A' + A $i\mathcal{M}$ A'
F F F
+ A $i\mathcal{M}$ $i\mathcal{M}$ A' + ⋯
F F F
iK + iK iK + ⋯
3
$\langle \pi\pi, \mathrm{out}|\mathcal{O}^\dagger|0\rangle$
$\langle 0|\mathcal{O}|\pi\pi, \mathrm{in}\rangle$

Review...

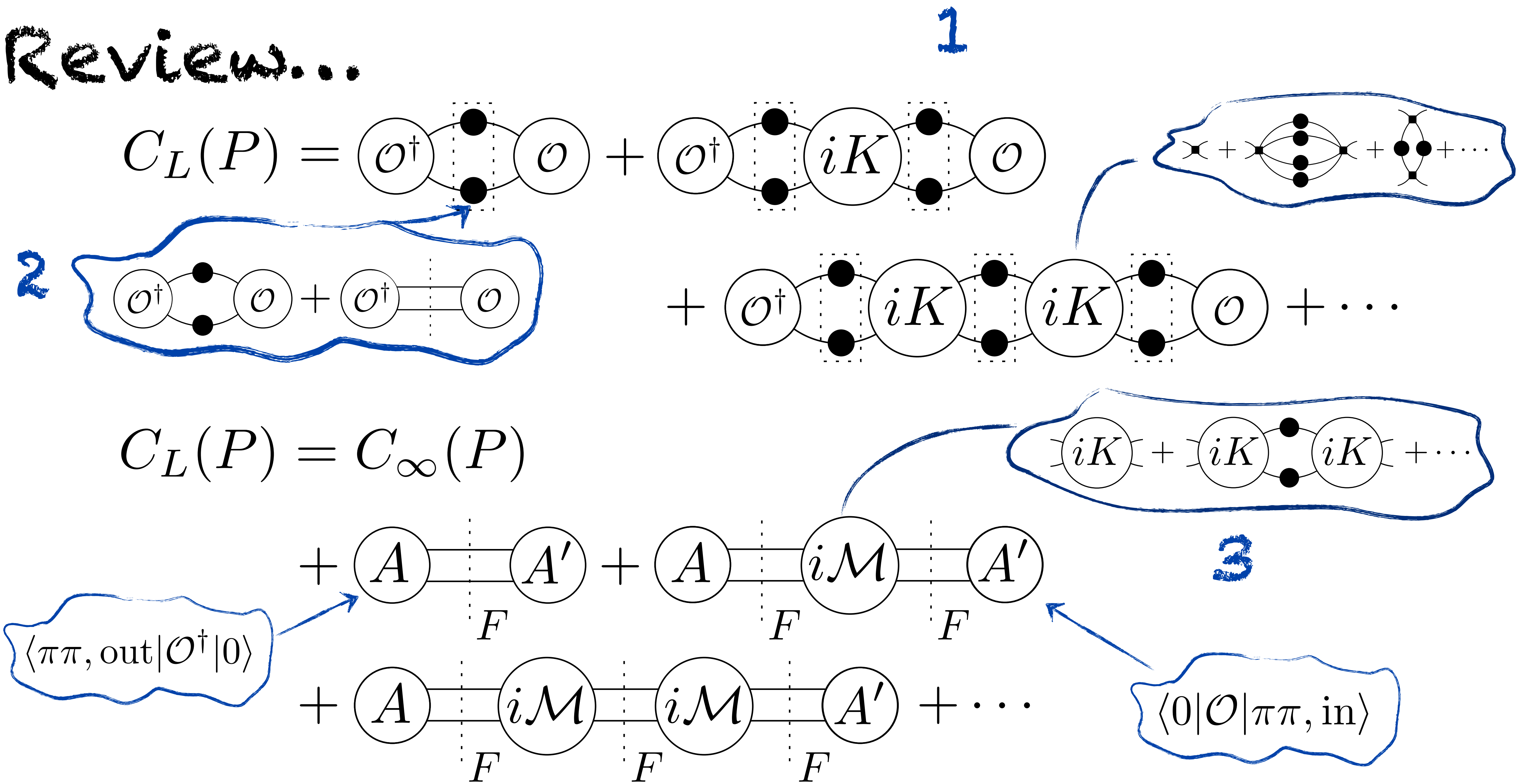
1



We deduce...

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to 2} F} A$$

Review...



We deduce...

poles are in here

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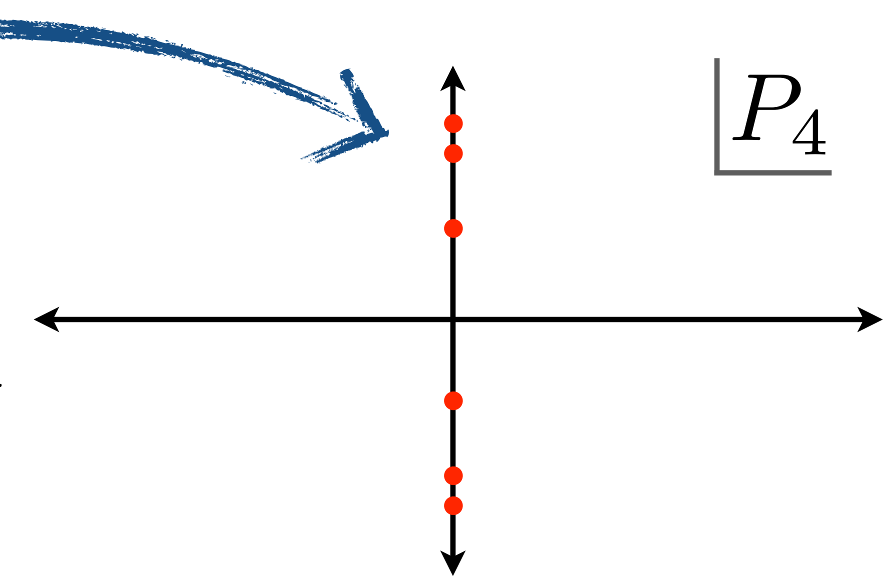

Two-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det[\mathcal{M}_{2\to2}^{-1} + F] = 0$

Rummukainen and Gottlieb, *Nucl. Phys.* B450, 397 (1995)
Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Matrices defined using angular-momentum states

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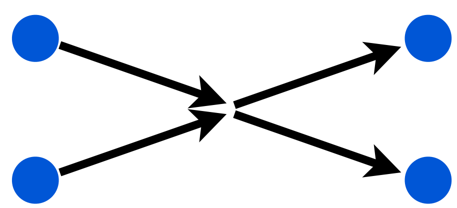
$\mathcal{M}_{2\to2} \equiv$  **diagonal matrix, parametrized by** $\delta_\ell(E^*)$

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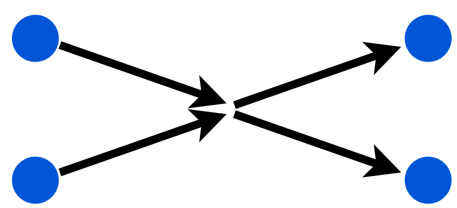
$\mathcal{M}_{2\to2} \equiv$  **diagonal matrix, parametrized by** $\delta_\ell(E^*)$

$F \equiv$ **non-diagonal matrix of known geometric functions**

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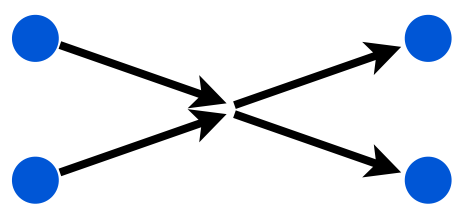
$\mathcal{M}_{2\to 2} \equiv$  **diagonal matrix, parametrized by** $\delta_\ell(E^*)$

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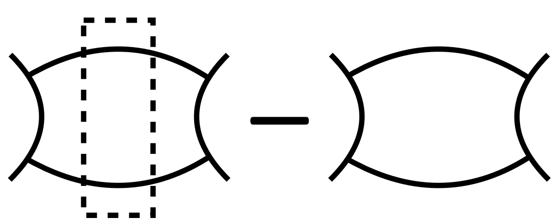
$\equiv$  difference of two-particle loops in finite and infinite volume

depends on $L, E, \vec{P}$

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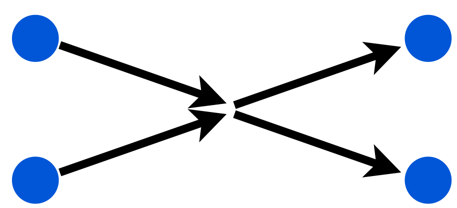
$\mathcal{M}_{2\to2} \equiv$  **diagonal matrix, parametrized by** $\delta_\ell(E^*)$

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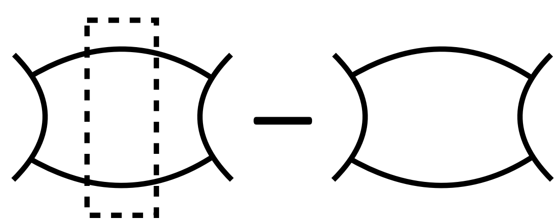
$\equiv$ 

difference of two-particle loops in finite and infinite volume

depends on $L, E, \vec{P}$

At low energies, lowest partial waves dominate $\mathcal{M}_{2\to2}$

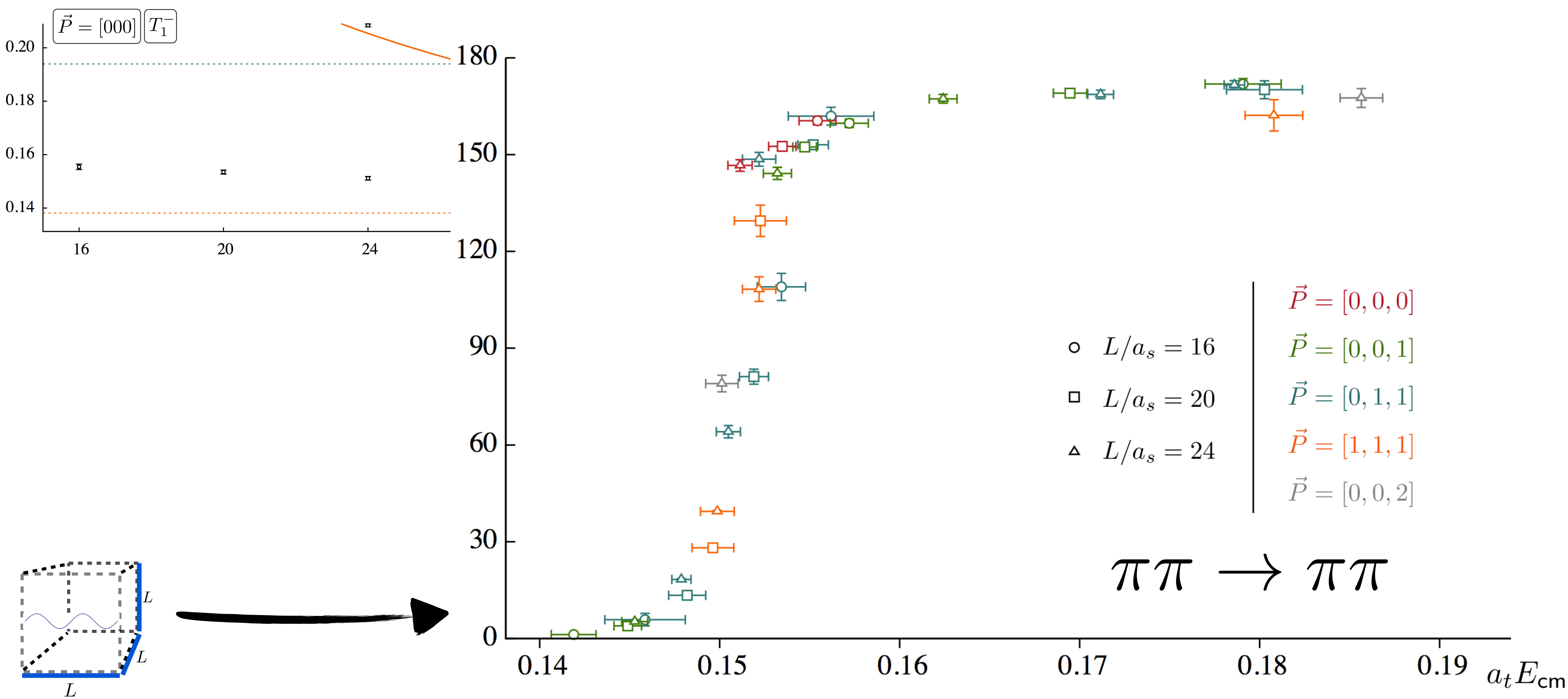
e.g. s-wave only with some rearranging

$$\longrightarrow \cot\delta(E_n^*) + \cot\phi(E_n, \vec{P}, L) = 0$$

scattering phase — known function

Using the result (p-wave)

$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

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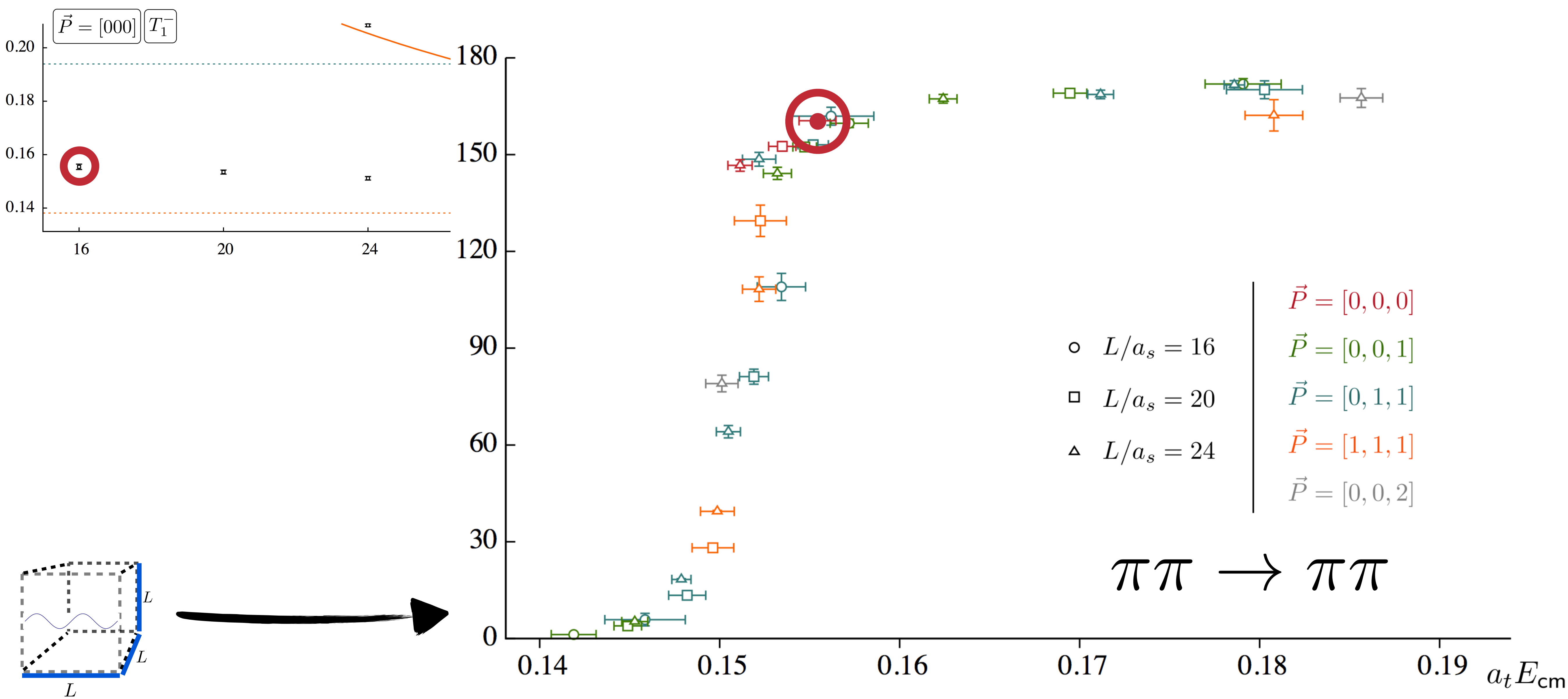

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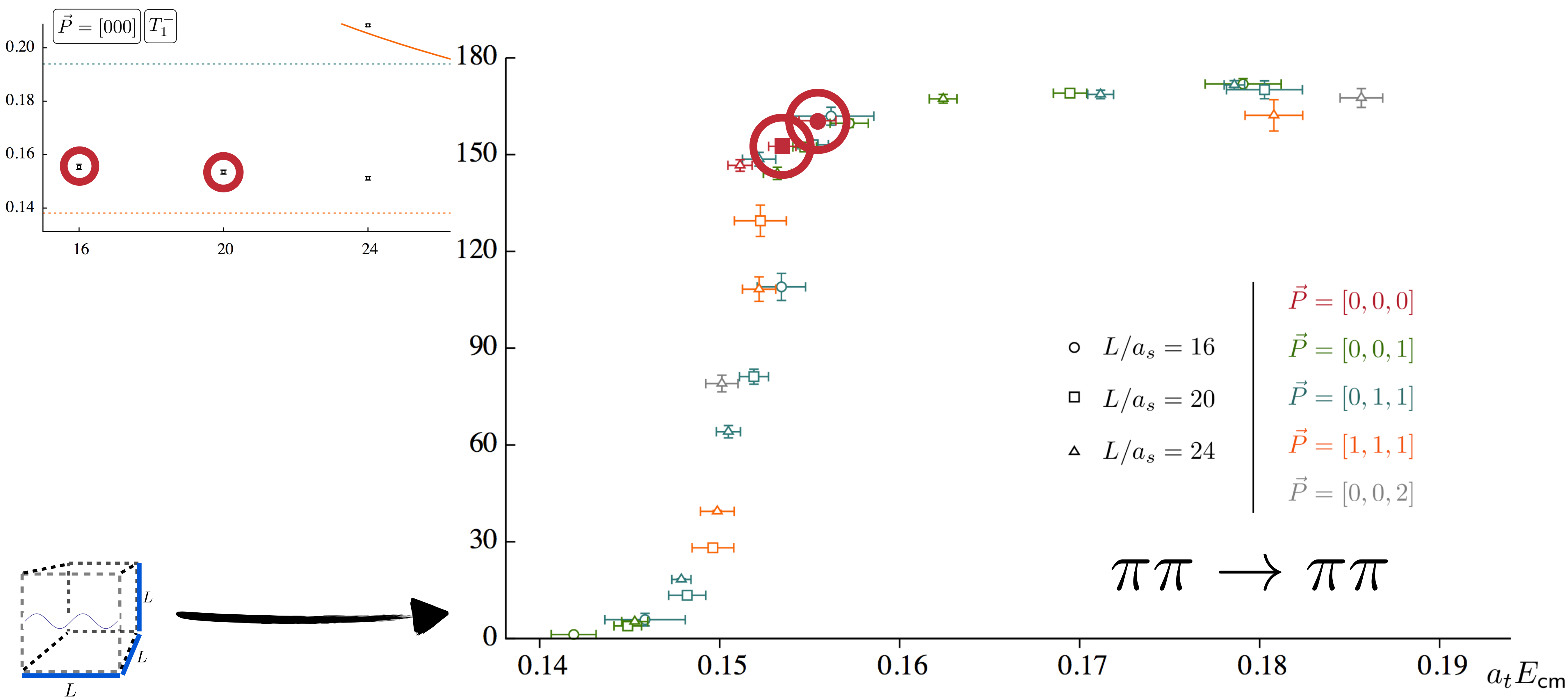

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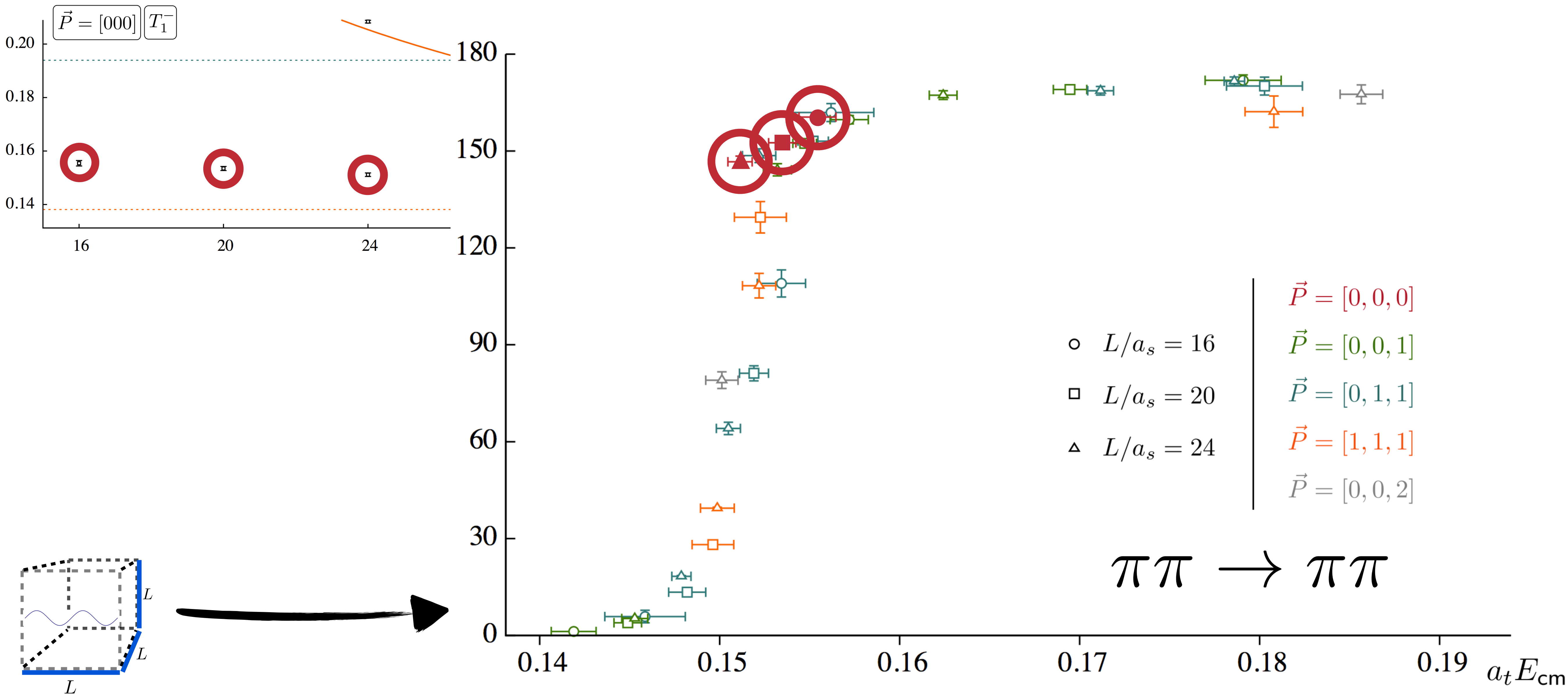

from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

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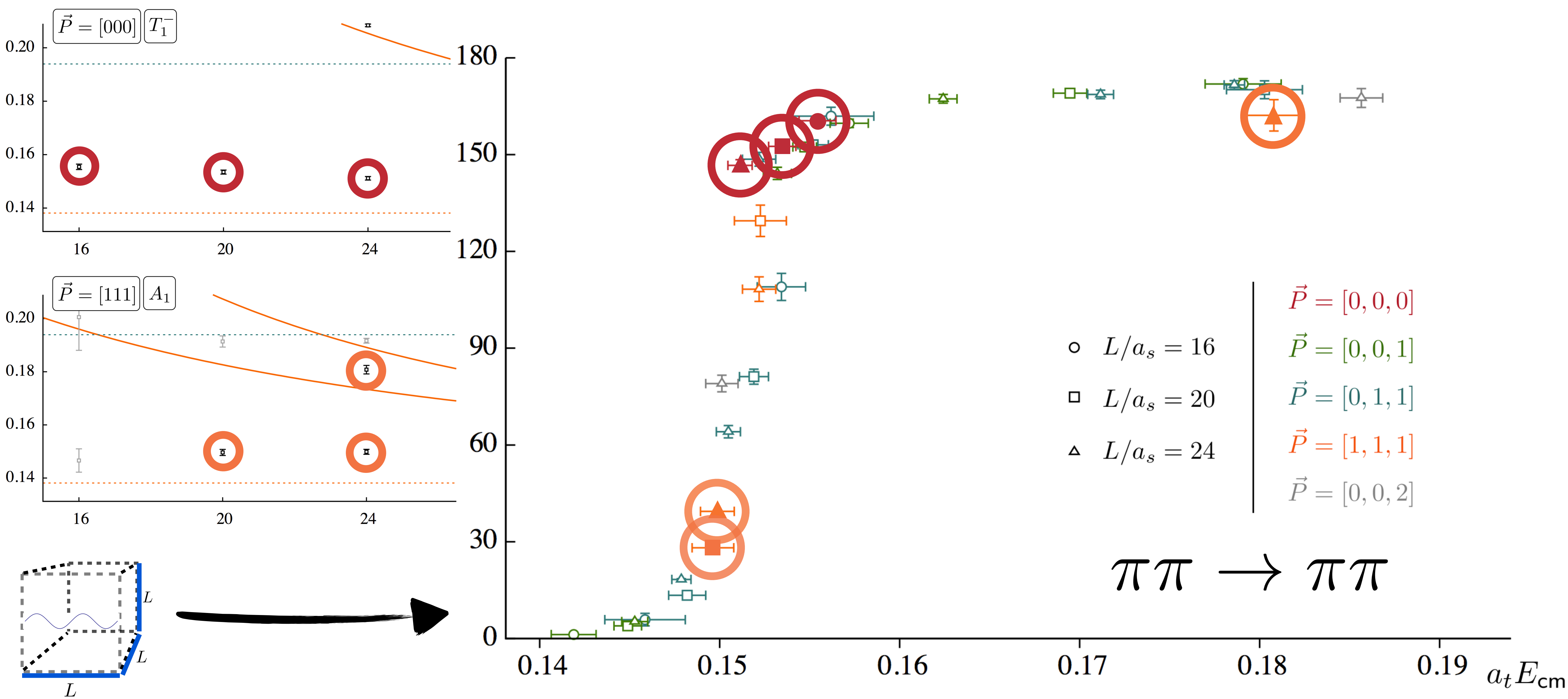

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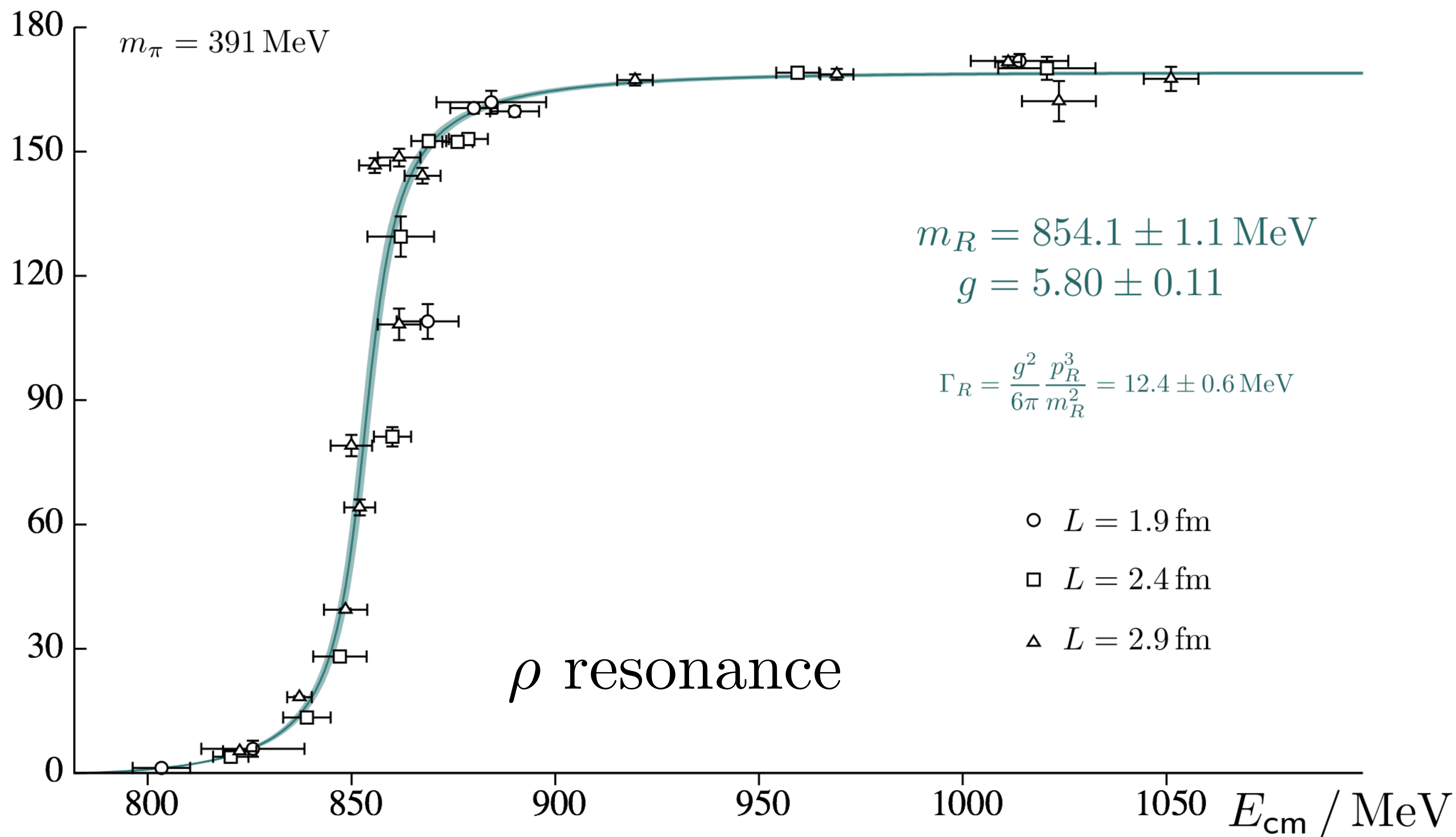
$\pi\pi \to \pi\pi$



$m_\pi = 391$ MeV

from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

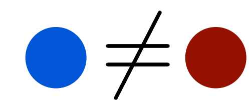
Two-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

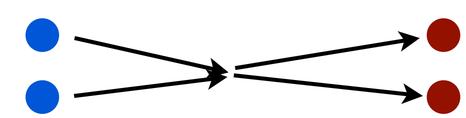
$$\det[\mathcal{M}_{2\to2}^{-1} + F] = 0$$

Has since been generalized to include…

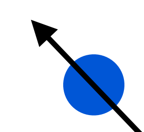
non-indentical particles



multiple two-particle channels



particles with spin



MTH and Sharpe, *Phys.Rev. D86* (2012) 016007
Briceño and Davoudi, *Phys.Rev. D88* (2013) 094507
Briceño, *Phys. Rev.* D 89, 074507 (2014)

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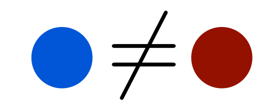

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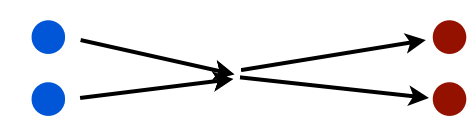

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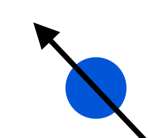

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Briceño and Davoudi, *Phys.Rev. D88* (2013) 094507
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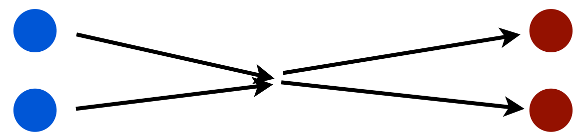
The basic form of the equation stays the same, but the matrix space and definition of F change

Multiple two-particle channels



Must now include a channel index

MTH and Sharpe/Briceño and Davoudi

$$\det\left[\begin{pmatrix}\mathcal{M}_{a\to a} & \mathcal{M}_{a\to b}\\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b}\end{pmatrix}^{-1} + \begin{pmatrix}F_a & 0\\ 0 & F_b\end{pmatrix}\right] = 0$$

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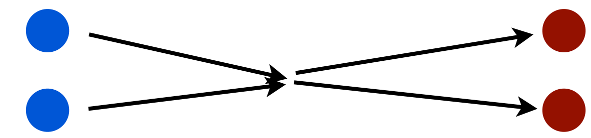

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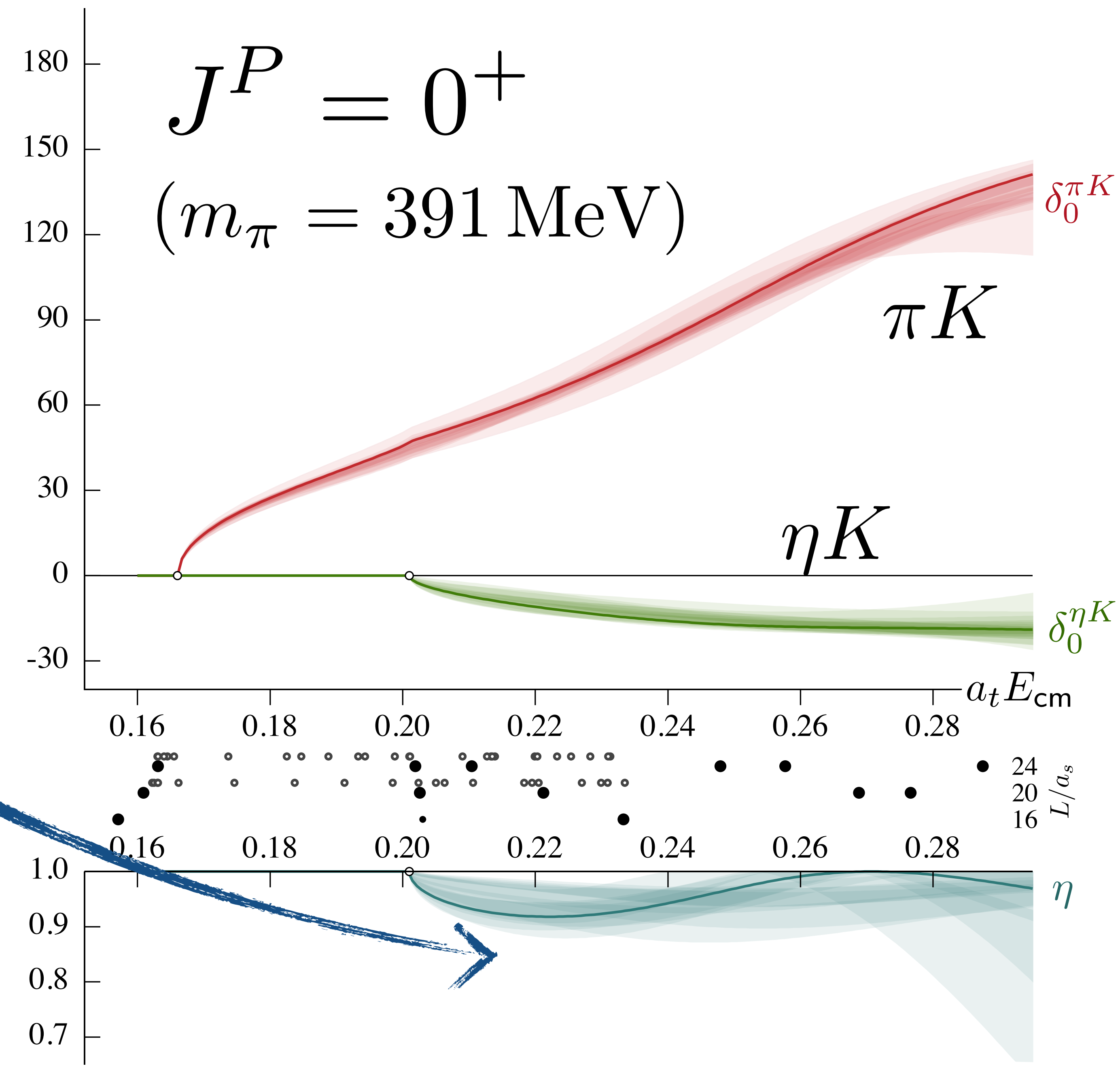
MTH and Sharpe/Briceño and Davoudi

Already used in JLab study of

$$\pi K,\ \eta K$$

$$\mathcal{M}(\pi K\to\eta K)\sim\sqrt{1-\eta^2}$$

Wilson, Dudek, Edwards, Thomas,
Phys. Rev. D 91, 054008 (2015)
arXiv: 1411.2004



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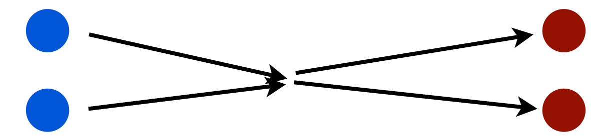

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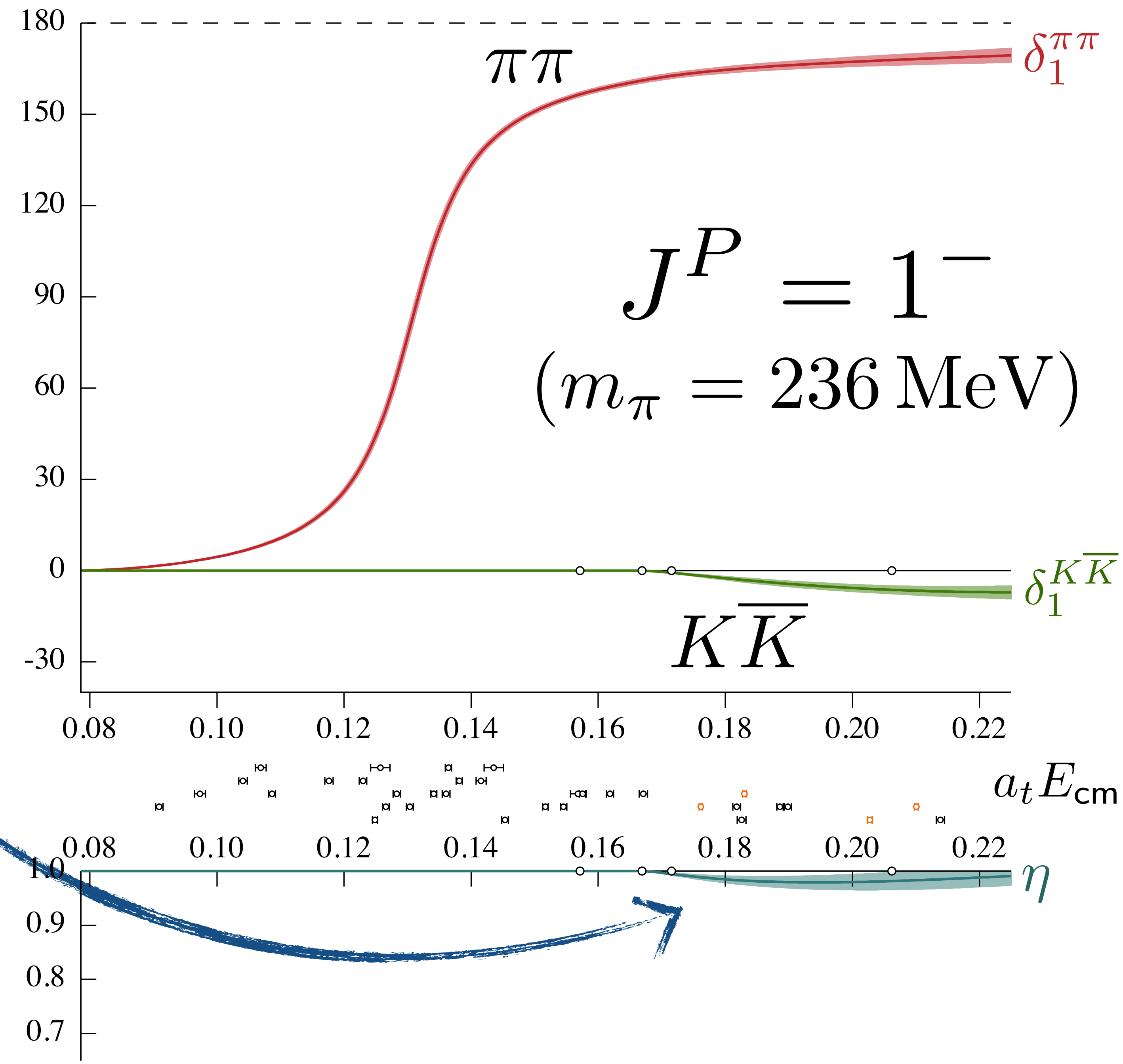
As well as JLab rho study with

$$\pi\pi,\ K\overline{K}$$

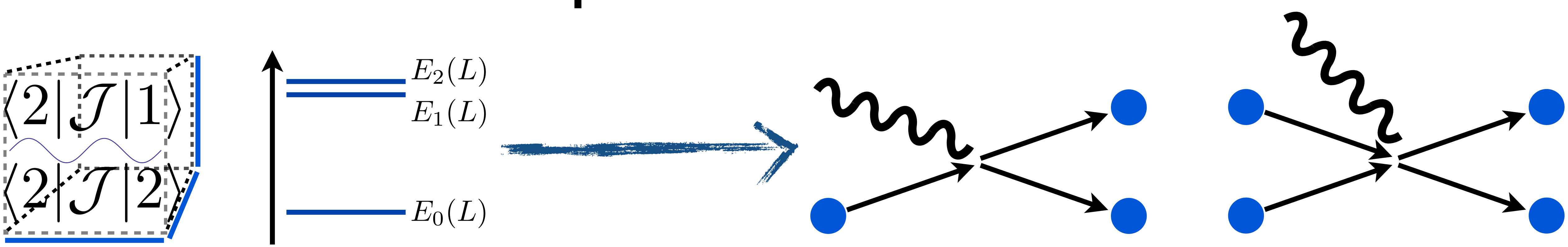
$$\mathcal{M}(\pi\pi \to K\overline{K}) \sim \sqrt{1-\eta^2}$$



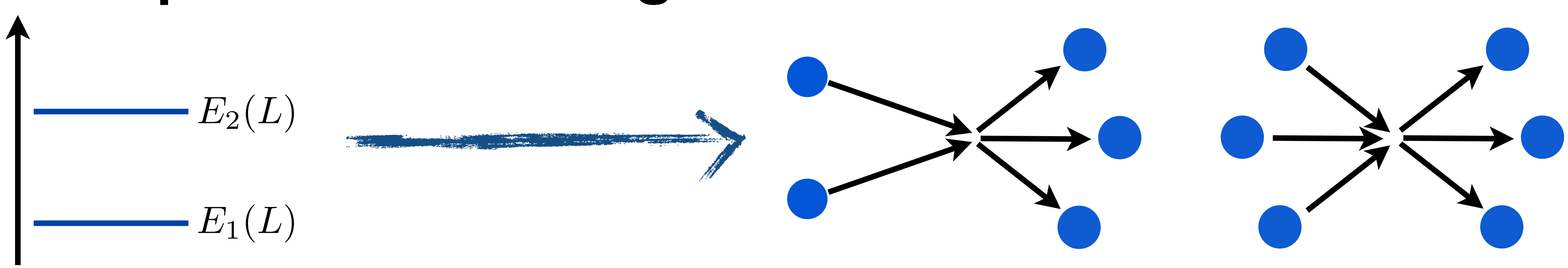
Wilson, Briceño, Dudek, Edwards, Thomas, arXiv:1507:02599

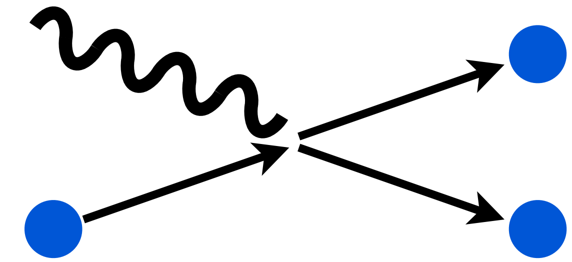
Two-particle scattering

Photo- and electroproduction



Three-particle scattering



Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

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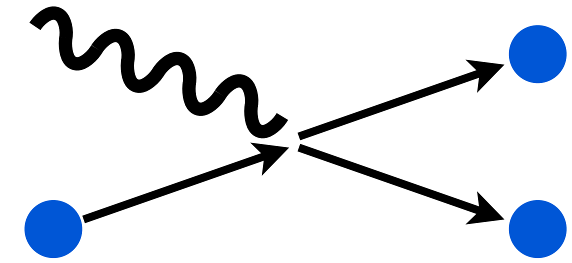
Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

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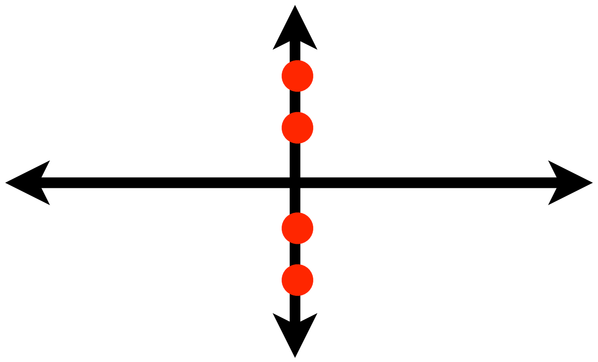
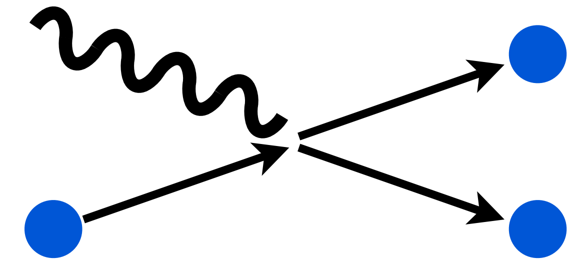
Why did we expect $C_L(P)$ *to have poles?*



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

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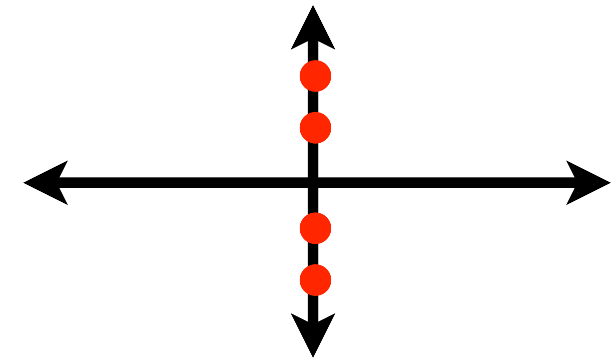

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Insert a complete set finite-volume of states

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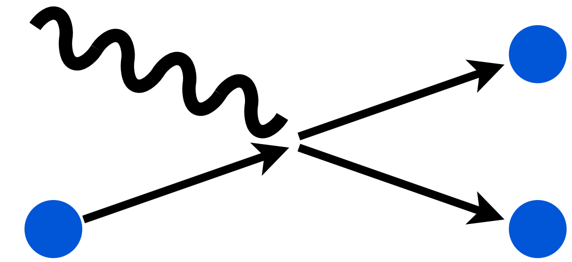

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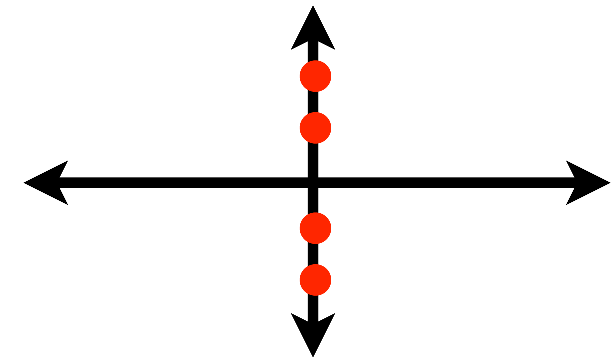

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$$C_L(P) \xrightarrow[P_4 \to iE_n]{} \frac{L^3 \langle 0|\mathcal{O}(0)|n, \vec{P}, L\rangle\langle n, \vec{P}, L|\mathcal{O}^\dagger(0)|0\rangle}{(E_n + iP_4)}$$

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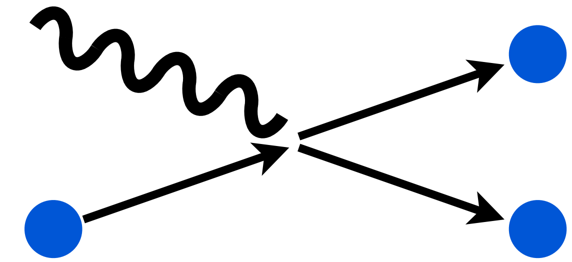

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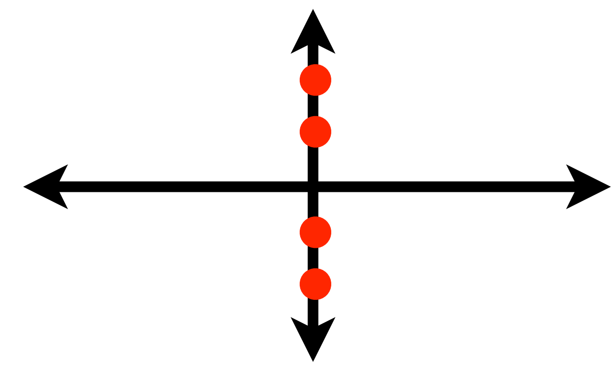

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Now compare this to our factorized result

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to 2} F} A$$

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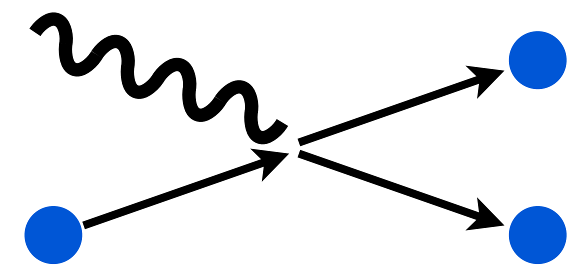

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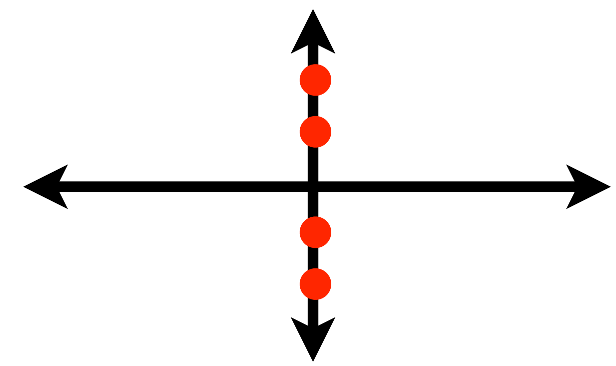

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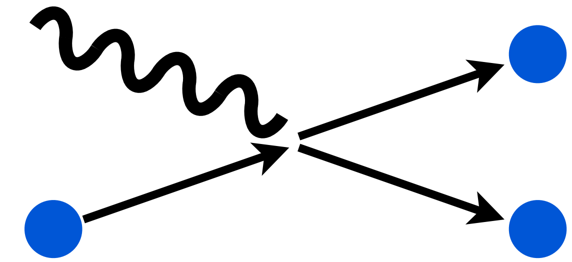

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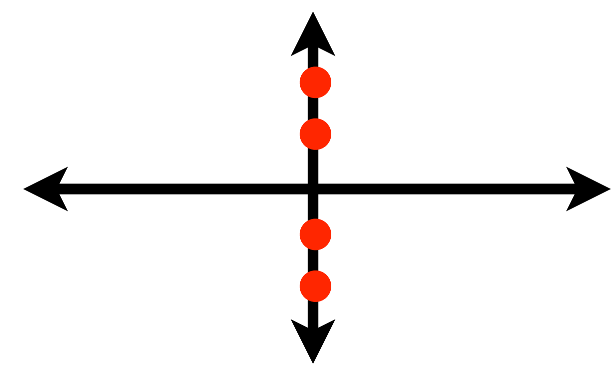

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$\mathcal{R}$ is the residue of this matrix

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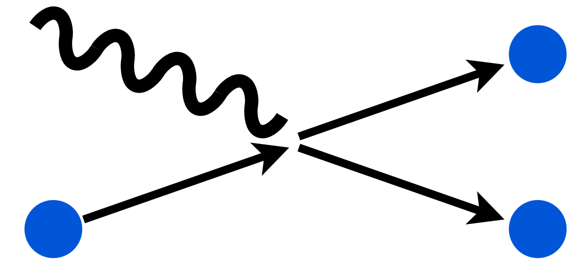

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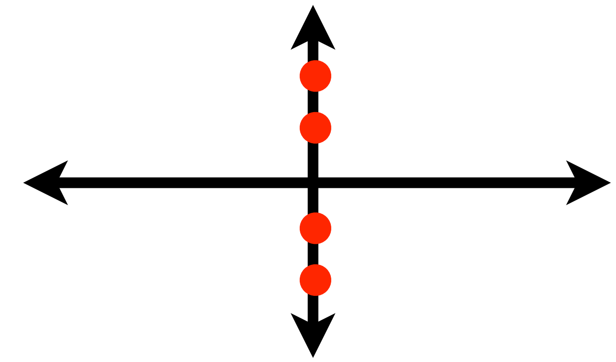

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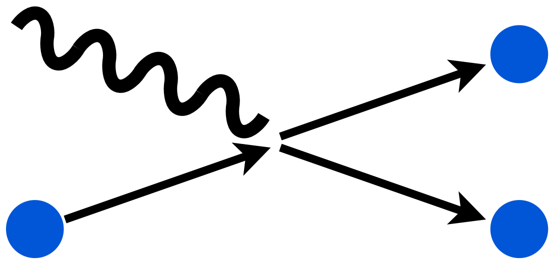
Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

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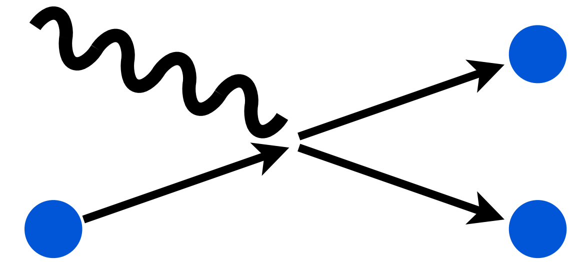
Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

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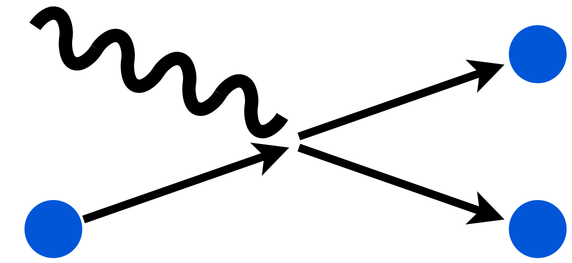
Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

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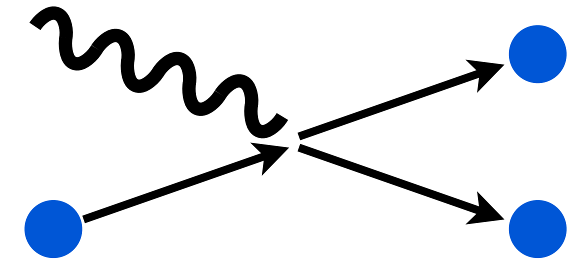
R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev.* D91, 034501 (2015)
R. A. Briceño, MTH, *Phys. Rev.* D92, 074509 (2015)

Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

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get this from the lattice

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experimental observable

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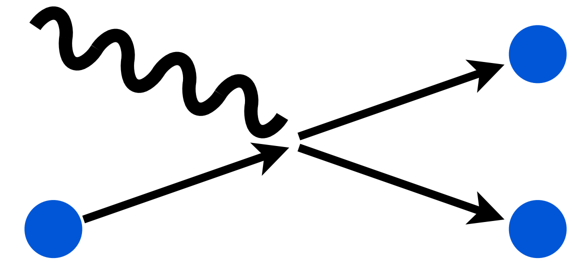
Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

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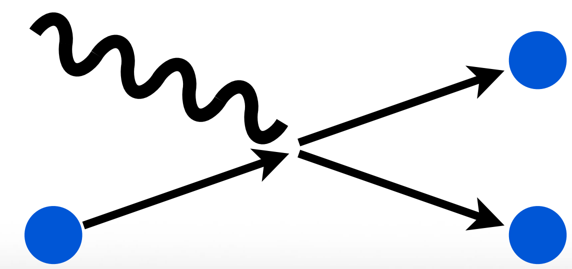
$$\mathcal{R}(E_n, \vec{P}, L) = -\underset{E_n}{\mathrm{Residue}} \left[\frac{1}{F^{-1} + \mathcal{M}_{2\to 2}} \right]$$

R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev.* D91, 034501 (2015)
R. A. Briceño, MTH, *Phys. Rev.* D92, 074509 (2015)

Photoproduction $\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$



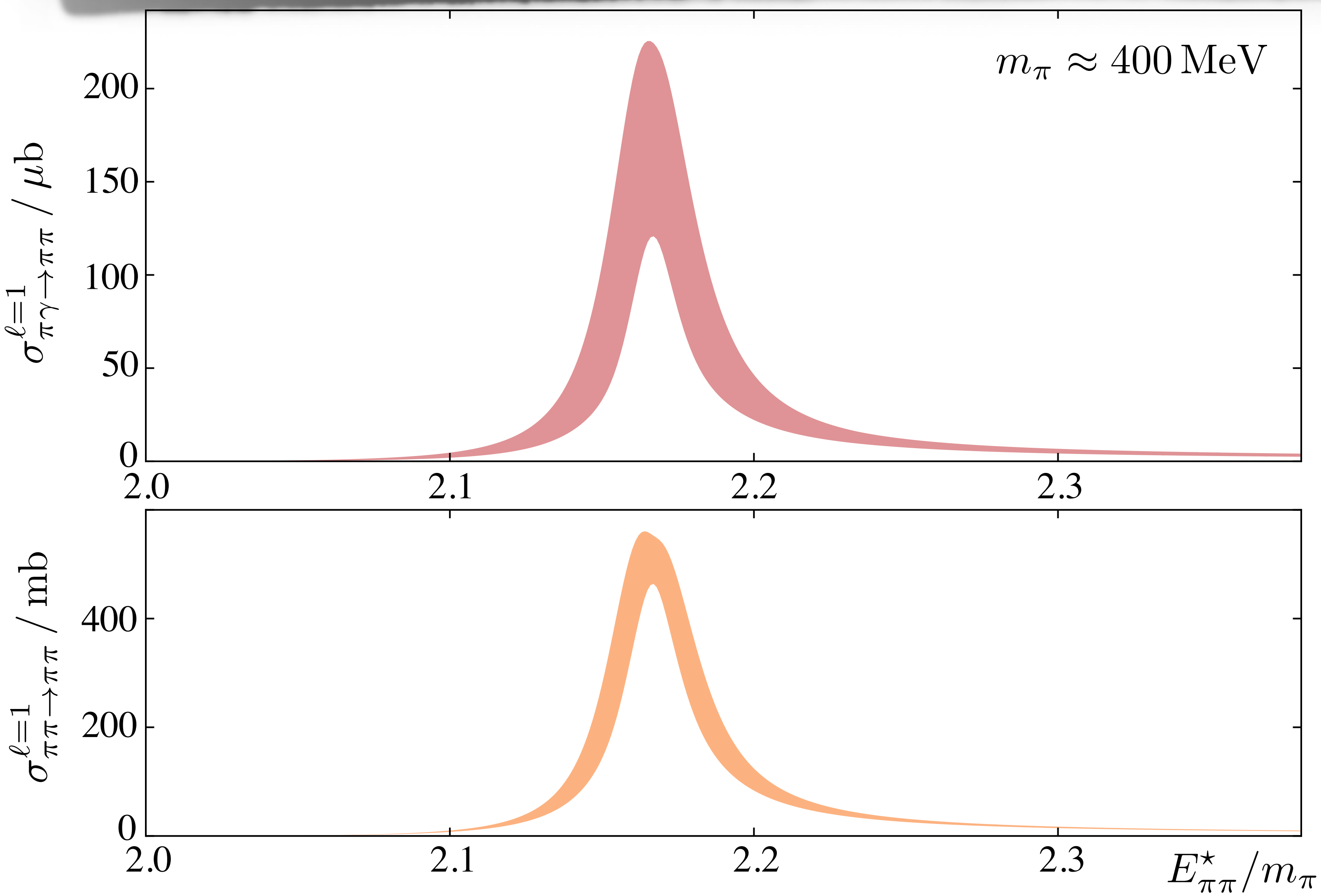
get this from the lattice

$$2\omega_\pi L^6 |\langle n, \vec{P}, L | \mathcal{J}_\mu(0) | \pi, L \rangle|^2 =$$

experimental observable

$$\langle \pi | \mathcal{J}_\mu(0) | \pi\pi, \mathrm{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu(0) | \pi \rangle$$

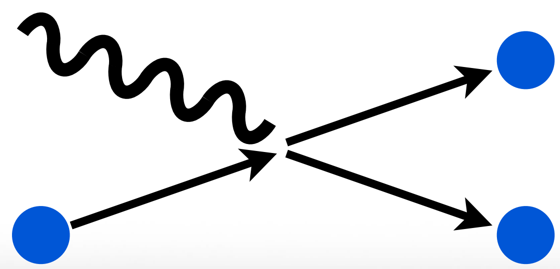
Briceño, MTH, Walker-Loud/Briceño, MTH



Photoproduction in the rho channel

Briceño, Dudek, Edwards, Schultz, Thomas, Wilson
arXiv: 1507.6622

Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$



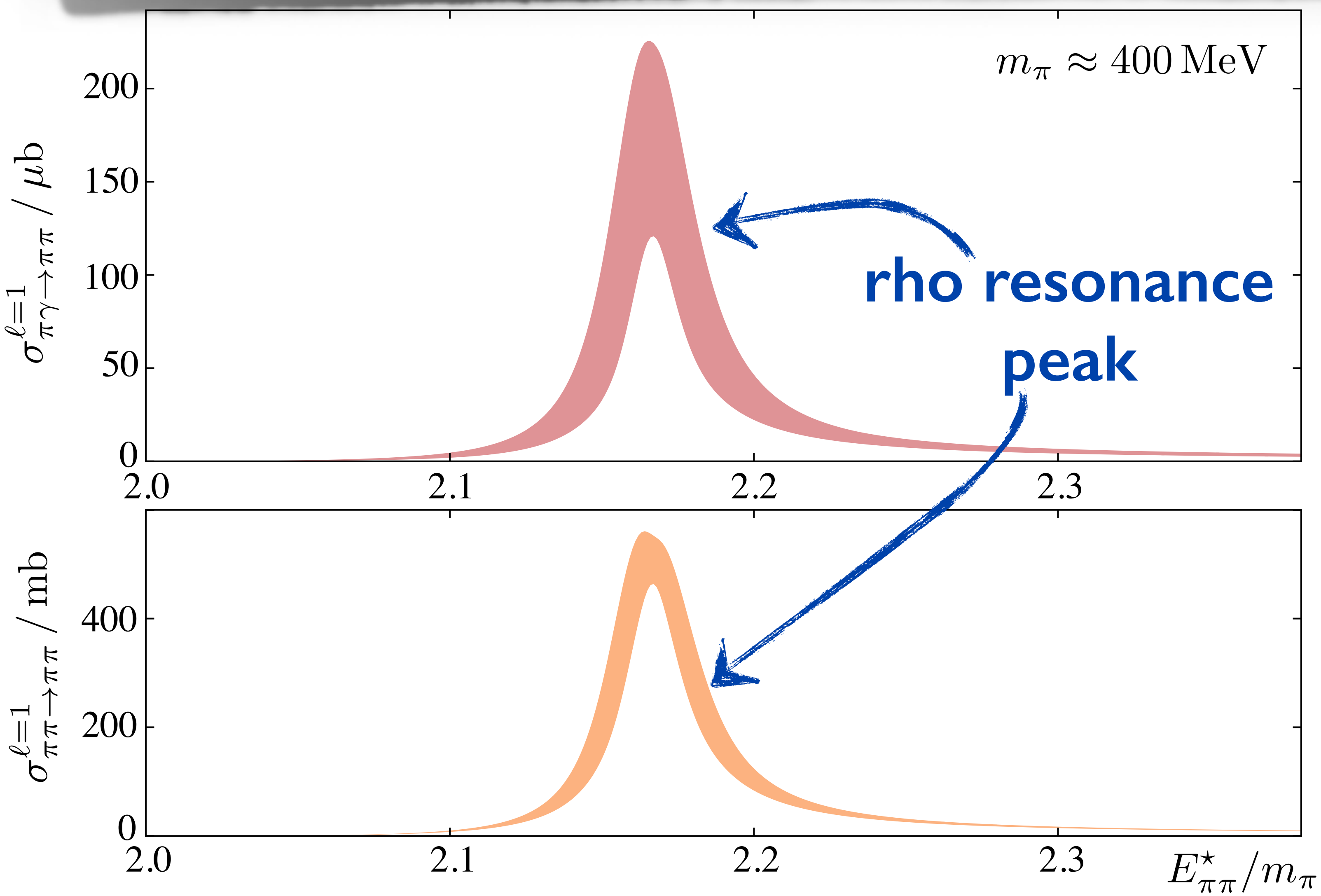
get this from the lattice

$$2\omega_\pi L^6 |\langle n, \vec{P}, L|\mathcal{J}_\mu(0)|\pi, L\rangle|^2 =$$

$$\langle \pi|\mathcal{J}_\mu(0)|\pi\pi, \mathrm{in}\rangle \mathcal{R}(E_n, \vec{P}, L)\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu(0)|\pi\rangle$$

experimental observable

Briceño, MTH, Walker-Loud/Briceño, MTH



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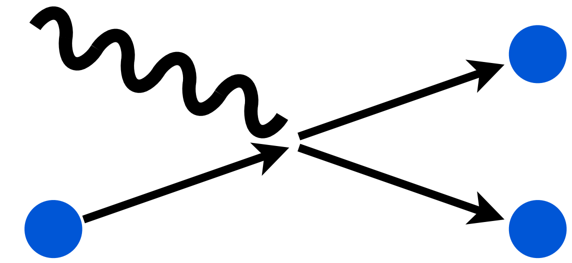
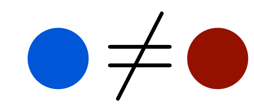
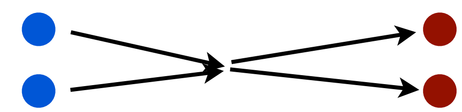
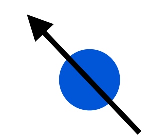
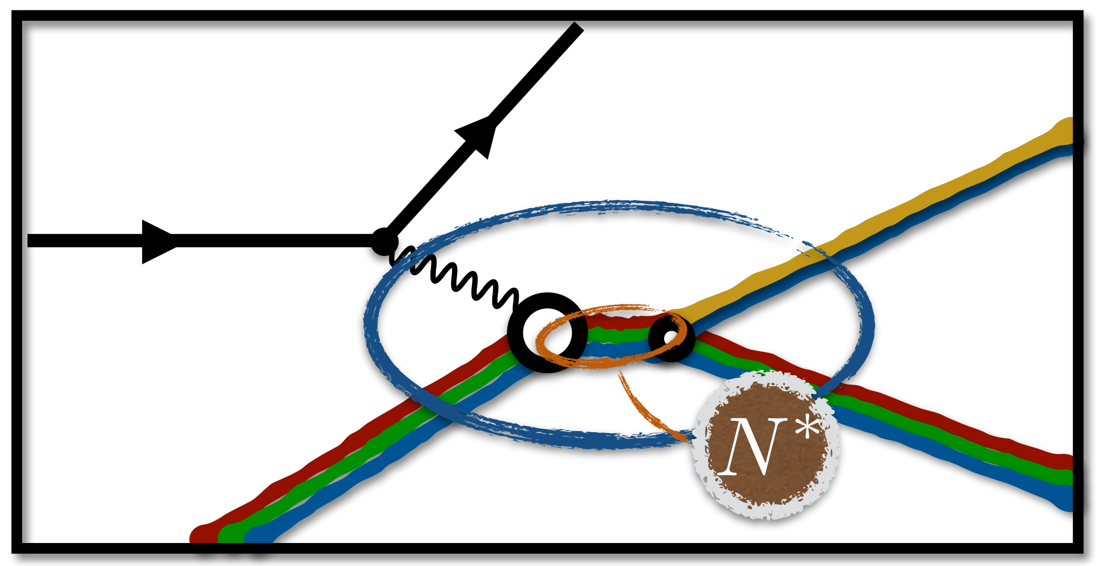
Photoproduction $\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

Result is very general

non-indentical particles 

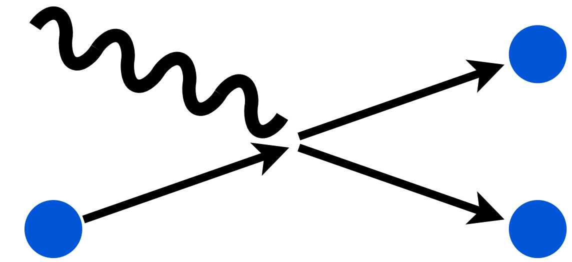
multiple two-particle channels 

particles with spin 

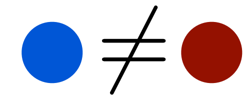


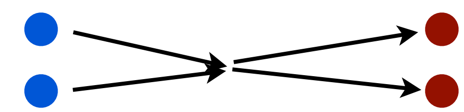
R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev.* D91, 034501 (2015)
R. A. Briceño, MTH, *Phys. Rev.* D92, 074509 (2015)

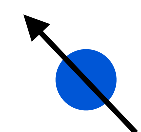
Photoprojection

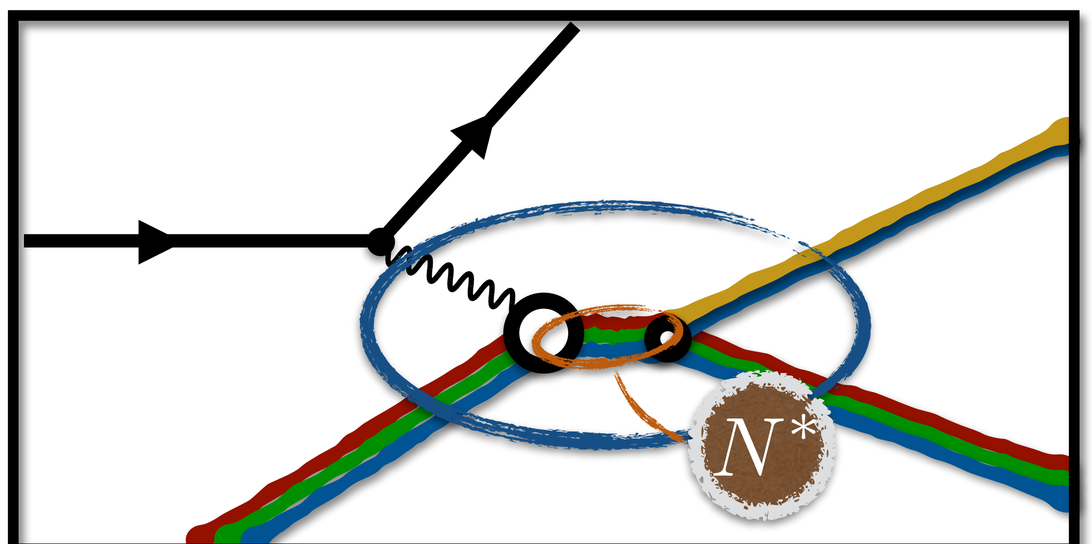
$\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi \rangle \equiv$ 

Result is very general

non-indentical particles 

multiple two-particle channels 

particles with spin 

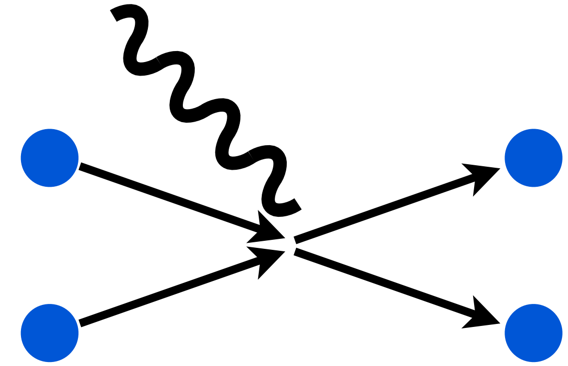


Formalism is in place to give Lattice QCD predictions of this process (ignoring three particles)

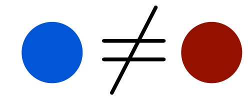
R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev.* D91, 034501 (2015)

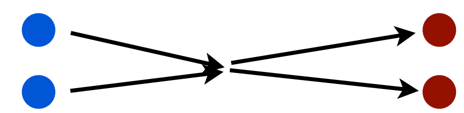
R. A. Briceño, MTH, *Phys. Rev.* D92, 074509 (2015)

Two-to-two transitions

$$\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi\pi, \mathrm{in} \rangle \equiv$$ 

Formalism complete

non-indentical particles  

multiple two-particle channels  

R. A. Briceño, MTH, arXiv: 1509.08507

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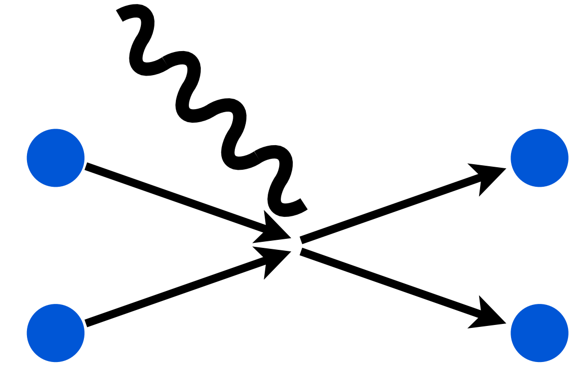
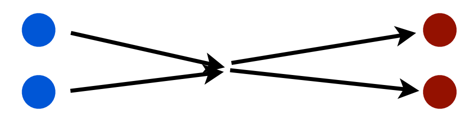
$$\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi\pi, \mathrm{in} \rangle \equiv$$ 

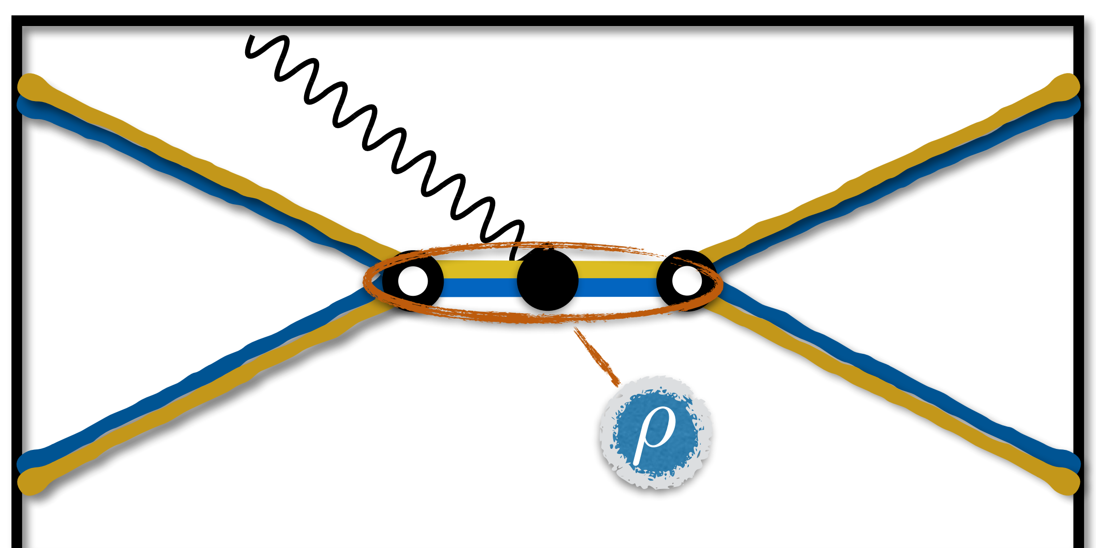
Formalism complete

non-indentical particles $\neq$ ✓

multiple two-particle channels  ✓

R. A. Briceño, MTH, arXiv: 1509.08507

Required to extract resonance form factors



Two-particle scattering

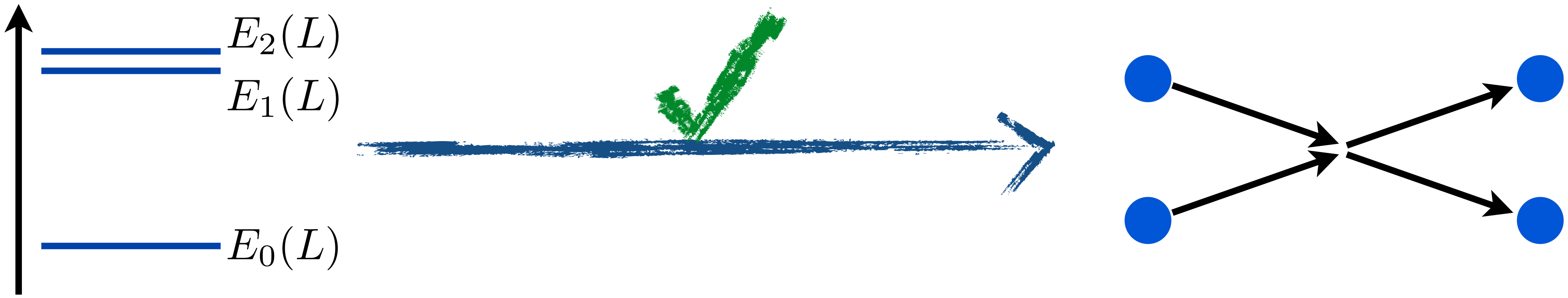
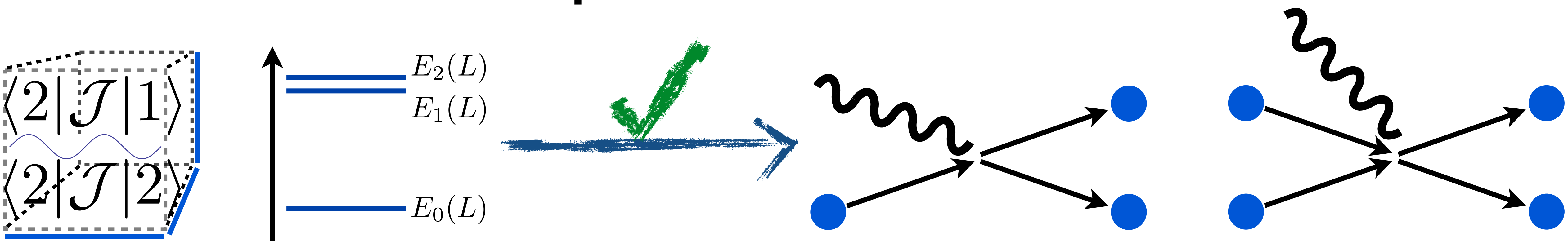
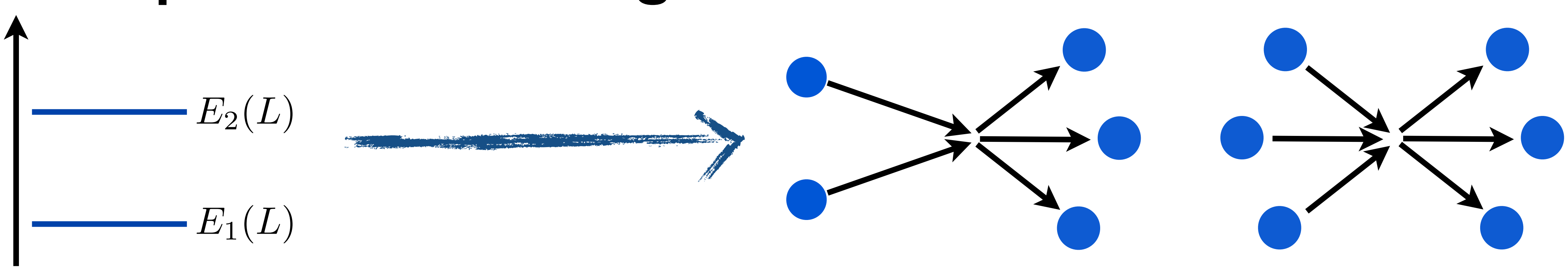


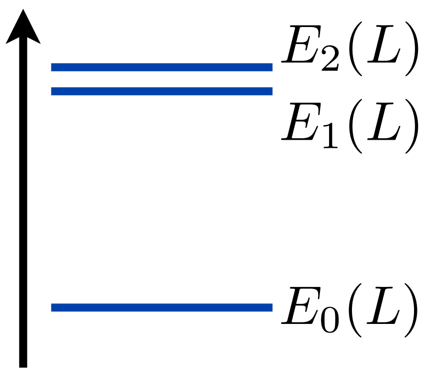
Photo- and electroproduction

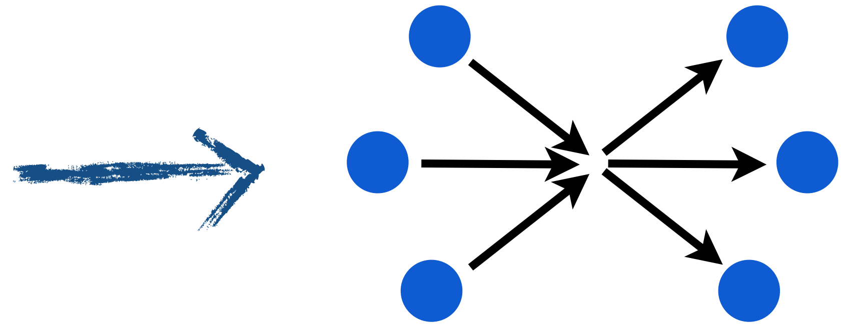


Three-particle scattering



Three-to-three scattering

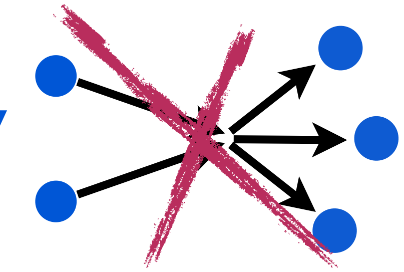




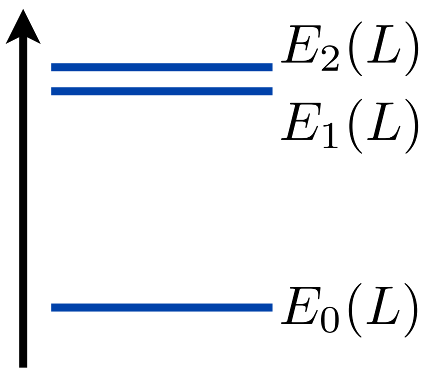
For now assume…

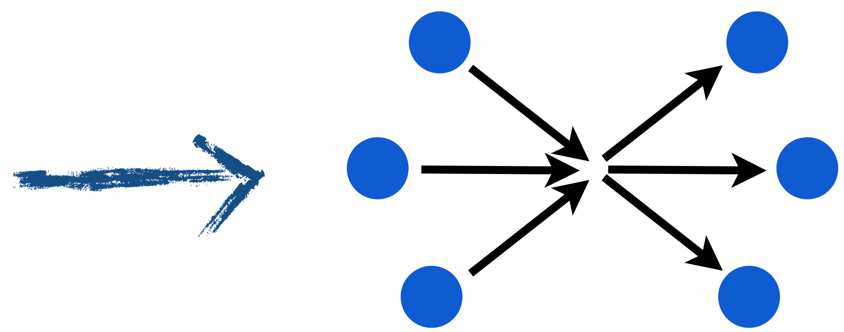
identical scalars, mass m

$\mathbb{Z}_2$ **symmetry**



Three-to-three scattering





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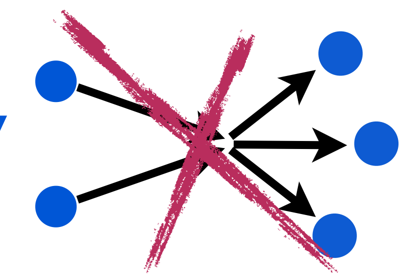


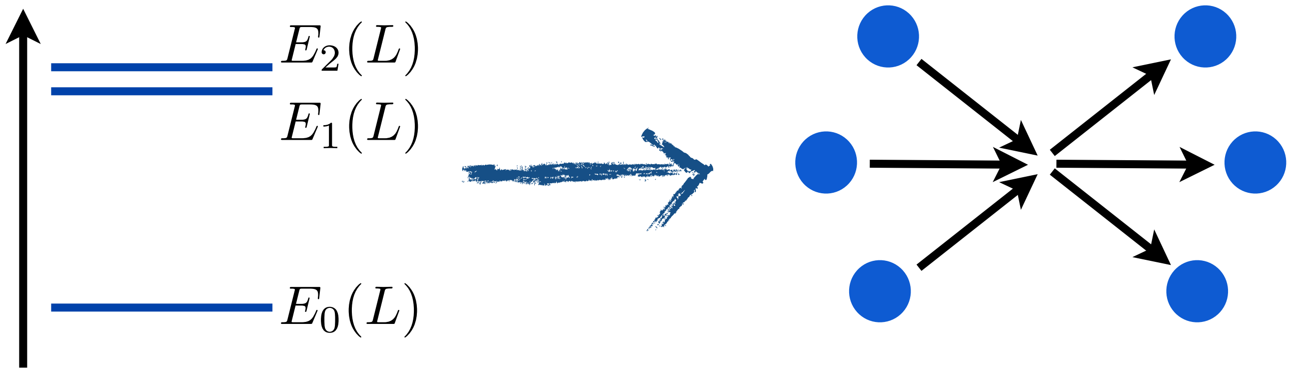
$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

three-pion interpolator

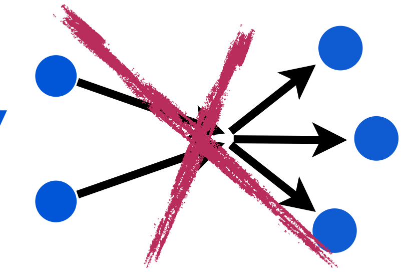
Three-to-three scattering

For now assume…

identical scalars, mass m

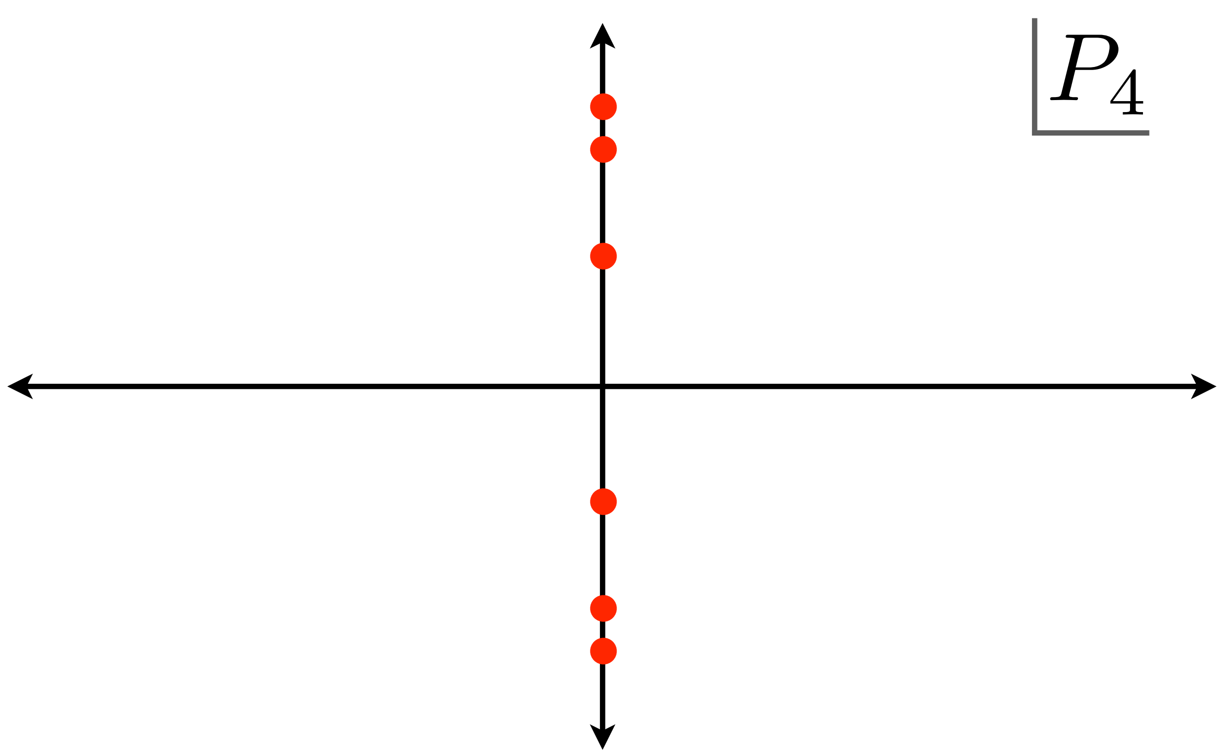


$\mathbb{Z}_2$ **symmetry**



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

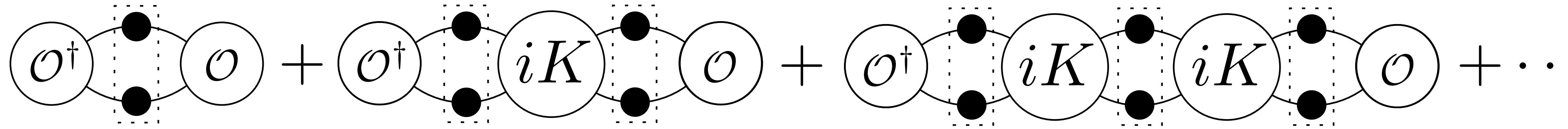
three-pion interpolator



Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.

Require $m < E^* < 5m$ **to isolate three-particle states**

Recall for two particles we started with a "skeleton expansion"

$C_L(P) =$ 

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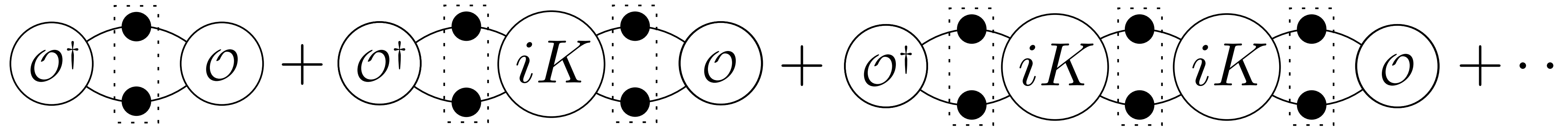
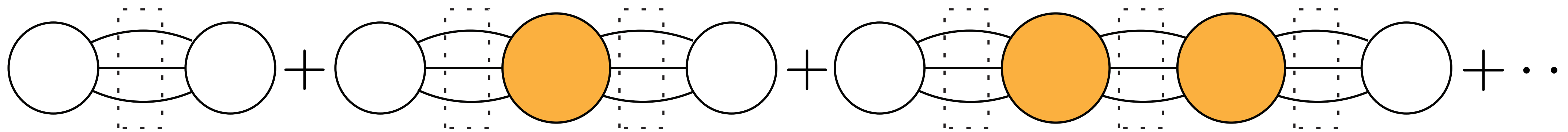


So now we need the same for three…

$C_L(E, \vec{P}) \overset{?}{=}$



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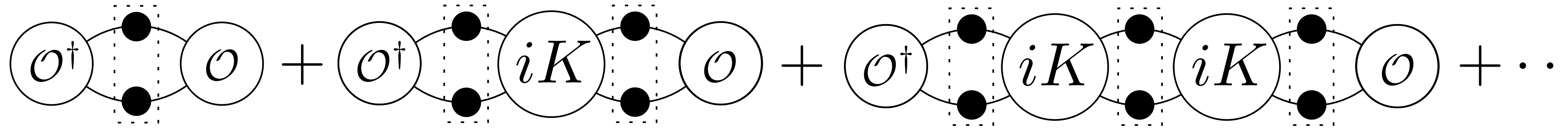
$C_L(P) =$ 

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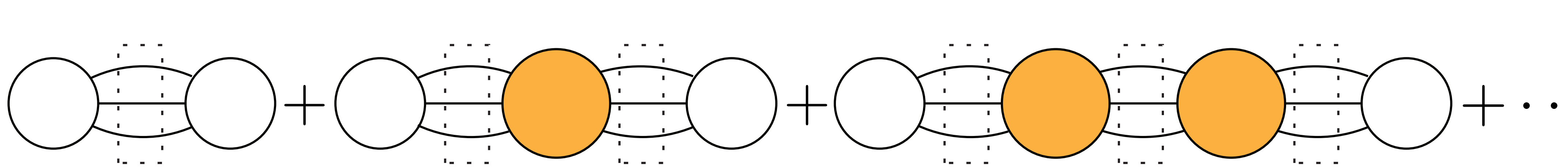
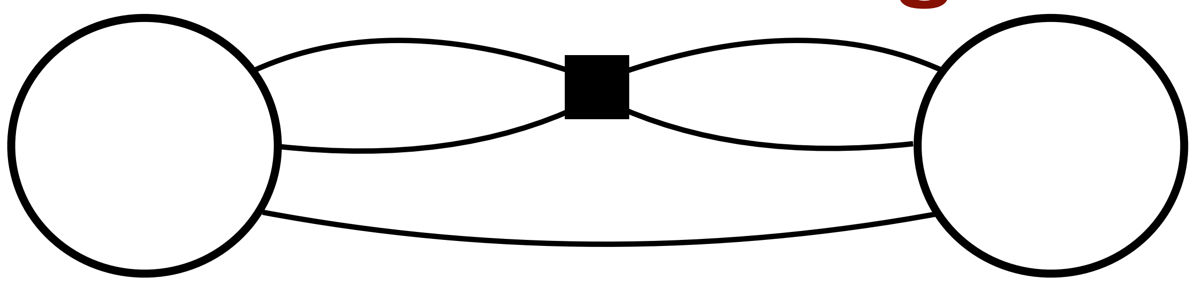
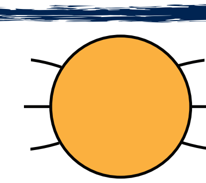
$C_L(E, \vec{P}) \stackrel{?}{=}$ 

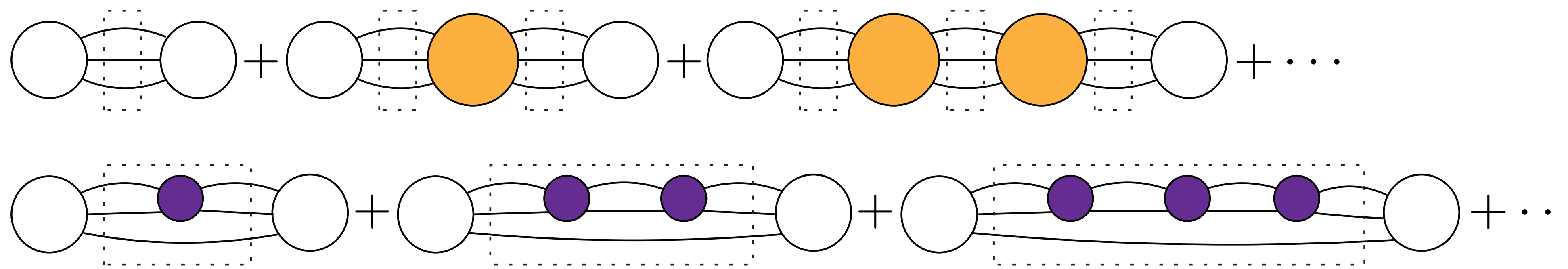
No! We also need diagrams like



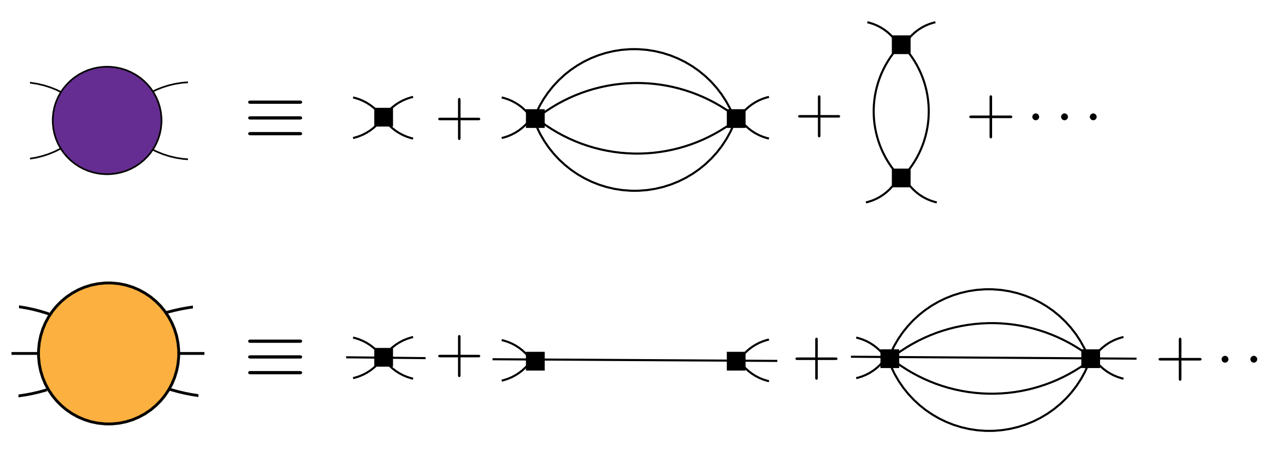
Disconnected diagrams in  **lead to singularities that invalidate the derivation**

New skeleton expansion

$C_L(E, \vec{P}) =$



Kernel definitions:



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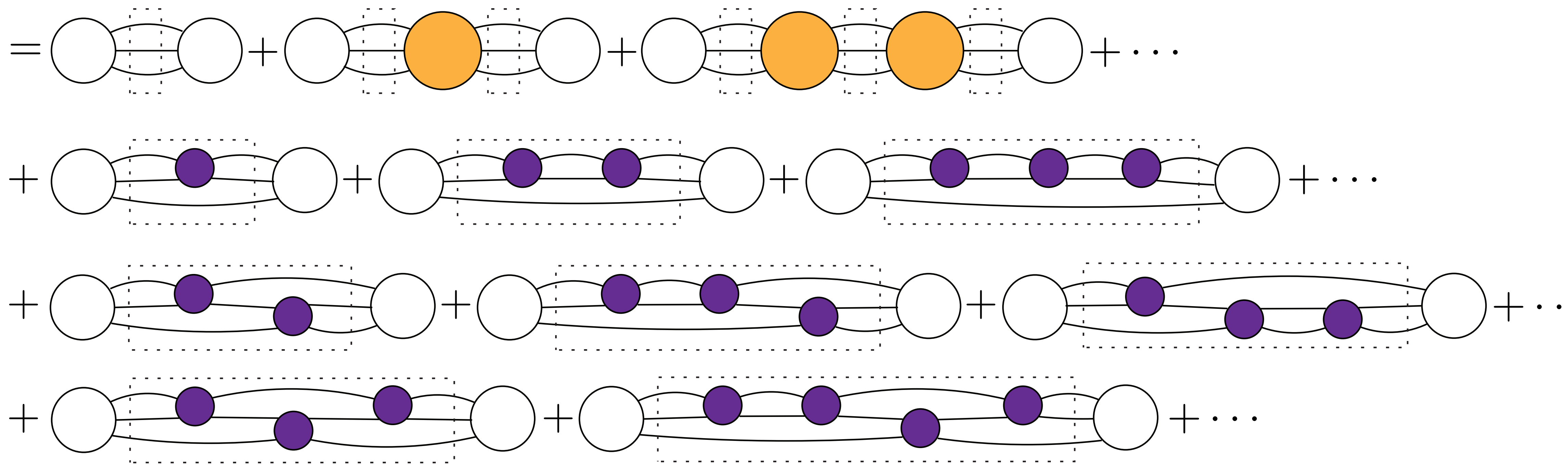

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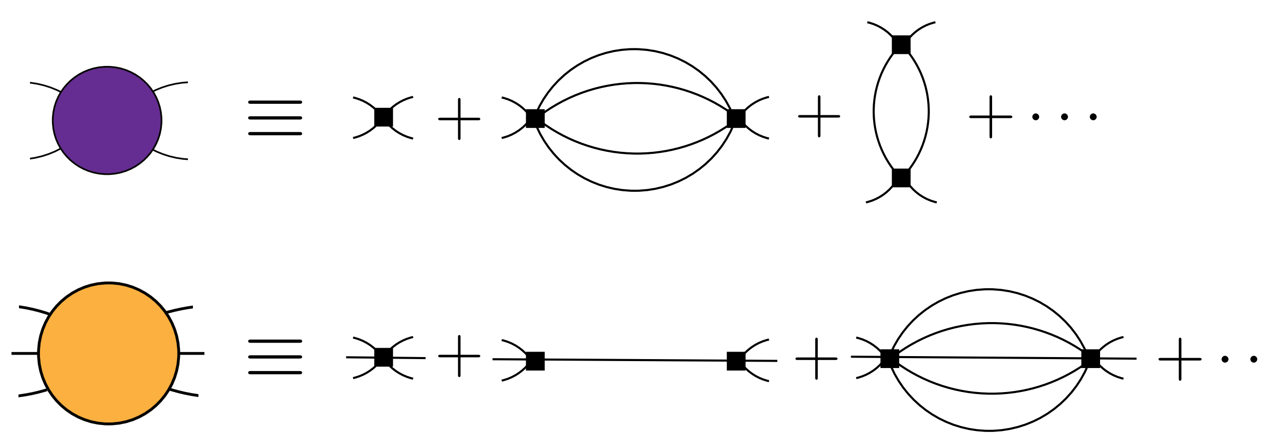
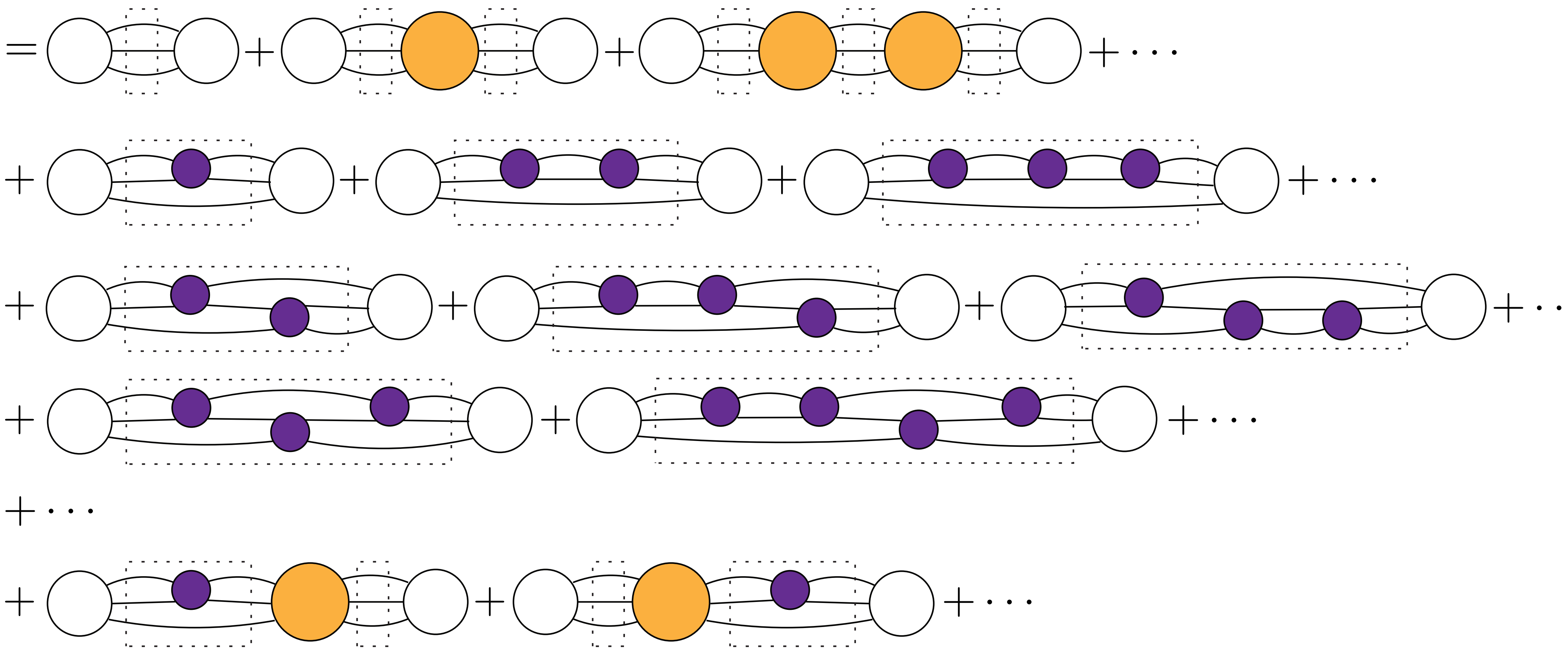


New skeleton expansion

$C_L(E, \vec{P}) =$



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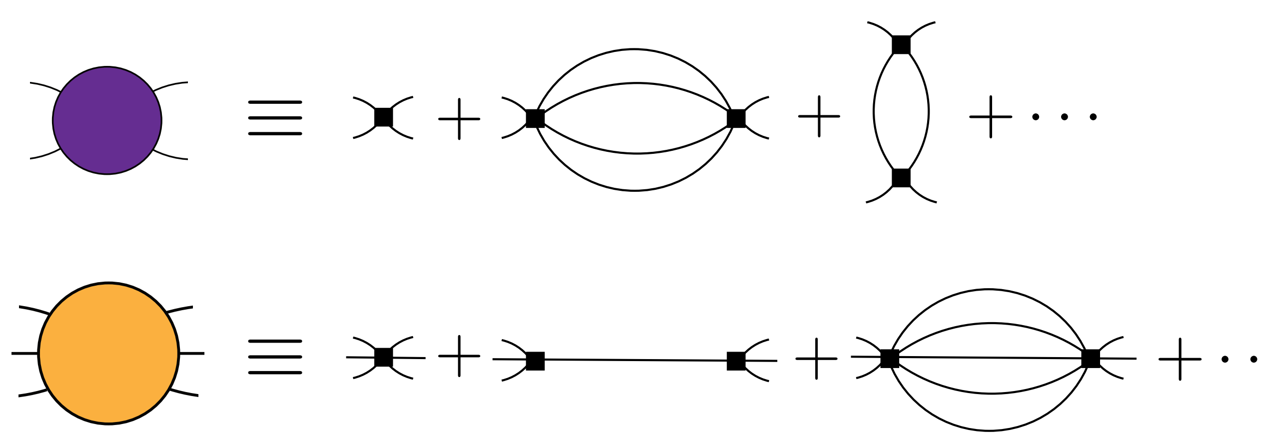

New skeleton expansion

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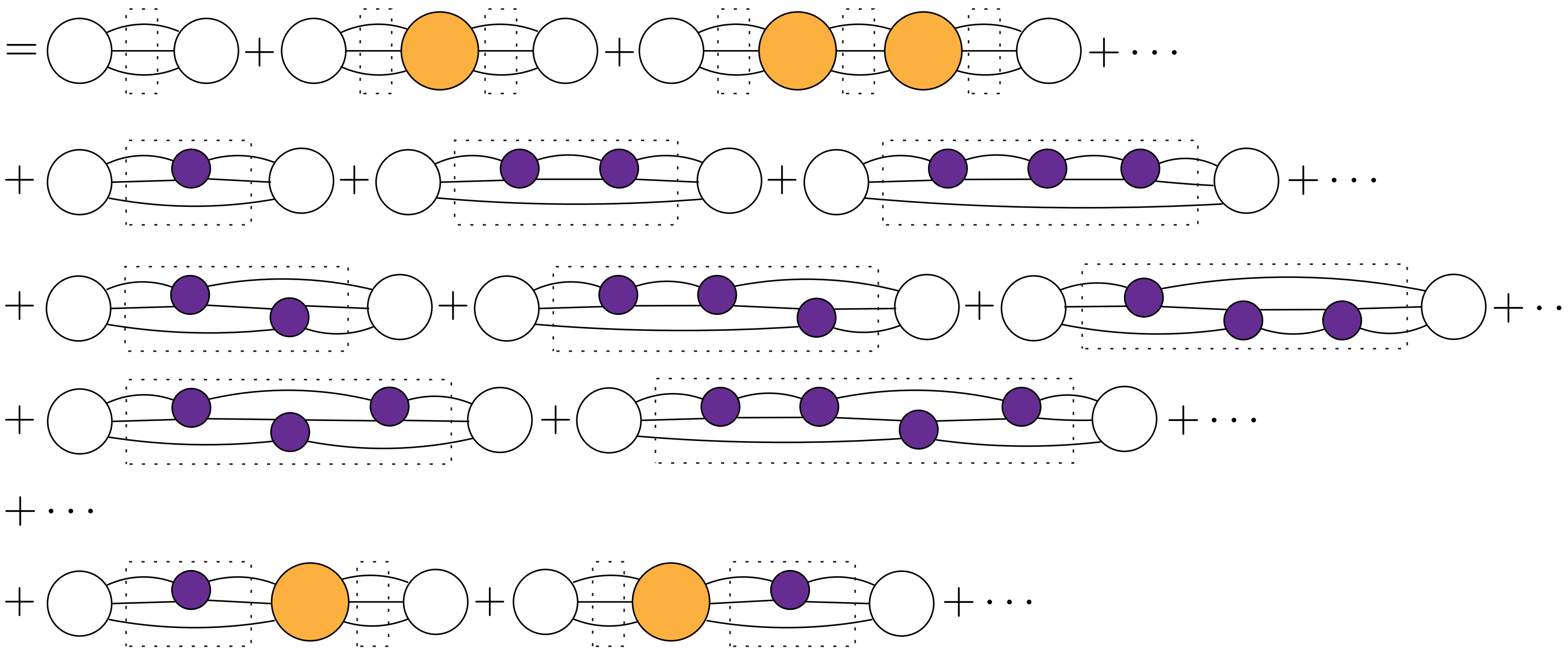

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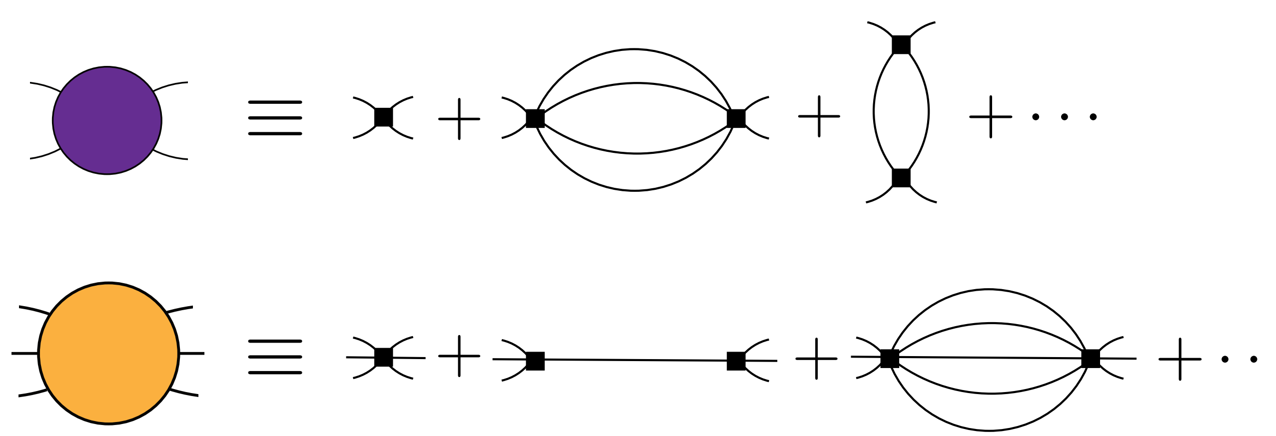
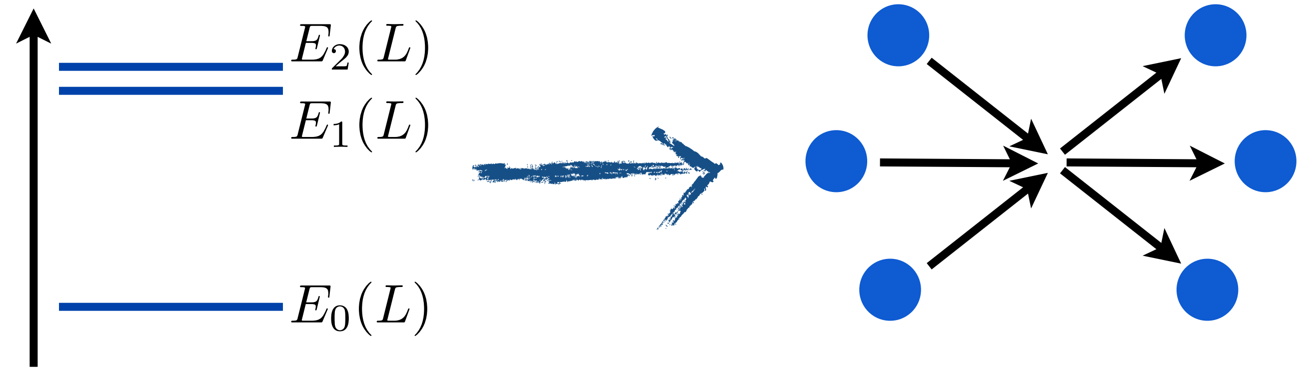
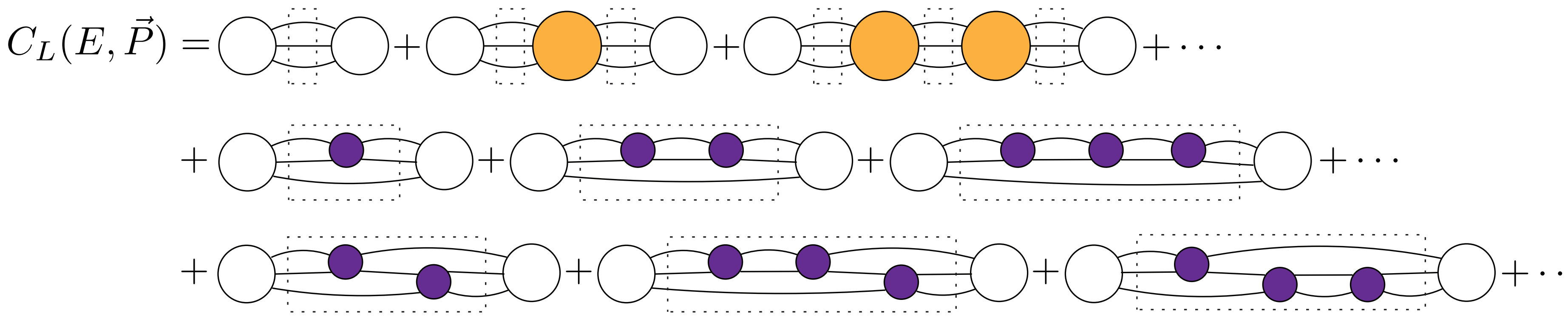


Significantly more complicated than two-particle story

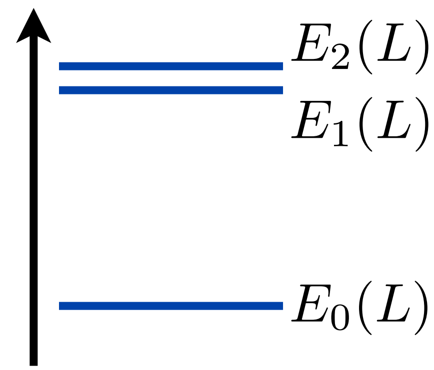
Three-to-three scattering

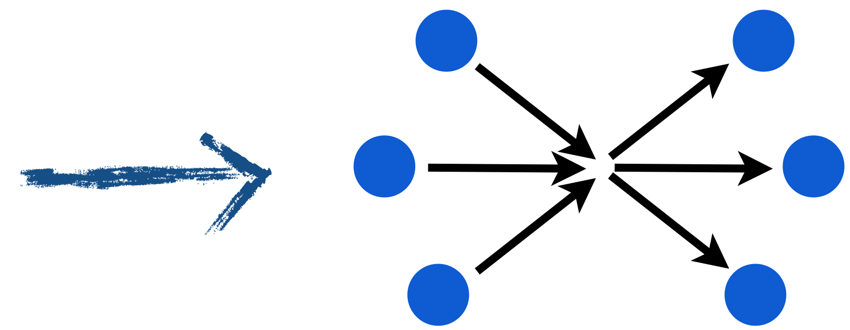


1. Work out the three particle skeleton expansion



Three-to-three scattering





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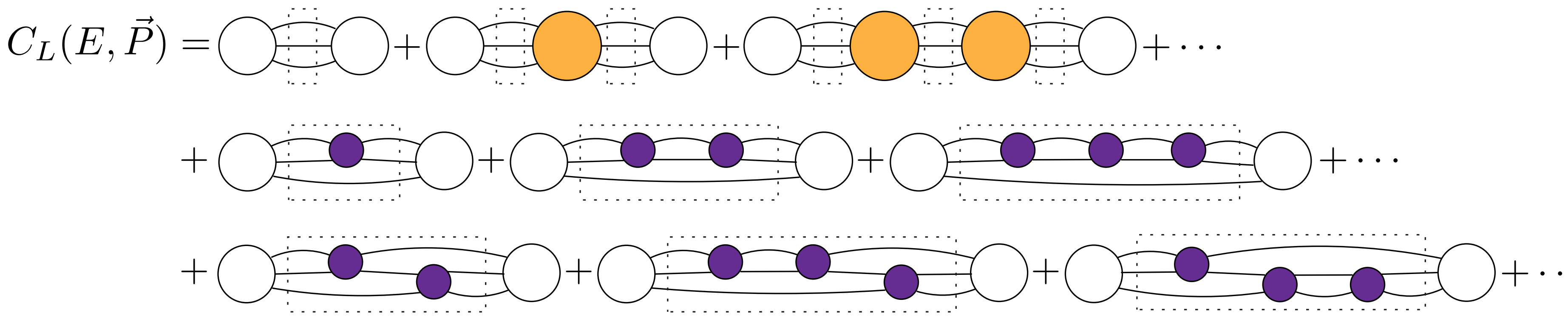
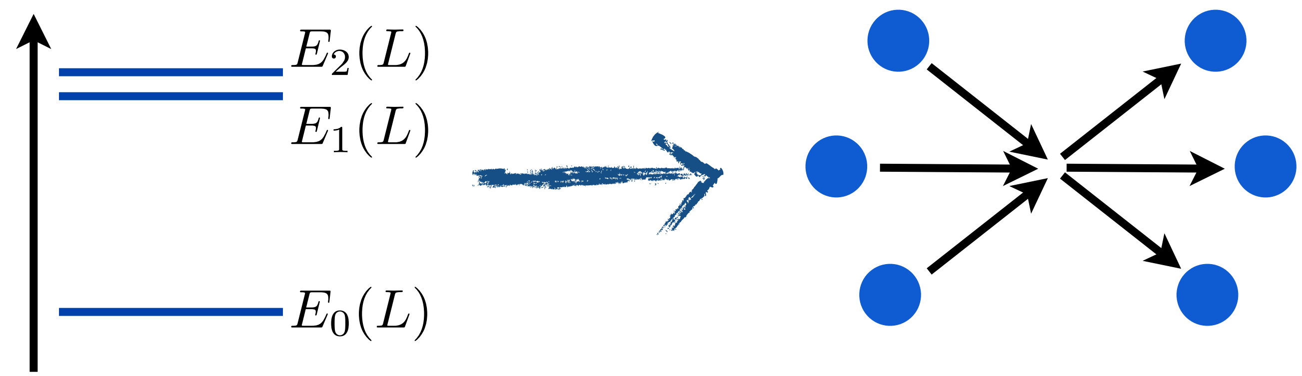


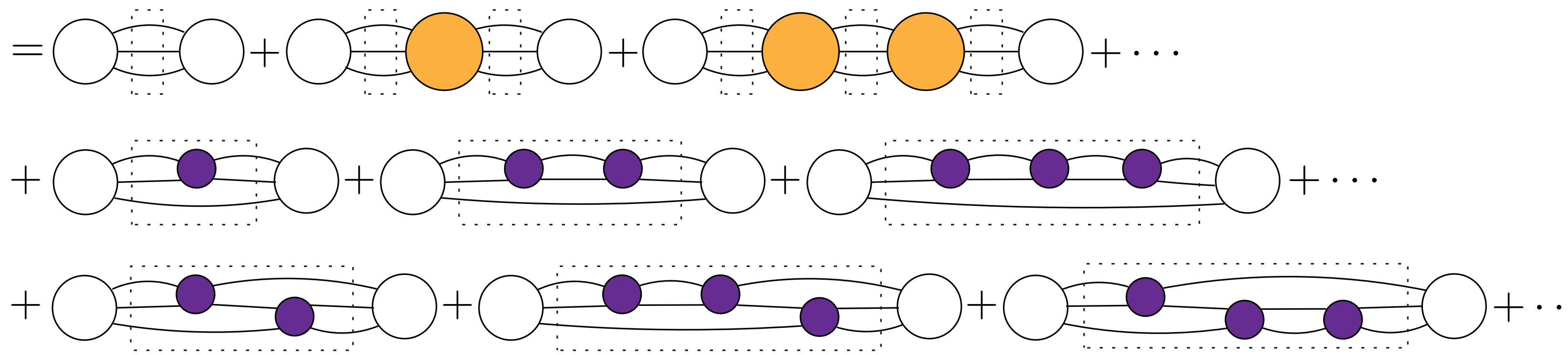
2. Break diagrams into finite- and infinite-volume parts

Three-to-three scattering



1. Work out the three particle skeleton expansion

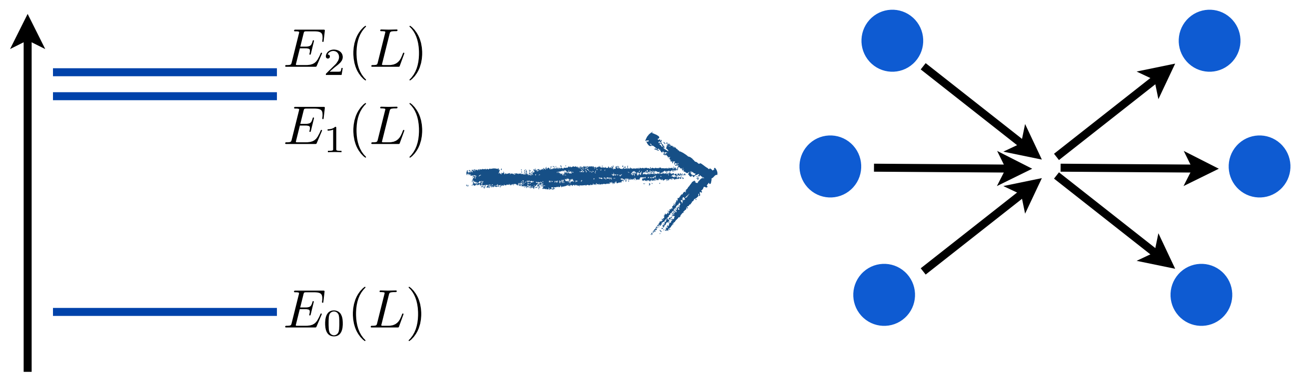
$C_L(E, \vec{P}) =$



2. Break diagrams into finite- and infinite-volume parts
3. Organize and sum terms to identify

infinite-volume observables

Three-to-three scattering



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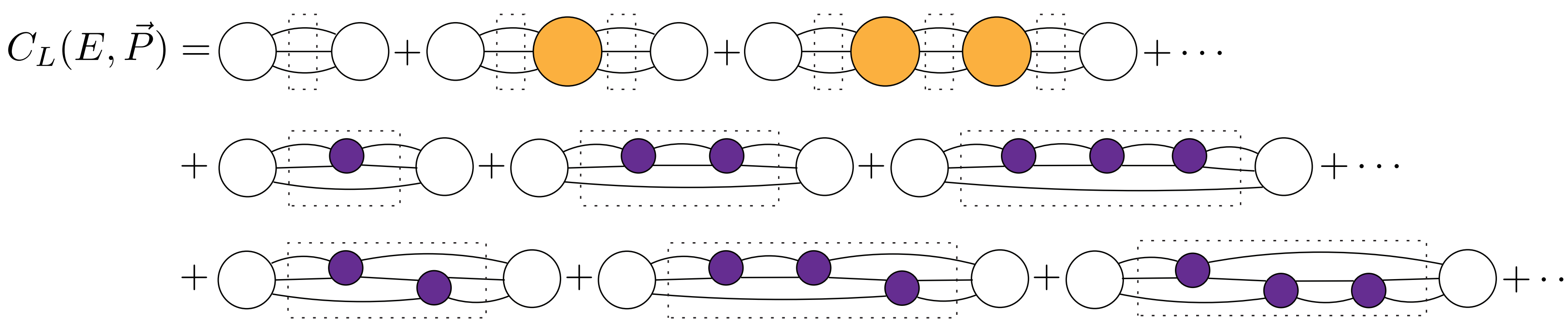


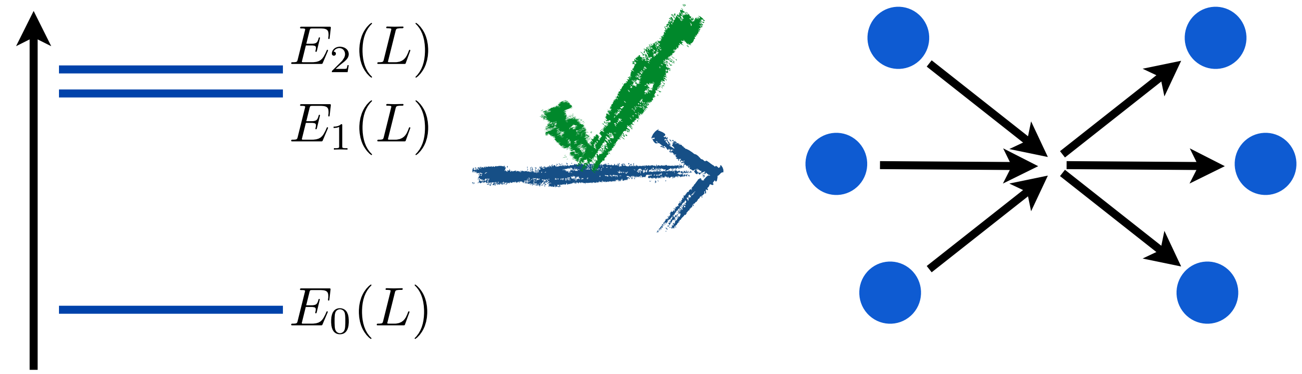
2. Break diagrams into finite- and infinite-volume parts

3. Organize and sum terms to identify
infinite-volume observables

Major complicating factors:

More diagram topologies, more degrees of freedom, three-to-three amplitude contains "long distance" kinematic poles

Three-to-three scattering



Current status:

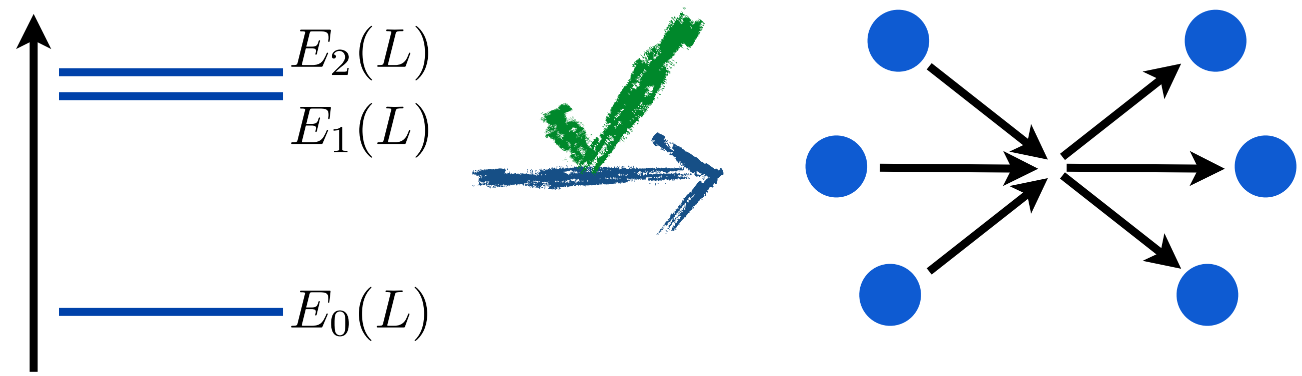
Formalism is complete for the simplest three-scalar system

General, model-independent relation between finite-volume energies and three-to-three scattering amplitude

Derived using a generic relativistic field theory

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)
MTH and Sharpe, *Phys. Rev.* D92, 114509 (2015)

Three-to-three scattering



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Important caveats:

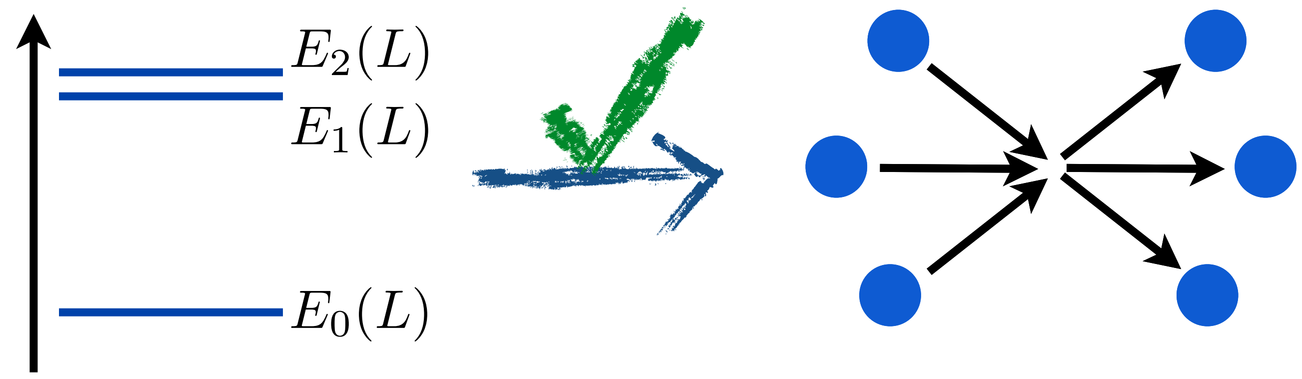
Identical particles with no two-to-three transitions

$$\pi\pi\pi \longrightarrow \pi\pi\pi$$

Requires that two-particle scattering phase is bounded

$$|\delta_\ell(E)| < \pi/2$$

Three-to-three scattering



To check the result we have expanded the lowest three-particle energy in powers of $1/L$.

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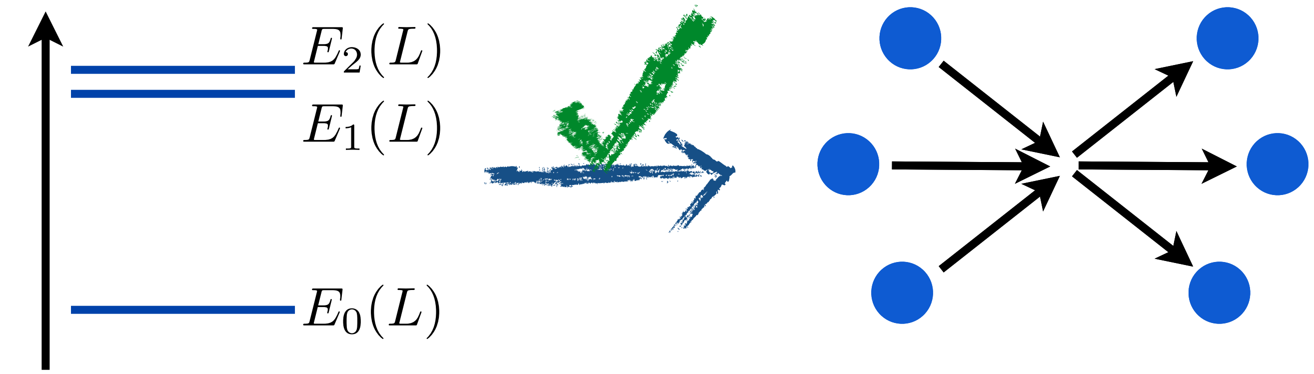
Three-to-three scattering



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These terms were already known.

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K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775

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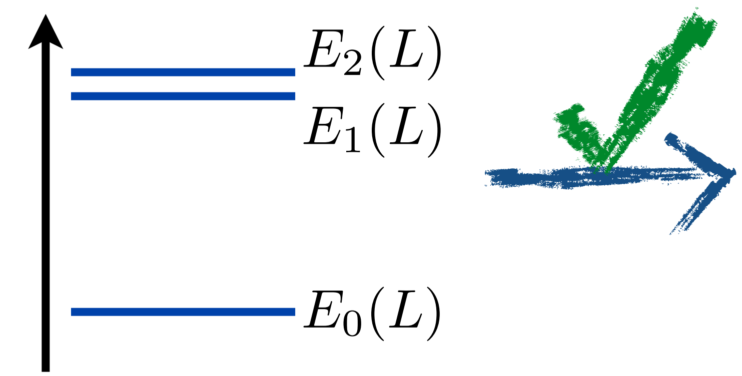

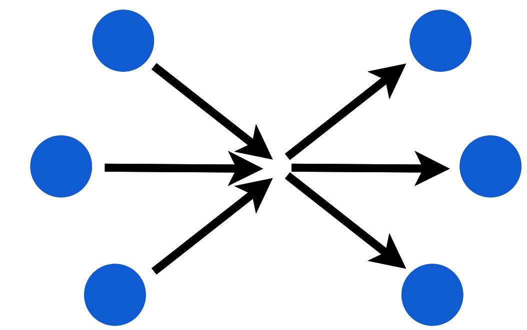

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$$- 37.25\frac{a^2}{m} + \frac{3\pi a}{m^2} + 6\pi r a^2 - \frac{\mathcal{M}_{3,\mathrm{thr}}}{48 m^3 a_3}$$

MTH and Sharpe, arXiv:1602.00324

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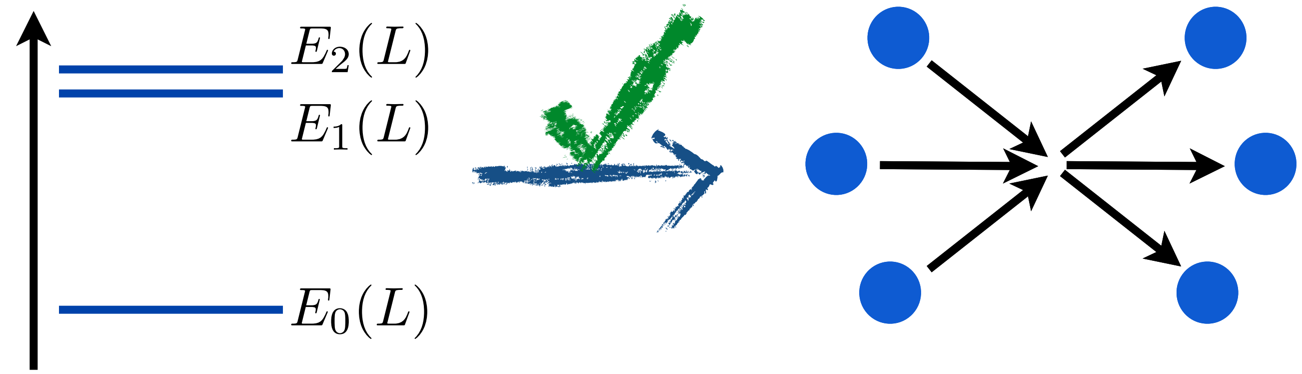

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checked independently in $\lambda\phi^4$ theory through $\mathcal{O}(\lambda^3)$

MTH and Sharpe, *Phys. Rev.* D 93, 014506 (2016)

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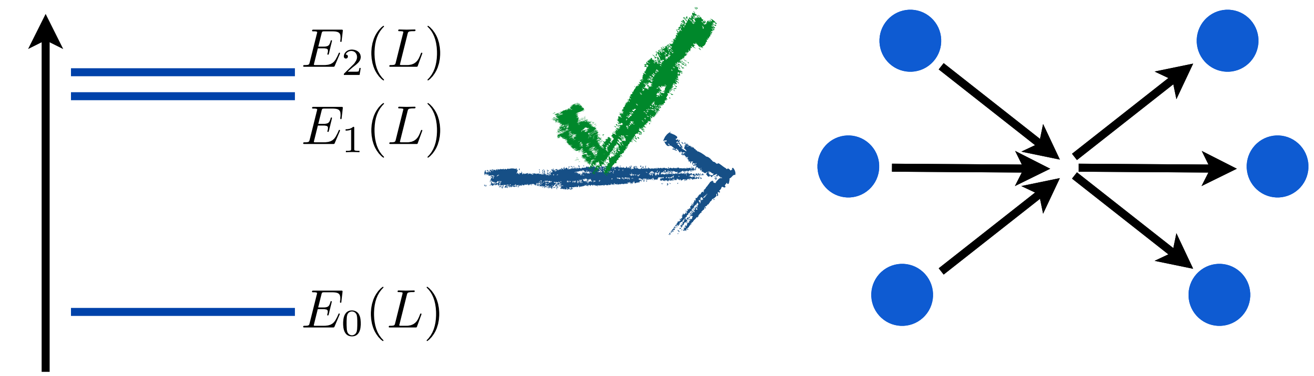

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relativistic three-particle observable

Add in a known "long distance" piece to get the standard amplitude.

MTH and Sharpe, arXiv:1602.00324

Currently underway:

Relax all simplifying assumptions:

Allow all particle types, allow two-to-three couplings, remove bound on phase shift

$K\pi \to K\pi\pi$ $\qquad$ $N\pi \to N\pi\pi$ $\qquad$ $NNN \to NNN$

Briceño, MTH, Sharpe, *in development*

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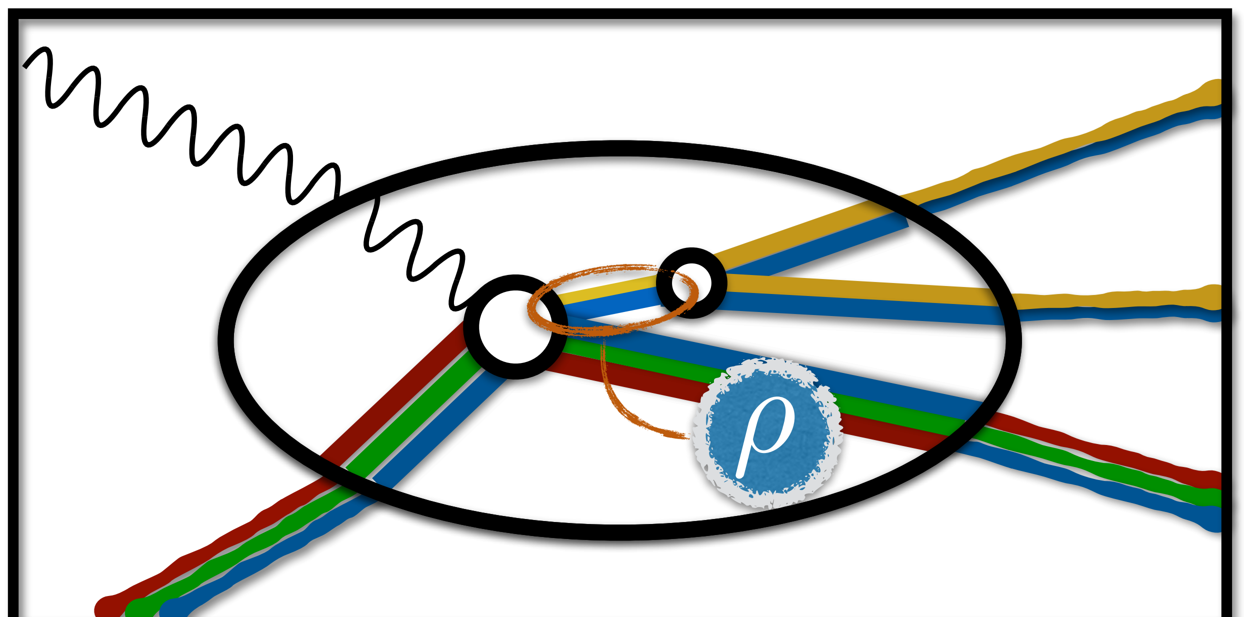
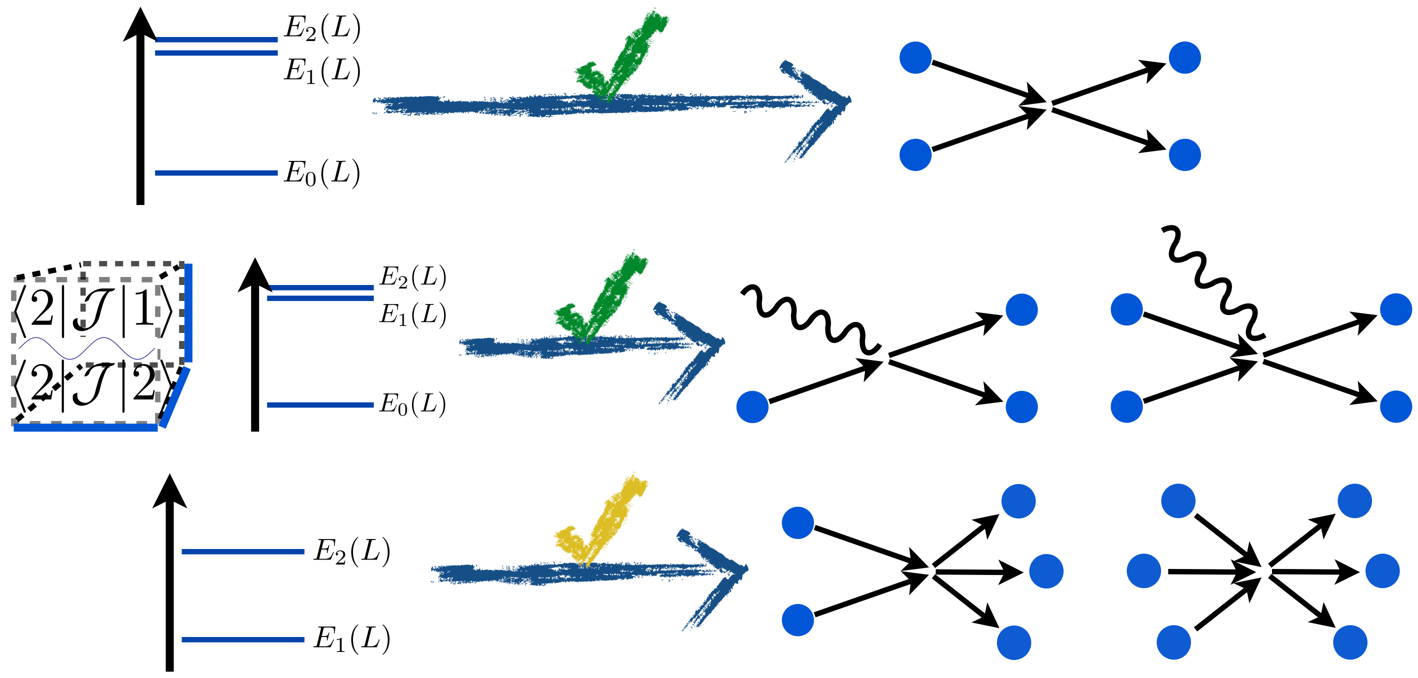
Use matching trick to recover transition amplitudes



$p\gamma \to N\rho \to N\pi\pi$

Summary

Reviewed methods to map finite-volume observables into physically observable scattering and transition amplitudes



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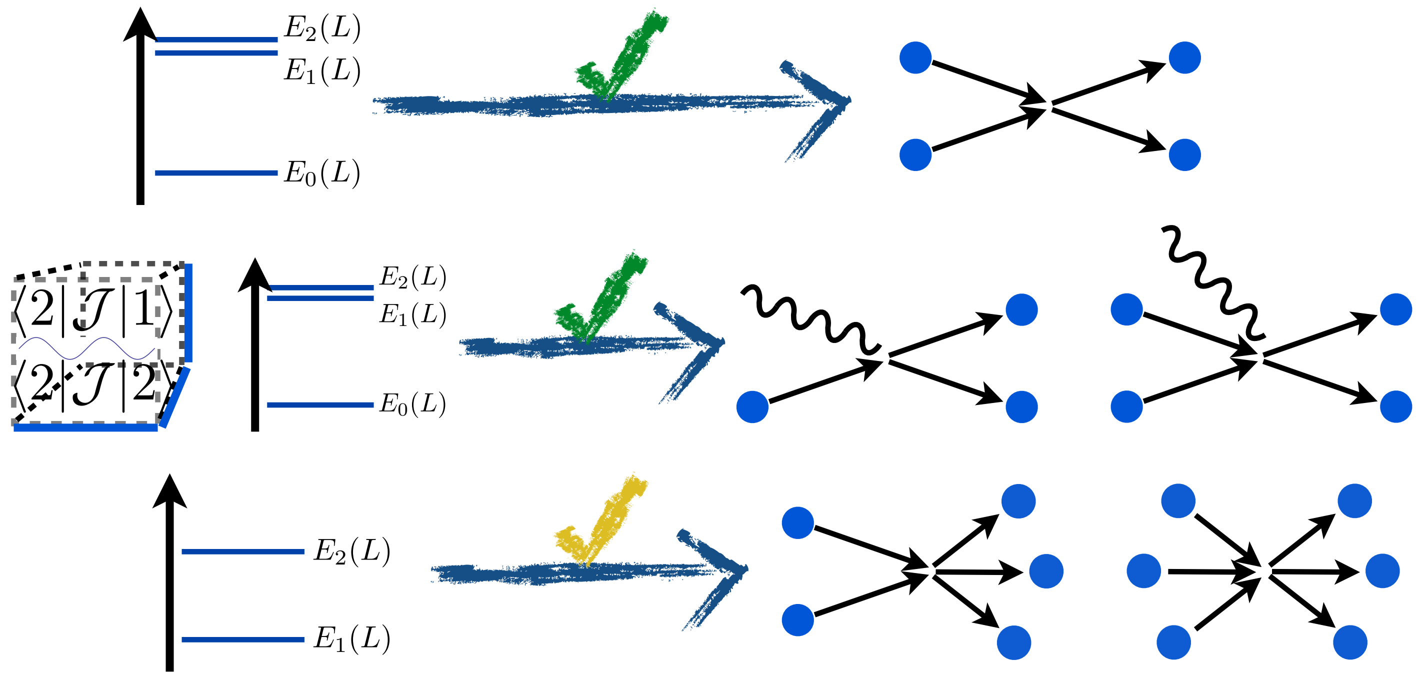

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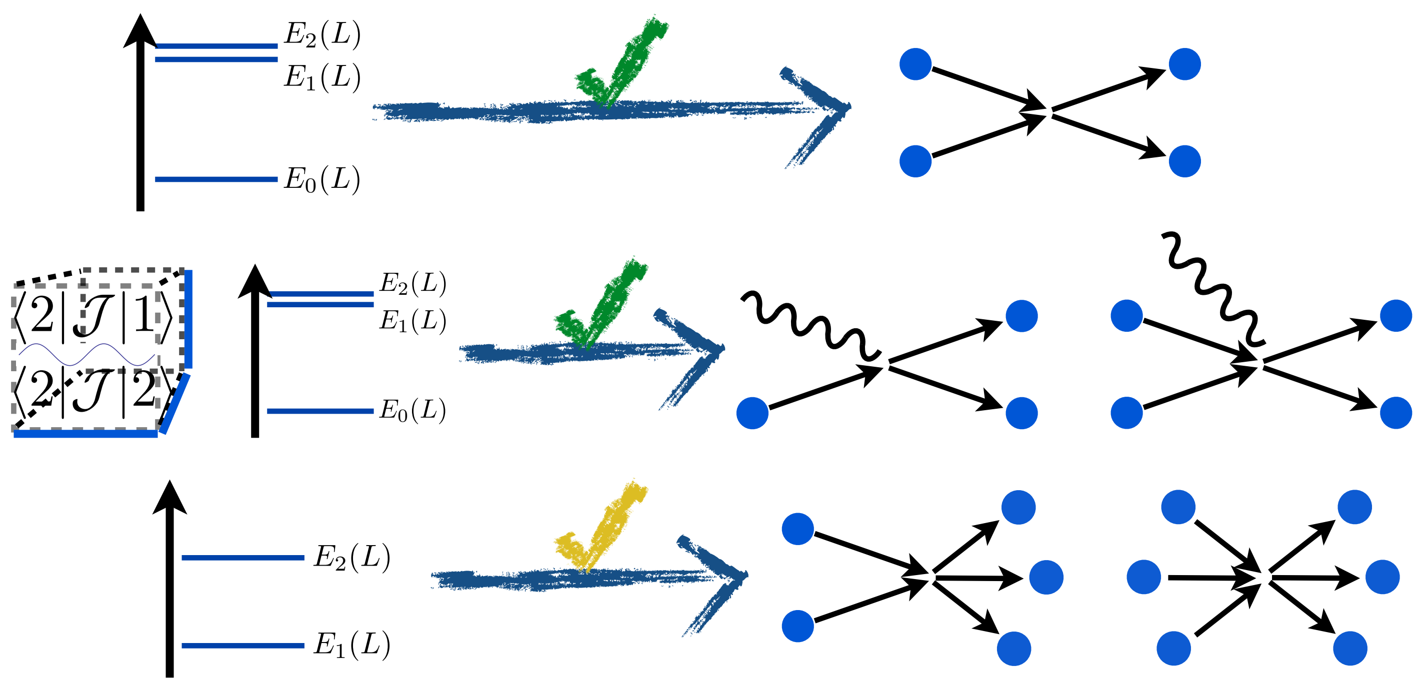

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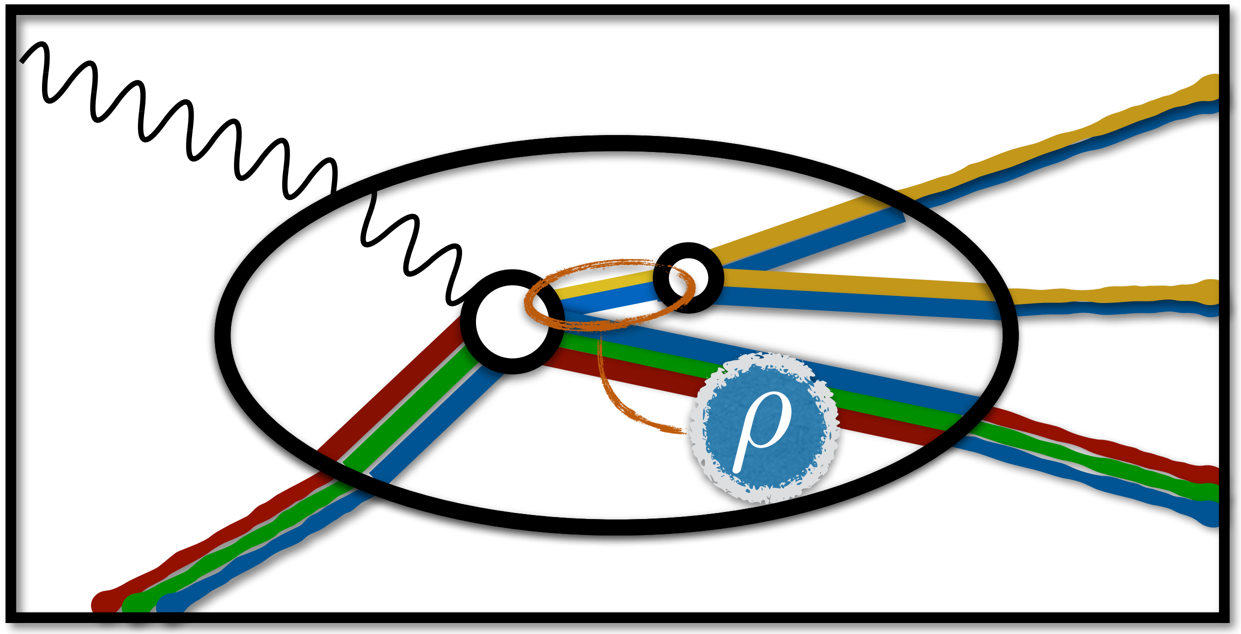
Can the scattering and transition amplitudes of QCD be extracted from Lattice QCD in a general, model independent way?

So far all signs point to yes!

My work at JLab:



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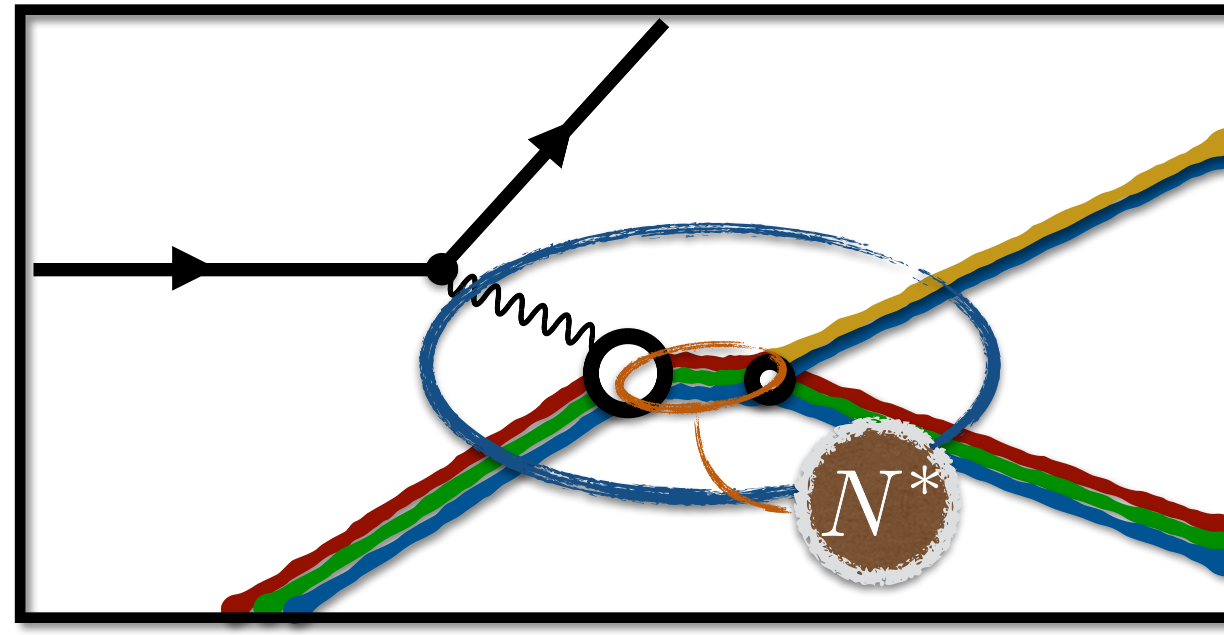

$p\gamma^* \to N^* \to N\pi, N\eta$

Experimental groups at JLab are measuring exactly the kinds of processes accommodated by this formalism.

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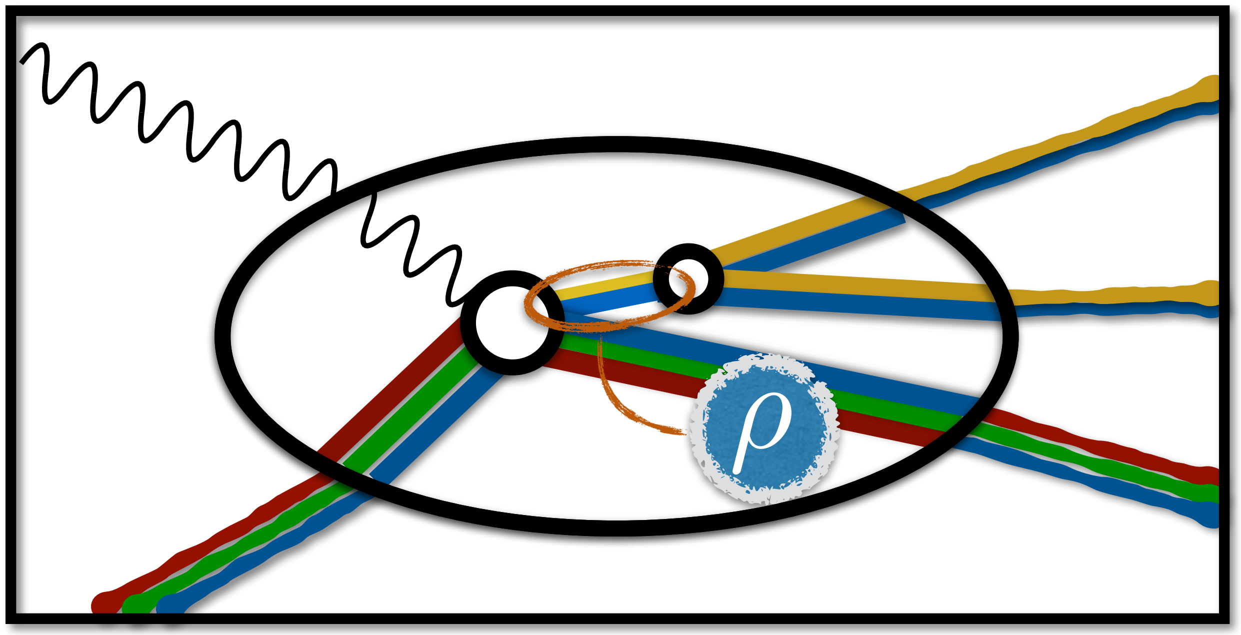

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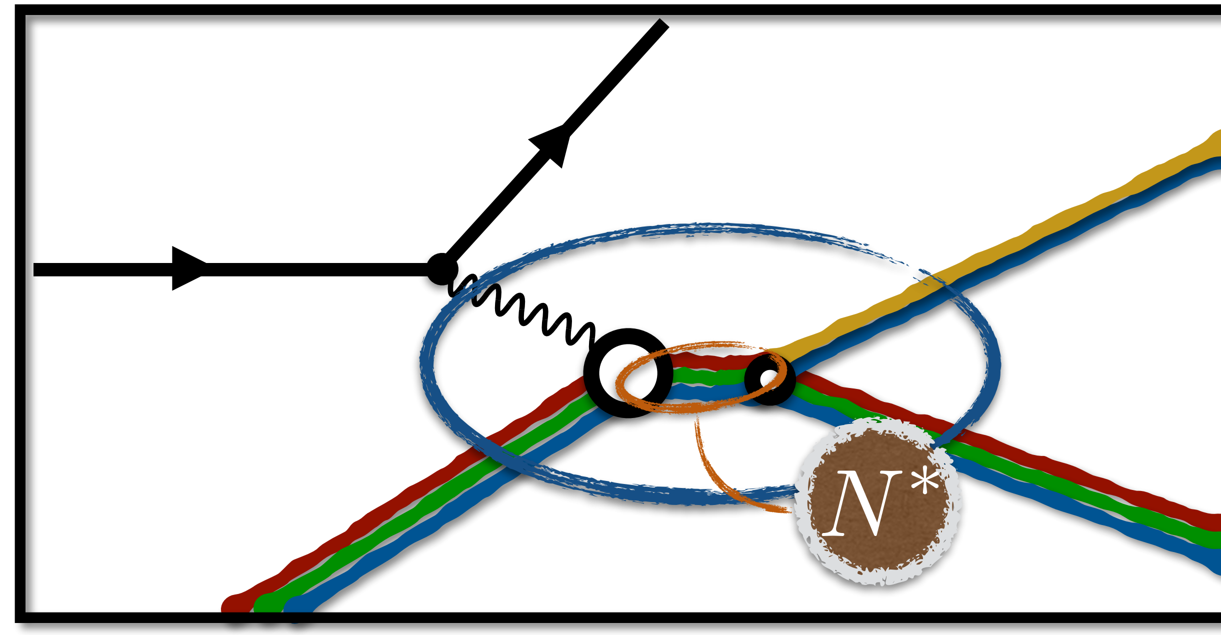

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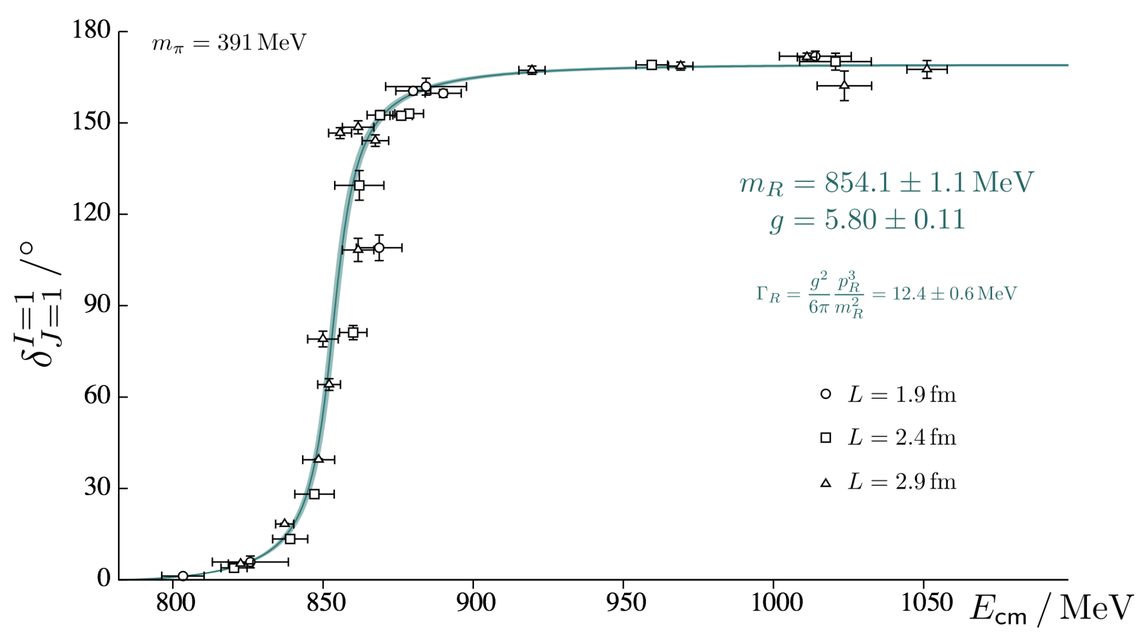
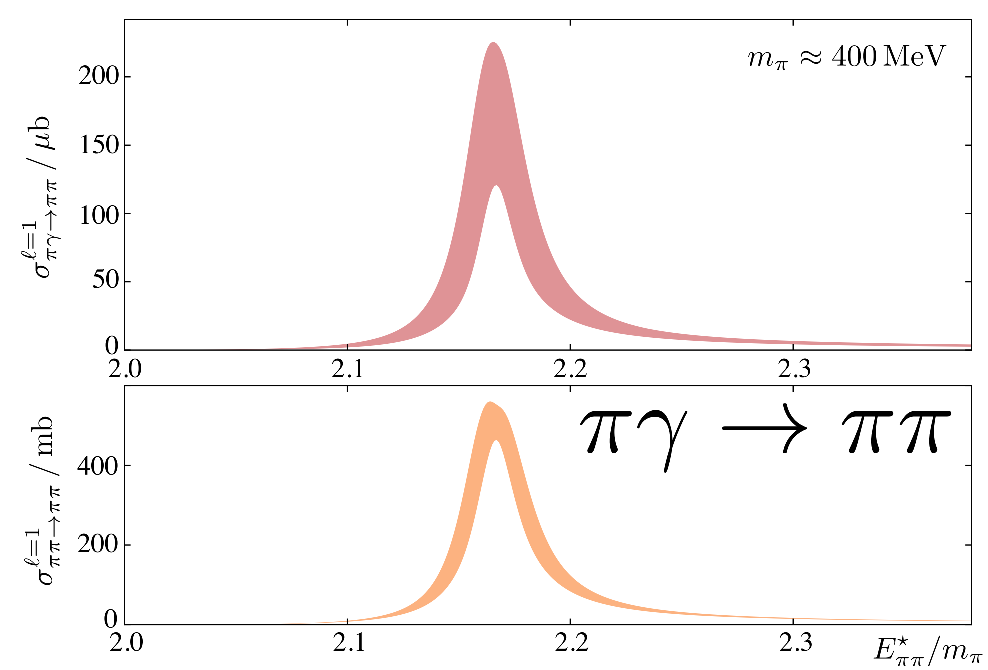
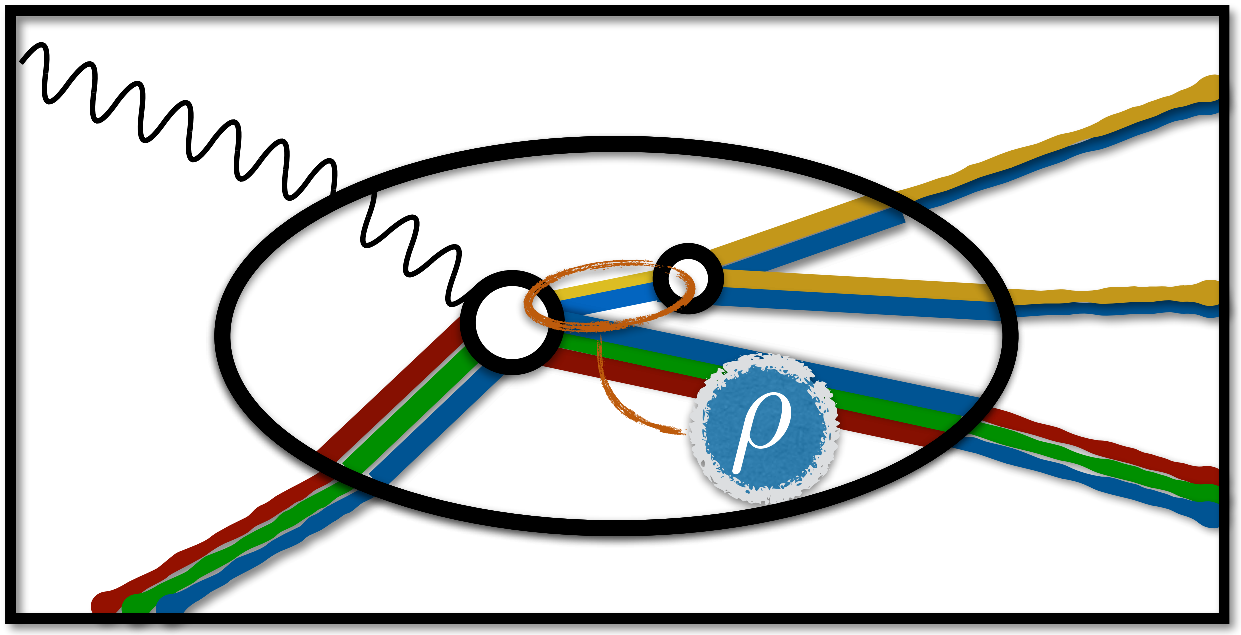




Lattice group at JLab leads the field in applying this kind of formalism

My work at JLab:



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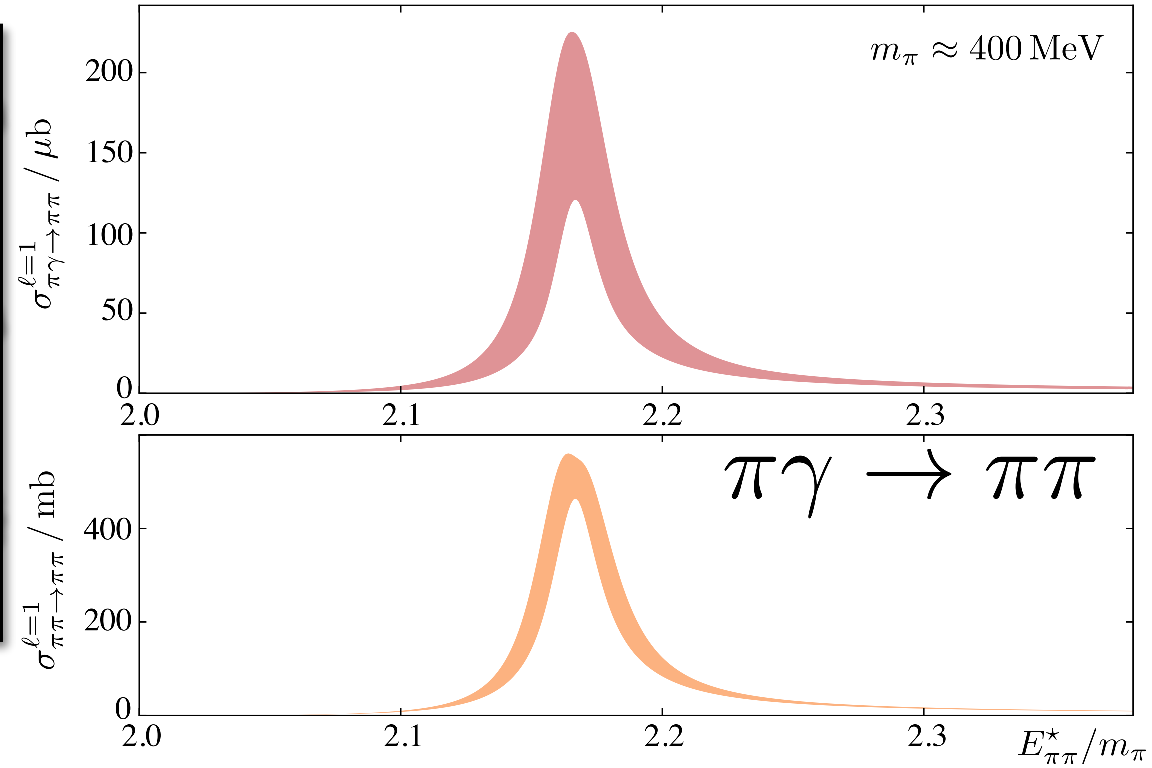

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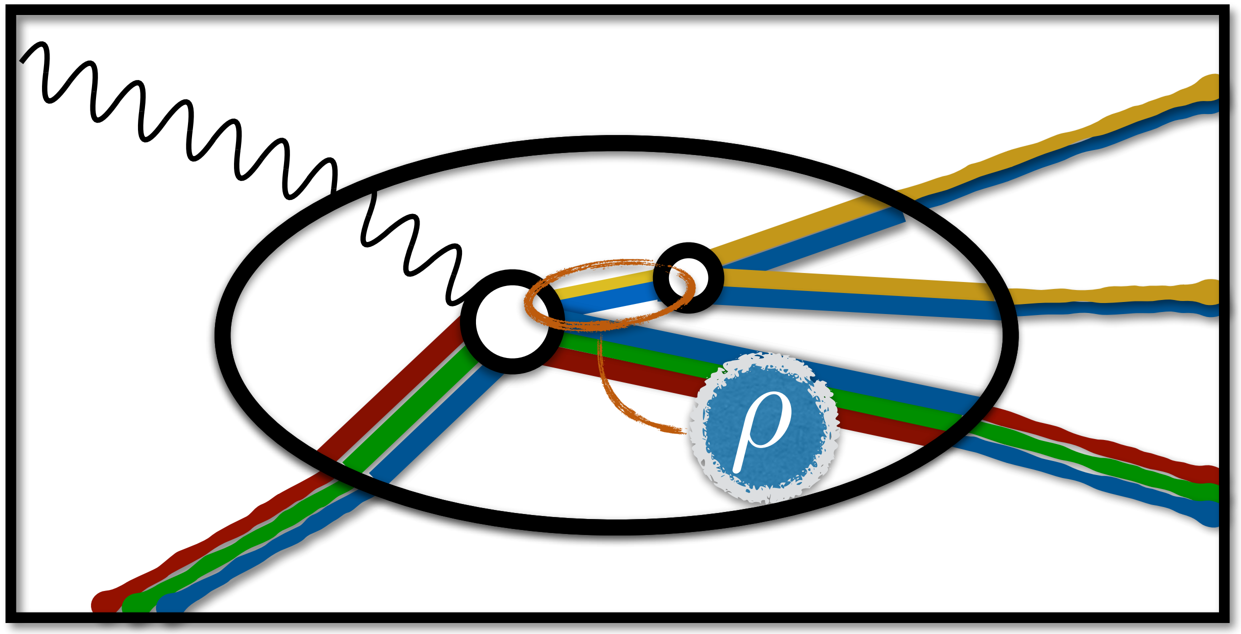

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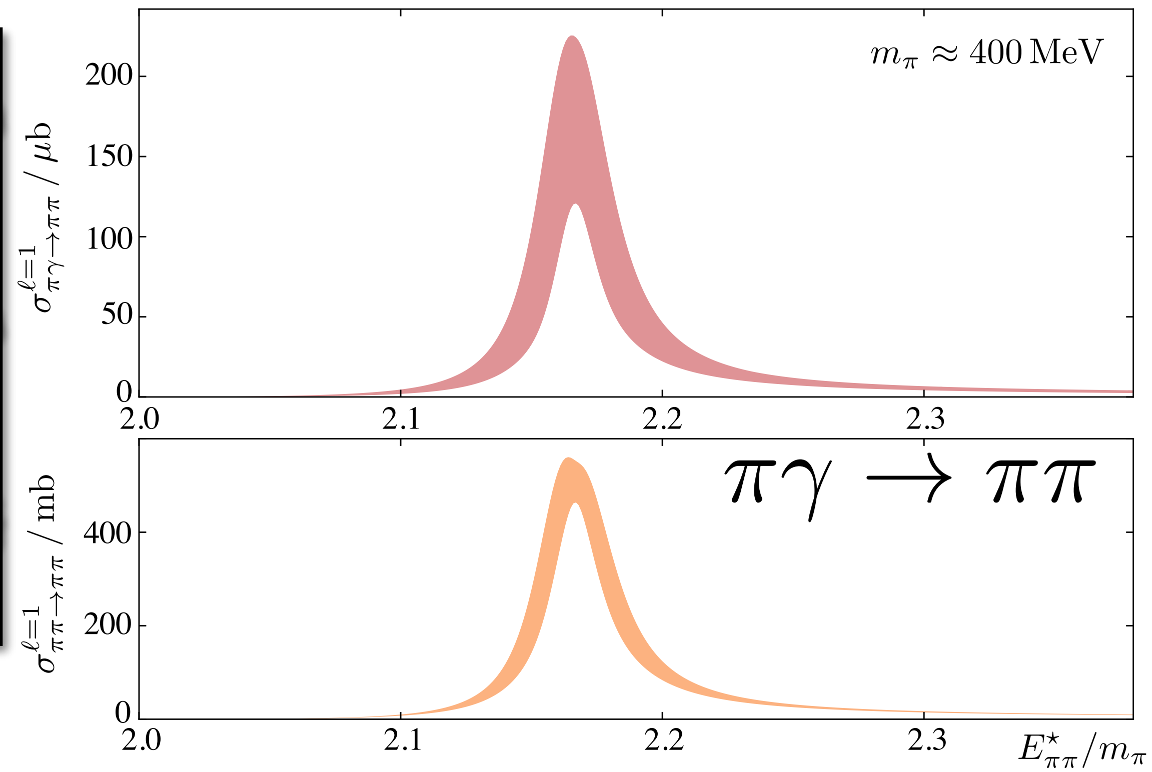

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The ideal scenario...

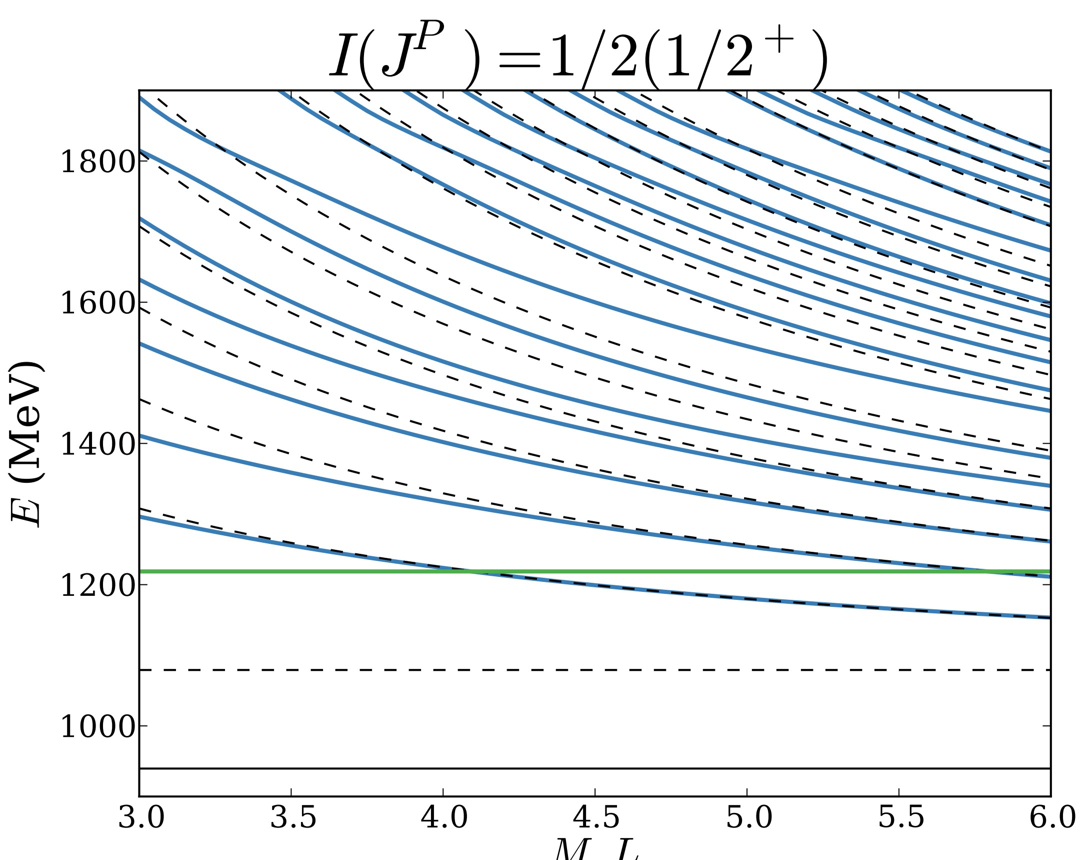
Regular interaction with experimental, lattice and theory groups:

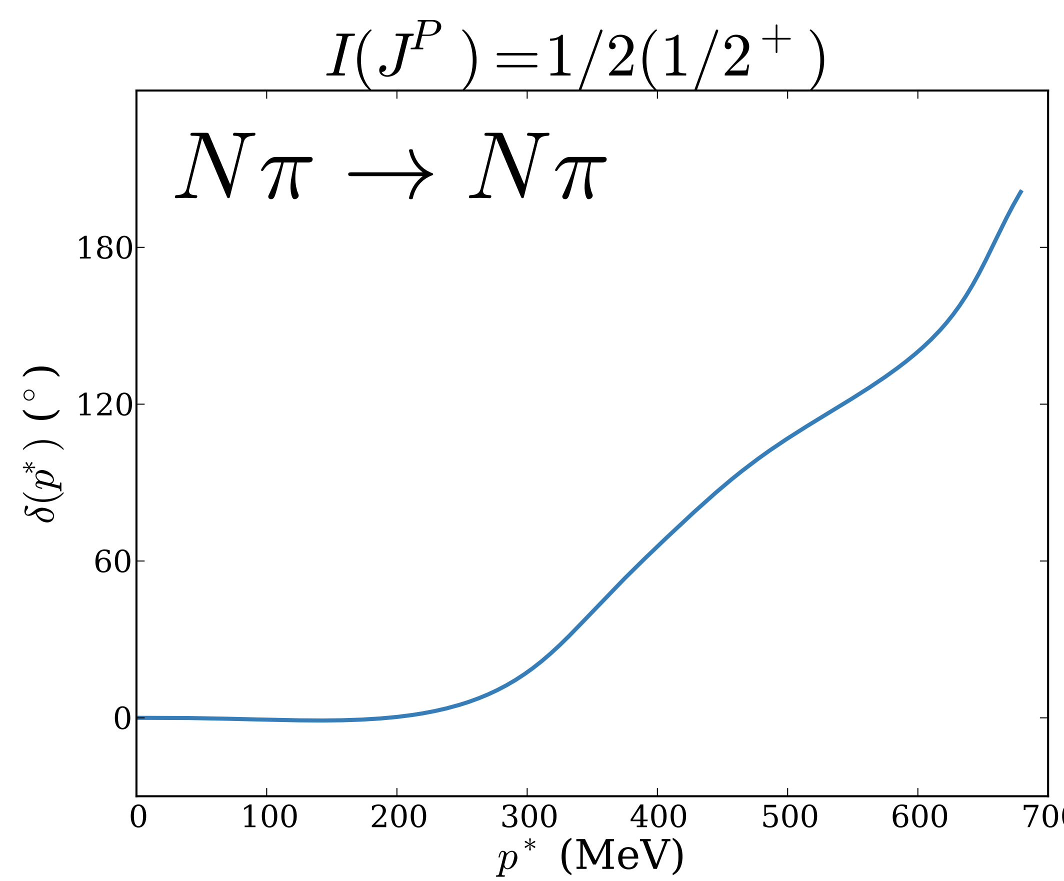
Identifying the most relevant observables,
Developing formalism to extract these,
Performing the calculations

My work at JLab:

One example of symbiosis...

Formalism can also be applied in the "other direction" to gain insight on lattice observables





GW partial-wave data base (solution WI08)

MTH and Meyer, *to appear*

My work at JLab:

More concretely...

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In one to two years:

The formalism needed for $N\pi \to N\pi\pi$ **and** $N\gamma \to N^* \to N\pi\pi$ **expected to be complete.**

First lattice studies of three-particle systems

$$K\pi \to K\pi\pi \qquad \omega \to \pi\pi\pi$$

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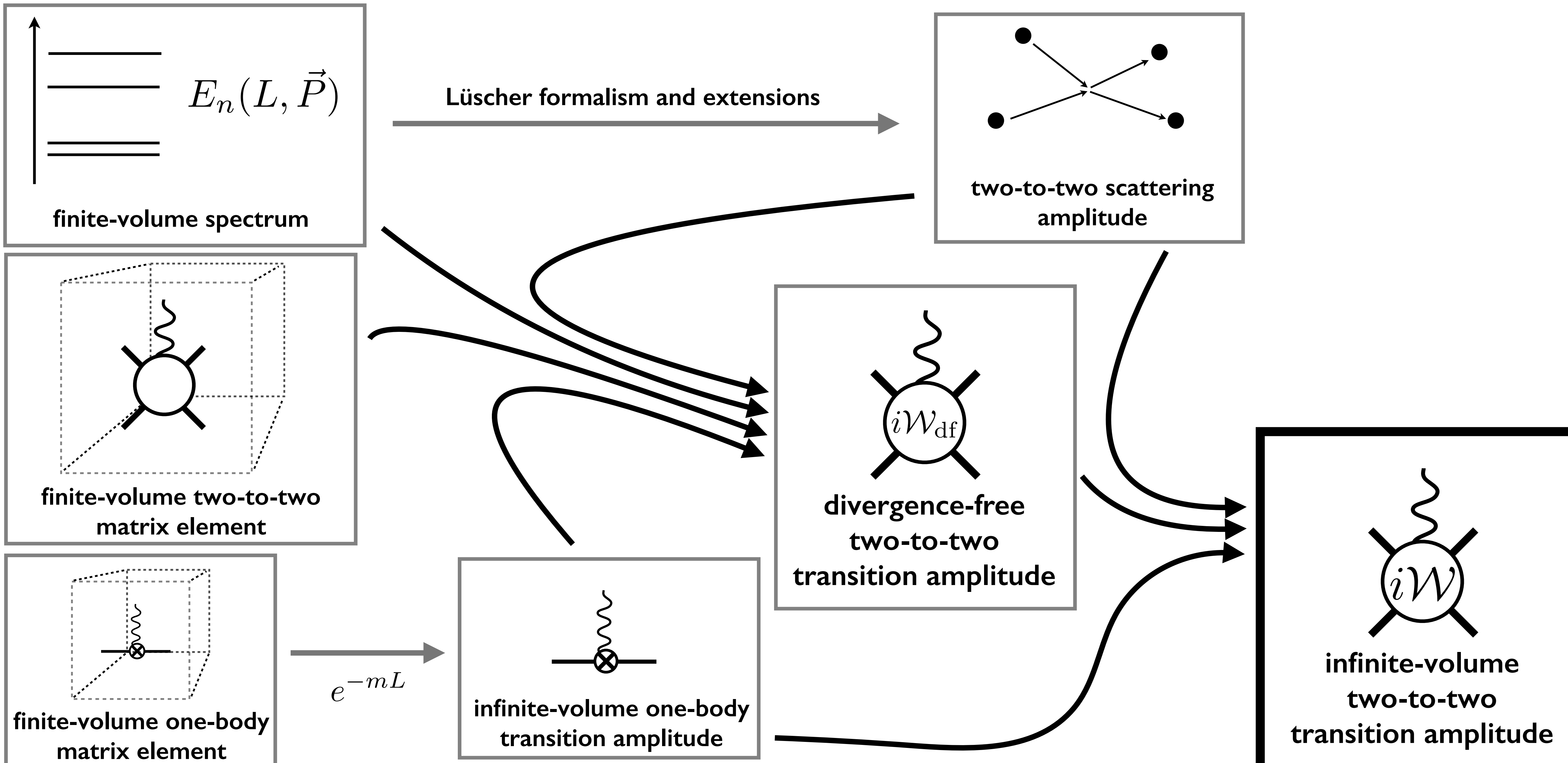
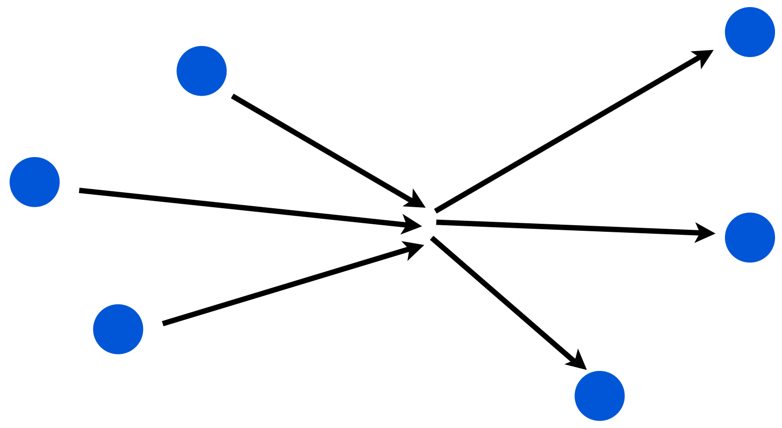
Thanks for listening!

Backup Slides

Two-to-two transition amplitudes

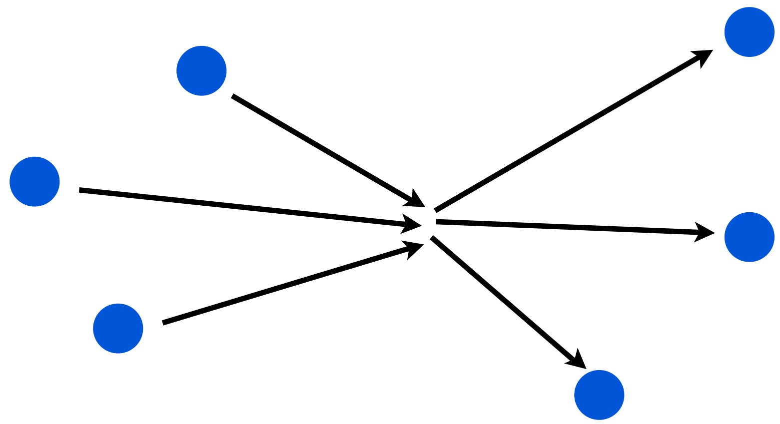


Infinite volume



$i\mathcal{M}_{3\to 3} \equiv$ **Sum of all connected Feynman diagrams with six external legs**

Infinite volume



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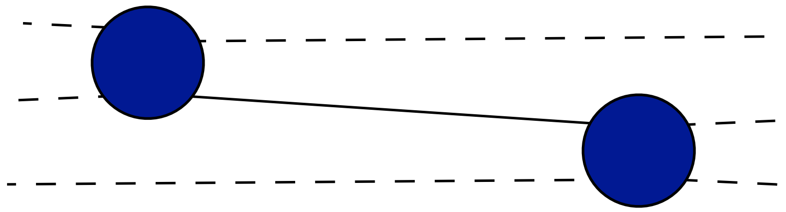
$$=$$  $$+\cdots$$

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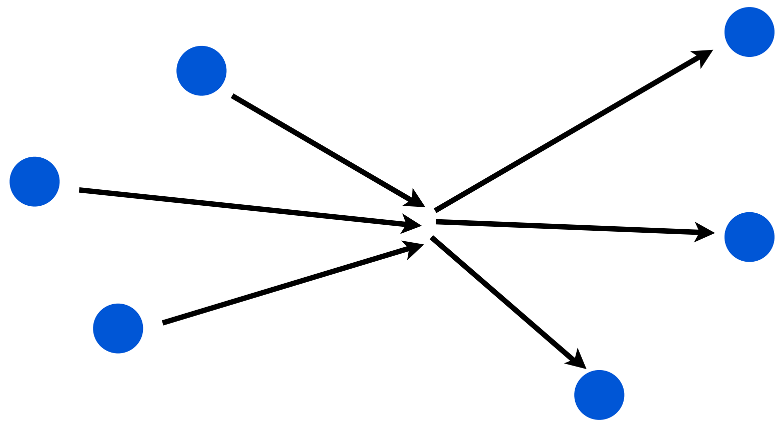

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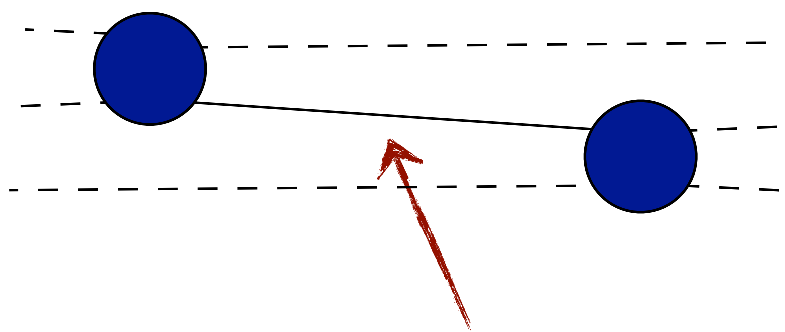
$$=$$  $$+\cdots$$

Certain external momenta put this on-shell!

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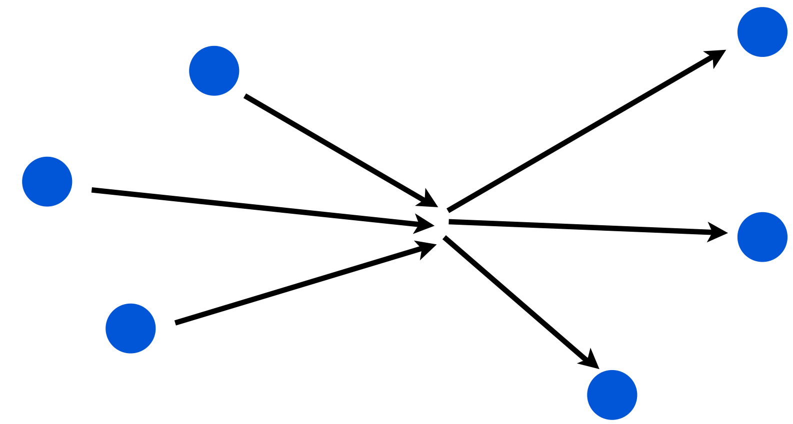

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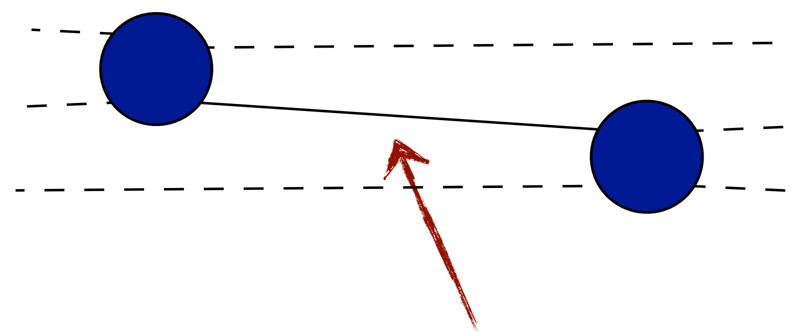


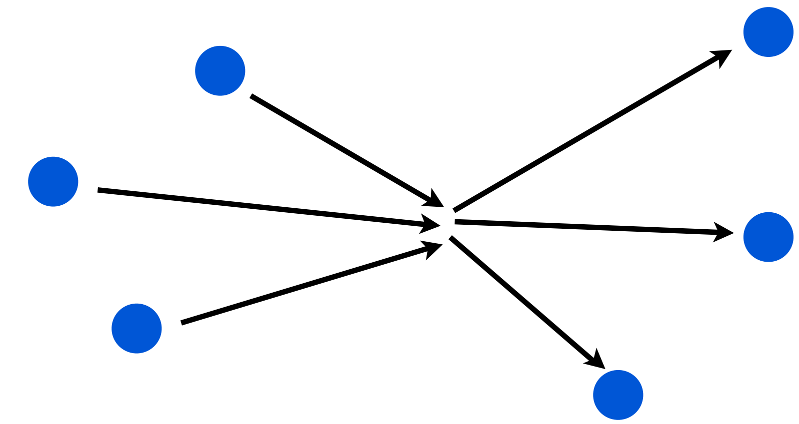
$$+\cdots$$

Certain external momenta put this on-shell!

$\mathcal{M}_{3\to3}$ has kinematic singularities at certain momenta

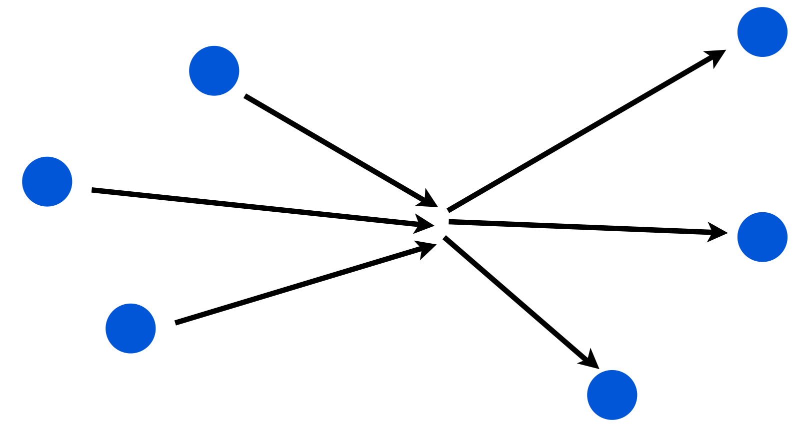
No dominance of lowest partial waves

Infinite volume



Degrees of freedom for three on-shell particles with $(E, \vec{P})$

Infinite volume

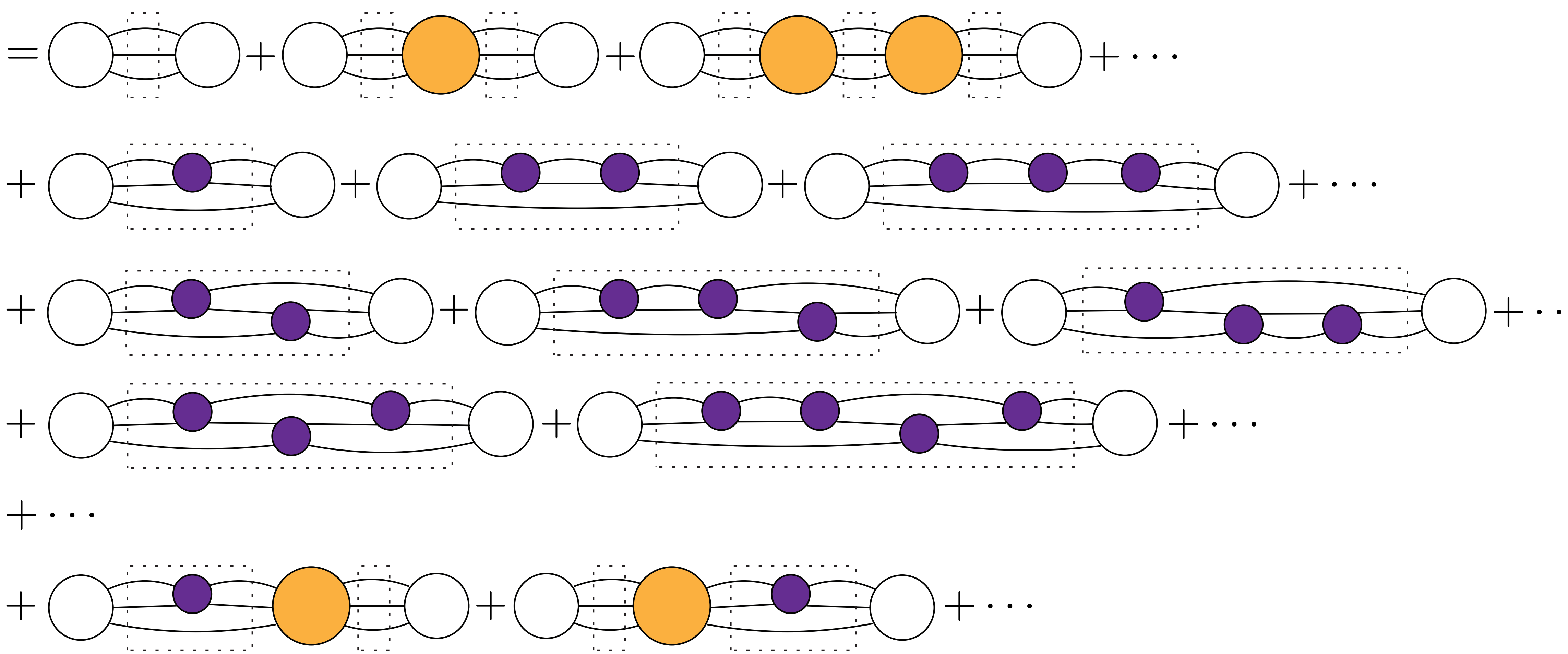


Degrees of freedom for three on-shell particles with $(E, \vec{P})$

$$\vec{k}, \ell, m$$

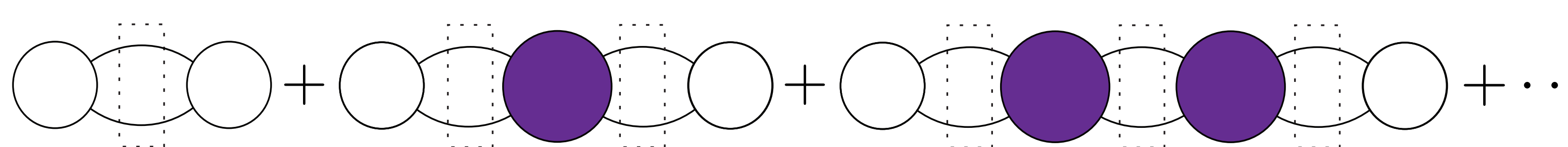
New skeleton expansion

$C_L(E, \vec{P}) =$



Compare to two-particle skeleton expansion

$C_L(E, \vec{P}) =$



What is new here?

1. Degrees of freedom are different

two particles

two-particle angular momentum

three particles

$\vec{k}$ + **two-particle angular momentum**

Our result only depends on finite-volume momentum $\vec{k} = \frac{2\pi}{L}\vec{n}$

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Our result only depends on finite-volume momentum $\vec{k} = \frac{2\pi}{L}\vec{n}$

Quantization condition expressed using matrices with indices

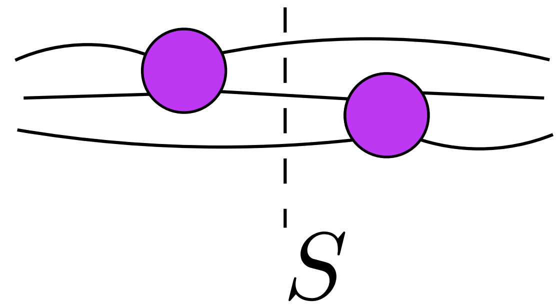
$$\vec{k}, \ell, m$$

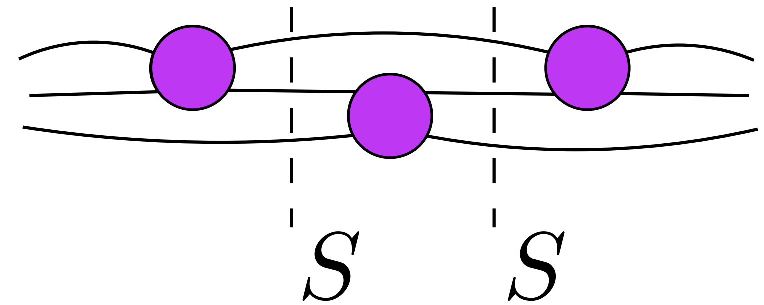
What is new here?

2. Three particle divergences

Define $i\mathcal{M}_{\mathrm{df},3\to3}$

$$\equiv i\mathcal{M}_{3\to3} - \left[i\mathcal{M}_{2\to2} S i\mathcal{M}_{2\to2} + \int i\mathcal{M}_{2\to2} S i\mathcal{M}_{2\to2} S i\mathcal{M}_{2\to2} + \cdots \right]$$





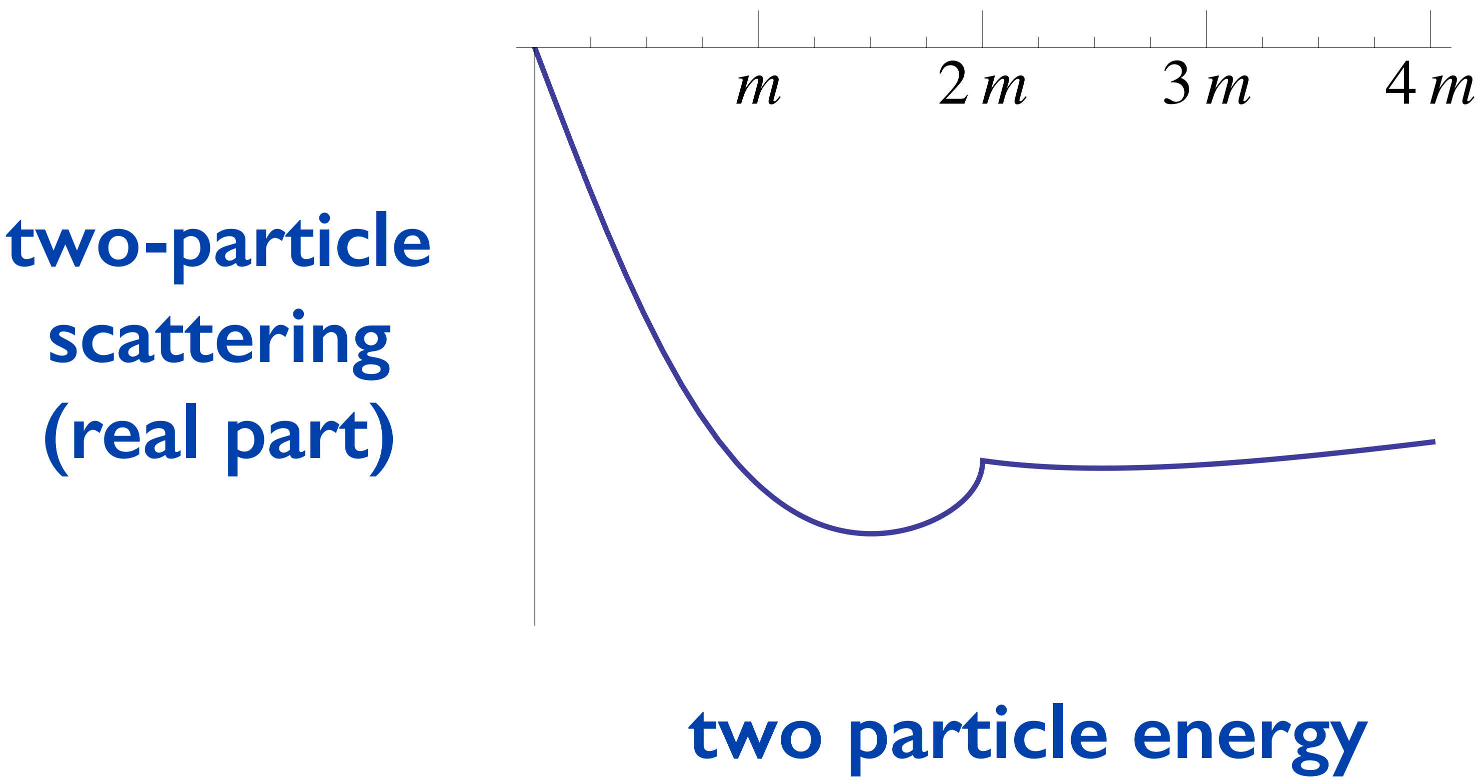
only on-shell amplitudes here

infinite series built with factors of $Si\mathcal{M}_{2\to2}$

This subtraction emerges naturally in our finite-volume analysis

What is new here?

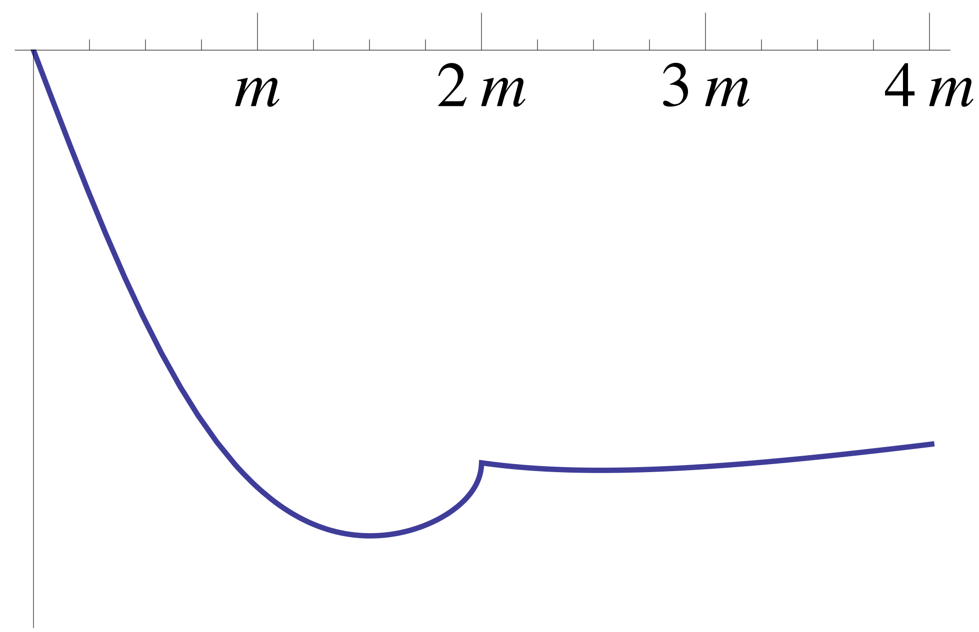
3. Must now worry about sum crossing two-particle unitary cusp



What is new here?

3. Must now worry about sum crossing two-particle unitary cusp

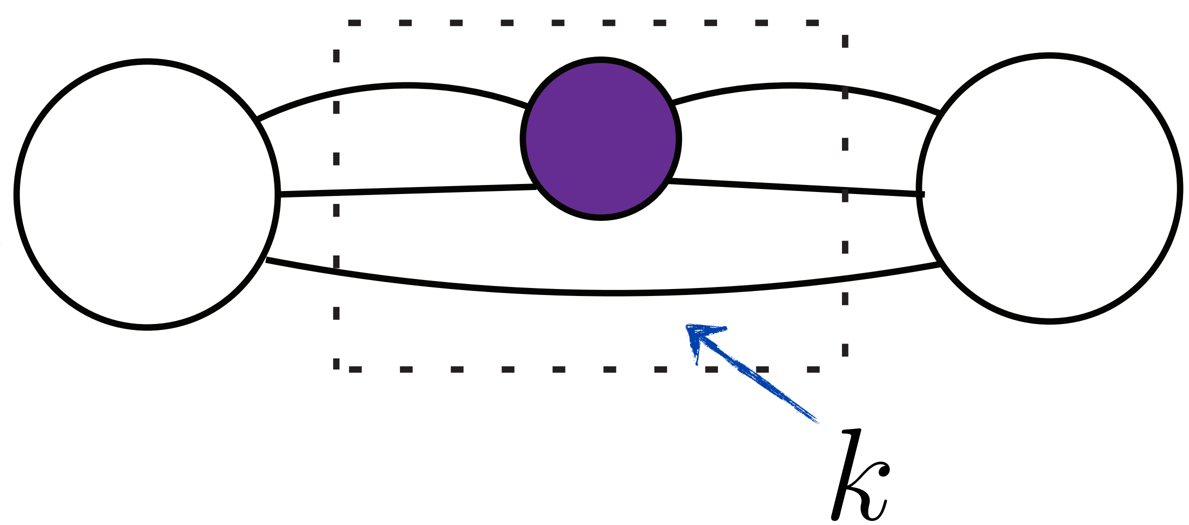
two-particle scattering (real part)



depends on k

two particle energy

$$\frac{1}{L^3}\sum_{\vec{k}}$$



What is new here?

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To remove cusp

$i\epsilon$ prescription → principal value $\widetilde{\mathrm{PV}}$

Analytically continue principal value below threshold then interpolate to prescription-free subthreshold form

Polejaeva, K. and Rusetsky, A. *Eur. Phys. J.* A48 (2012) 67

What is new here?

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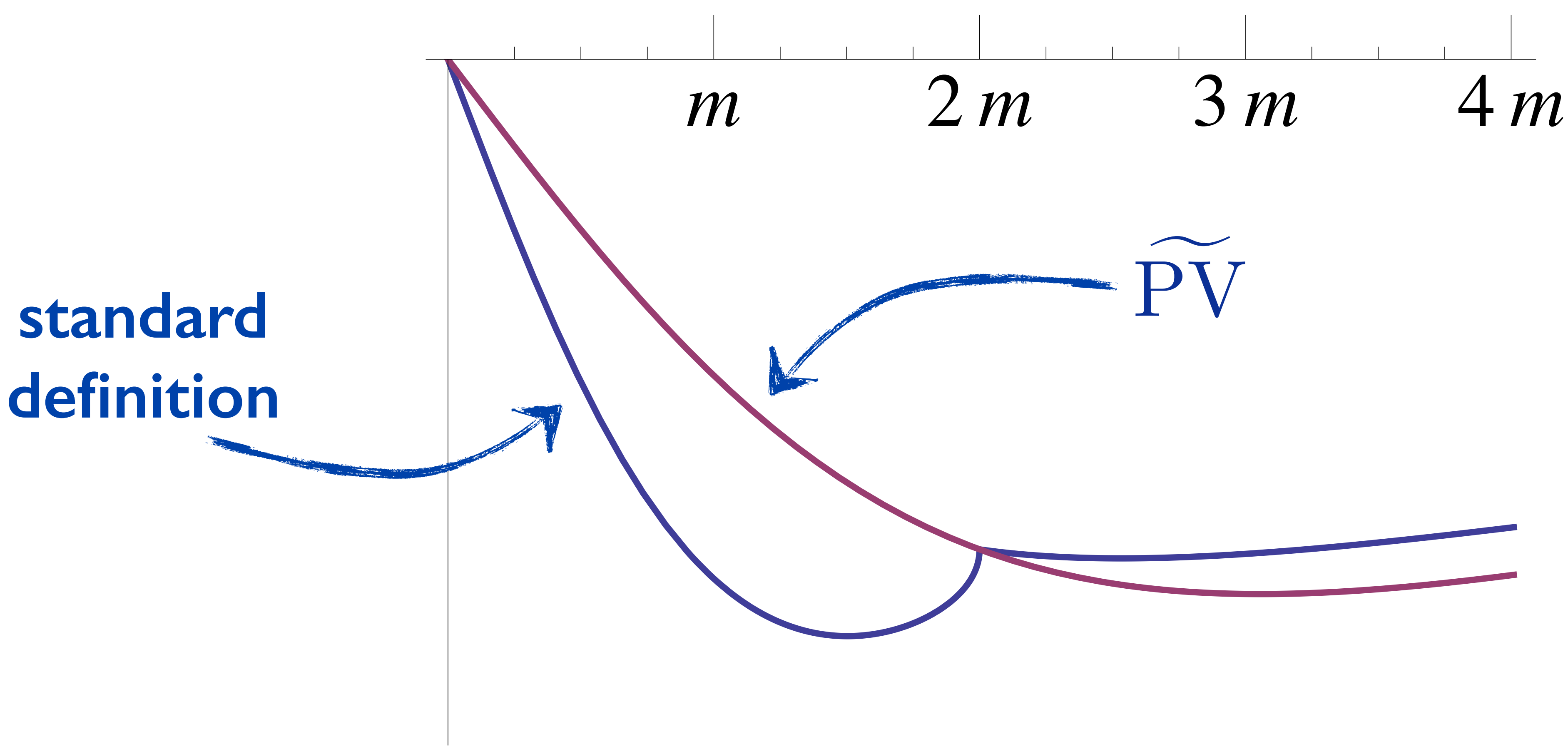

What is new here?

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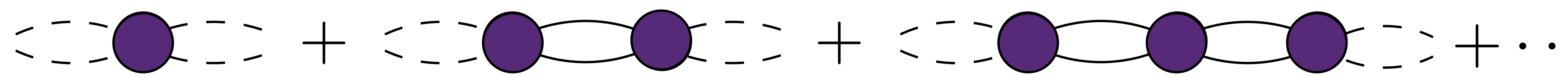
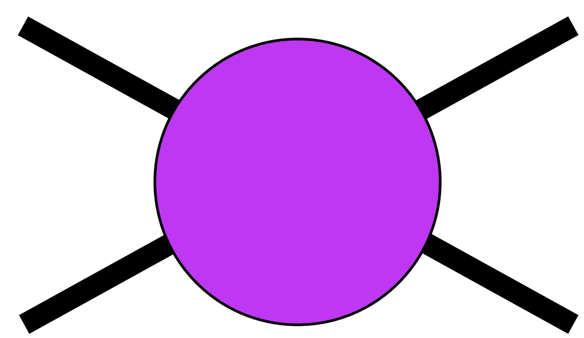
has a cusp

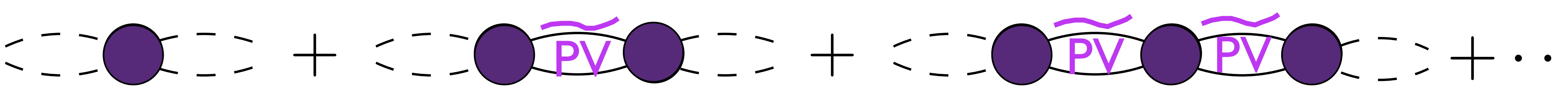
$i\mathcal{M}_{2\to 2} =$



$i\widetilde{\mathcal{K}}_{2\to 2} =$



$=$



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3. Must now worry about sum crossing two-particle unitary cusp

$$i\mathcal{M}_{2\to2}$$

$$i\mathcal{M}_{\mathrm{df},3\to3}$$

$$i\mathcal{K}_{2\to2}$$

$$i\mathcal{K}_{\mathrm{df},3\to3}$$

We relate these infinite-volume quantities to the finite-volume spectrum

Three-particle result

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + A_3' iF_3 \frac{1}{1 - i\mathcal{K}_{\mathrm{df},3\to3}\; iF_3} A_3$$

$$iF_3 \equiv \frac{iF}{2\omega L^3}\left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\to2}\; iG}\; i\mathcal{M}_{L,2\to2}\; iF\right]$$

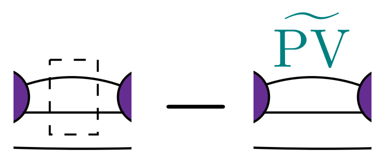
$$i\mathcal{M}_{L,2\to2} \equiv i\mathcal{K}_{2\to2}\frac{1}{1 - iFi\mathcal{K}_{2\to2}}$$

All factors are matrices with indices $\vec{k}, \ell, m$

Three-particle result

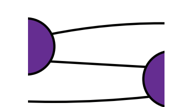
$$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) + A_3' iF_3 \frac{1}{1 - i\mathcal{K}_{\mathrm{df},3\to3}\, iF_3} A_3$$

sum-integral difference



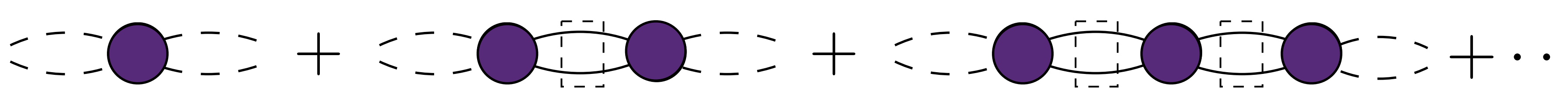
$$iF_3 \equiv \frac{iF}{2\omega L^3}\left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\to2}\, iG}\, i\mathcal{M}_{L,2\to2}\, iF\right]$$

encodes switches



$$i\mathcal{M}_{L,2\to2} \equiv i\mathcal{K}_{2\to2}\frac{1}{1 - iF i\mathcal{K}_{2\to2}}$$

sum of all two-particle loops (with summed momenta)



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Three-particle result

At fixed $(L, \vec{P})$, finite-volume spectrum is all solutions to

$$\det\left[1 - i\mathcal{K}_{\mathrm{df},3\to3} iF_3\right] = 0$$

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MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

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Assumes two-particle phase shift is bounded by $\pi/2$

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Model independent general result of relativistic scalar field theory

Assumes two-particle phase shift is bounded by $\pi/2$

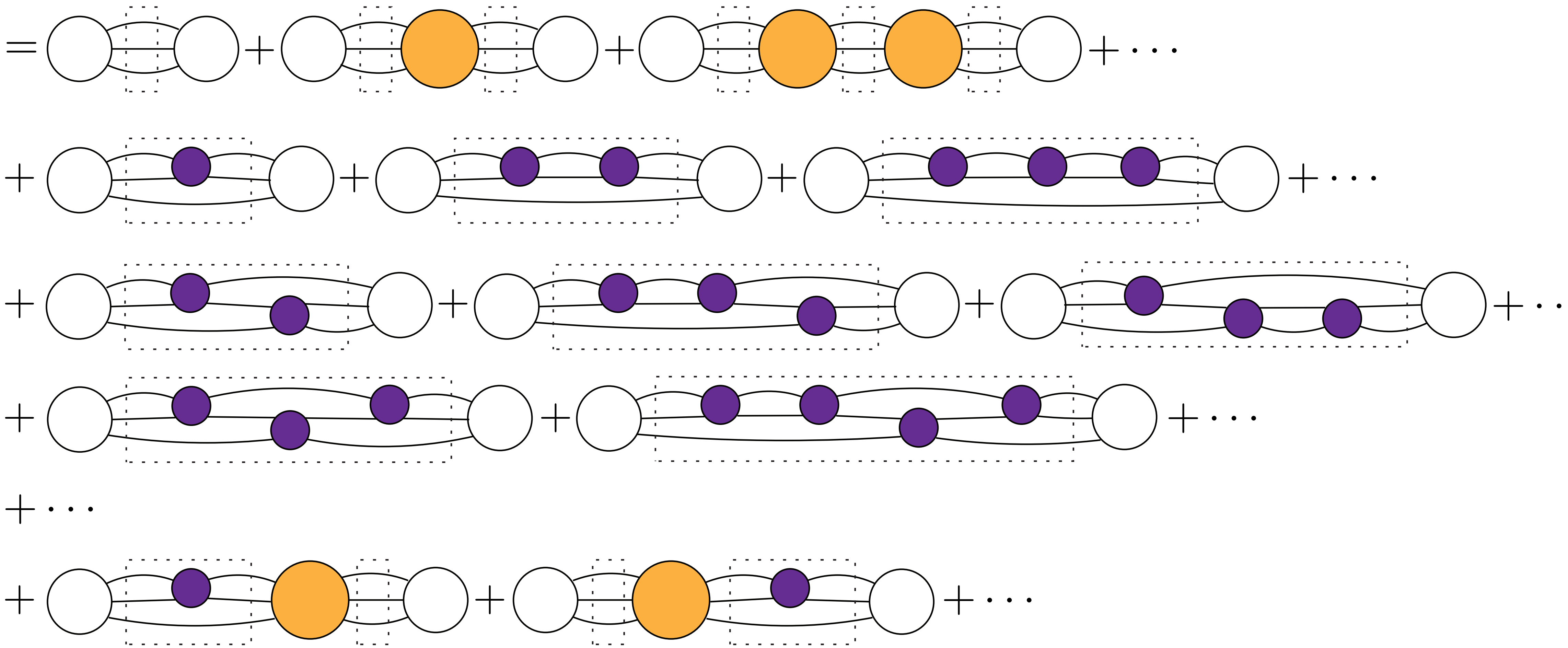
Infinite matrices truncate if we truncate in angular momentum

Strongest truncation is the isotropic limit, gives simple result

$$\mathcal{K}_{\mathrm{df},3\to3}(E_n^*) = -[F_{3,\mathrm{iso}}(E_n, \vec{P}, L)]^{-1}$$

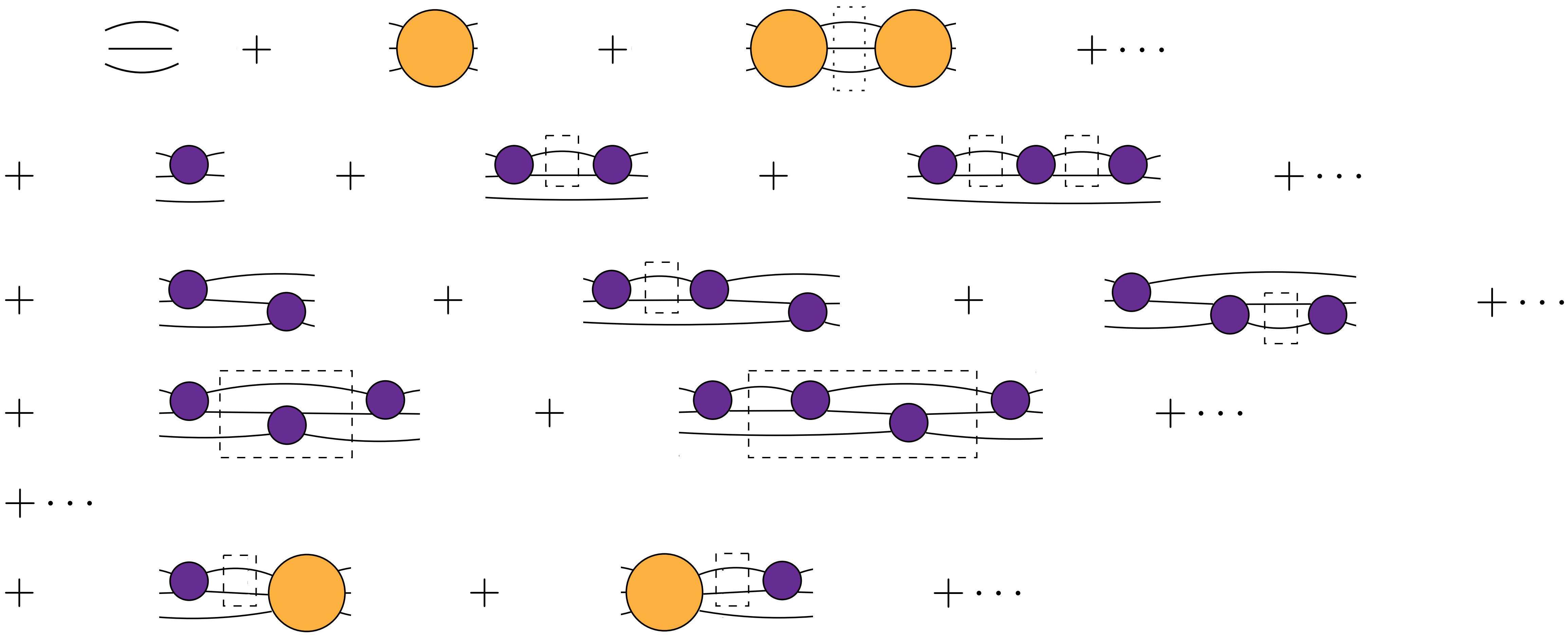
Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

$C_L(E, \vec{P}) =$



First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3\to3}$

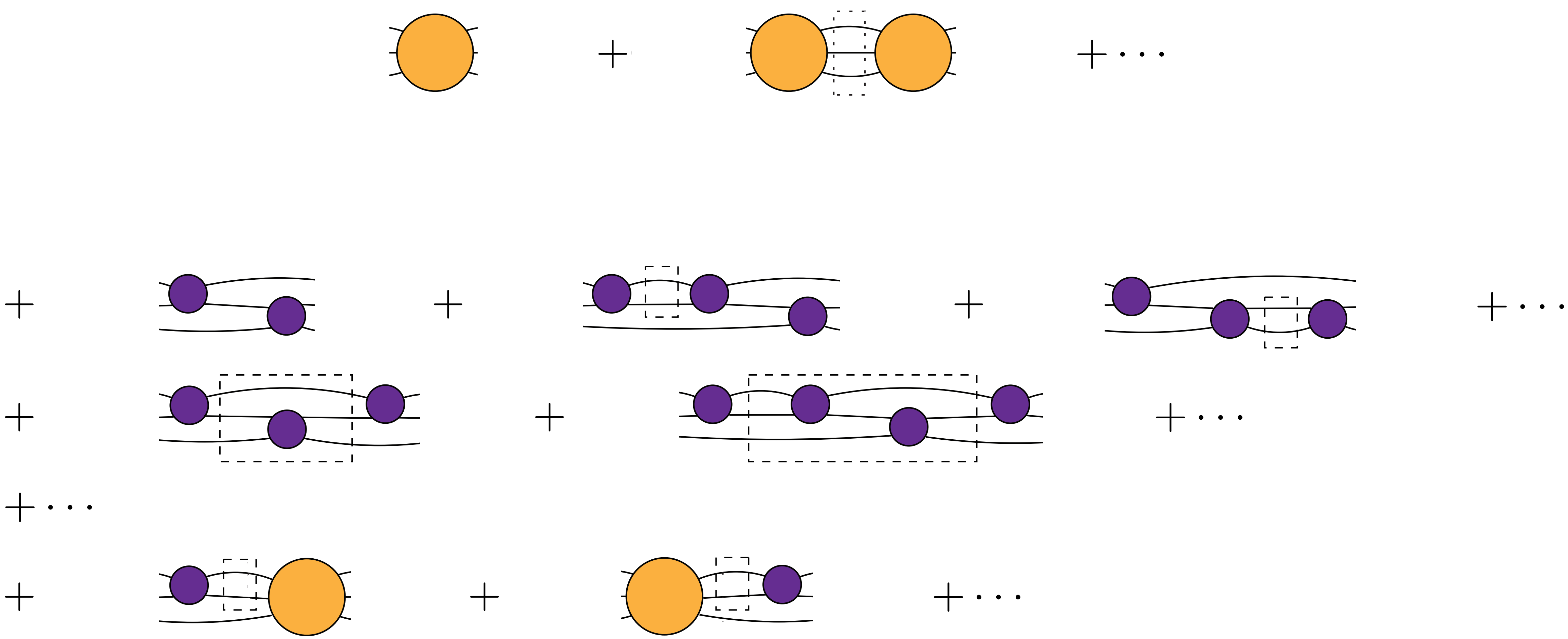
Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$



First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3\to3}$

1. Amputate interpolating fields

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

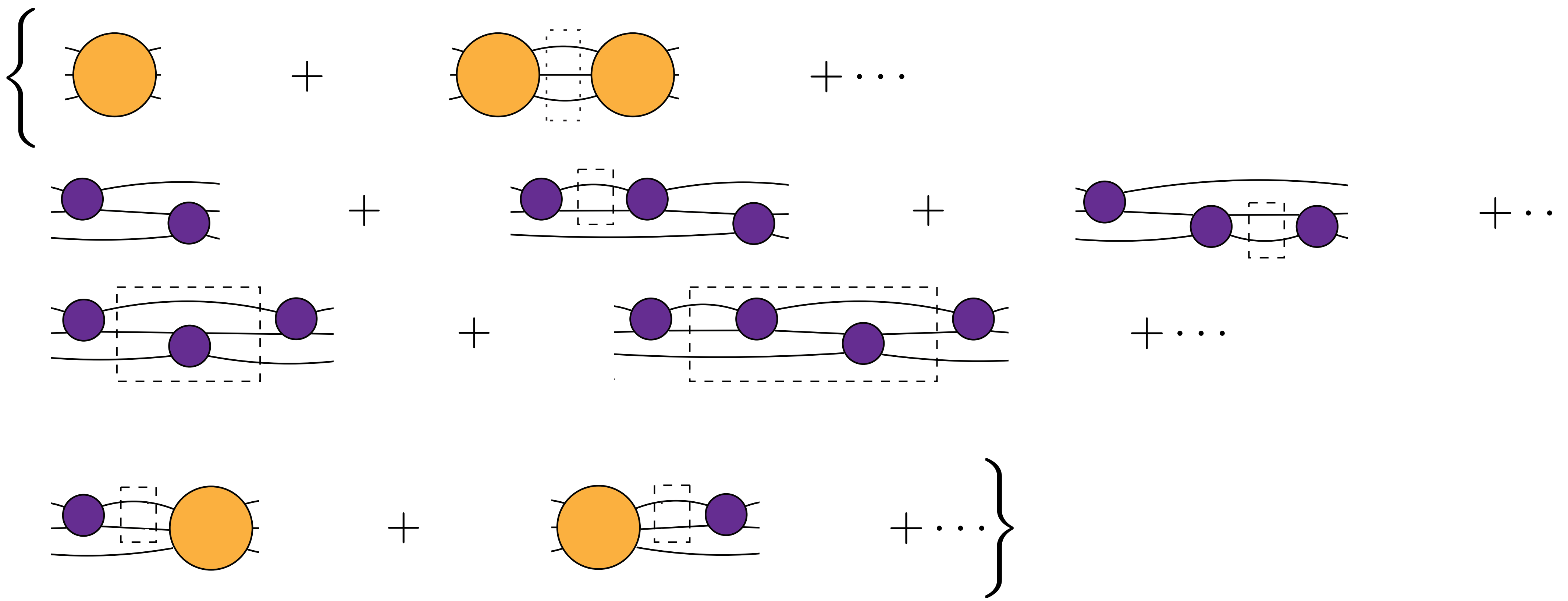


First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3\to3}$

2. Drop disconnected diagrams

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

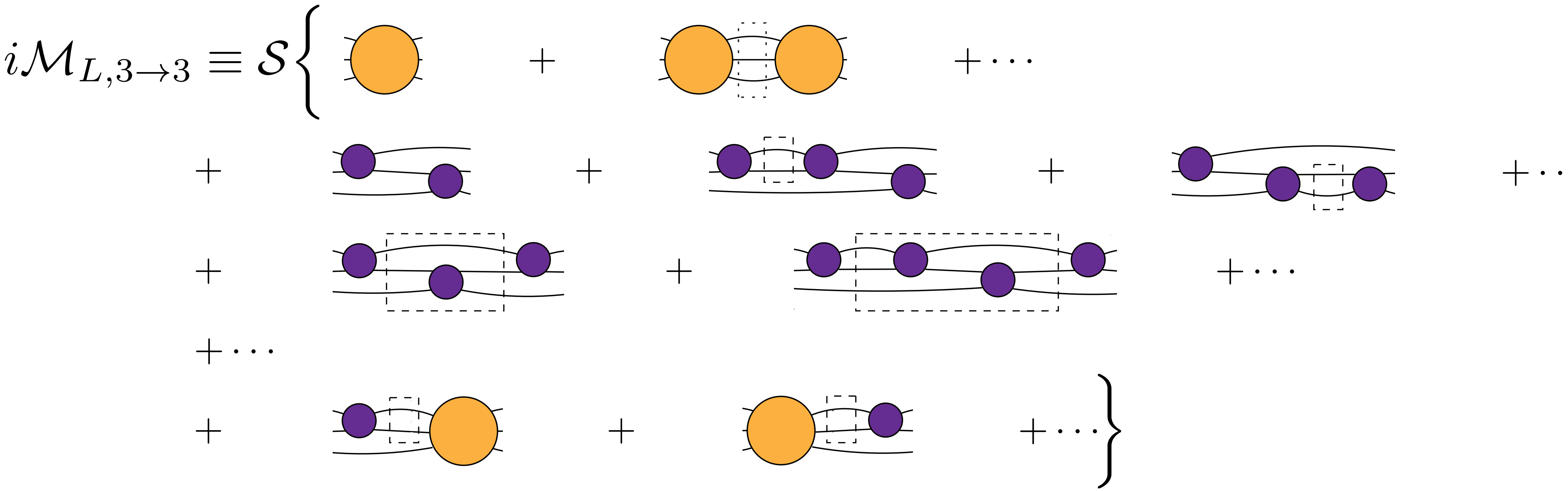
$i\mathcal{M}_{L,3\to3} \equiv \mathcal{S}\{\ \cdots\ \}$



First we modify $C_L(E,\vec{P})$ to define $i\mathcal{M}_{L,3\to3}$

3. Symmetrize

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$



Replacing all loop momentum sums with i-epsilon prescription integrals gives physical three-to-three scattering amplitude

$$i\mathcal{M}_{3\to3} = \lim_{L\to\infty}\Big|_{i\epsilon} i\mathcal{M}_{L,3\to3}$$

Relating $i\mathcal{K}_{\mathrm{df},3\to3}$ to $i\mathcal{M}_{3\to3}$

$$i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \; i\mathcal{K}_{\mathrm{df},3\to3} \frac{1}{1 - iF_3 \; i\mathcal{K}_{\mathrm{df},3\to3}} \; \mathcal{R}_L\right]$$

$$i\mathcal{M}_{3\to3} = \lim_{L\to\infty}\Big|_{i\epsilon} \; i\mathcal{M}_{L,3\to3}$$

MTH and Sharpe, *Phys. Rev.* D 92, 114509 (2015)

Gives integral equation relating $i\mathcal{K}_{\mathrm{df},3\to3}$ **to** $i\mathcal{M}_{3\to3}$

Completes formal story (for the setup considered!)

Relation only depends on on-shell scattering quantities

$1/L$ expansions

In 1957, Huang and Yang determined energy shift for n identical bosons in a box

K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775

$$E_0(n,L) = \frac{4\pi\, a}{M\, L^3}\left\{\binom{n}{2} - \left(\frac{a}{\pi\, L}\right)\binom{n}{2}\mathcal{I} + \left(\frac{a}{\pi\, L}\right)^2\left\{\binom{n}{2}\mathcal{I}^2 - \left[\binom{n}{2}^2 - 12\binom{n}{3} - 6\binom{n}{4}\right]\mathcal{J}\right\}\right\} + \mathcal{O}\left(L^{-6}\right)$$

where a is the two-particle scattering length and

$$\mathcal{I} = \lim_{\Lambda\to\infty}\sum_{\mathbf{i}\neq\mathbf{0}}^{|\mathbf{i}|\leq\Lambda}\frac{1}{|\mathbf{i}|^2} - 4\pi\Lambda = -8.91363291781 \qquad \mathcal{J} = \sum_{\mathbf{i}\neq\mathbf{0}}\frac{1}{|\mathbf{i}|^4} = 16.532315959$$

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In 2007 Beane, Detmold and Savage pushed the order to $1/L^6$ and the latter two calculated to $1/L^7$ the next year

Beane, S., Detmold, W. & Savage, M. *Phys. Rev.* D76 (2007) 074507
Detmold, W. & Savage, M. *Phys. Rev.* D77 (2008) 057502

At $1/L^6$ a three-particle contact term appears

$1/L$ expansions

Last year Detmold and Flynn performed a similar calculation for matrix elements

Detmold and Flynn, *Phys. Rev.* D91, 074509 (2015)

$$\begin{aligned}\langle n|J|n\rangle \;=\;& n\alpha_1 + \frac{n\alpha_1 a^2}{\pi^2 L^2}\binom{n}{2}\mathcal{J} + \frac{\alpha_2}{L^3}\binom{n}{2} \\ &+ \frac{2n\alpha_1 a^3}{\pi^3 L^3}\binom{n}{2}\left\{\mathcal{K}\binom{n}{2} - \left[\mathcal{I}\mathcal{J} + 4\mathcal{K}\binom{n-2}{1} + \mathcal{K}\binom{n-2}{2}\right]\right\} - \frac{2\alpha_2 a}{\pi L^4}\binom{n}{2}\mathcal{I} \\ &+ \frac{n\alpha_1 a^4}{\pi^4 L^4}\left[3\mathcal{I}^2\mathcal{J} + \mathcal{L}\left(186 - \frac{241n}{2} + \frac{29}{2}n^2\right) + \mathcal{J}^2\left(\frac{n^2}{4} + \frac{3n}{4} - \frac{7}{2}\right)\right. \\ &\left. + \mathcal{I}\mathcal{K}(4n-14) + \mathcal{U}(32n-64) + \mathcal{V}(16n-32)\right] + \mathcal{O}(1/L^5)\,.\end{aligned}$$

Here $\mathcal{I}, \mathcal{J}, \cdots$ are known geometric constants and α_1, α_2 are one- and two-boson current couplings

Nonperturbative and non-relativistic

Non-relativistic Faddeev analysis

In 2012, Polejaeva and Rusetsky derived a Lüscher-like result using

non-relativistic Faddeev equations

Polejaeva and Rusetsky, *Eur. Phys. J.* A48, 67 (2012)

Demonstrates that on-shell S-matrix determines spectrum

Difficult to extract scattering from the formalism

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Dimer formalism

In 2013, Briceño and Davoudi studied three-particles in finite-volume using the Dimer formalism

Briceño and Davoudi, *Phys. Rev.* D87, 094507 (2013)

Recovered Lüscher result when two of the three become bound

$$k \cot \delta = -k \cot \phi + \eta \frac{e^{-\gamma L}}{L}$$

Final result involves an integral equation that one needs to solve numerically

Three-particle bound state

This year Meißner, Rios and Rusetsky determined the finite-volume energy shift to a three-body bound state

$$\Delta E = c\frac{\kappa^2}{m}\frac{|A|^2}{(kL)^{3/2}} \ \exp(-2\kappa L/\sqrt{3}) + \cdots$$

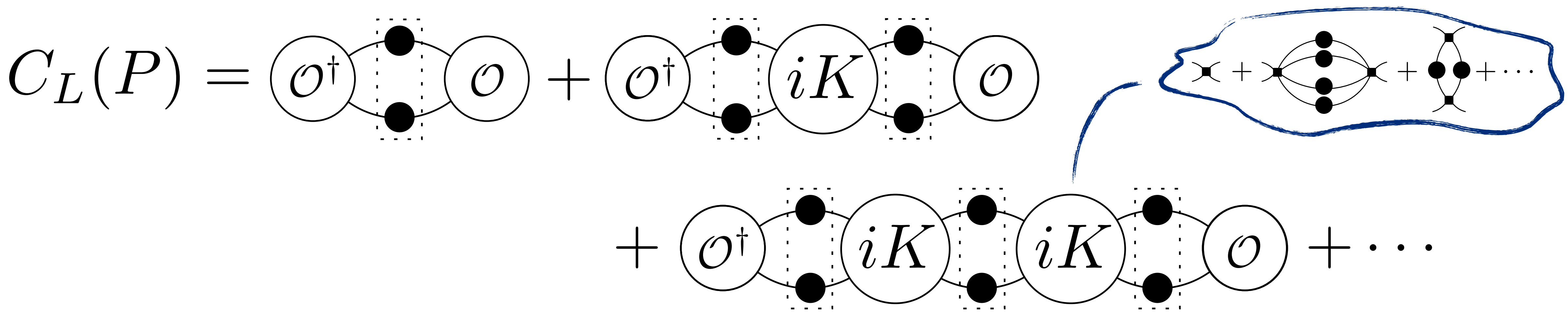
Meißner, Rios and Rusektsky. *Phys. Rev. Lett.* 114, 091602 (2015)

Assumes the unitary limit for two-particle scattering

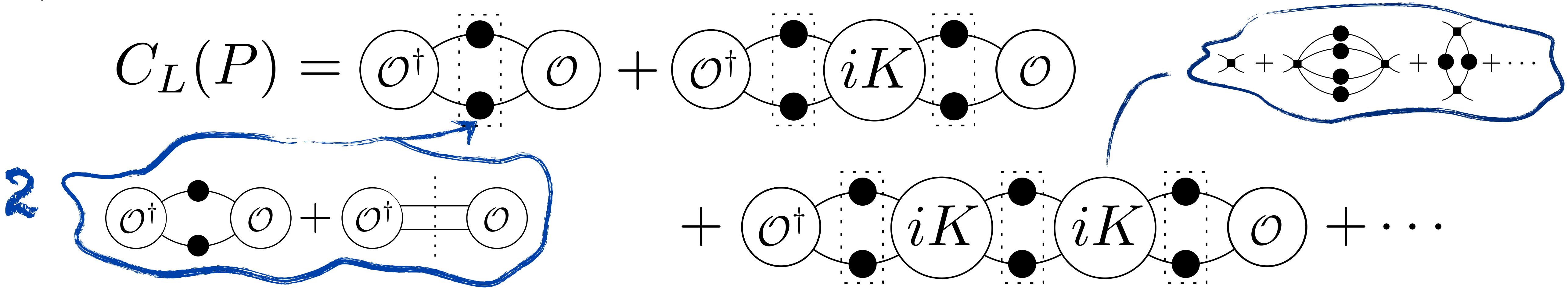
Result derived using non-relativistic quantum mechanics

Review...

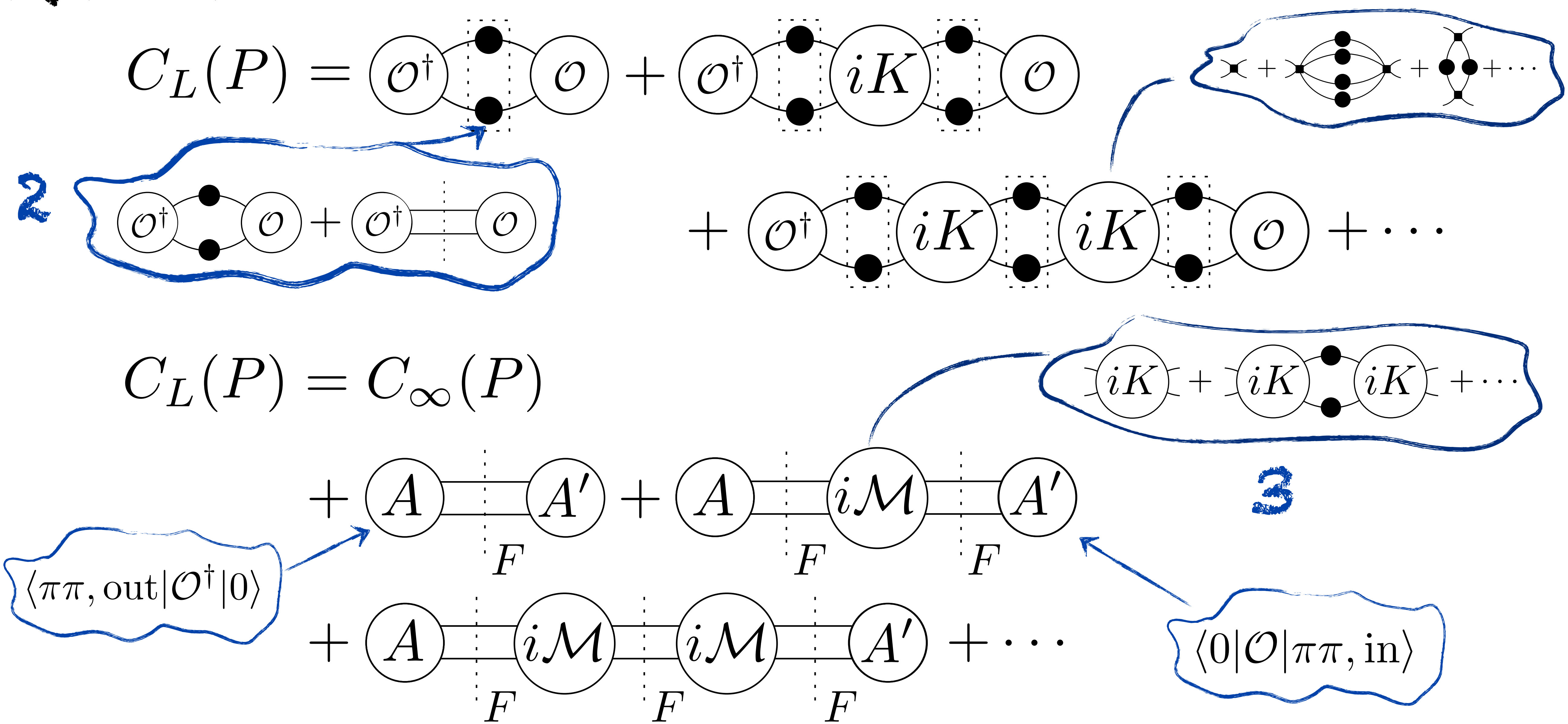
Review...



Review...

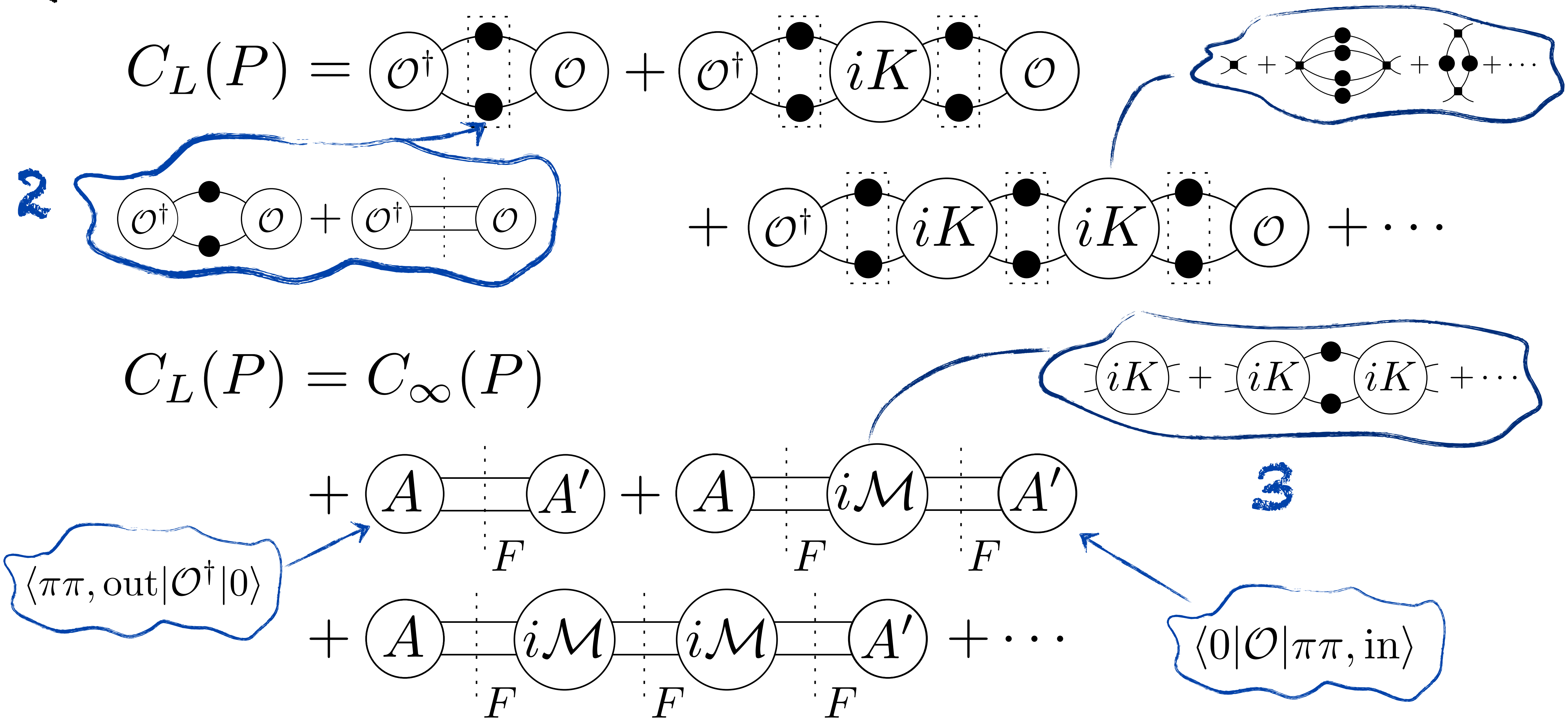


Review...



1
$C_L(P) =$
$\mathcal{O}^\dagger$
$\mathcal{O}$
iK
2
3
$C_L(P) = C_\infty(P)$
A
A'
$i\mathcal{M}$
F
$\langle \pi\pi, \mathrm{out}|\mathcal{O}^\dagger|0\rangle$
$\langle 0|\mathcal{O}|\pi\pi, \mathrm{in}\rangle$

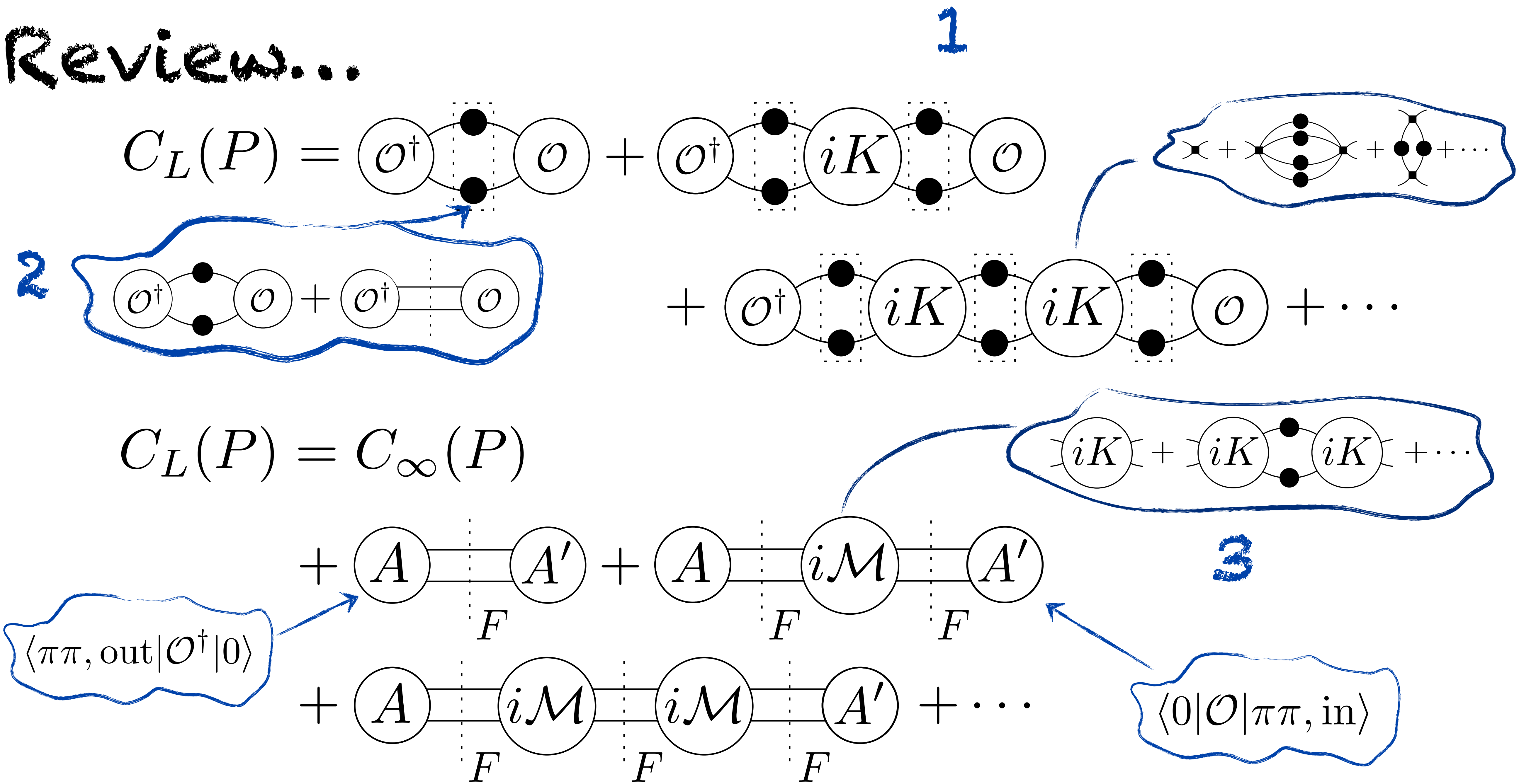
Review...



We deduce...

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to 2} F} A$$

Review...



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poles are in here

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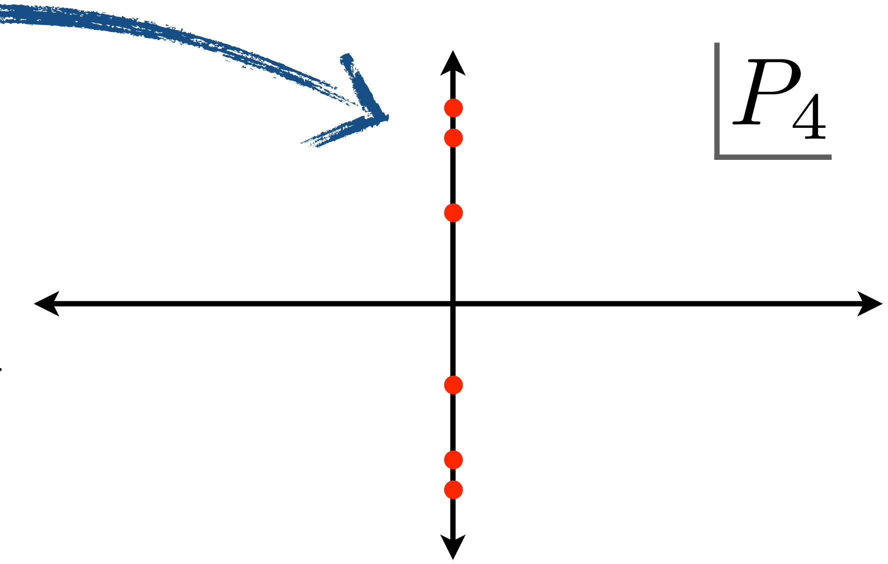
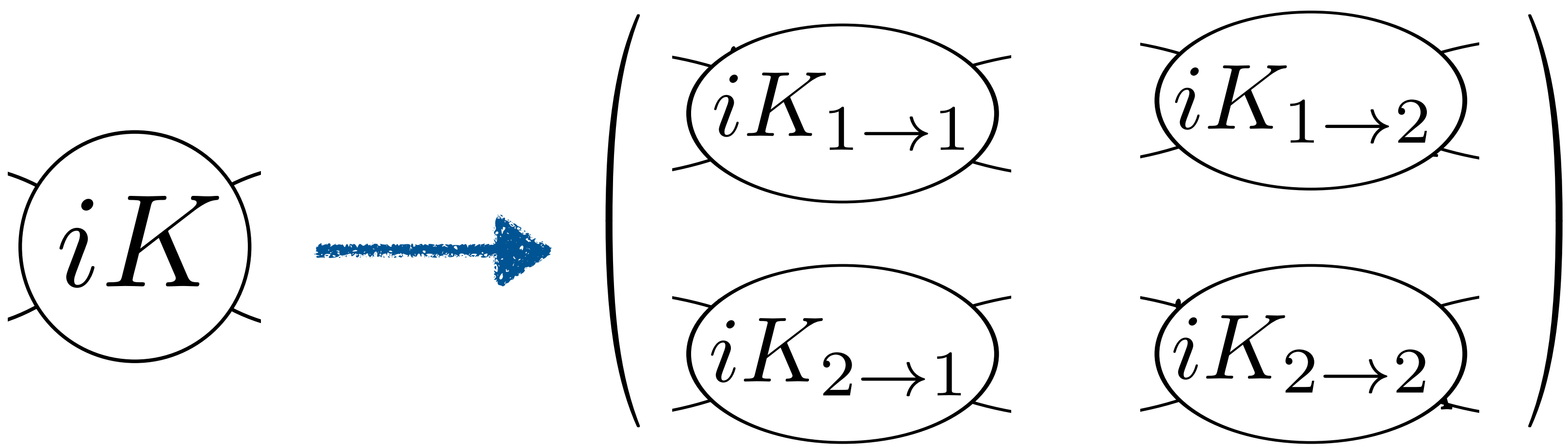
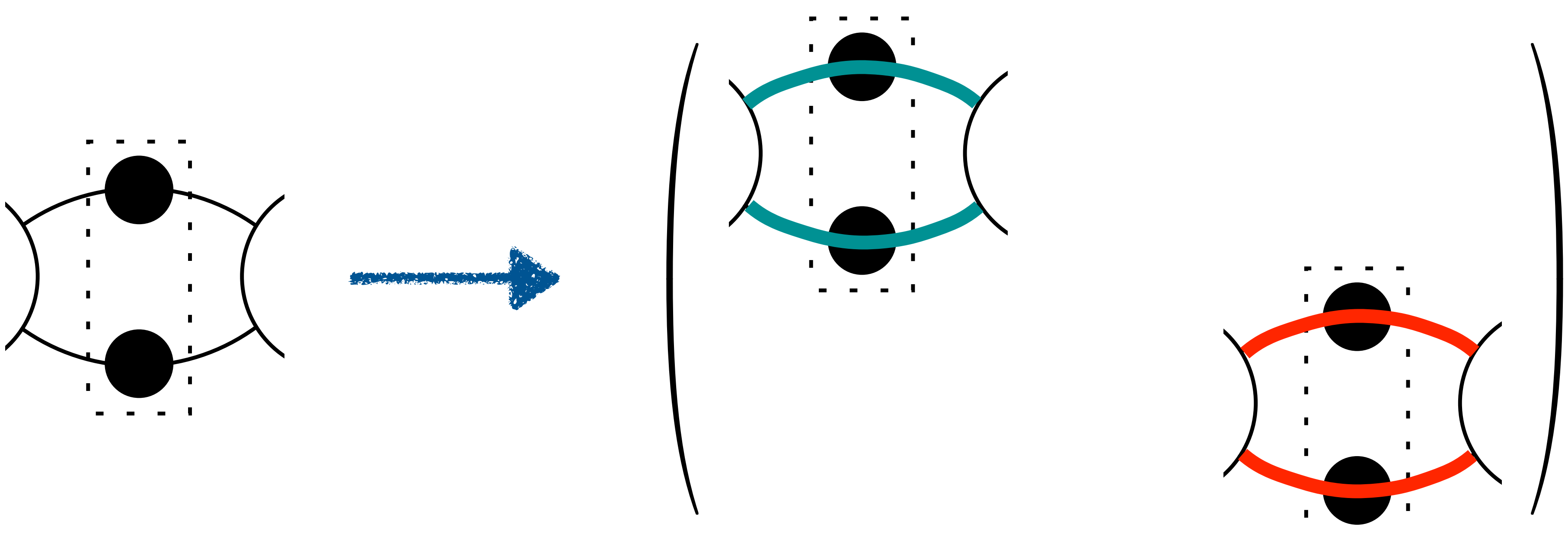


Scattering of multiple two-particle channels

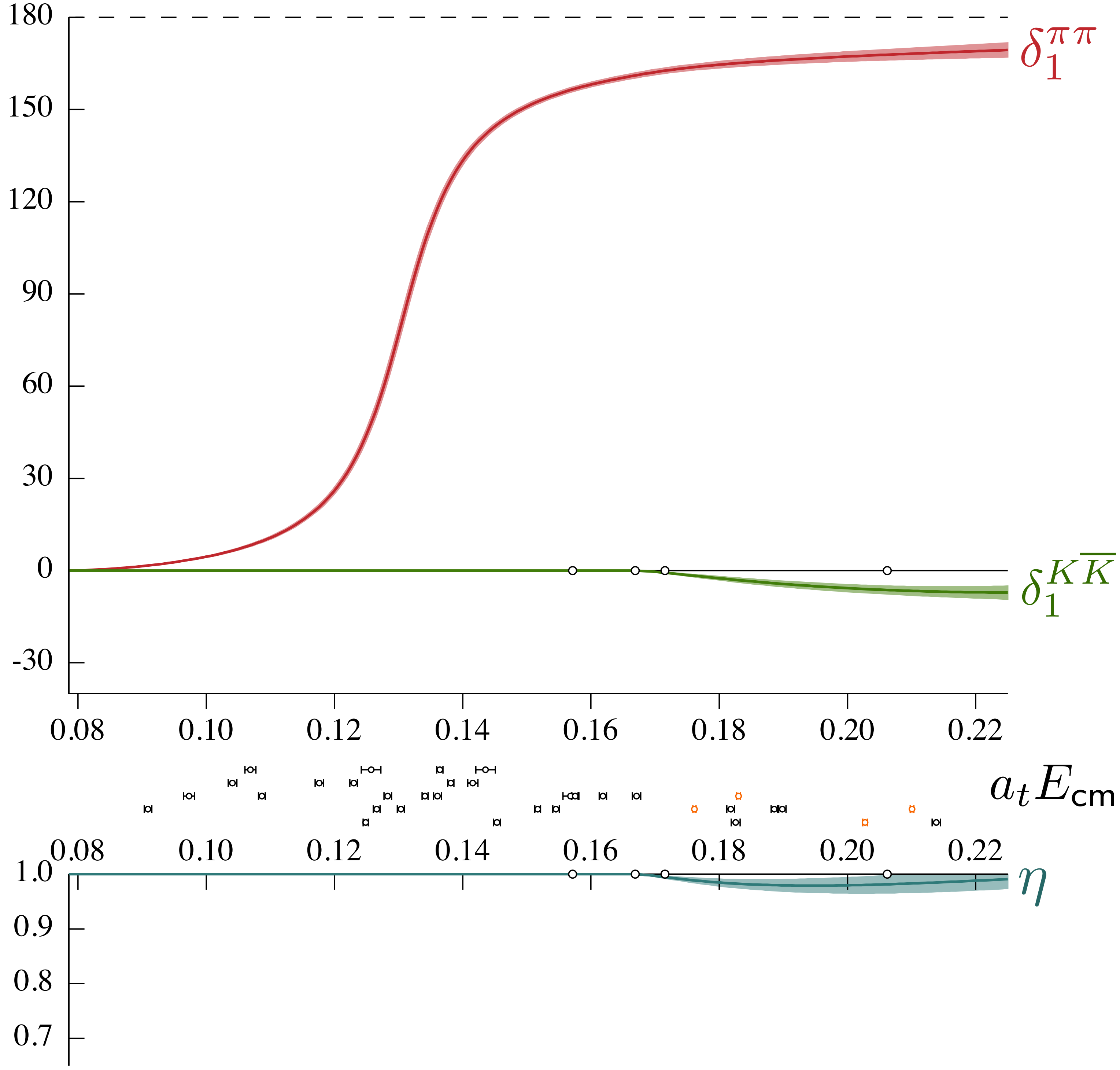
$$\pi\pi \to \overline{K}K \qquad \pi K \to \eta K$$

Make following replacements





And also for the rho meson



Wilson, Briceño, Dudek, Edwards, Thomas, arXiv:1507:02599

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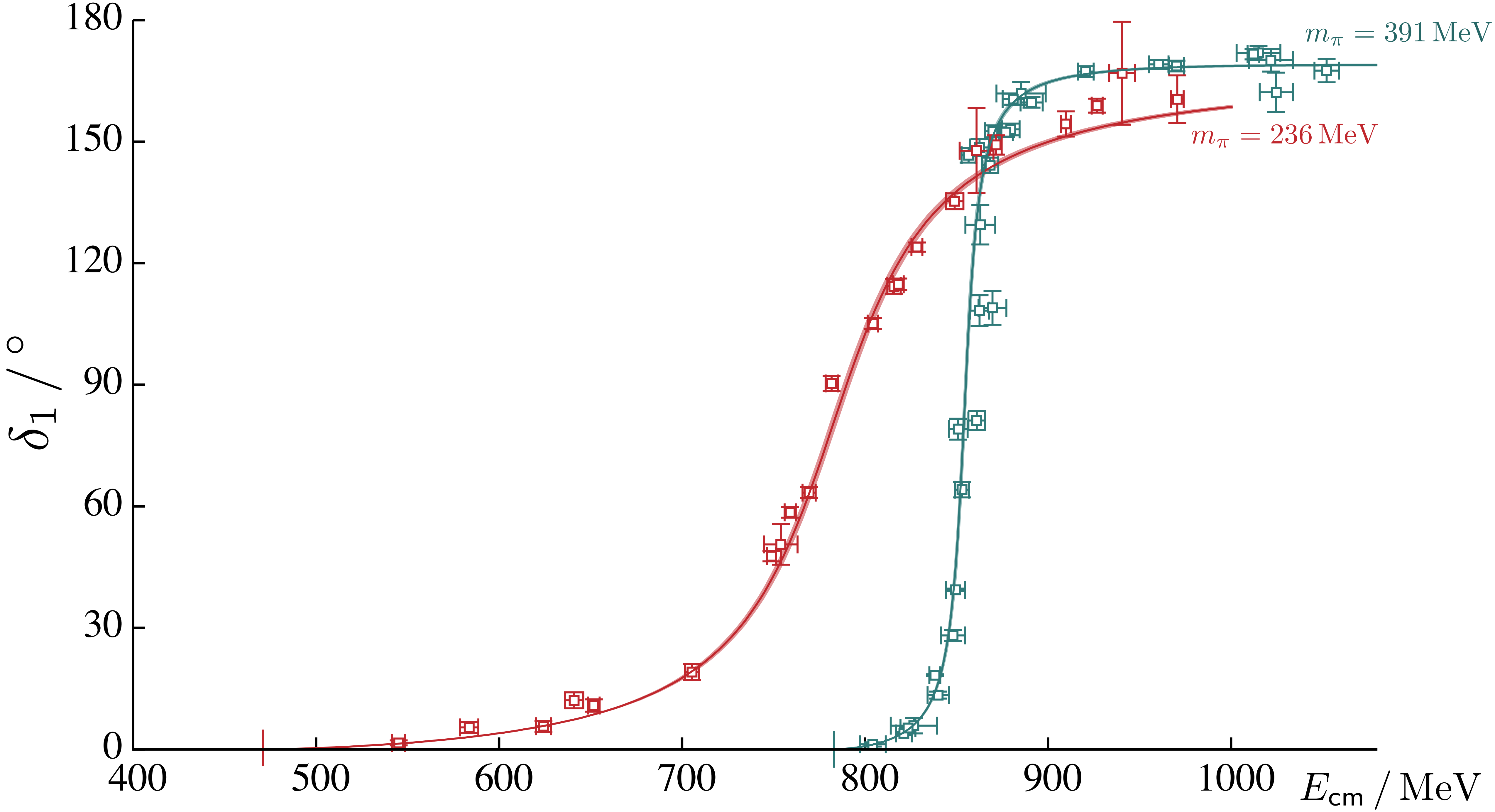

Wilson, Briceño, Dudek, Edwards, Thomas, arXiv:1507:02599