

Extracting scattering and resonance properties from the lattice

Maxwell T. Hansen

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Mainz, Germany

February 10th, 2016

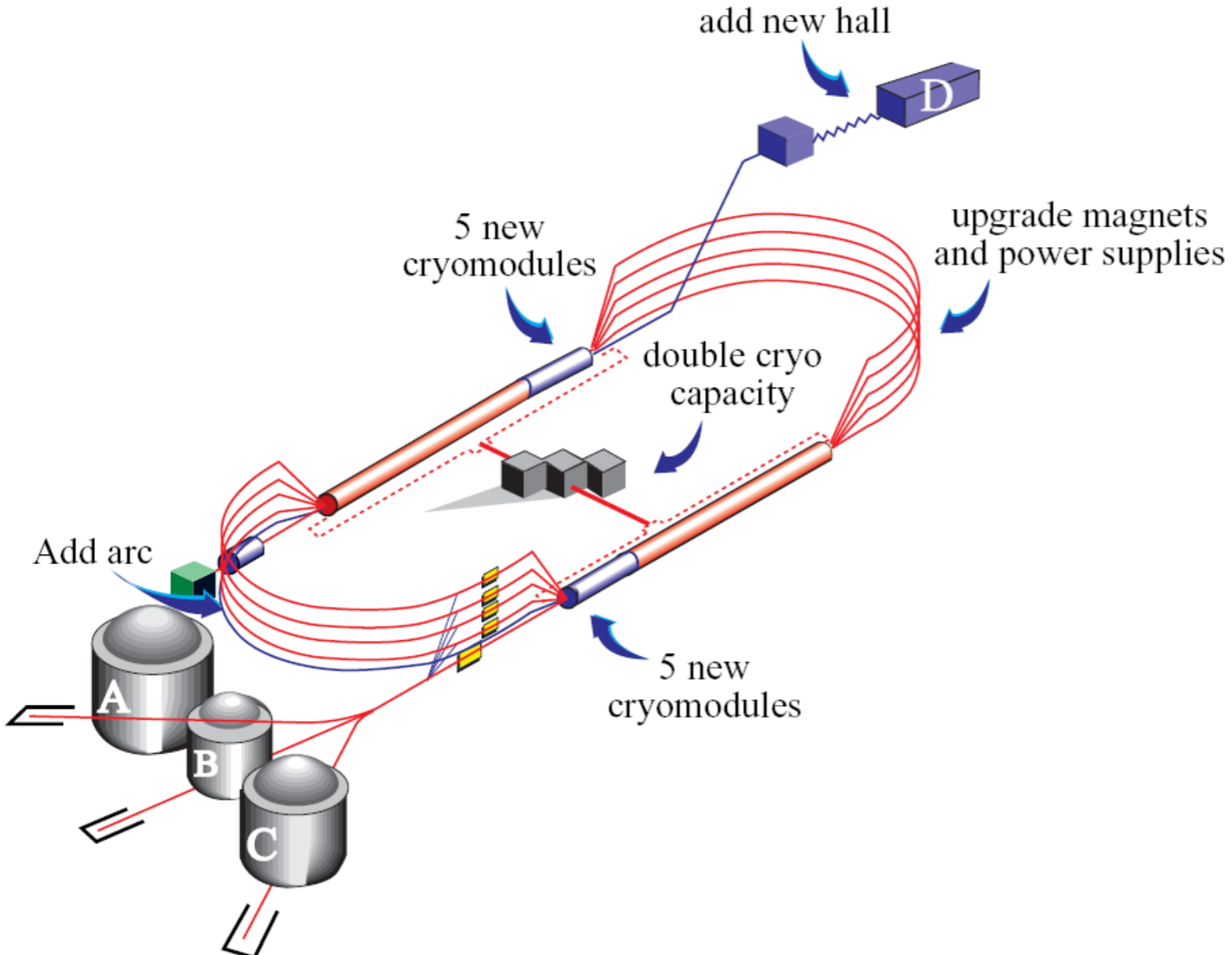


HIM

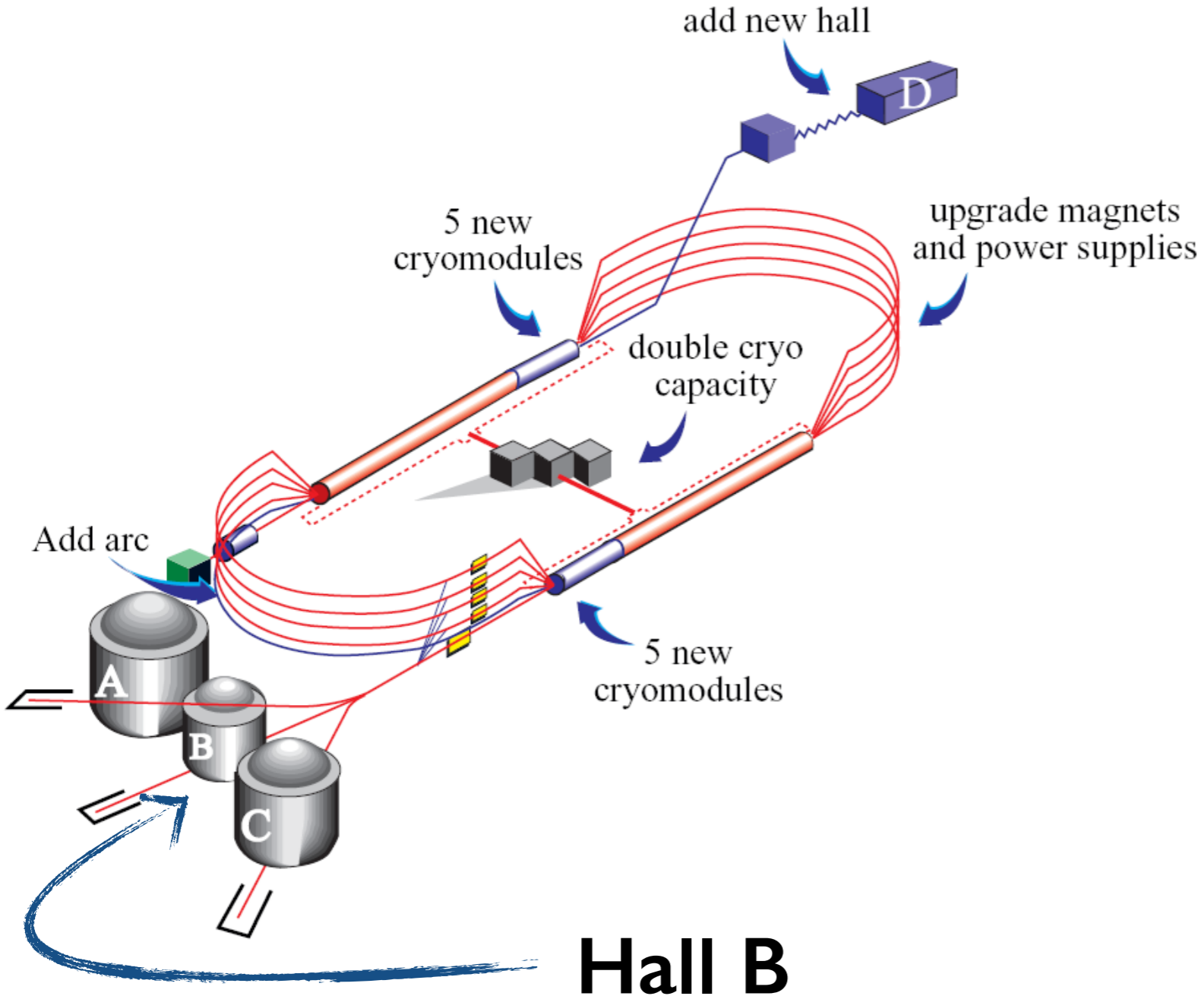
Helmholtz-Institut Mainz



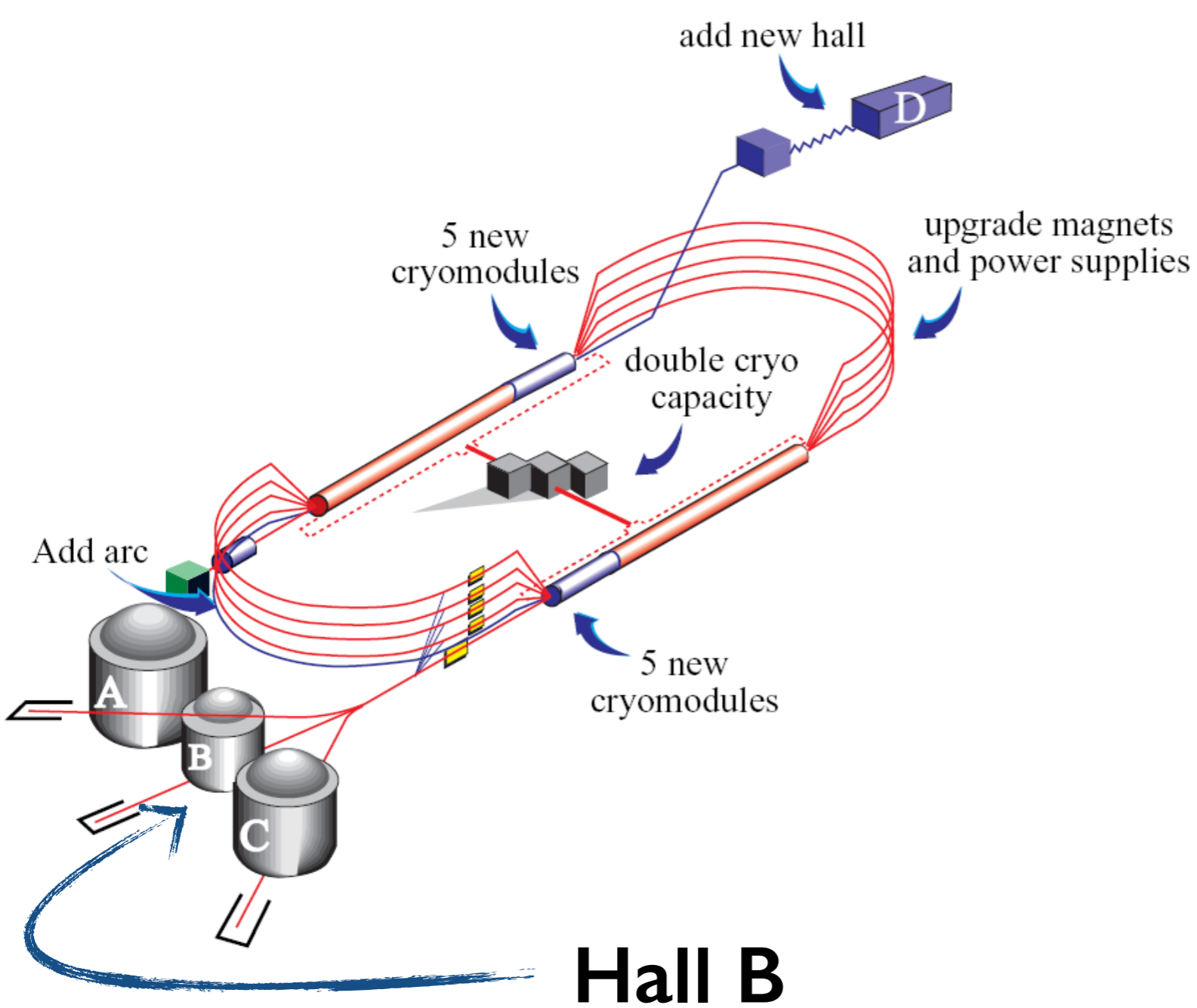
JLab Physics



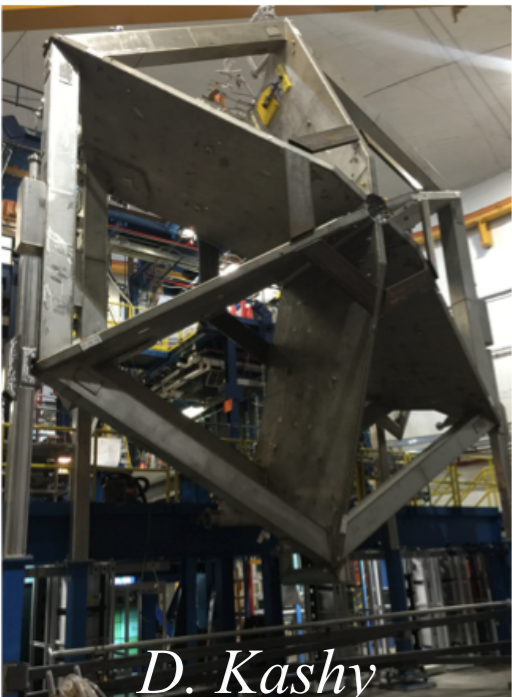
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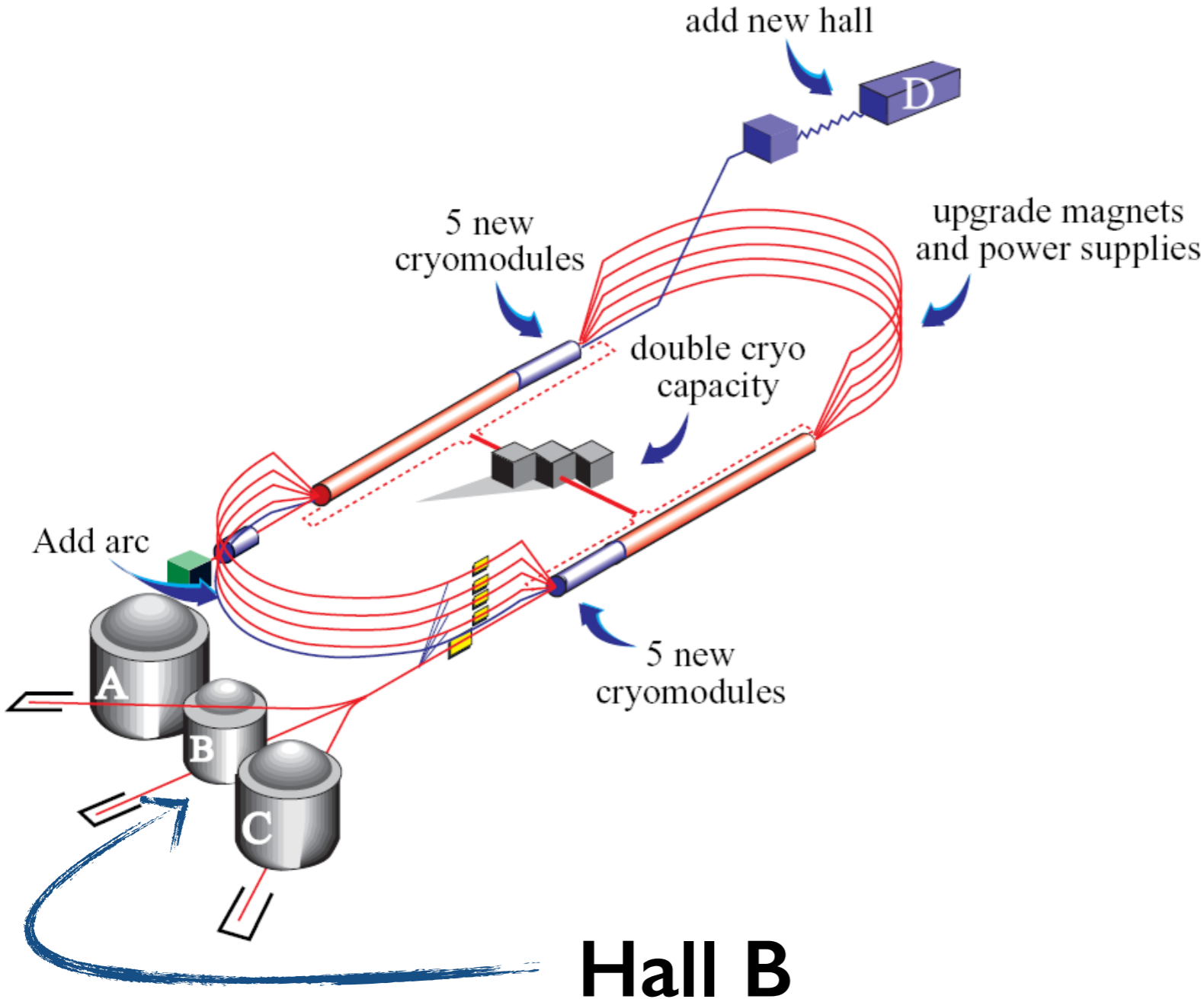


CLAS12 Torus Magnet complete



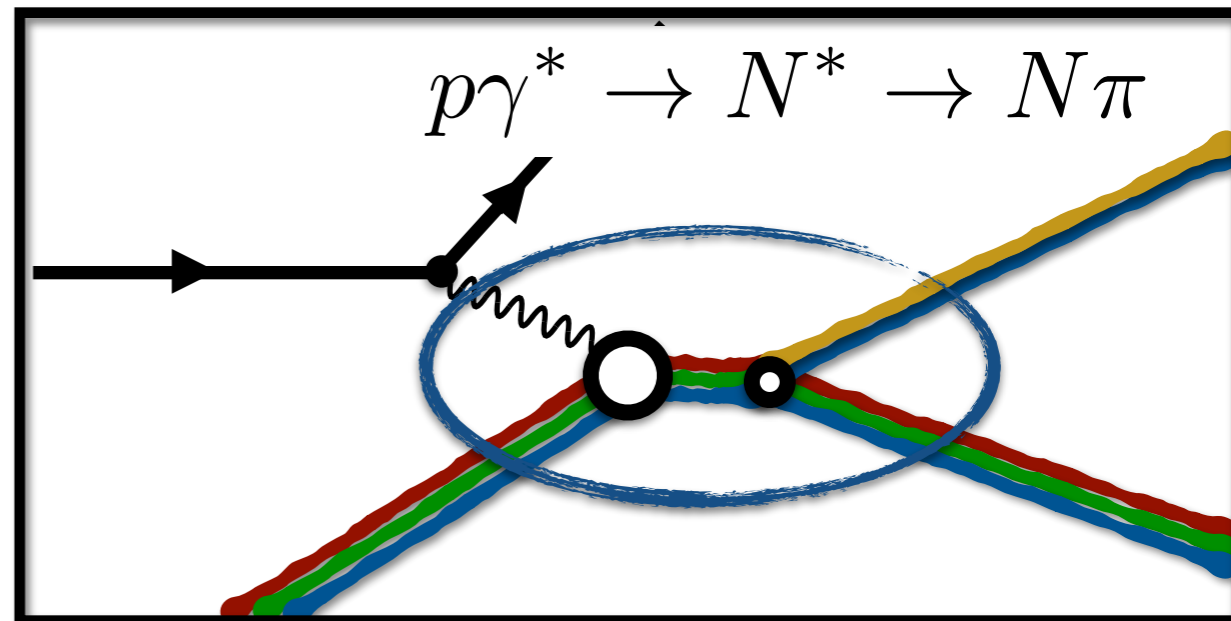
D. Kashy

JLab Physics



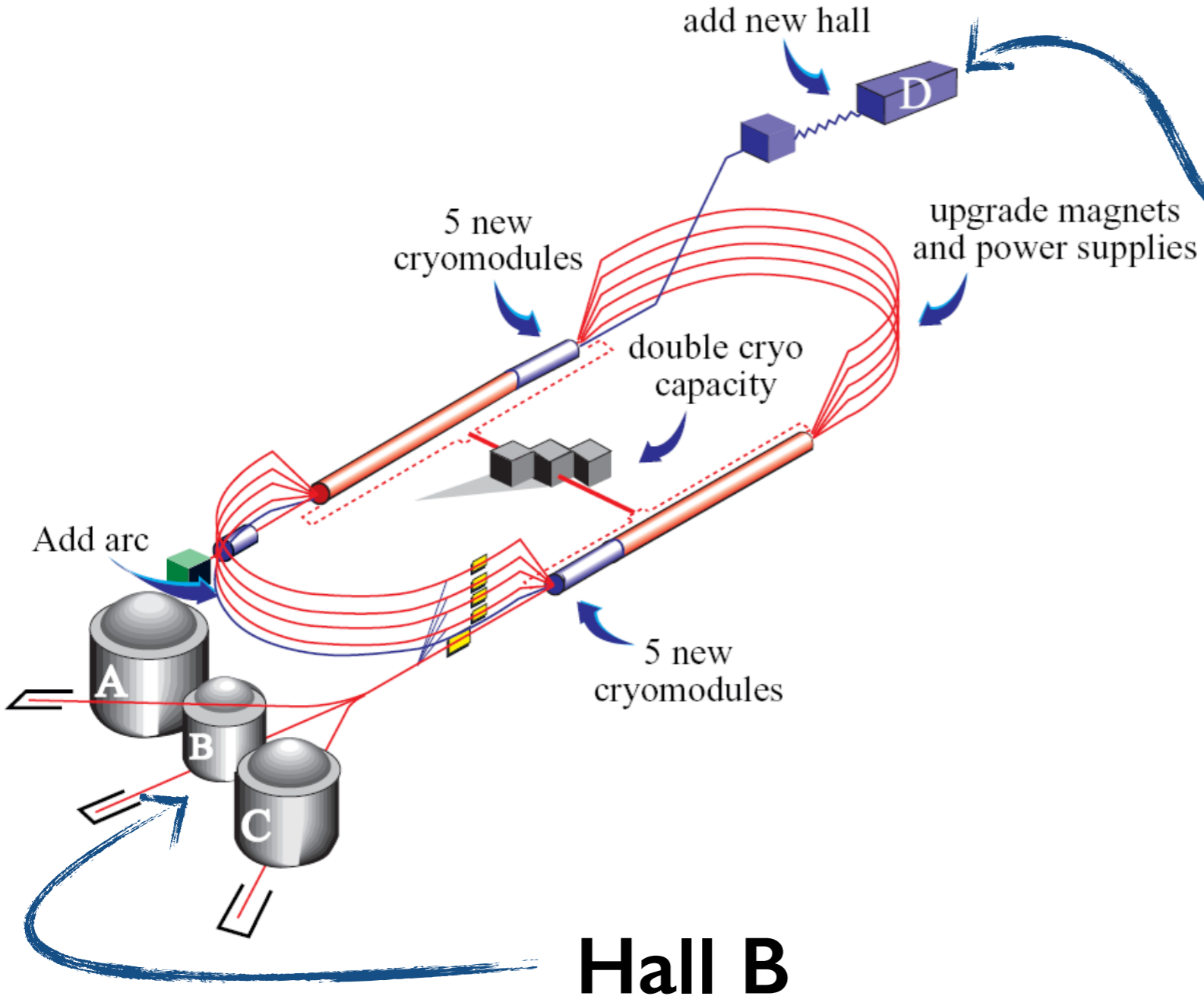
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Start taking data next year!



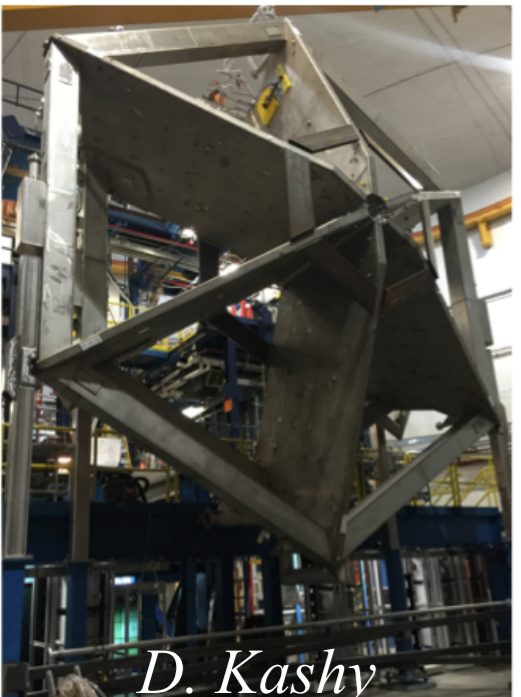
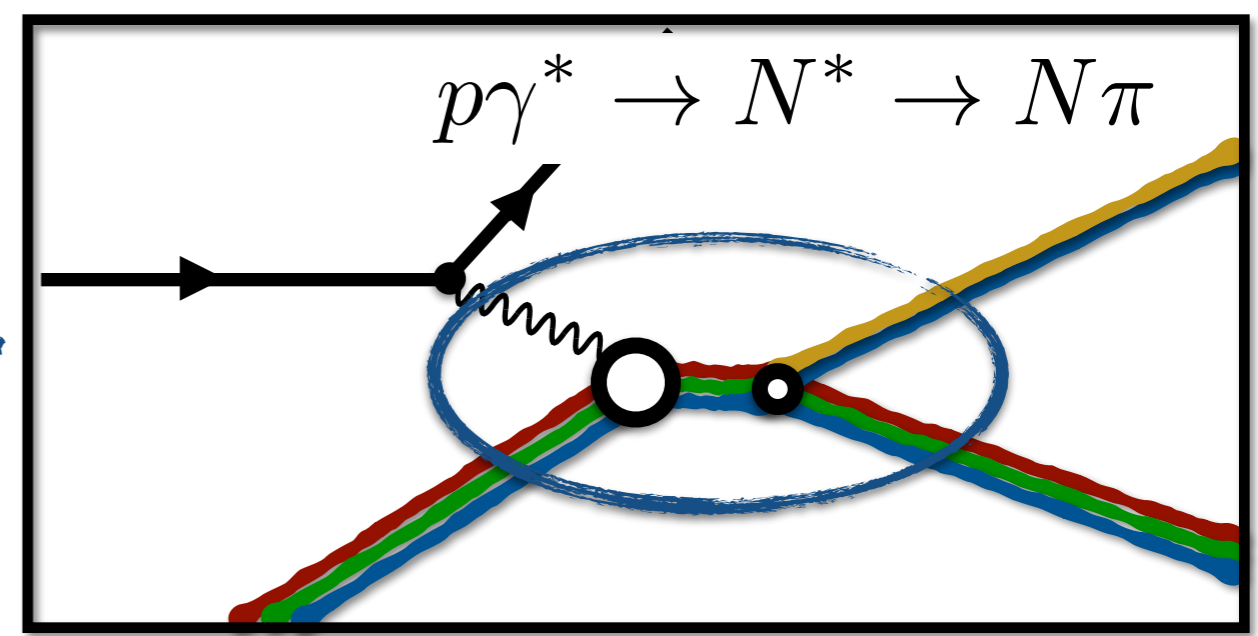
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Hall D (GlueX)



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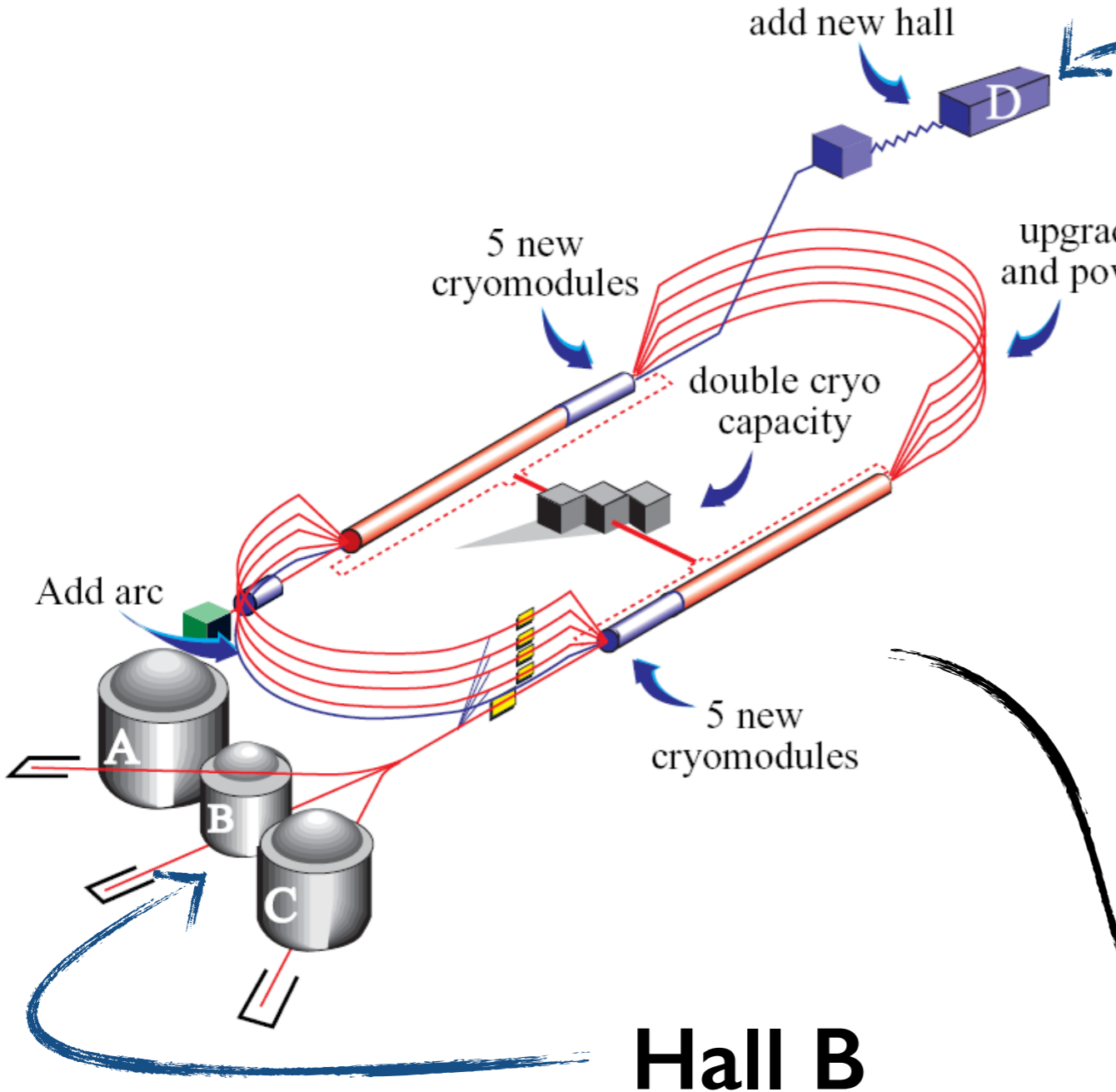
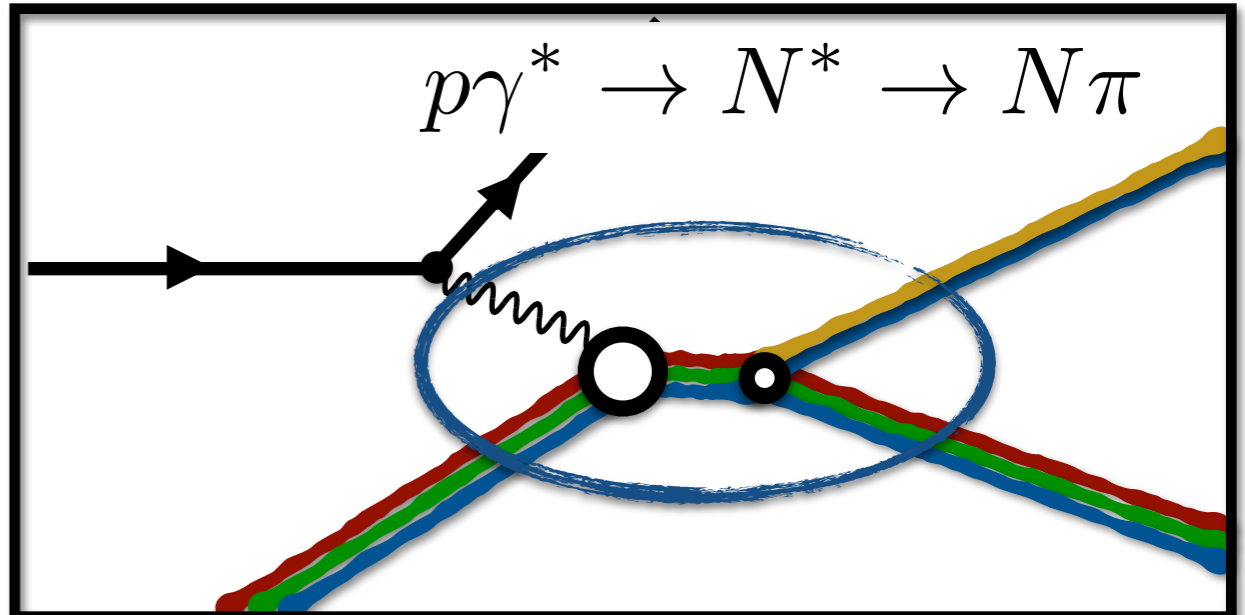
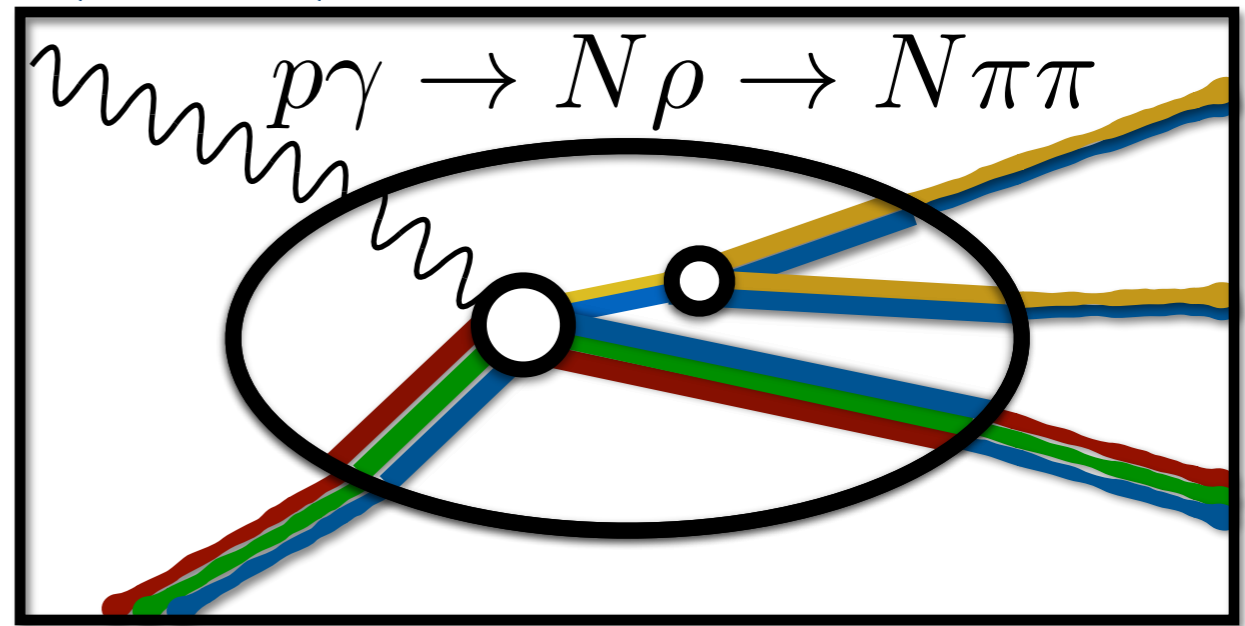
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JLab Physics

Hall D (GlueX)

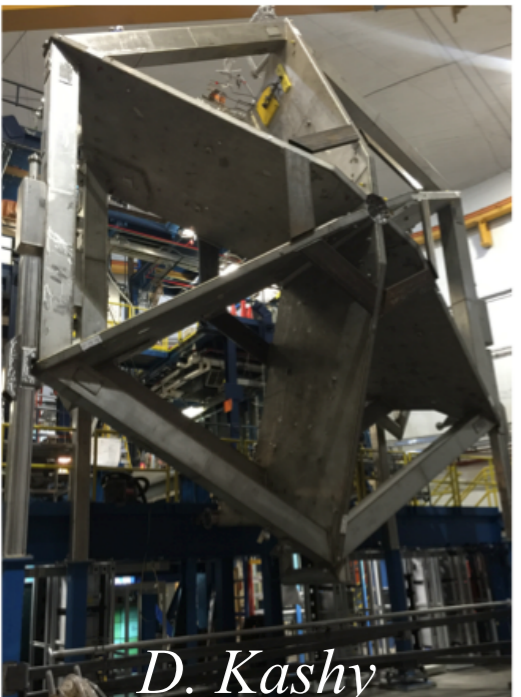
Observed polarized rho photoproduction in 2015!



Hall B

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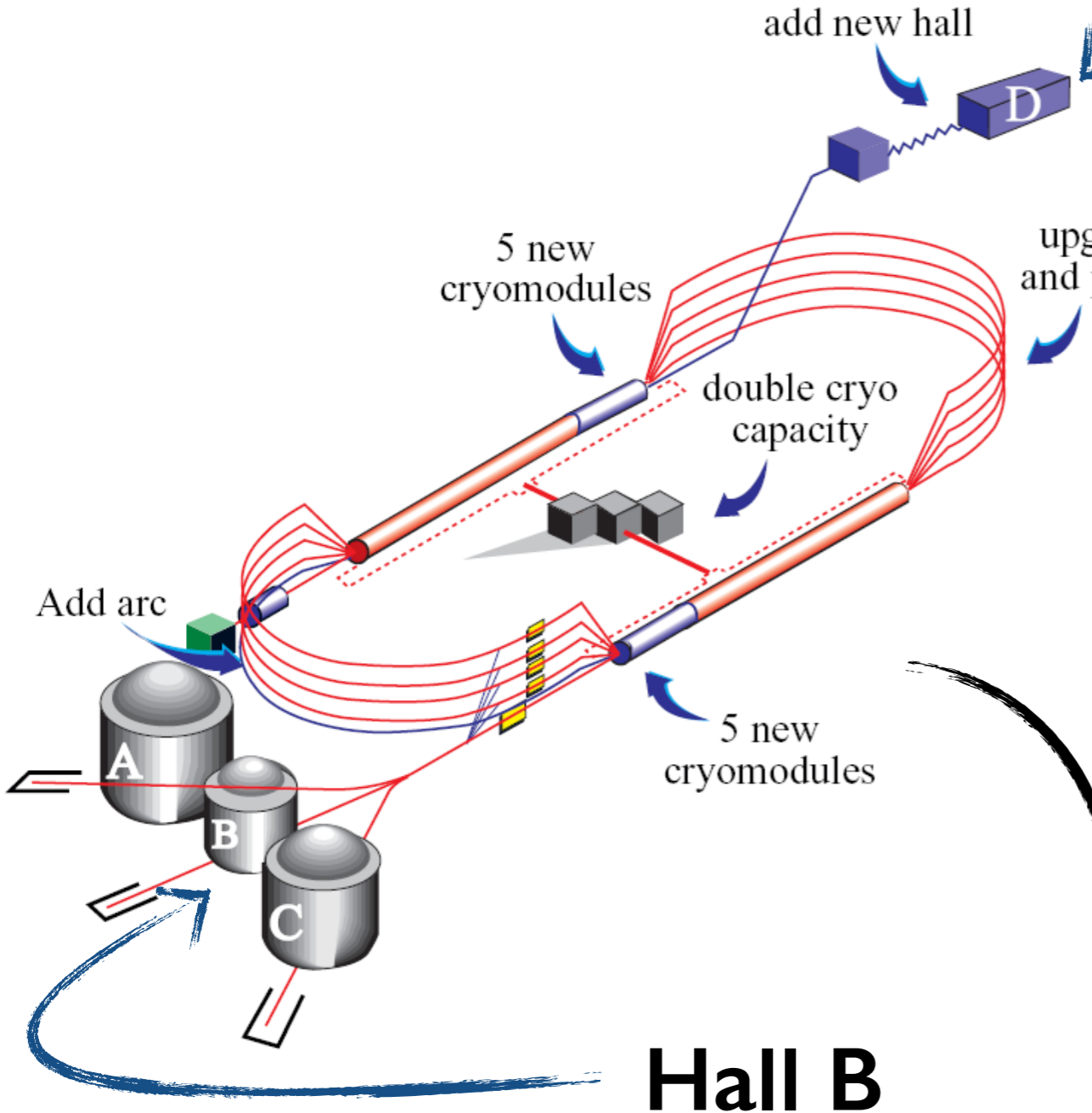
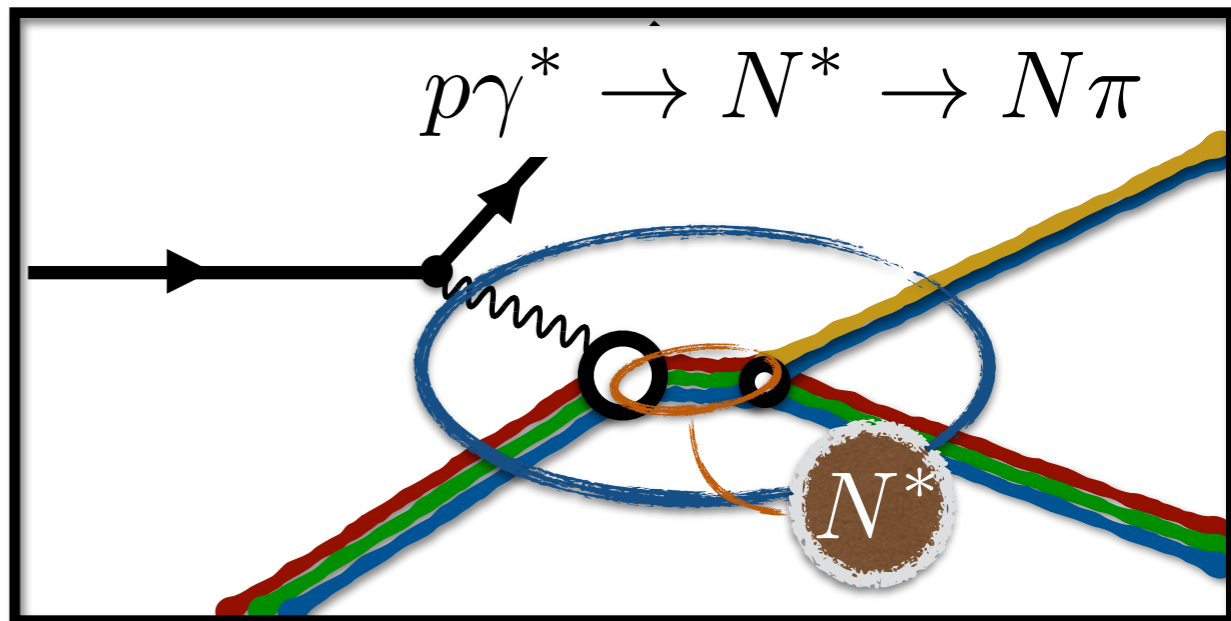
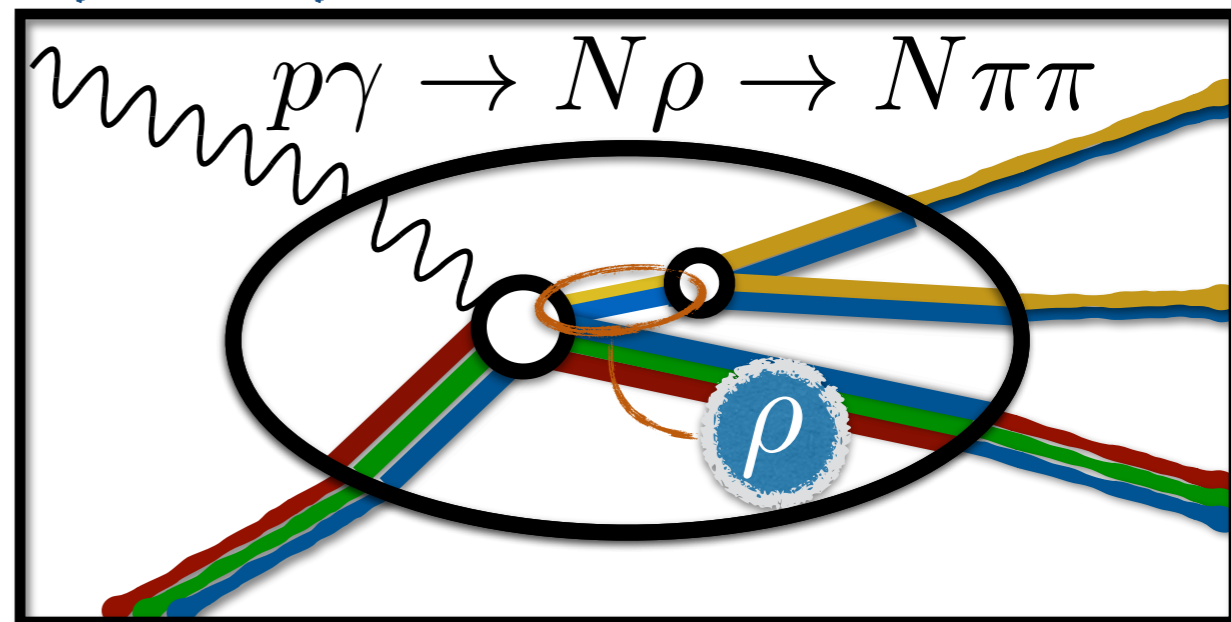


D. Kashy

JLab Physics

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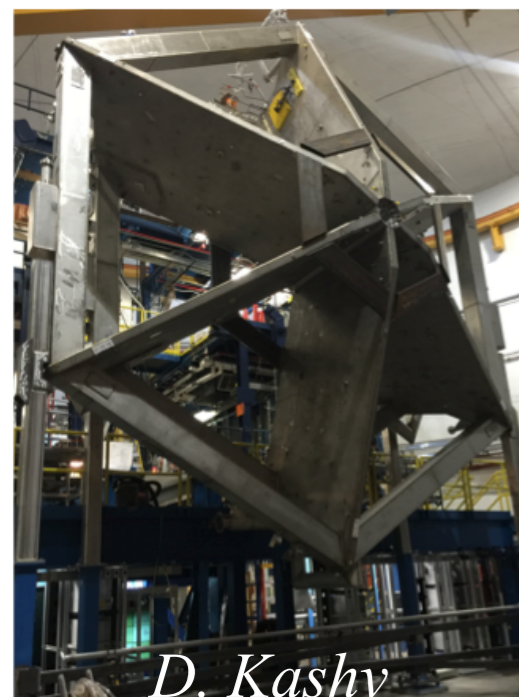
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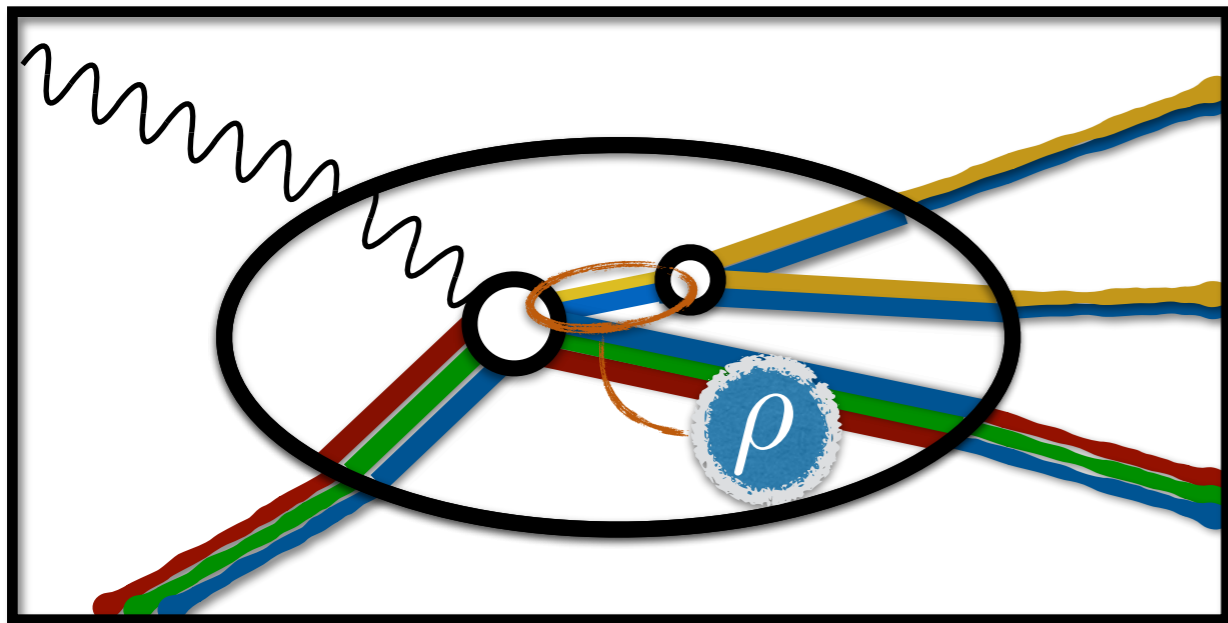
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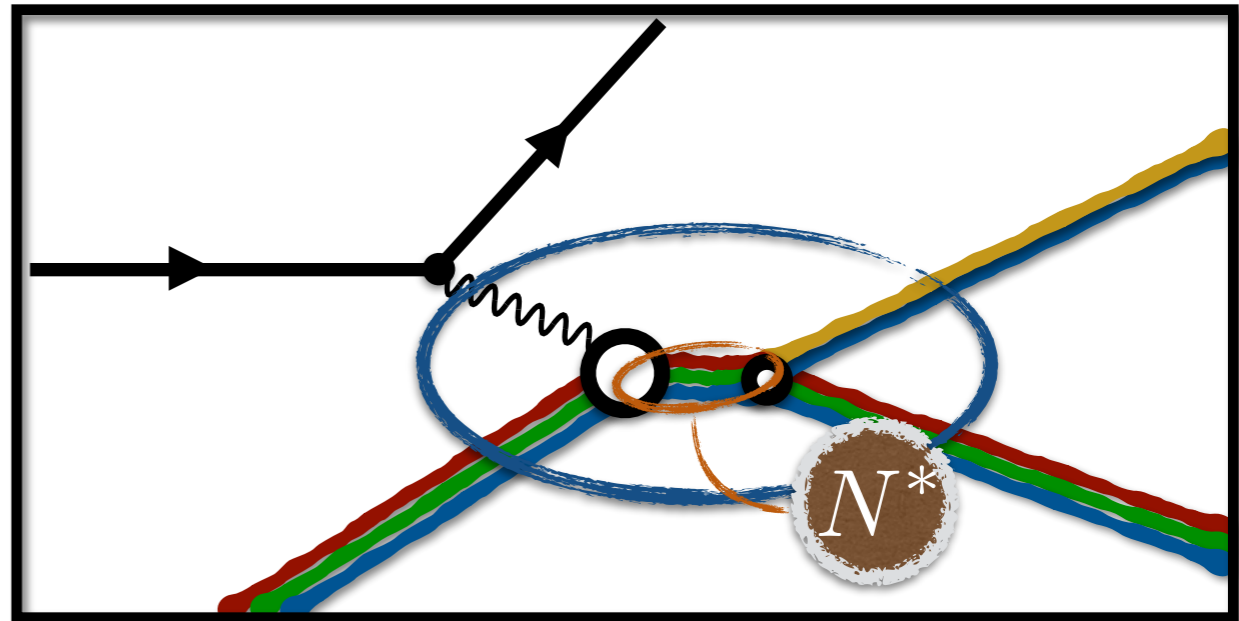
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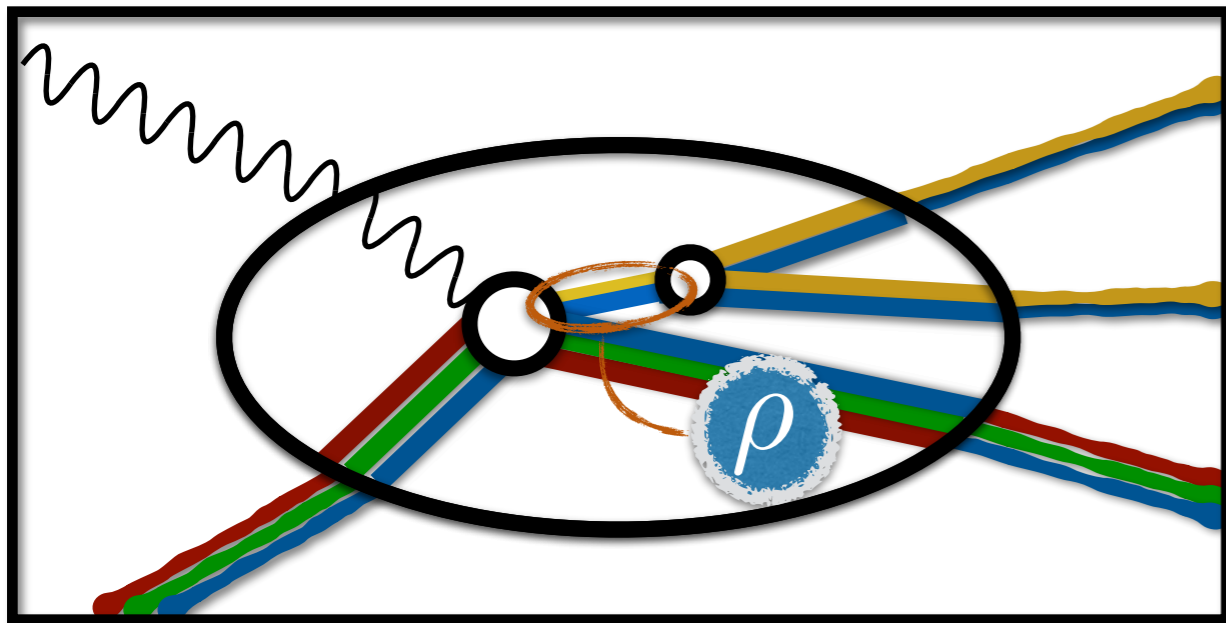
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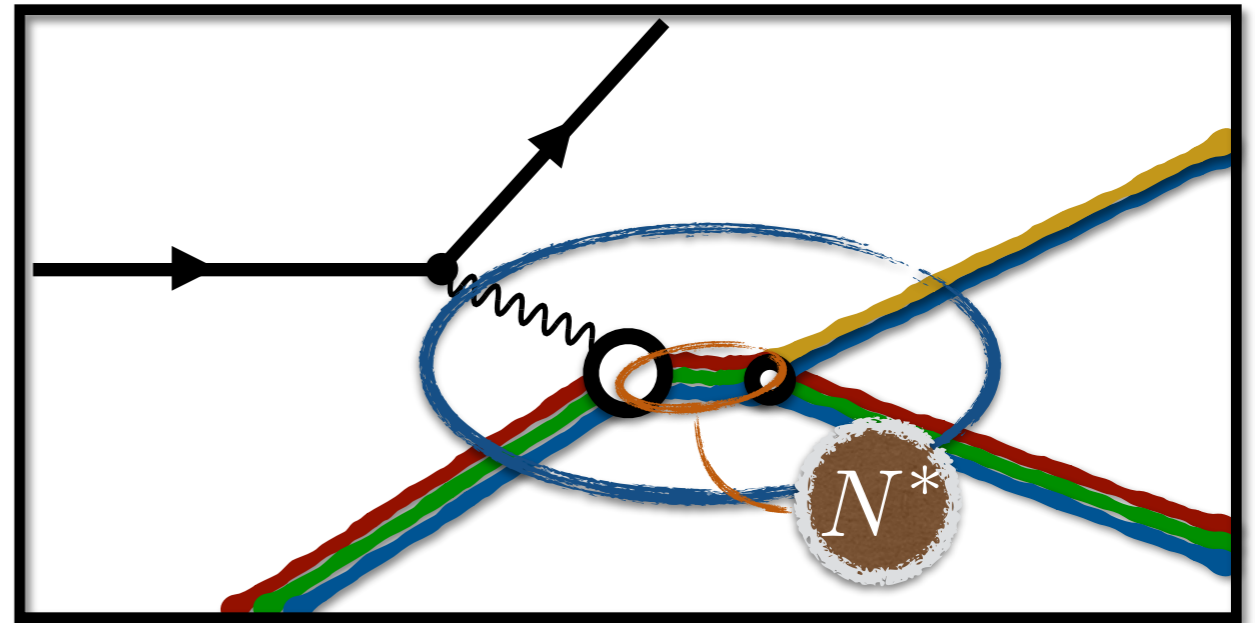
$$p\gamma \rightarrow N\rho \rightarrow N\pi\pi$$



$$p\gamma^* \rightarrow N^* \rightarrow N\pi, N\eta$$



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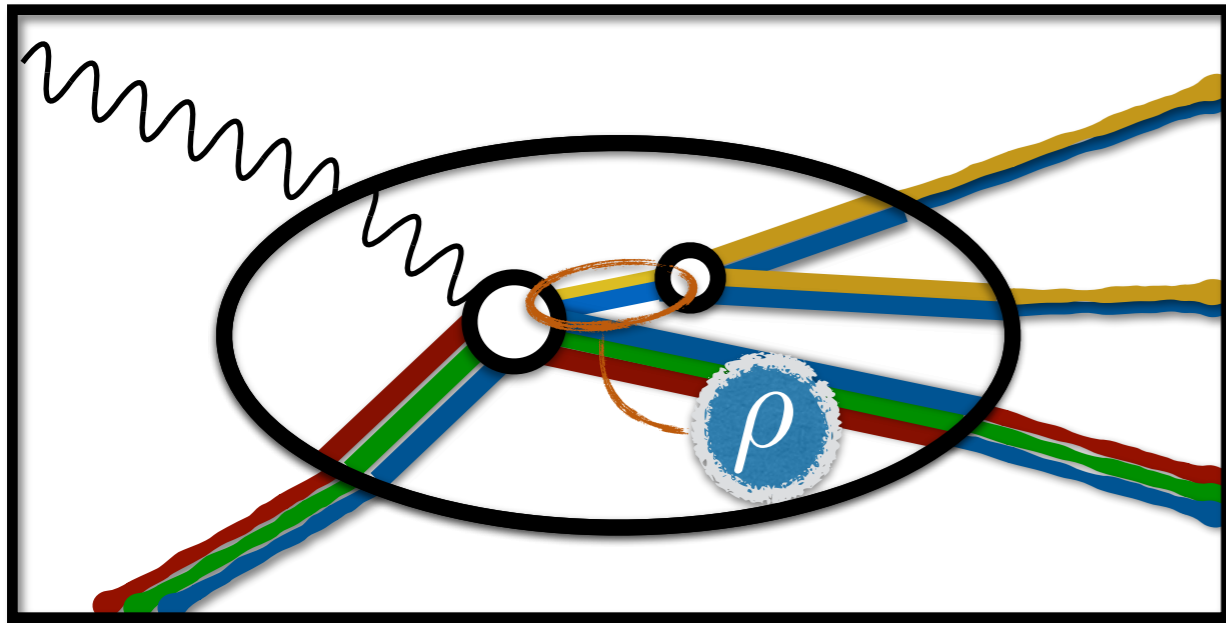


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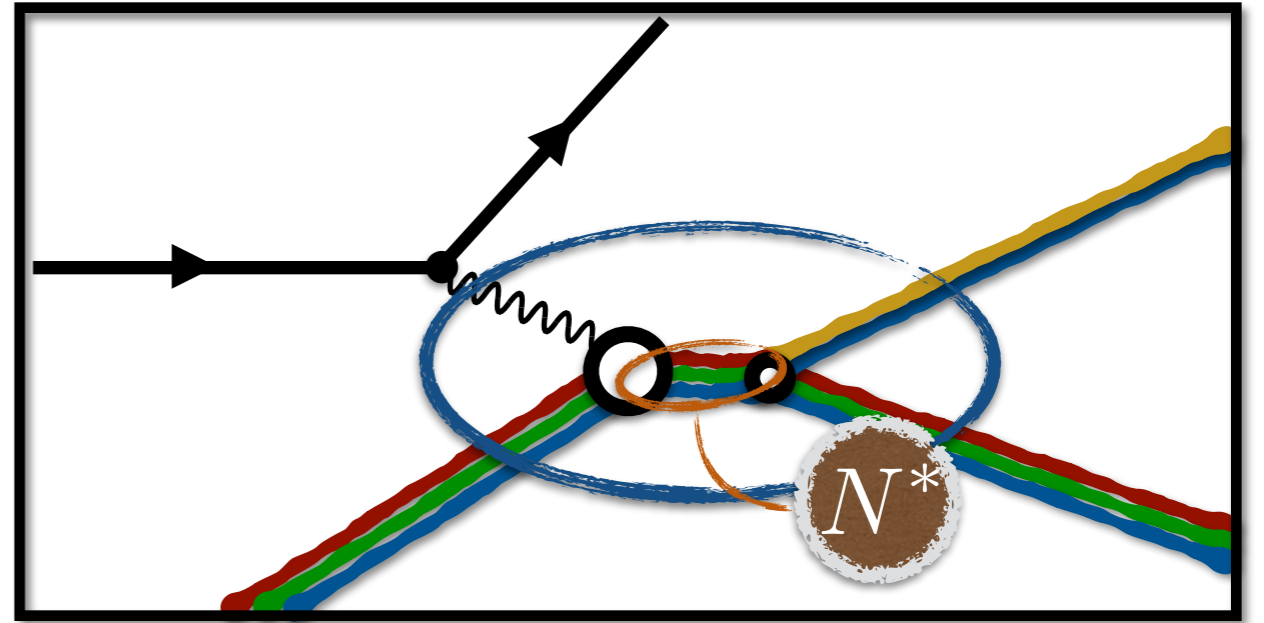
Resonances are not directly detected.

Outgoing hadrons are used to reconstruct resonance properties.

It is thus highly valuable to predict these transition amplitudes from the underlying theory of QCD



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It is thus highly valuable to predict these transition amplitudes from the underlying theory of QCD

Combining accurate, model-independent predictions with experiment will lead to a deeper understanding of QCD's rich resonance structure

What can we extract from the lattice?

We are trying to evaluate a difficult integral numerically

$$\text{observable} = \int \mathcal{D}\phi e^{iS} \left[\begin{array}{l} \text{interpolator} \\ \text{for observable} \end{array} \right]$$

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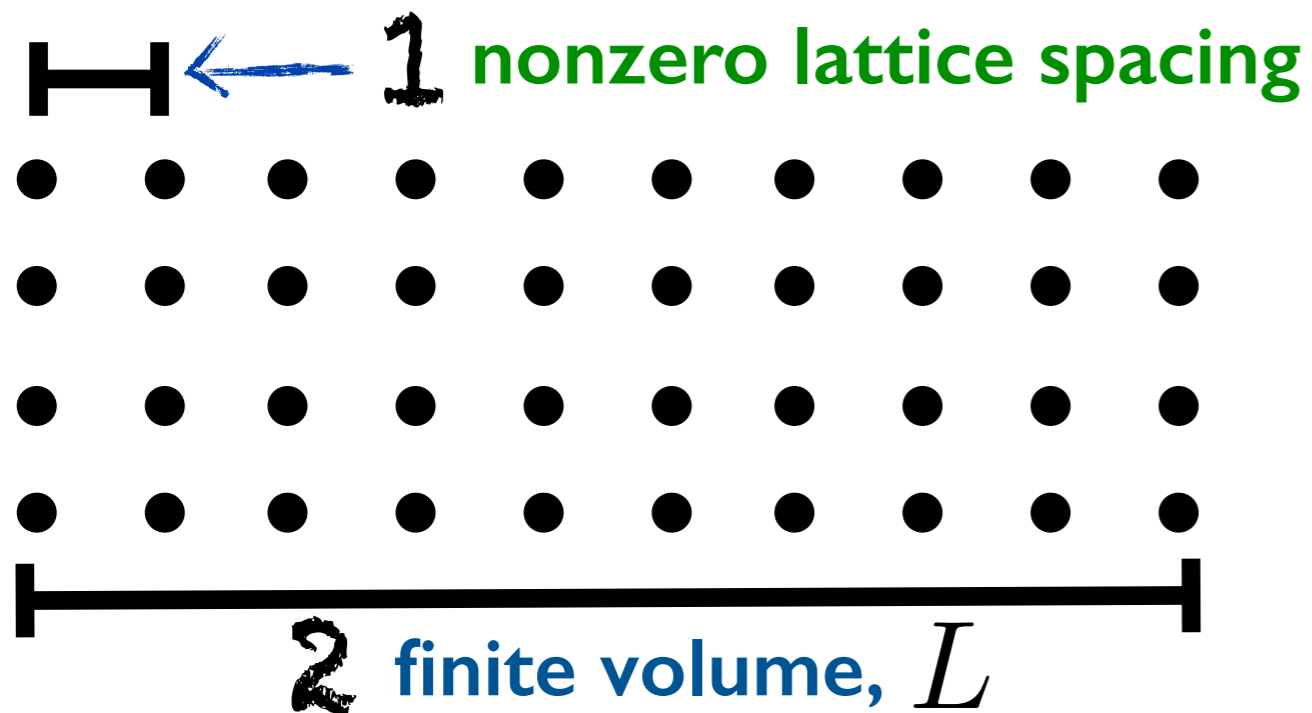
To do so we have to make four compromises

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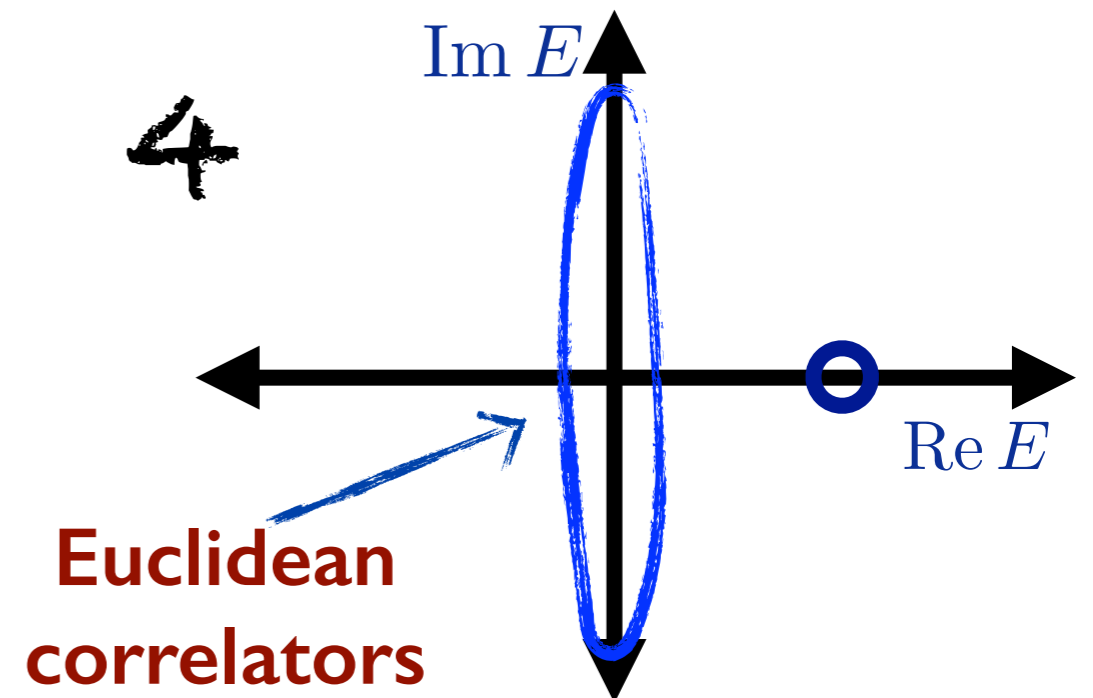
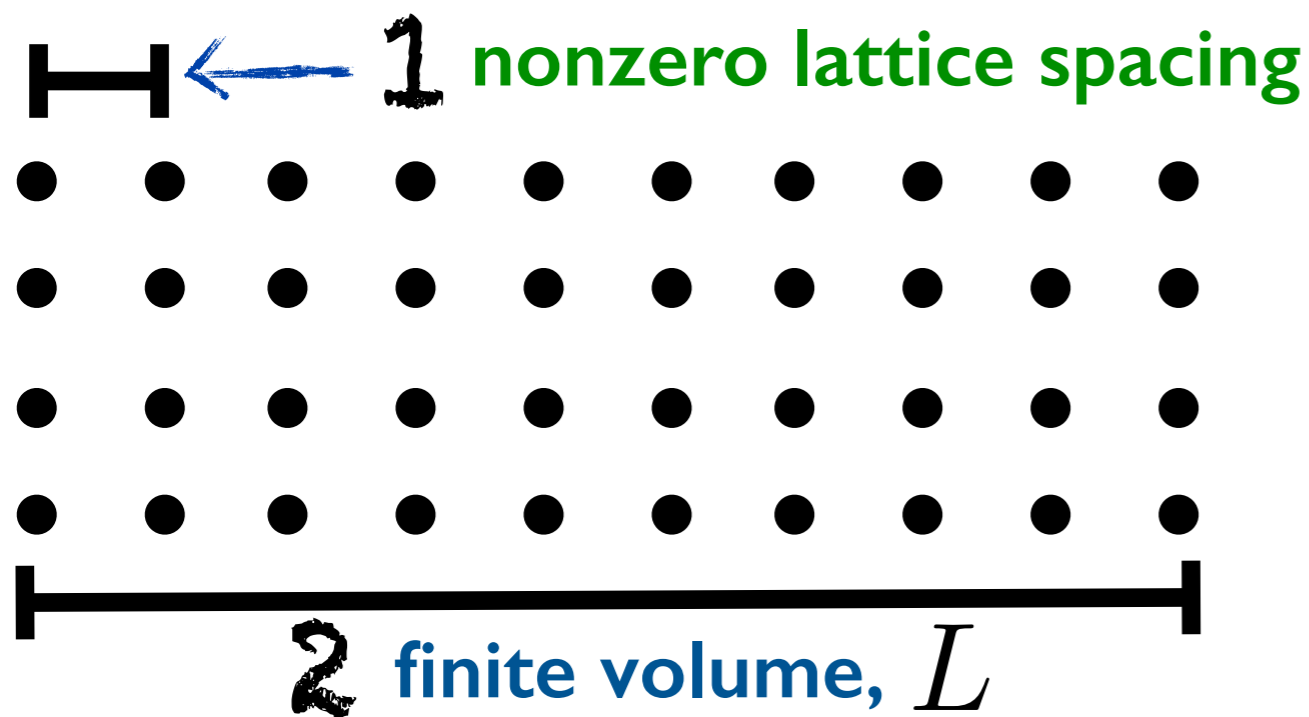


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3 Unphysical pion masses $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

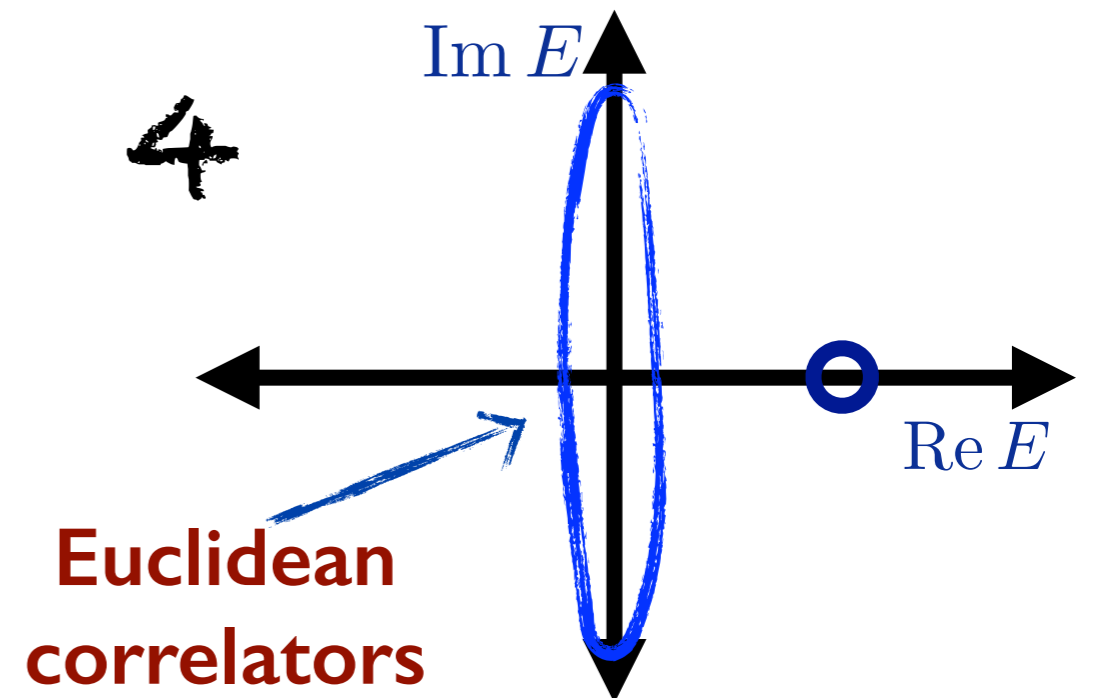
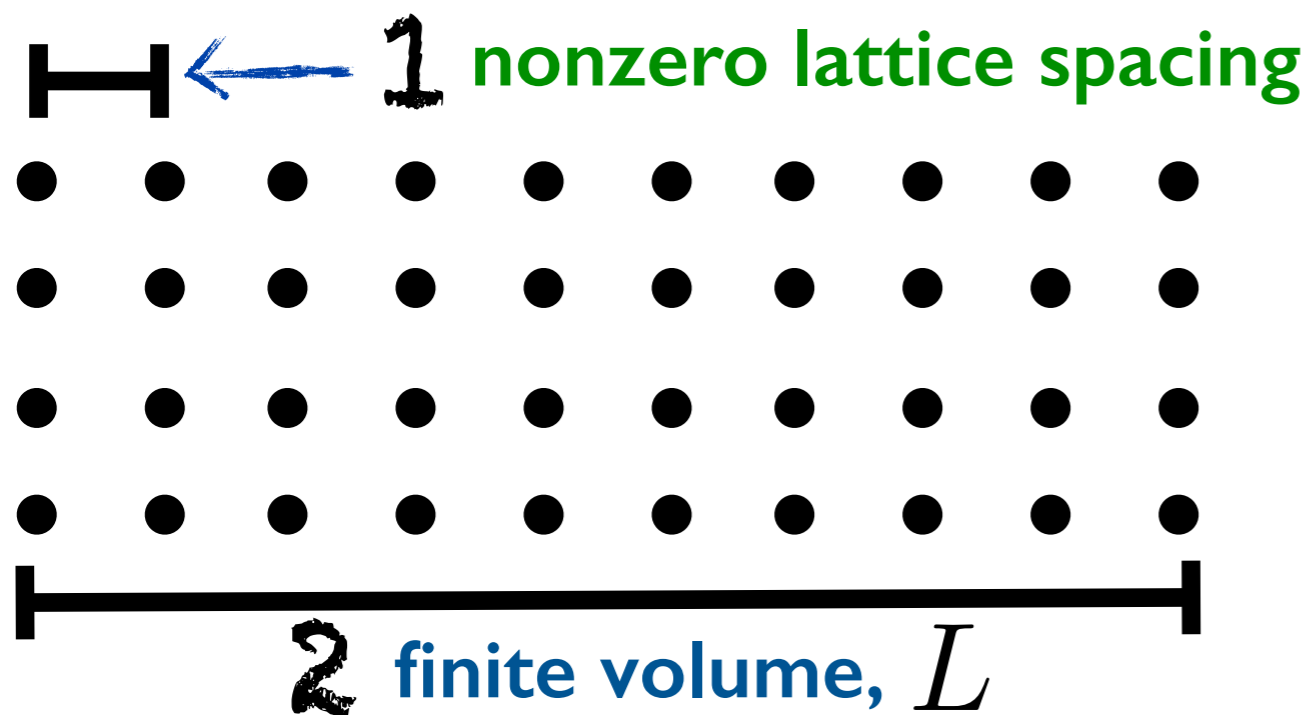
But calculations at the physical pion mass do now exist

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$$\left(\begin{array}{l} \text{observable?} \\ \text{discretized, finite volume,} \\ \text{Euclidean, heavy pions} \end{array} \right) = \int \prod_i^N d\phi_i e^{-S} \left[\begin{array}{l} \text{interpolator} \\ \text{for observable} \end{array} \right]$$

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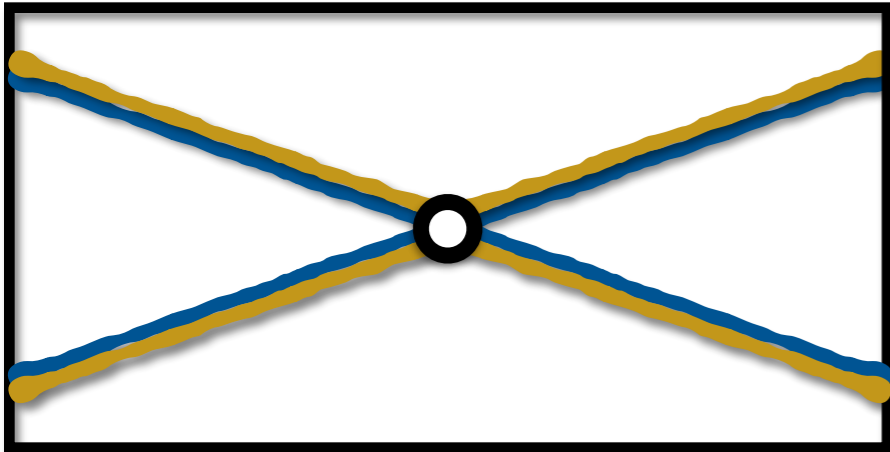


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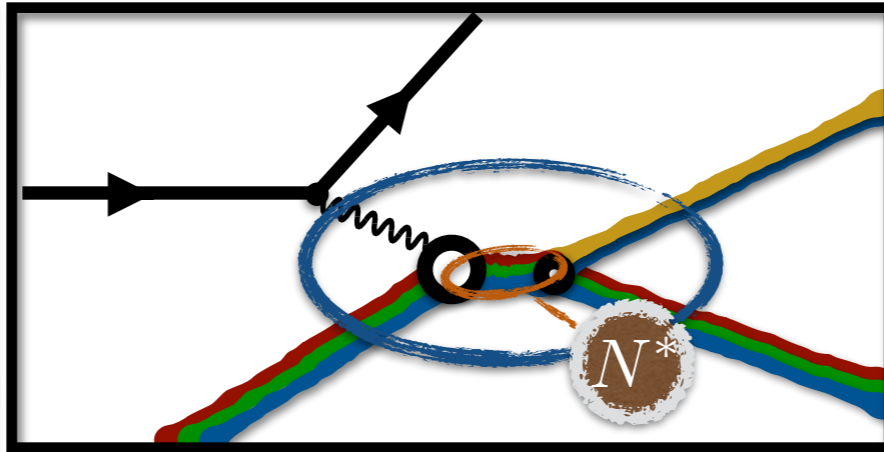
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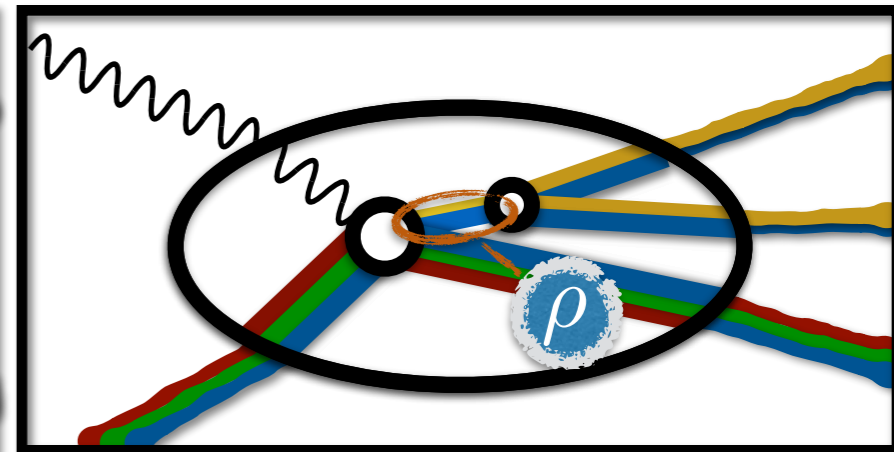
Not possible to directly calculate



$$\langle \pi\pi | \pi\pi \rangle$$



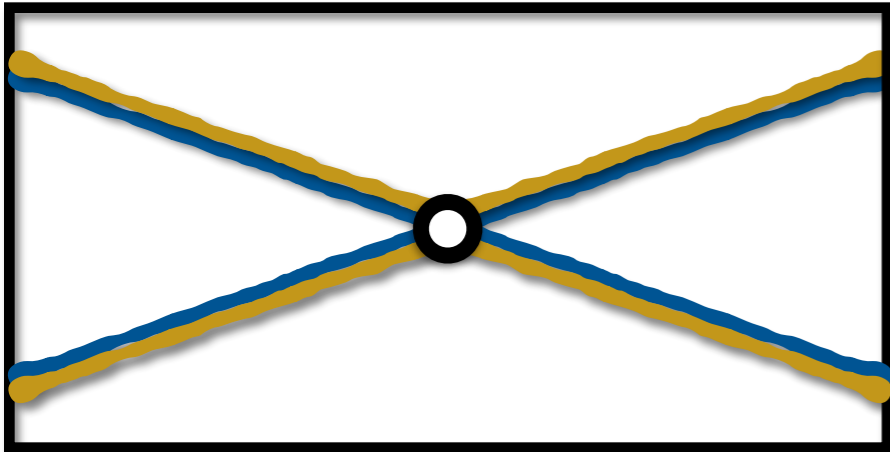
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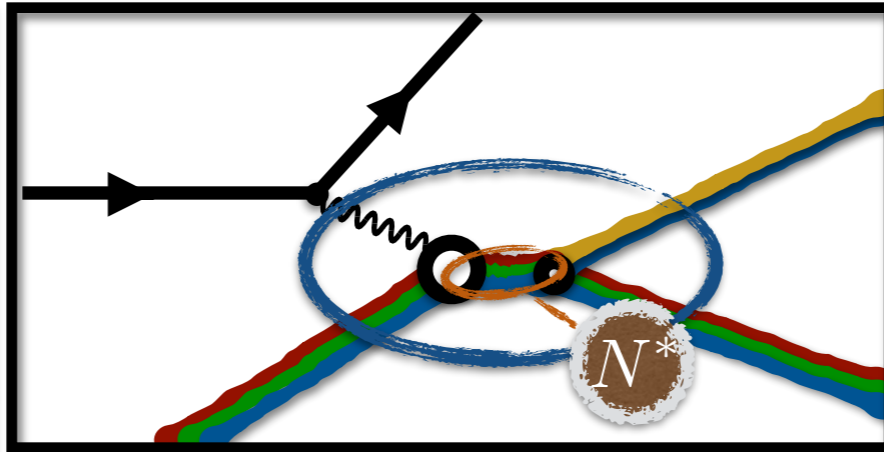
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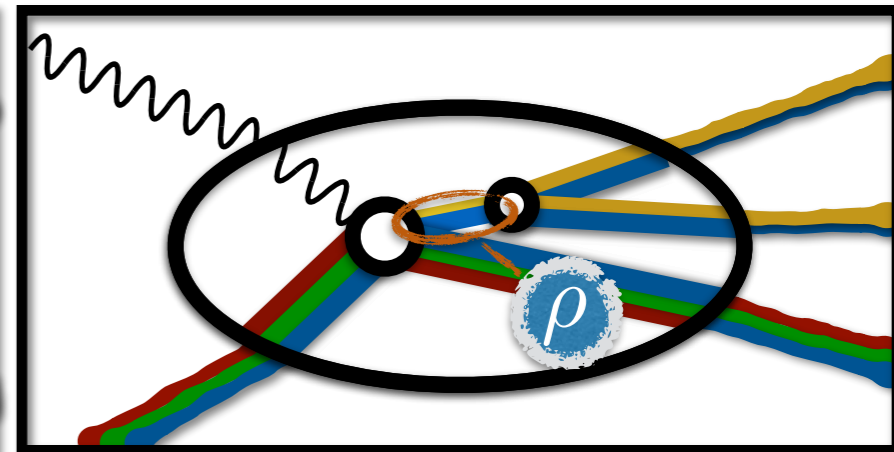
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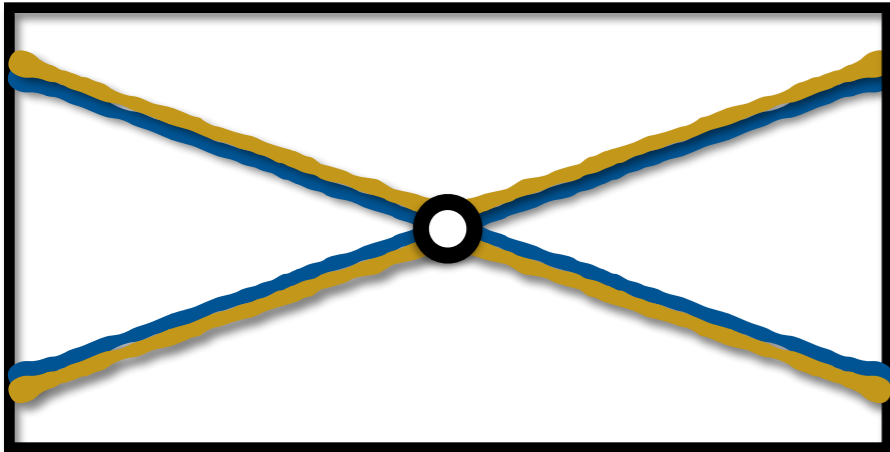


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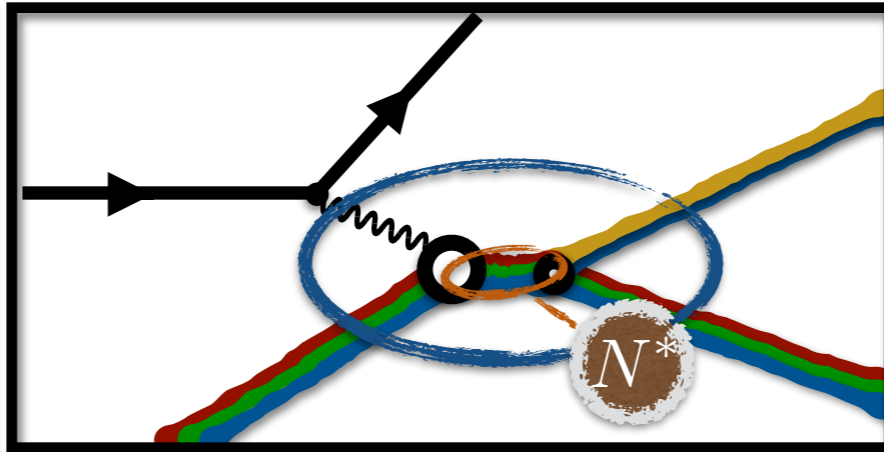
multi-particle in- and outstates

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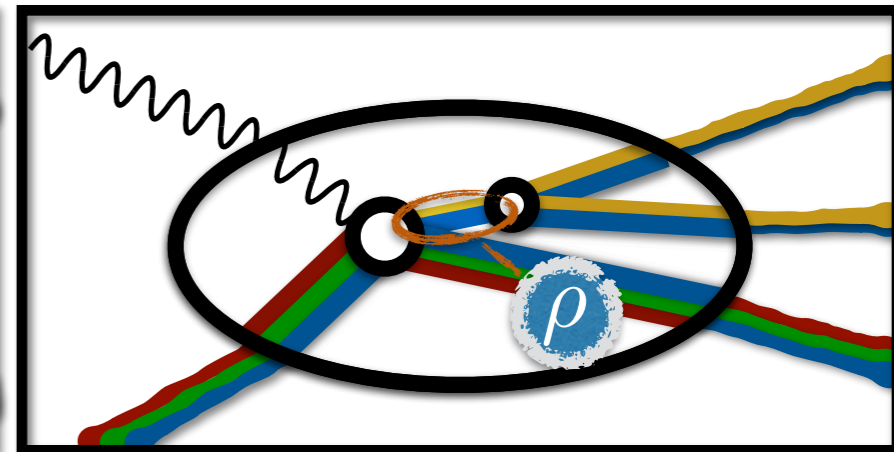
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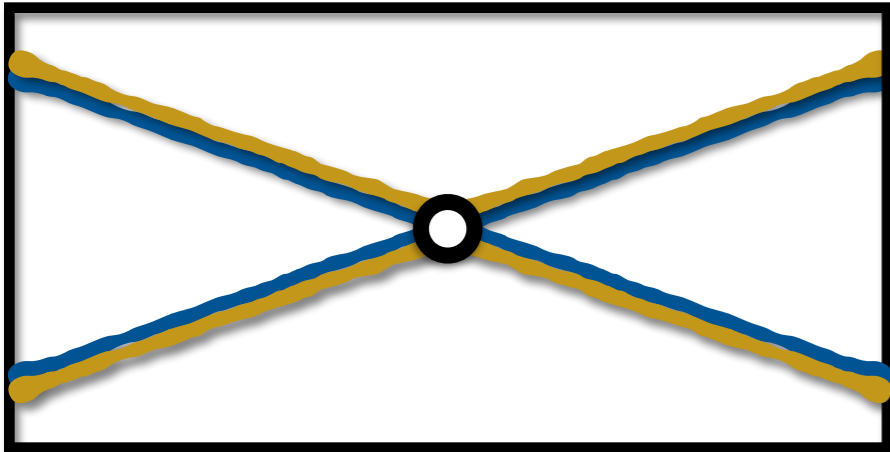
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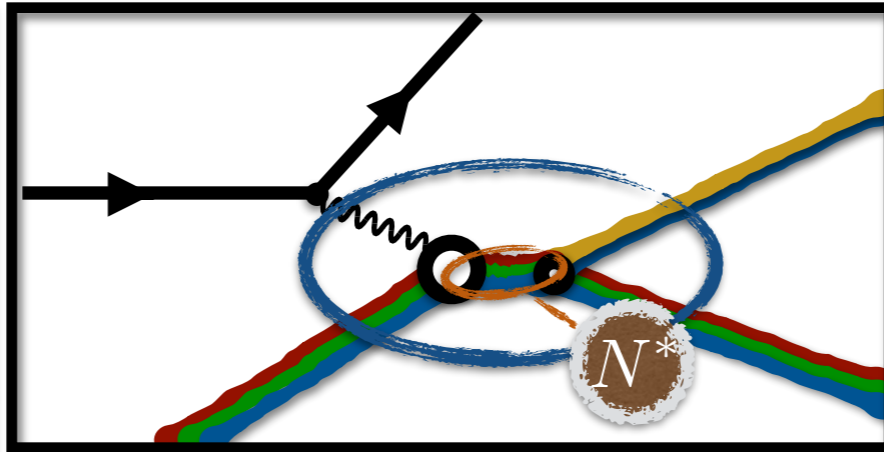
$$\langle \pi\pi, \text{out} | \pi\pi, \text{in} \rangle = \frac{\text{amputate and put on-shell}}{\langle 0 | \tilde{\pi}(p') \tilde{\pi}(k') \tilde{\pi}(p) \tilde{\pi}(k) | 0 \rangle}$$

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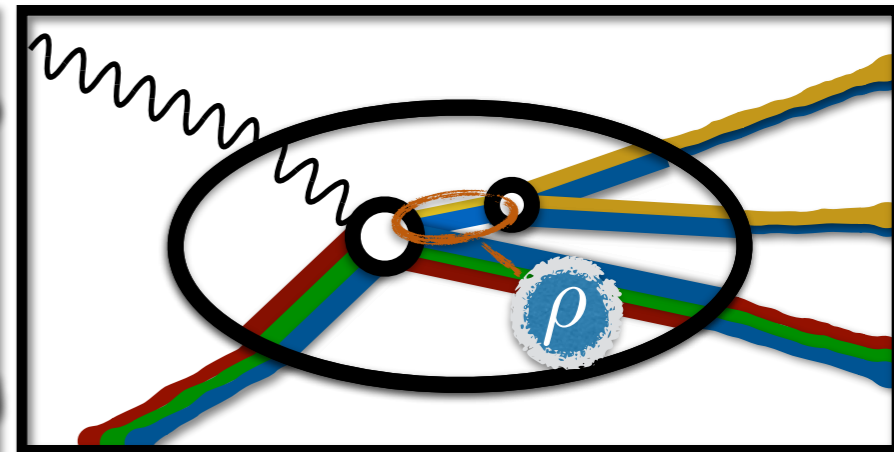
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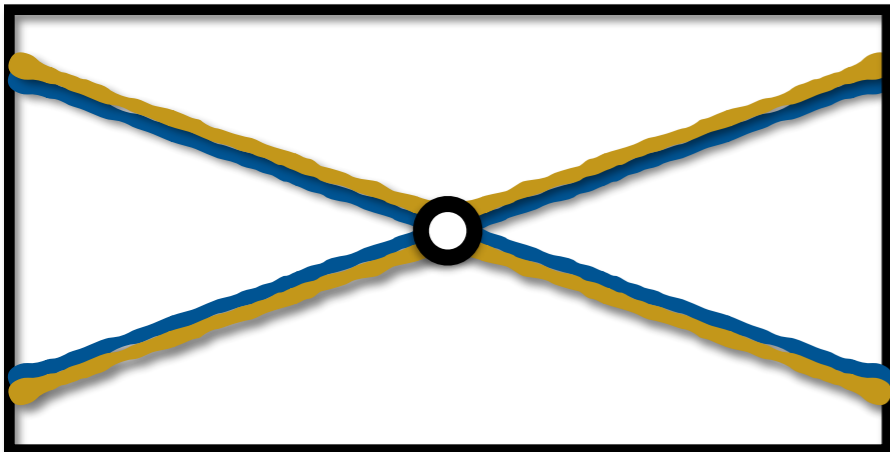
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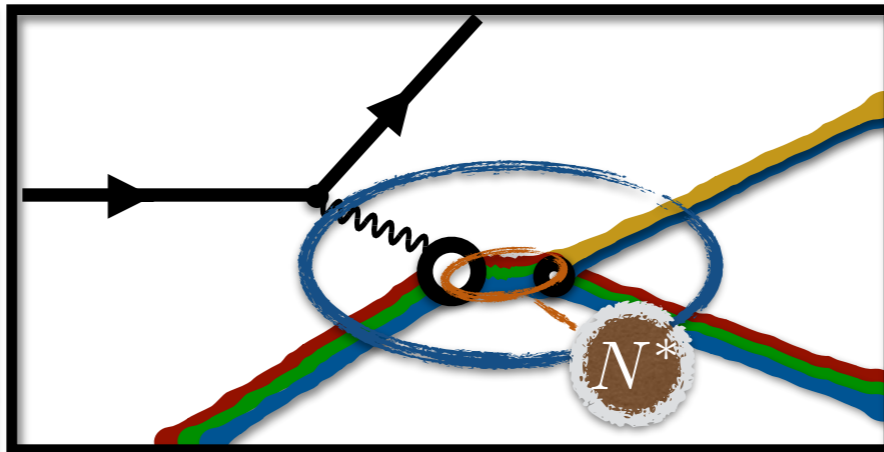
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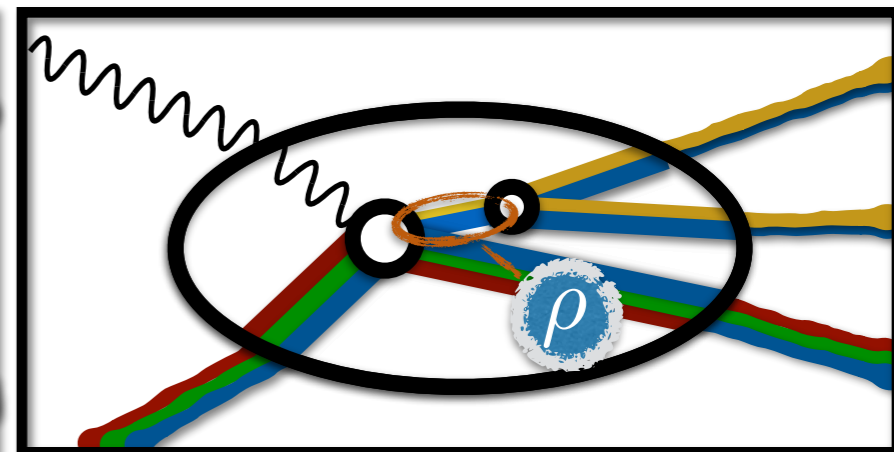
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Requires Minkowski momenta and infinite volume

What can we extract from the lattice?

Instead we can only access

$$H_{\text{QCD}}|n, L\rangle = |n, L\rangle \underline{E_n(L)} \quad \underline{\langle n, L, \text{"}N\pi\pi\text{"} | \mathcal{J}_\mu(x) | \text{"}N\text{"}, L \rangle}$$

finite-volume energies and matrix elements

labels in quotes indicate quantum numbers

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How can we determine

$$\underline{\langle \pi\pi, \text{out} | \pi\pi, \text{in} \rangle \quad \text{and} \quad \langle N\pi\pi, \text{out} | \mathcal{J}_\mu(x) | N \rangle}$$

from

$$\underline{E_n(L) \quad \text{and} \quad \langle n, L, "N\pi\pi" | \mathcal{J}_\mu(x) | "N", L \rangle ?}$$

**It is possible to derive relations between
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Two-particle scattering

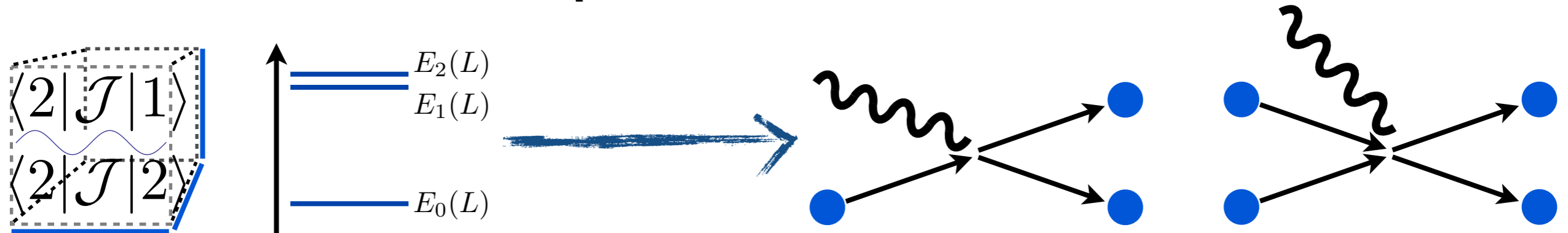


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Photo- and electroproduction

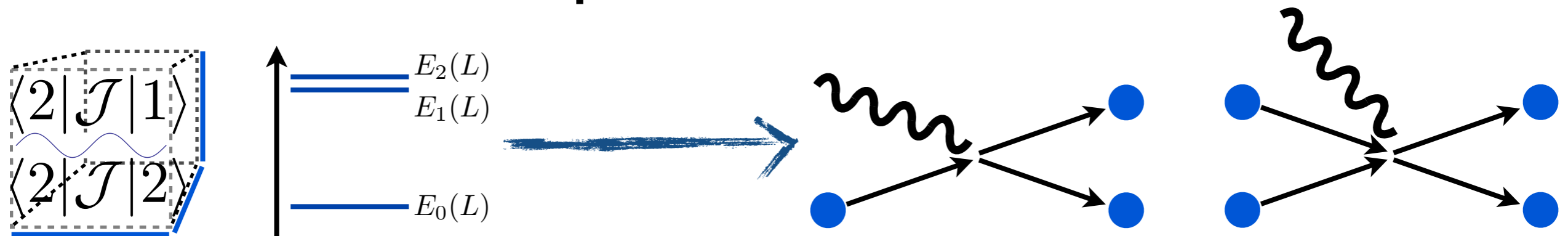


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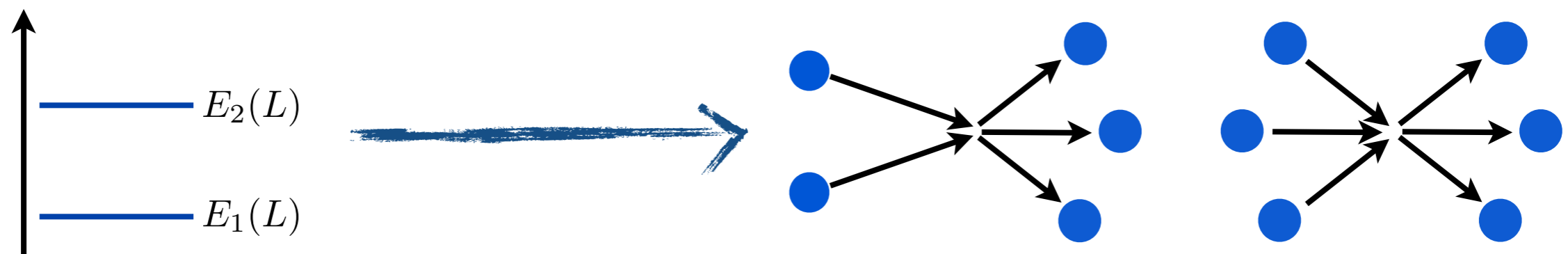
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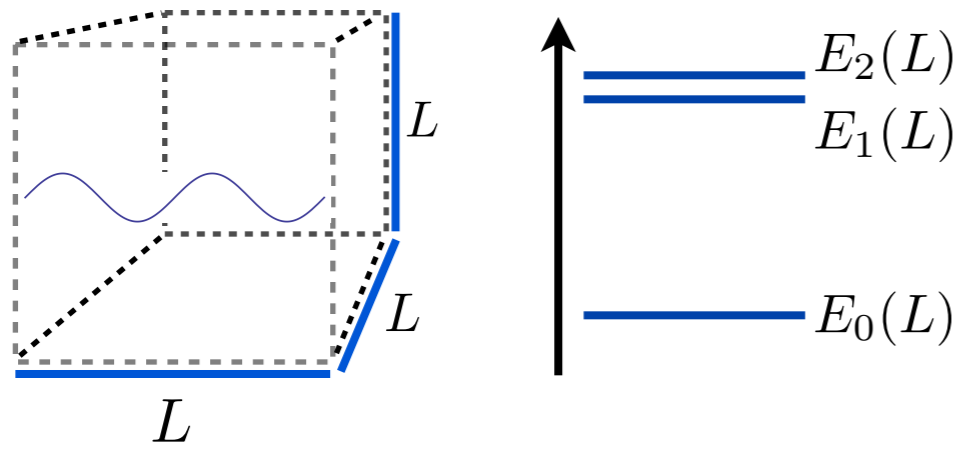
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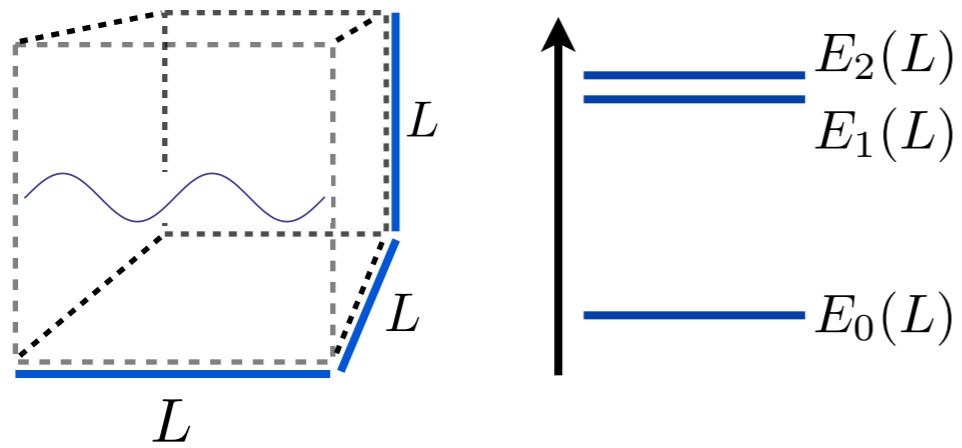
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Finite volume



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cubic, spatial volume (extent L)

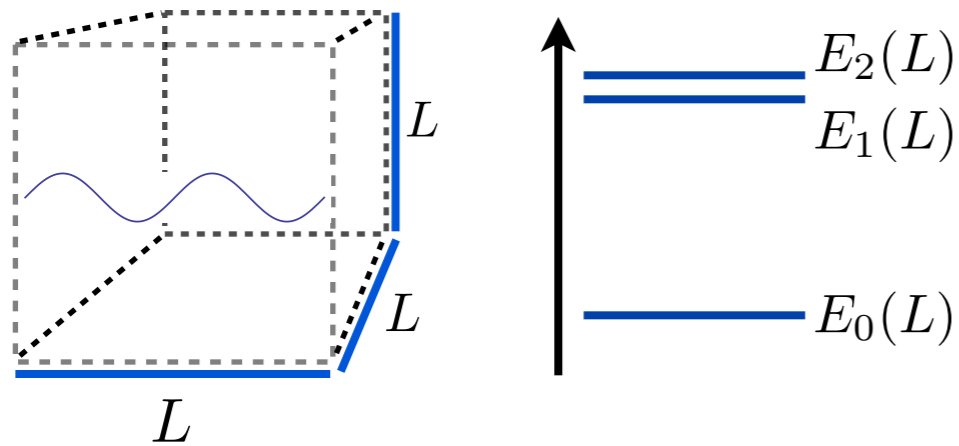
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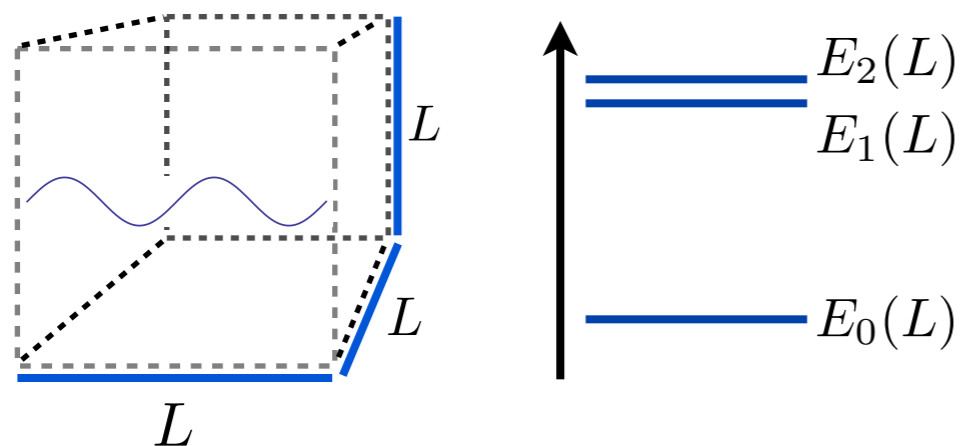
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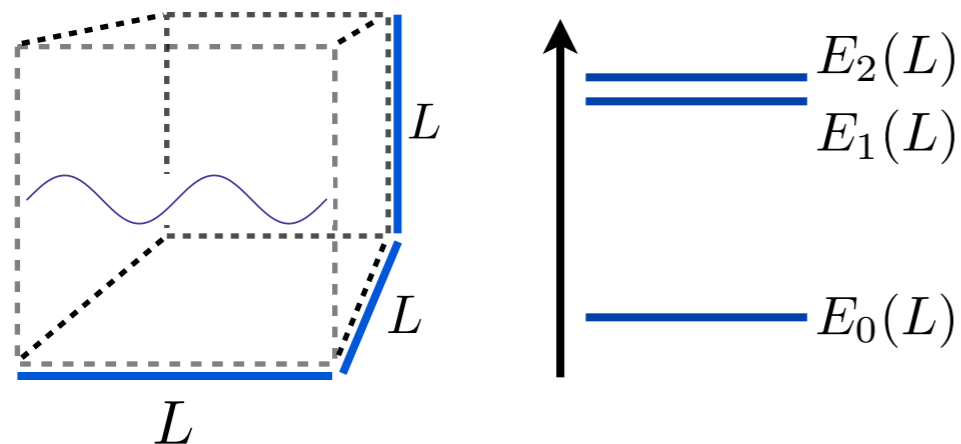
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Assume lattice effects are small and accommodated elsewhere

Work in continuum field theory throughout

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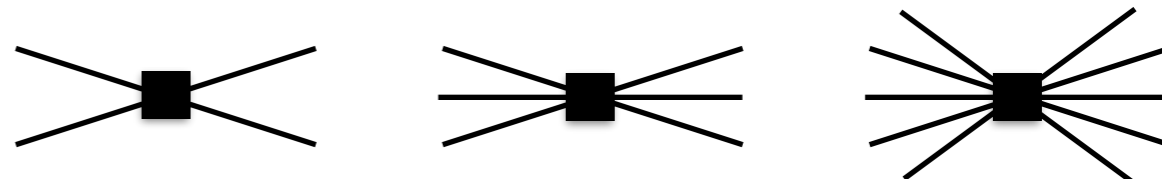
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quantum field theory

generic relativistic QFT

1. Include all interactions

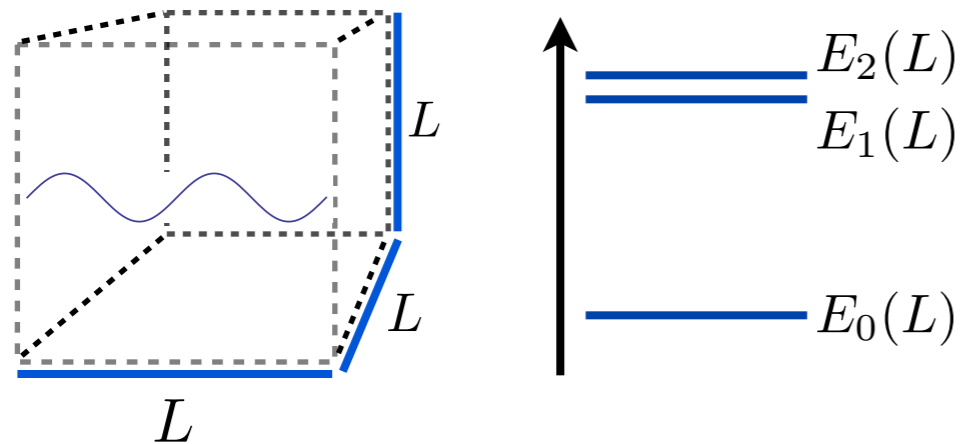


2. no power-counting scheme

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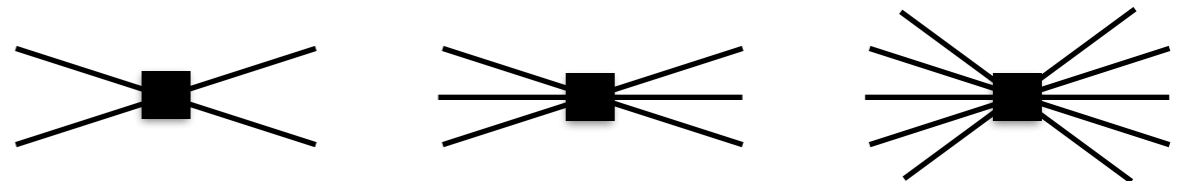


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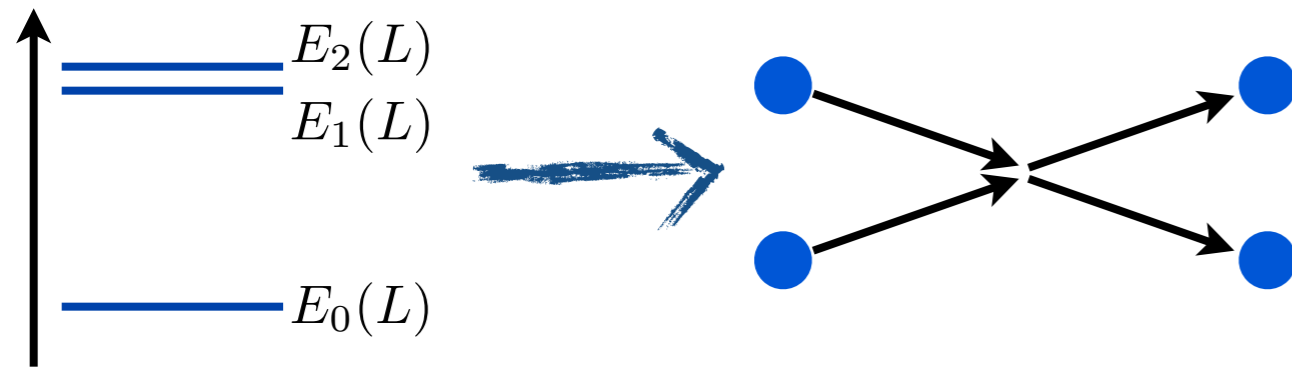
Not possible to directly calculate scattering observables to all orders

But it is possible to derive general, all-orders relations to finite-volume quantities

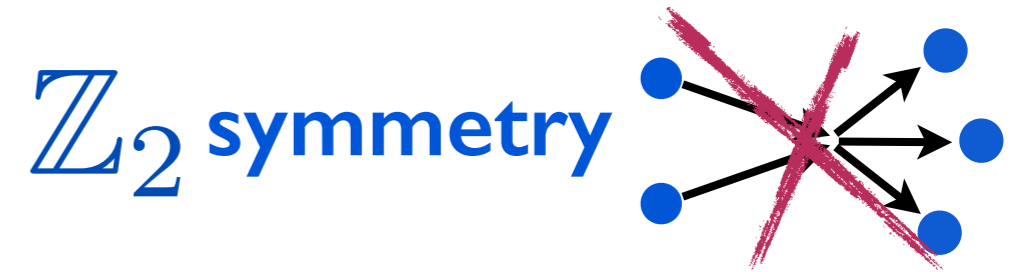
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Two-to-two scattering



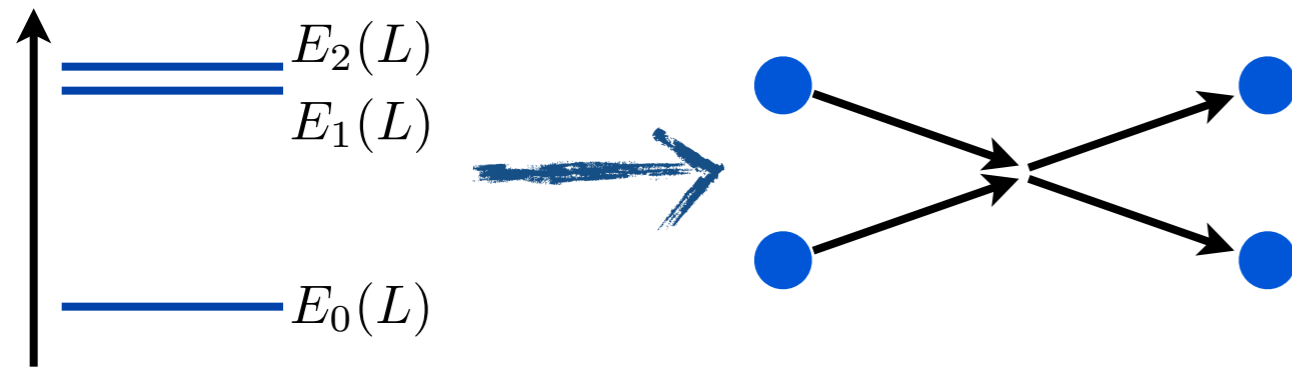
For now assume...
identical scalars, mass m



Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

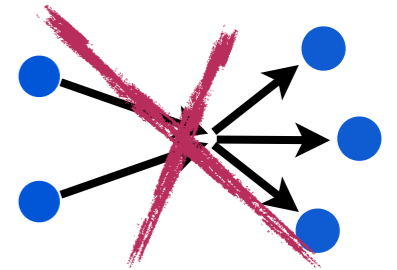
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Two-to-two scattering



For now assume...
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\mathbb{Z}_2 symmetry



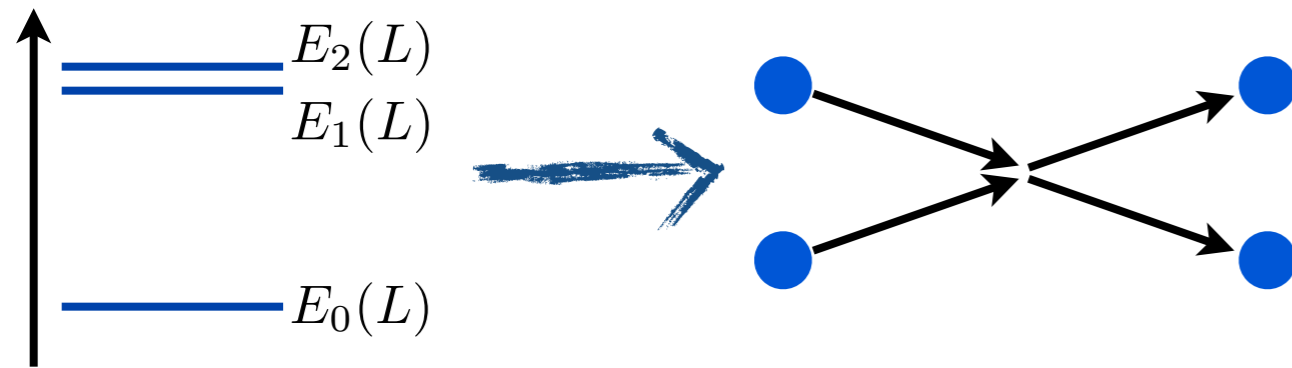
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two-particle interpolator

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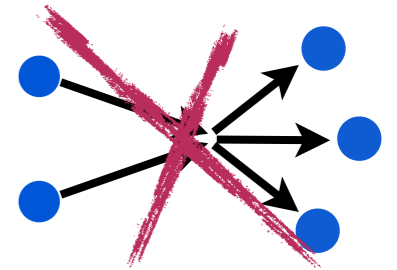
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Euclidean convention

two-particle interpolator

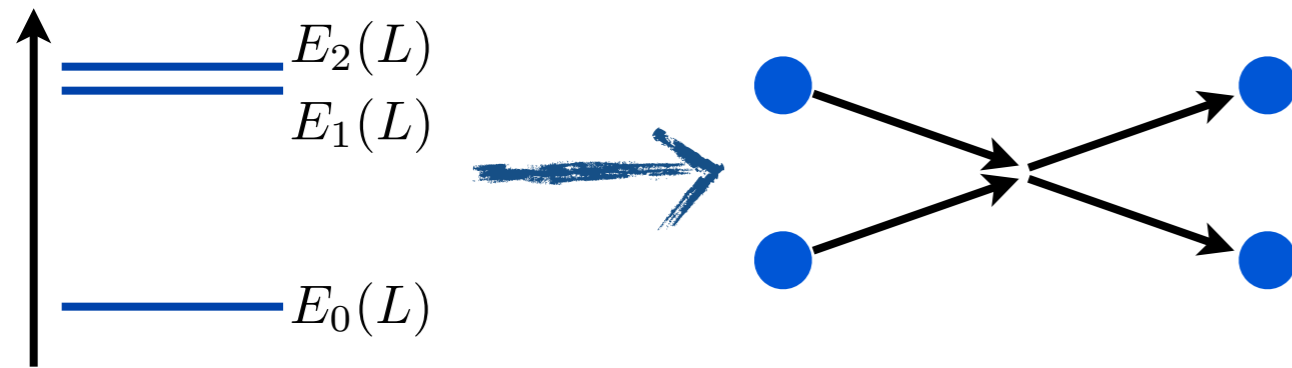
$$P = (P_4, \vec{P}) = (P_4, 2\pi\vec{n}/L)$$

but allow P_4 to be real or imaginary

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

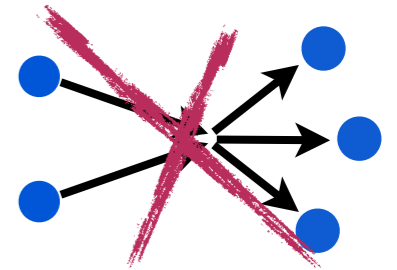
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Two-to-two scattering



For now assume...
identical scalars, mass m

\mathbb{Z}_2 symmetry



$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Euclidean convention

two-particle interpolator

$$P = (P_4, \vec{P}) = (P_4, 2\pi\vec{n}/L)$$

but allow P_4 to be real or imaginary

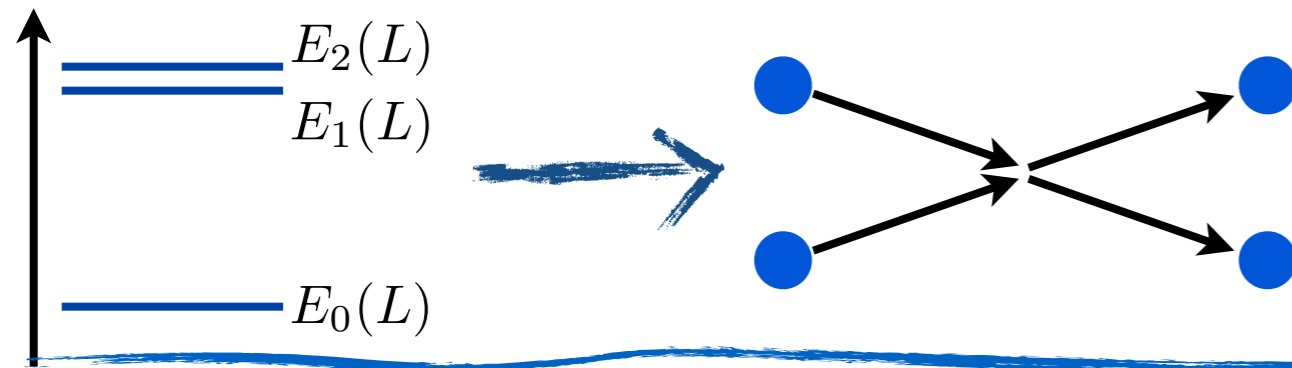
CM frame energy is then $E^{*2} = -P_4^2 - \vec{P}^2$

Require $E^* < 4m$ to isolate two-to-two scattering

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

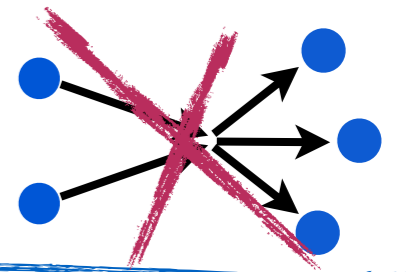
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Two-to-two scattering



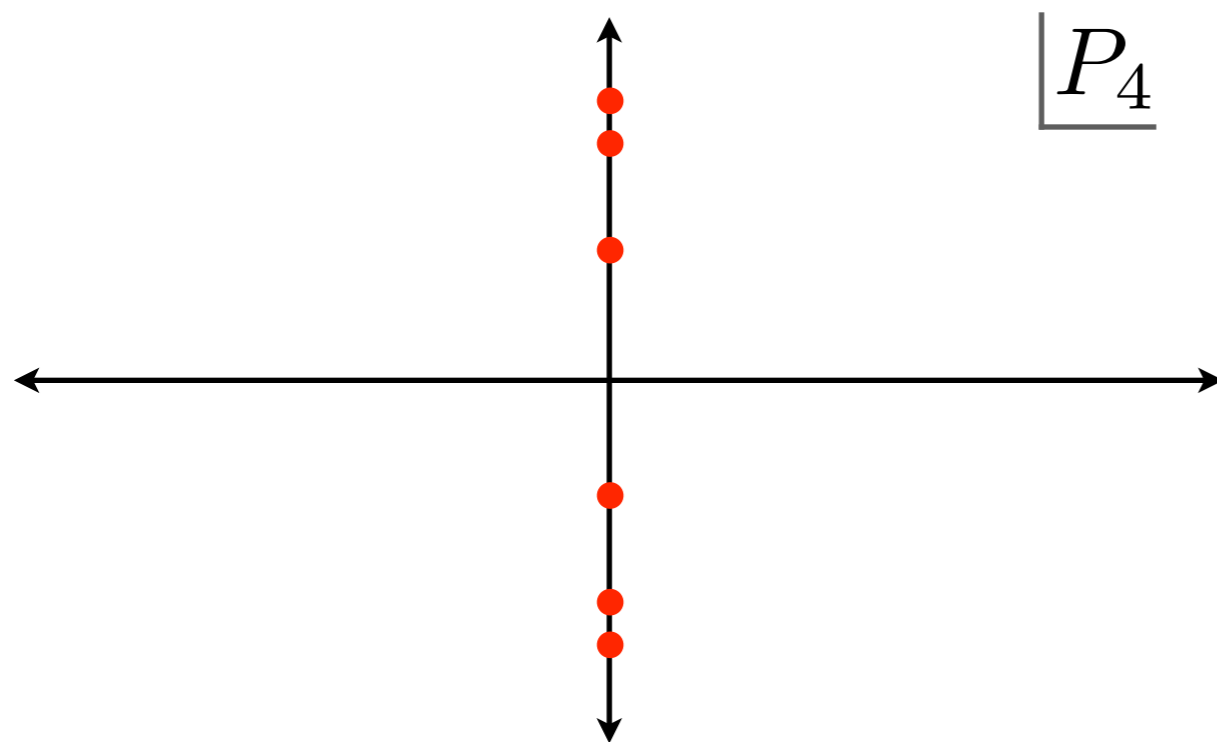
For now assume...
identical scalars, mass m

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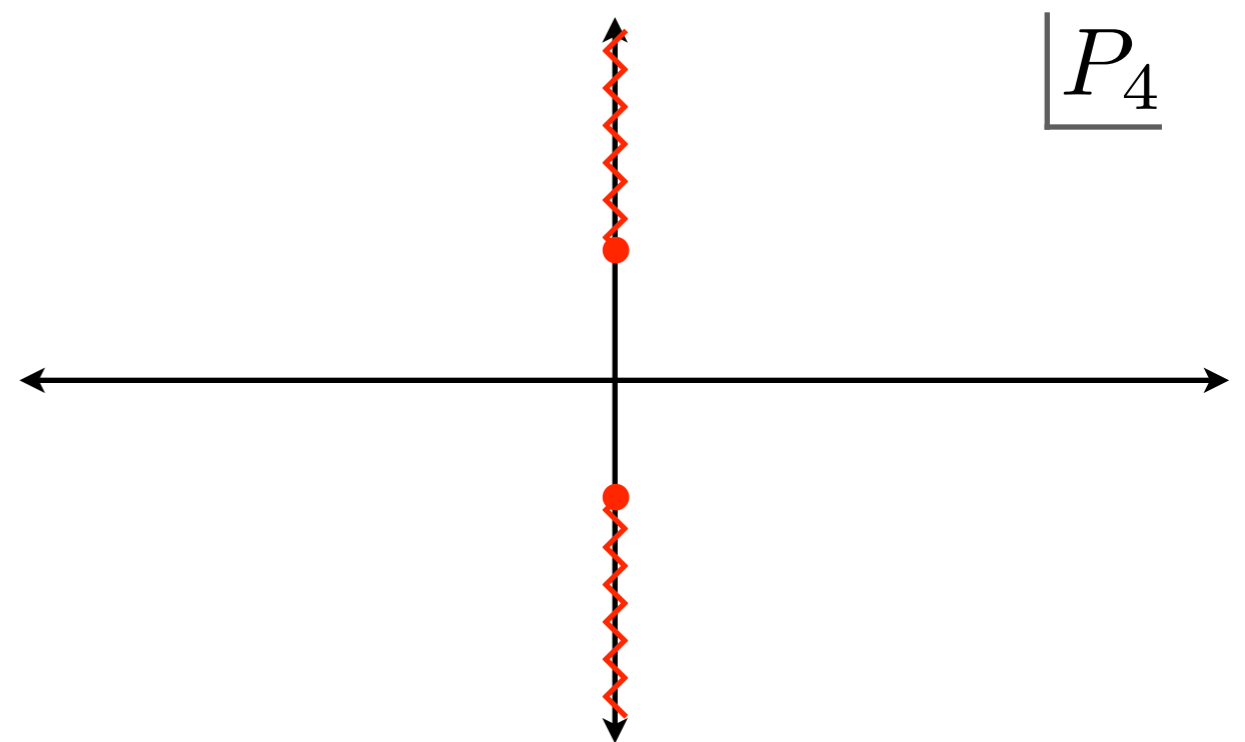


$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

At fixed L, \vec{P} , poles in C_L give finite-volume spectrum

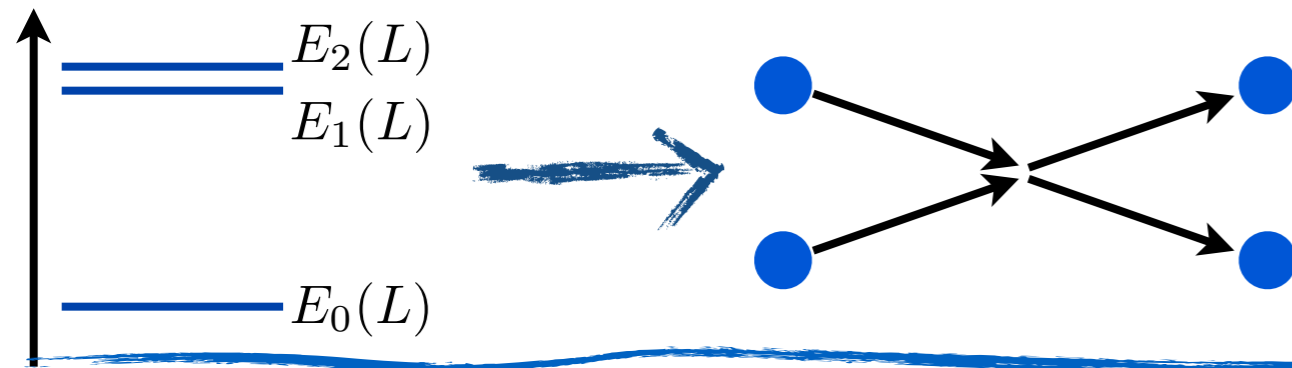


C_L analytic structure



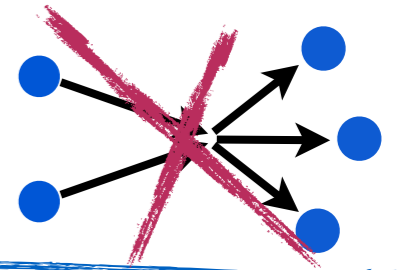
C_∞ analytic structure

Two-to-two scattering



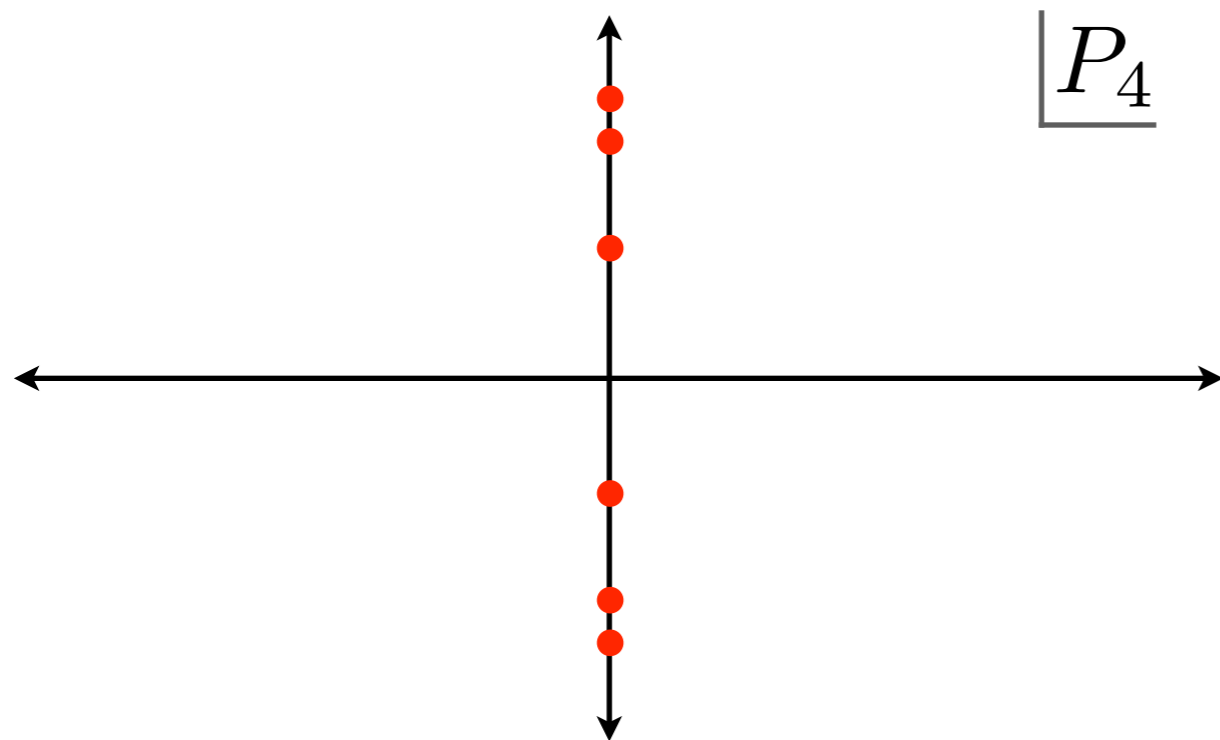
For now assume...
identical scalars, mass m

\mathbb{Z}_2 symmetry



$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

At fixed L, \vec{P} , poles in C_L give finite-volume spectrum



C_L analytic structure

Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.

$$\begin{aligned}
C_L(P) = & \text{Diagram 1} + \text{Diagram 2} \\
& + \text{Diagram 3} + \dots
\end{aligned}$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The diagram shows a series of terms in a sum. Each term consists of a sequence of circles connected by arcs. The first circle in each term is labeled \mathcal{O}^\dagger and the last is labeled \mathcal{O} . The first term has two black dots between \mathcal{O}^\dagger and \mathcal{O} . The second term has two black dots between \mathcal{O}^\dagger and a circle labeled iK , and two black dots between iK and \mathcal{O} . The third term has two black dots between \mathcal{O}^\dagger and iK , two between the two iK circles, and two between the second iK and \mathcal{O} . A blue dashed box highlights the two dots between \mathcal{O}^\dagger and iK in the second term, with a blue arrow pointing to a callout box.

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The first diagram shows a circle labeled \mathcal{O}^\dagger connected to a circle labeled \mathcal{O} . Between them are two black dots, one above and one below, connected by a vertical dashed line. The second diagram is similar but includes a circle labeled iK between the two dots. The third diagram includes two iK circles. Blue dashed boxes and an arrow highlight the first iK circle in the second diagram.

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$



Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

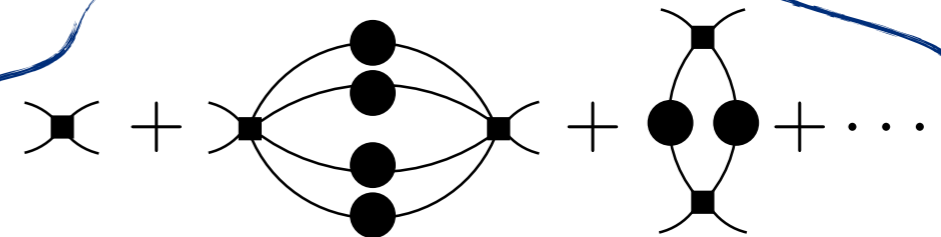
$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows a series of diagrams representing the correlation function $C_L(P)$. The first diagram consists of two circles, \mathcal{O}^\dagger and \mathcal{O} , connected by two lines, each with a black dot. The second diagram is similar but includes a circle labeled iK between the two lines. The third diagram includes two iK circles. Blue dashed boxes highlight the internal lines in each diagram, and blue arrows point from these boxes to the corresponding diagrams in the 'Bethe Salpeter kernel' section below.

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

$\Delta \equiv$ 
fully dressed
propagator



Bethe Salpeter kernel

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

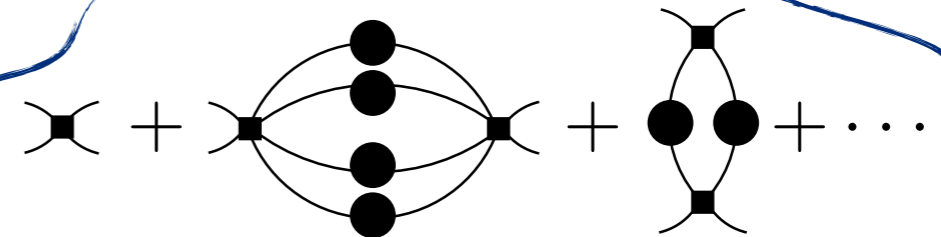
$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The diagrams consist of circles representing operators \mathcal{O}^\dagger and \mathcal{O} , and internal lines representing propagators and kernels. The first diagram shows \mathcal{O}^\dagger connected to \mathcal{O} via a single line with two vertices. The second diagram shows \mathcal{O}^\dagger connected to \mathcal{O} via a line with two vertices, then a kernel iK , then another line with two vertices, and finally \mathcal{O} . The third diagram shows \mathcal{O}^\dagger connected to \mathcal{O} via a line with two vertices, then two kernels iK in series, then another line with two vertices, and finally \mathcal{O} .

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

$\Delta \equiv \text{diagram}$
fully dressed
propagator



Bethe Salpeter kernel

if $E^* < 4m$ **then**

$$K_L = K_\infty + \mathcal{O}(e^{-mL})$$

$$\Delta_L = \Delta_\infty + \mathcal{O}(e^{-mL})$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

$\frac{1}{L^3} \sum_{\vec{k}} \int_{\vec{k}}$

contains all power-law corrections

Now we introduce an important identity.

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} \\ + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \dots$$

$$\frac{1}{L^3} \sum_{\vec{k}} \int_{\vec{k}} \underbrace{\hspace{10em}}_F$$

contains all power-law corrections

Now we introduce an important identity.

In  **all four-momenta are projected on shell.**

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows a series of diagrams representing terms in a sum. Each diagram consists of two large circles (representing source and sink operators) connected by two smaller circles (representing propagators). The first diagram has two black dots on the left propagator. The second diagram has two black dots on the left propagator and two black dots on the right propagator. The third diagram has two black dots on the left propagator, two on the right propagator, and two on a third propagator in the middle. Blue arrows point from the first two diagrams to the corresponding terms in the sum above.

$$\frac{1}{L^3} \sum_{\vec{k}} \text{diagram}_1 = \text{diagram}_2 + \underbrace{\text{diagram}_3}_F$$

The diagram on the left shows two large circles connected by two smaller circles, with two black dots on the left propagator. This is equal to the sum of two diagrams. The first diagram is identical to the left one. The second diagram shows two large circles connected by a single horizontal line, with a vertical dashed line through the center labeled F . A blue bracket under the F diagram is labeled "contains all power-law corrections".

Now we introduce an important identity.

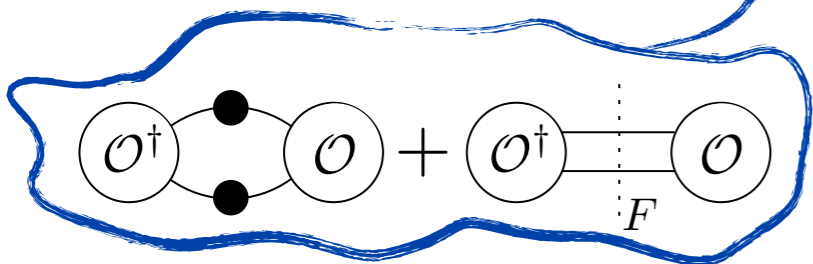
In  all four-momenta are projected on shell.

Physical, propagating states give dominate finite-volume effects.

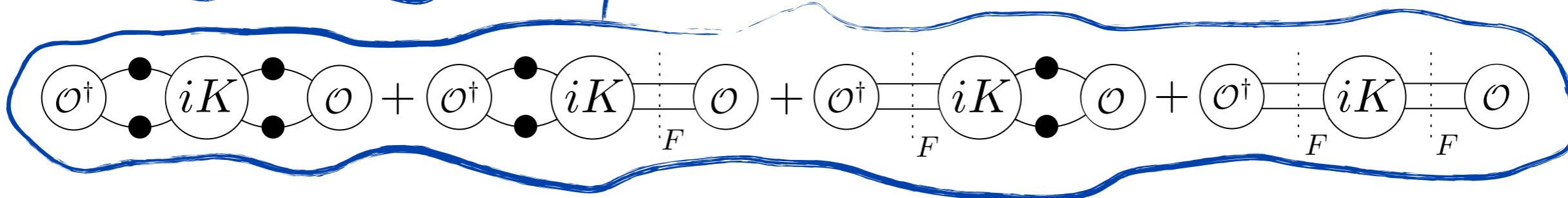
Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

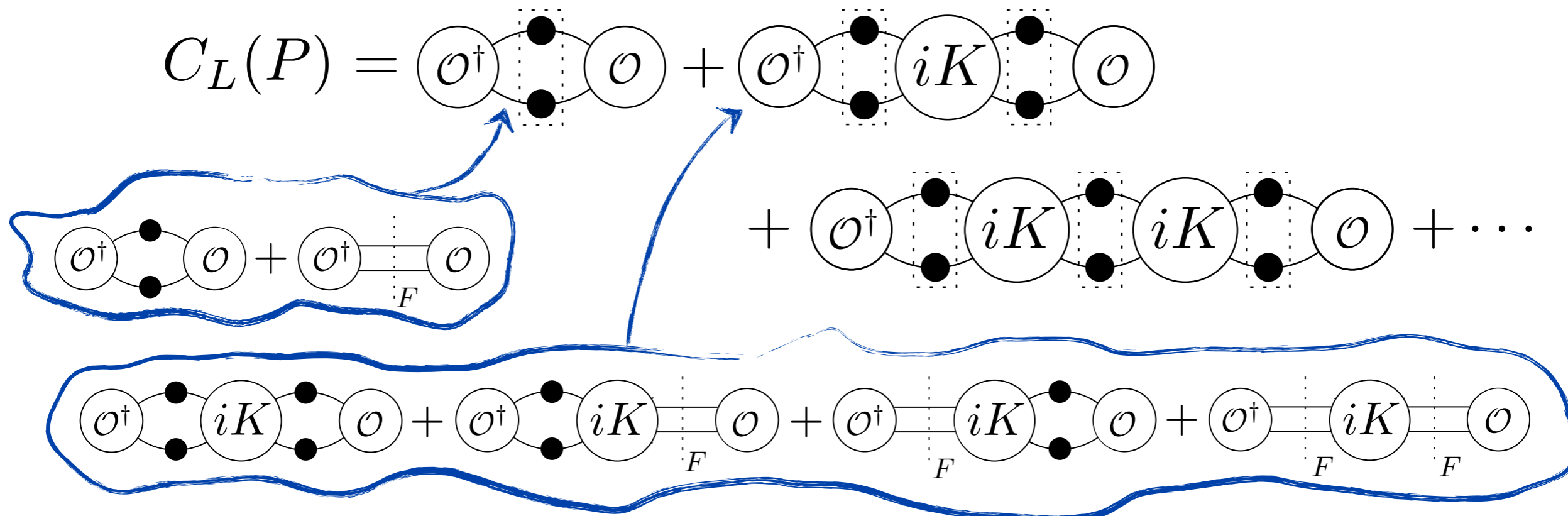
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \circ^\dagger \\ \circ \end{array} + \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \circ^\dagger \\ \circ \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} iK \\ \circ \end{array} + \dots$$



$$+ \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \circ^\dagger \\ \circ \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} iK \\ \circ \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} iK \\ \circ \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \circ^\dagger \\ \circ \end{array} + \dots$$

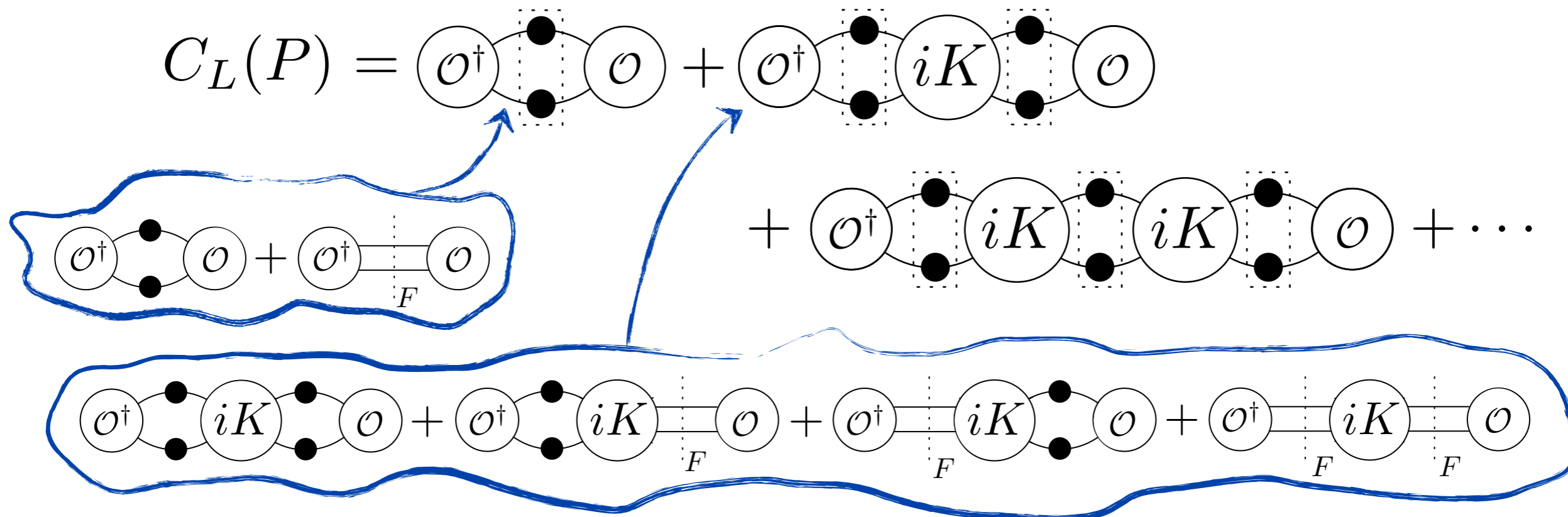




Now regroup by number of Fs

zero Fs

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) +$$



Now regroup by number of Fs

$$C_L(E, \vec{P}) = \overset{\text{zero Fs}}{C_\infty(E, \vec{P})} + \overset{\text{one F}}{\text{diagram with } A \text{ and } A' \text{ and } F} + \dots$$

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The first row shows the expansion of $C_L(P)$ as a sum of diagrams. The first diagram has two vertices \mathcal{O}^\dagger and \mathcal{O} with two internal lines. The second diagram has three vertices \mathcal{O}^\dagger , iK , and \mathcal{O} , with two internal lines between \mathcal{O}^\dagger and iK , and two between iK and \mathcal{O} . The third diagram has four vertices \mathcal{O}^\dagger , iK , iK , and \mathcal{O} , with two internal lines between each adjacent vertex.

A blue bracket groups the first two diagrams, with an arrow pointing to the first diagram of the second row. Another blue bracket groups the first three diagrams of the second row, with an arrow pointing to the first diagram of the third row.

The second row shows the first two diagrams of the expansion, with a vertical dashed line labeled F between the iK and \mathcal{O} vertices in the second diagram.

The third row shows the first three diagrams of the expansion, with vertical dashed lines labeled F between the iK and \mathcal{O} vertices in each diagram.

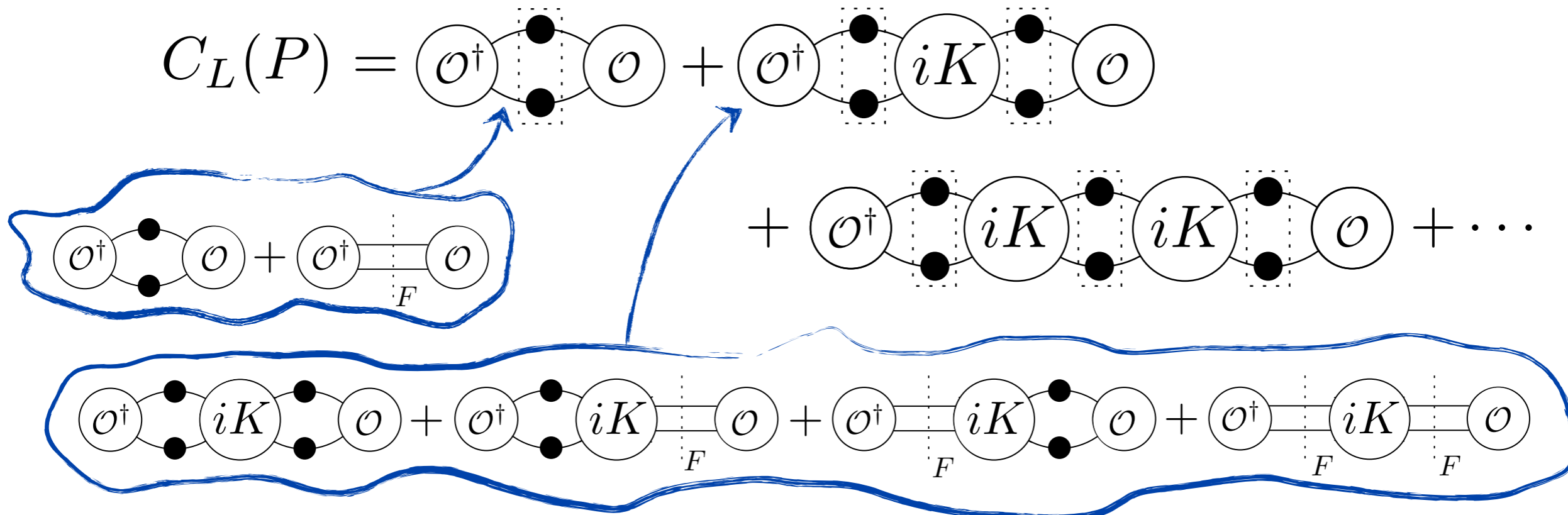
Now regroup by number of Fs

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{diagram}_4 + \dots$$

The diagram diagram_4 consists of two vertices A and A' connected by a horizontal line, with a vertical dashed line labeled F below it.

$$= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$$

A blue bracket groups the first two terms of the expansion $\mathcal{O}^\dagger + \mathcal{O}^\dagger \text{diagram}_1 + \dots$.



Now regroup by number of Fs

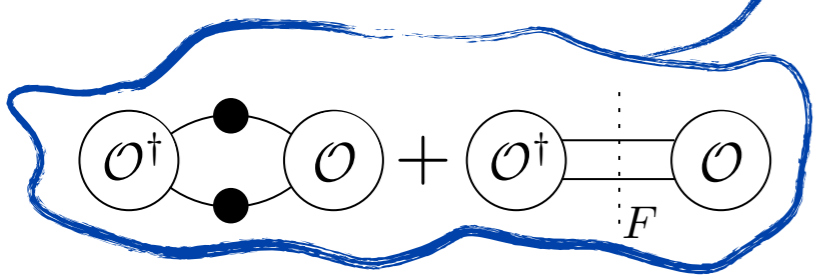
$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{one } F + \text{two } Fs + \dots$$

The diagram shows the expansion of $C_L(E, \vec{P})$ regrouped by the number of F lines. The terms are labeled "zero Fs", "one F", and "two Fs". Each term consists of a chain of vertices: A , A' , and iM . The vertices are connected by lines, and some are enclosed in dashed boxes.

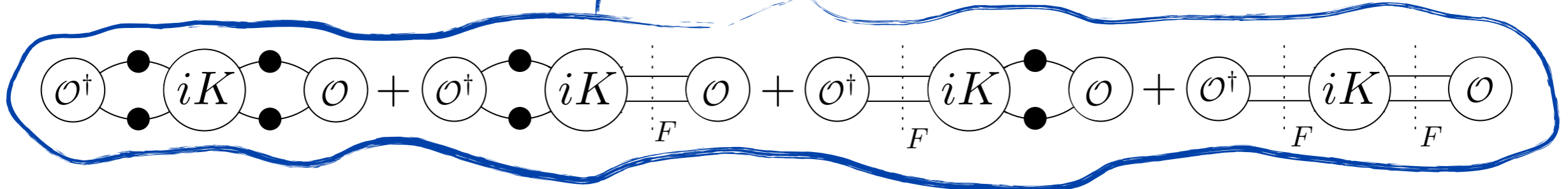
$$= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$$

The diagram shows a blue box containing the terms $\mathcal{O}^\dagger + \mathcal{O}^\dagger \cdot iK + \dots$, which is then equated to the vacuum expectation value $\langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$.

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2$$



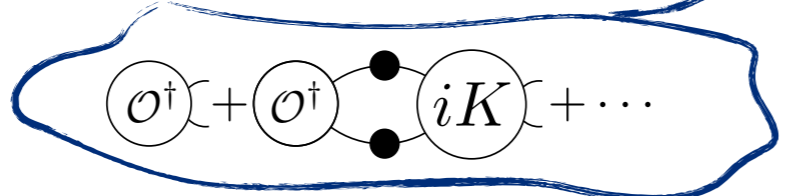
$$+ \text{diagram}_3 + \dots$$



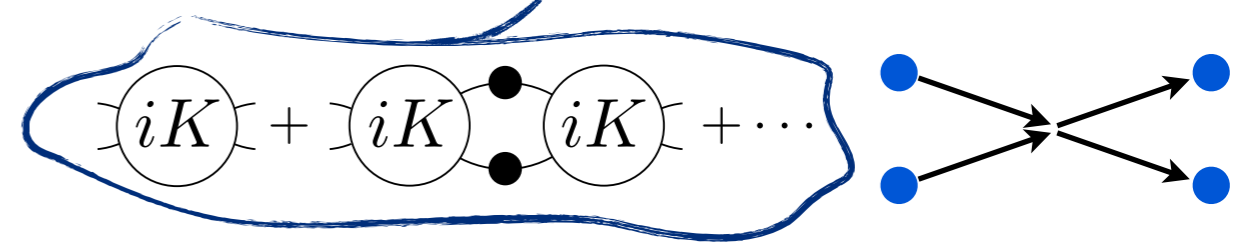
Now regroup by number of Fs

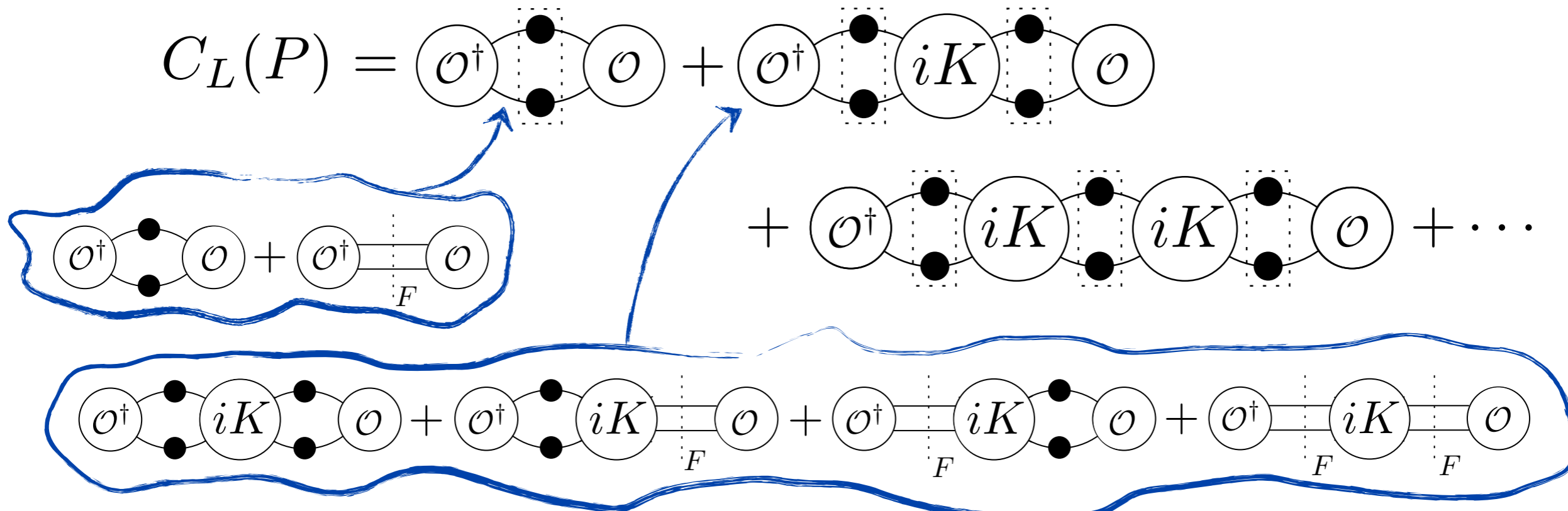
$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{diagram}_4 + \text{diagram}_5 + \dots$$

zero Fs
one F
two Fs



$$= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$$





Now regroup by number of Fs

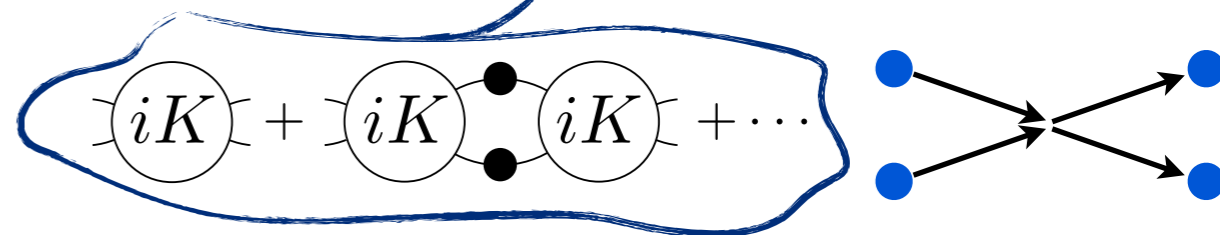
$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{diagram}_1 + \text{diagram}_2 + \dots$$

zero Fs
one F
two Fs

The diagram shows the expansion of $C_L(E, \vec{P})$ as a sum of terms. Each term consists of a chain of vertices: A , A' , and iM . The first term has two vertices. The second term has three vertices. The third term has four vertices. Labels 'zero Fs', 'one F', and 'two Fs' are placed above the diagrams. Blue arrows point from the first two terms to the first two terms of the expansion.

$$= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$$

The diagram shows the first part of the expansion, with vertices \mathcal{O}^\dagger and iK . Blue arrows point from the first two terms to the first two terms of the expansion.



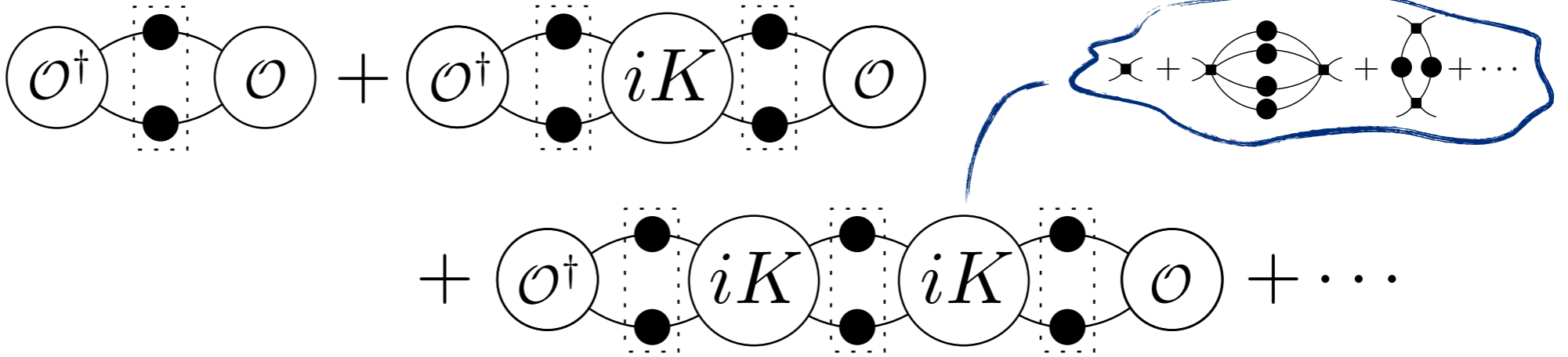
When we factorize diagrams and group infinite-volume parts...

physical observables emerge!

Review..

Review...

1

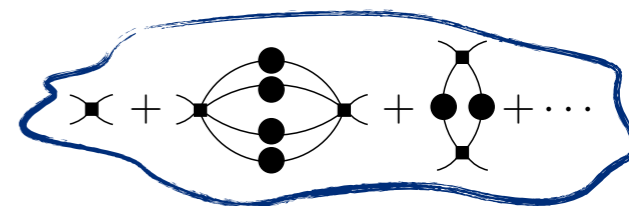
$$C_L(P) = \begin{array}{c} \textcircled{\mathcal{O}^\dagger} \textcircled{\mathcal{O}} + \textcircled{\mathcal{O}^\dagger} \textcircled{iK} \textcircled{\mathcal{O}} \\ + \textcircled{\mathcal{O}^\dagger} \textcircled{iK} \textcircled{iK} \textcircled{\mathcal{O}} + \dots \end{array}$$


Review...

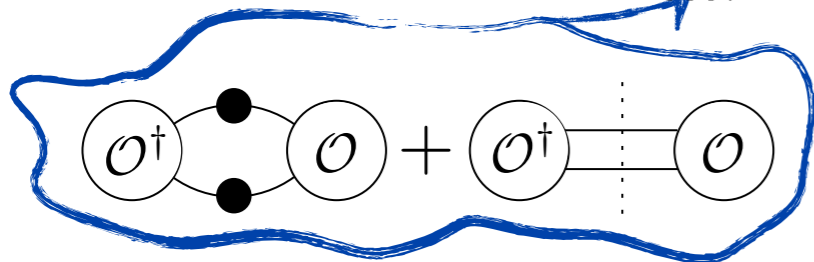
1

$$C_L(P) = \text{diagram 1} + \text{diagram 2}$$

The first diagram shows a circle labeled \mathcal{O}^\dagger on the left and a circle labeled \mathcal{O} on the right. Two black dots are positioned between them, enclosed in a vertical dashed rectangle. Two arcs connect the dots to the \mathcal{O}^\dagger circle, and two arcs connect the dots to the \mathcal{O} circle.



2



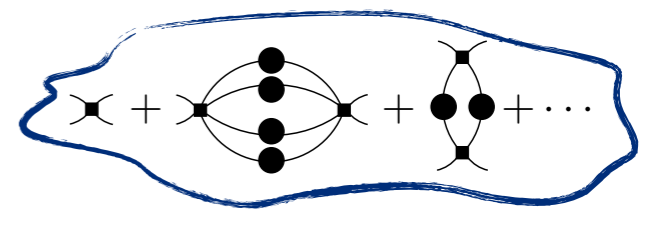
$$+ \text{diagram 3} + \text{diagram 4} + \dots$$

The third diagram shows a sequence of three circles: \mathcal{O}^\dagger , iK , and \mathcal{O} . Each circle is connected to the next by two arcs, and each of the two intermediate dots is enclosed in a vertical dashed rectangle. The fourth diagram shows a sequence of four circles: \mathcal{O}^\dagger , iK , iK , and \mathcal{O} . Each circle is connected to the next by two arcs, and each of the four intermediate dots is enclosed in a vertical dashed rectangle.

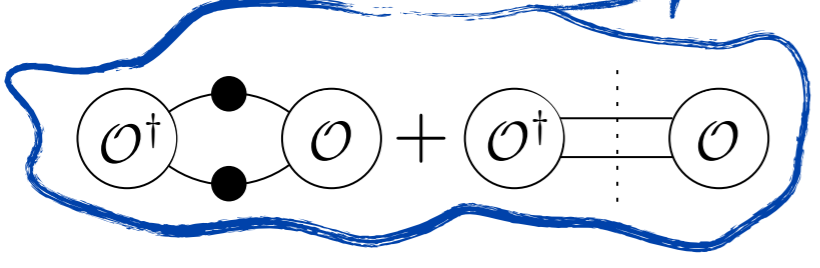
Review...

1

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \mathcal{O}$$

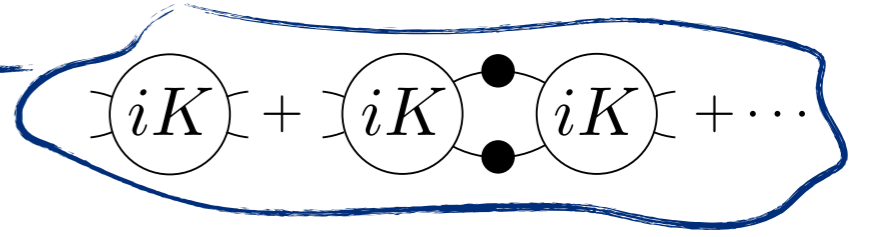


2



$$+ \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \mathcal{O} + \dots$$

$$C_L(P) = C_\infty(P)$$



3

$\langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$

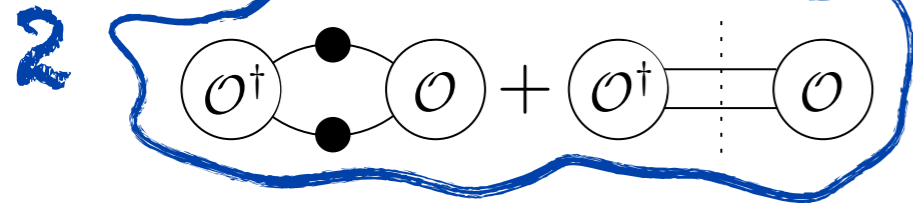
$$+ \begin{array}{c} A \\ \text{---} \\ F \\ \text{---} \\ A' \end{array} + \begin{array}{c} A \\ \text{---} \\ F \\ \text{---} \\ i\mathcal{M} \\ \text{---} \\ F \\ \text{---} \\ A' \end{array} + \dots$$

$$+ \begin{array}{c} A \\ \text{---} \\ F \\ \text{---} \\ i\mathcal{M} \\ \text{---} \\ F \\ \text{---} \\ i\mathcal{M} \\ \text{---} \\ F \\ \text{---} \\ A' \end{array} + \dots$$

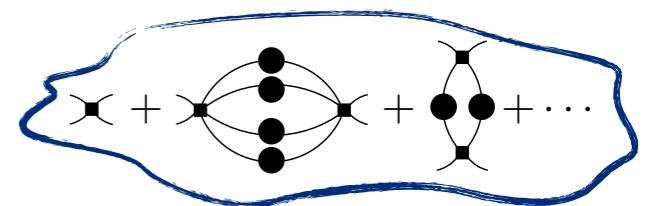
$\langle 0 | \mathcal{O} | \pi\pi, \text{in} \rangle$

Review...

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O}$$

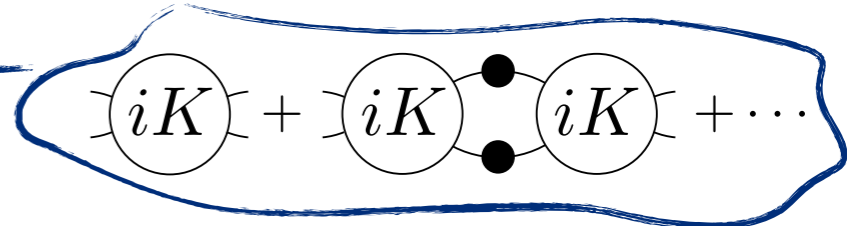
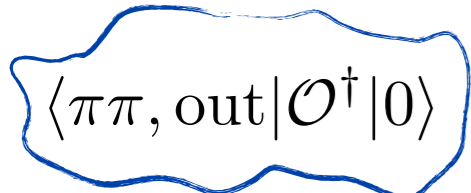


$$+ \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \dots$$

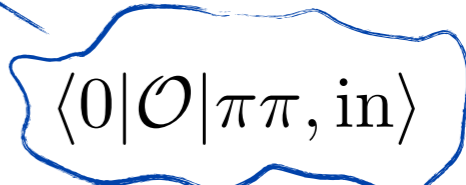


$$C_L(P) = C_\infty(P)$$

$$+ \begin{array}{c} A \\ \vdots \\ F \end{array} A' + \begin{array}{c} A \\ \vdots \\ F \end{array} i\mathcal{M} \begin{array}{c} \vdots \\ F \end{array} A'$$



$$+ \begin{array}{c} A \\ \vdots \\ F \end{array} i\mathcal{M} \begin{array}{c} \vdots \\ F \end{array} i\mathcal{M} \begin{array}{c} \vdots \\ F \end{array} A' + \dots$$

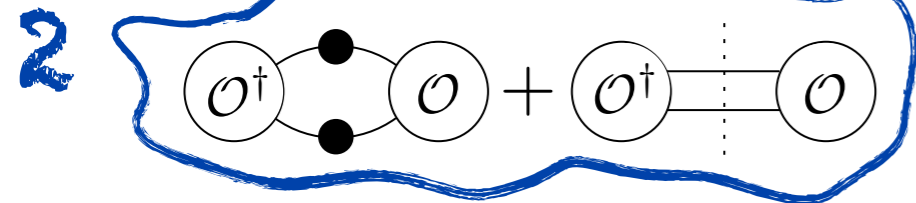


We deduce...

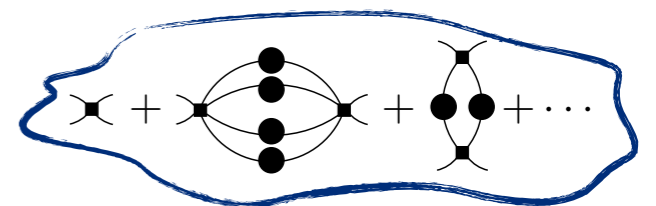
$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

Review...

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O}$$

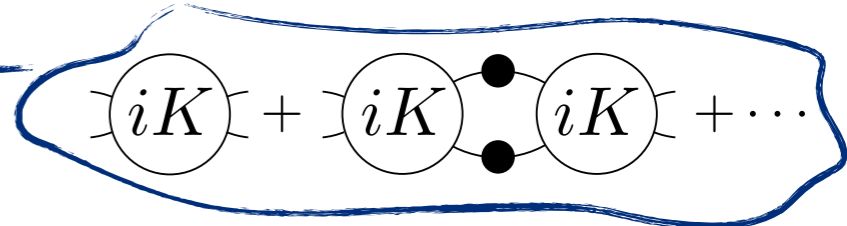
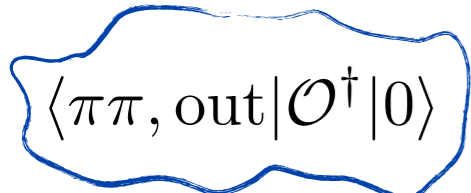


$$+ \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \dots$$

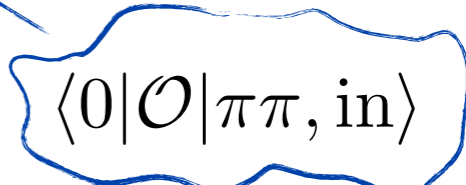


$$C_L(P) = C_\infty(P)$$

$$+ \begin{array}{c} A \\ \vdots \\ F \end{array} A' + \begin{array}{c} A \\ \vdots \\ F \end{array} i\mathcal{M} \begin{array}{c} \vdots \\ F \end{array} A'$$



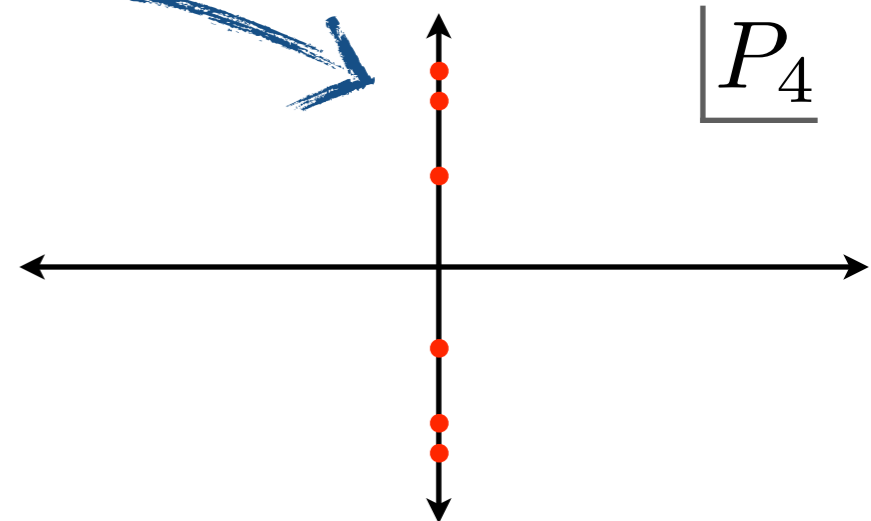
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We deduce...

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

poles are in here



Two-particle result

At fixed (L, \vec{P}) , finite-volume energies are solutions to $\det[\mathcal{M}_{2 \rightarrow 2}^{-1} + F] = 0$

Rummukainen and Gottlieb, *Nucl. Phys.* B450, 397 (1995)

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Matrices defined using angular-momentum states

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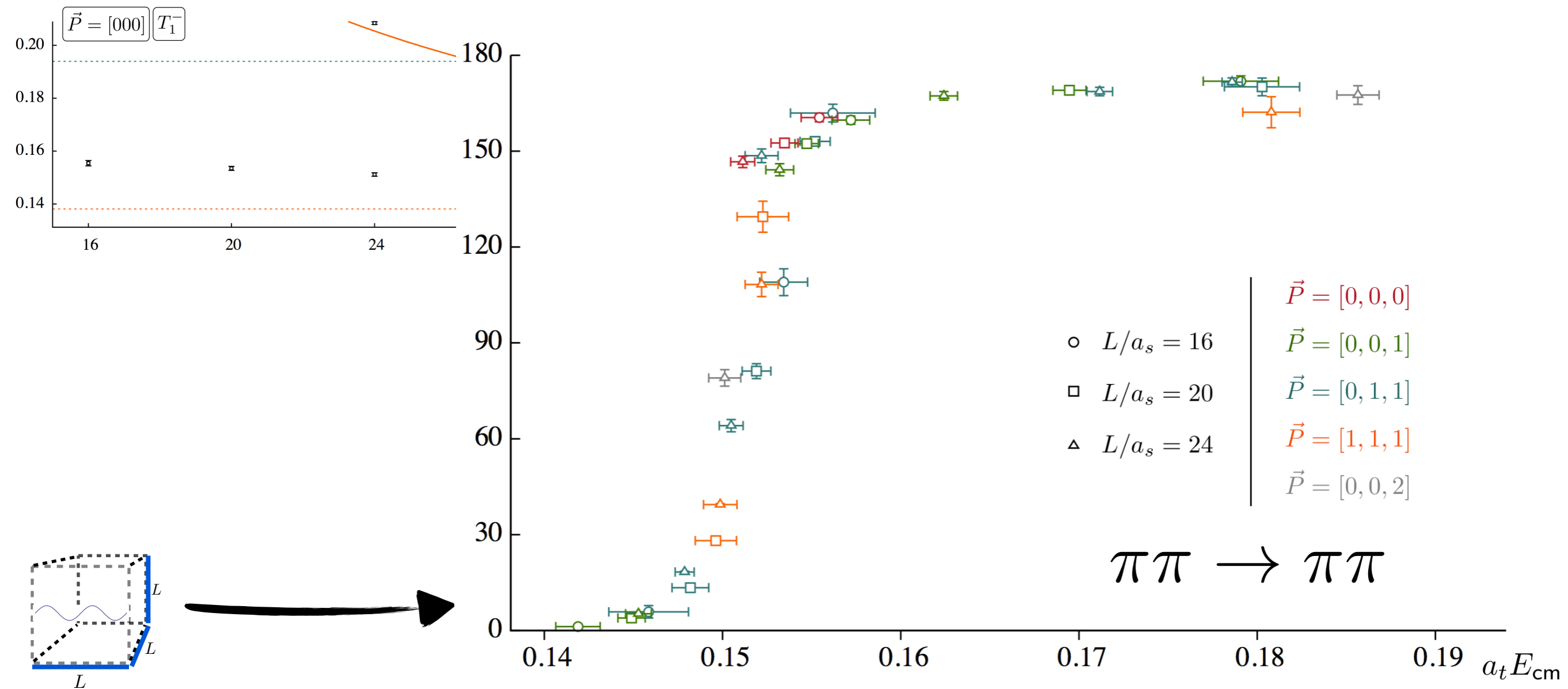
At low energies, lowest partial waves dominate $\mathcal{M}_{2 \rightarrow 2}$

e.g. s-wave only
 with some rearranging \rightarrow $\cot \delta(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$

scattering phase known function

Using the result (p-wave)

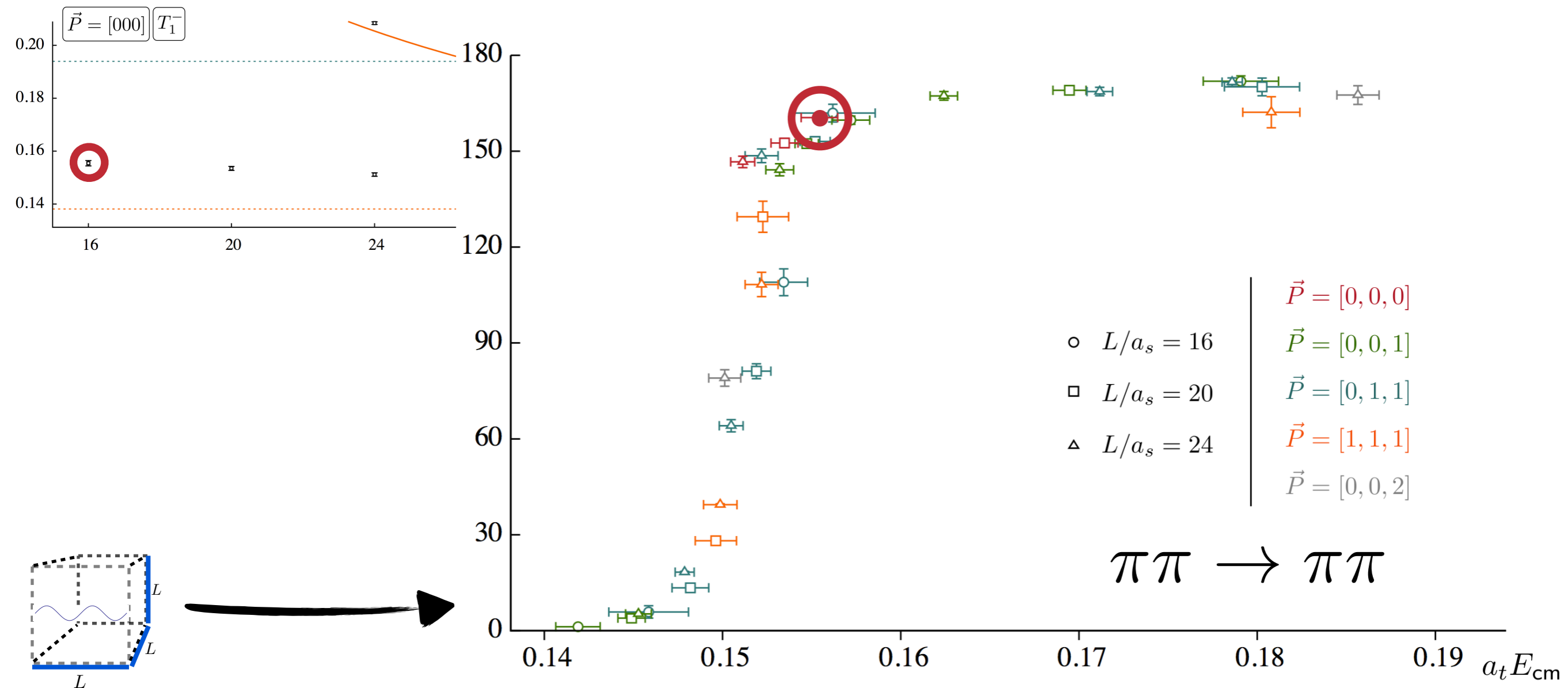
$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



from Dudek, Edwards, Thomas in *Phys.Rev. D87* (2013) 034505

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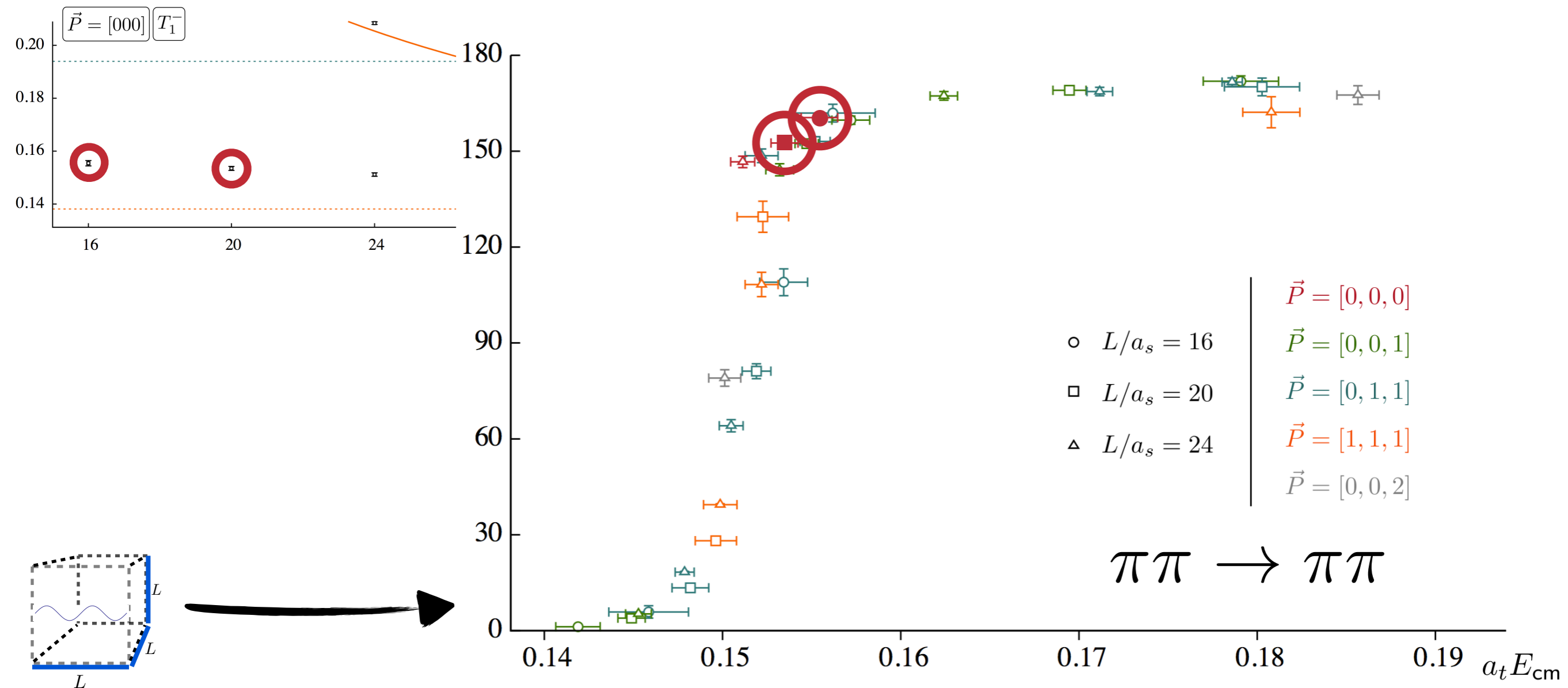
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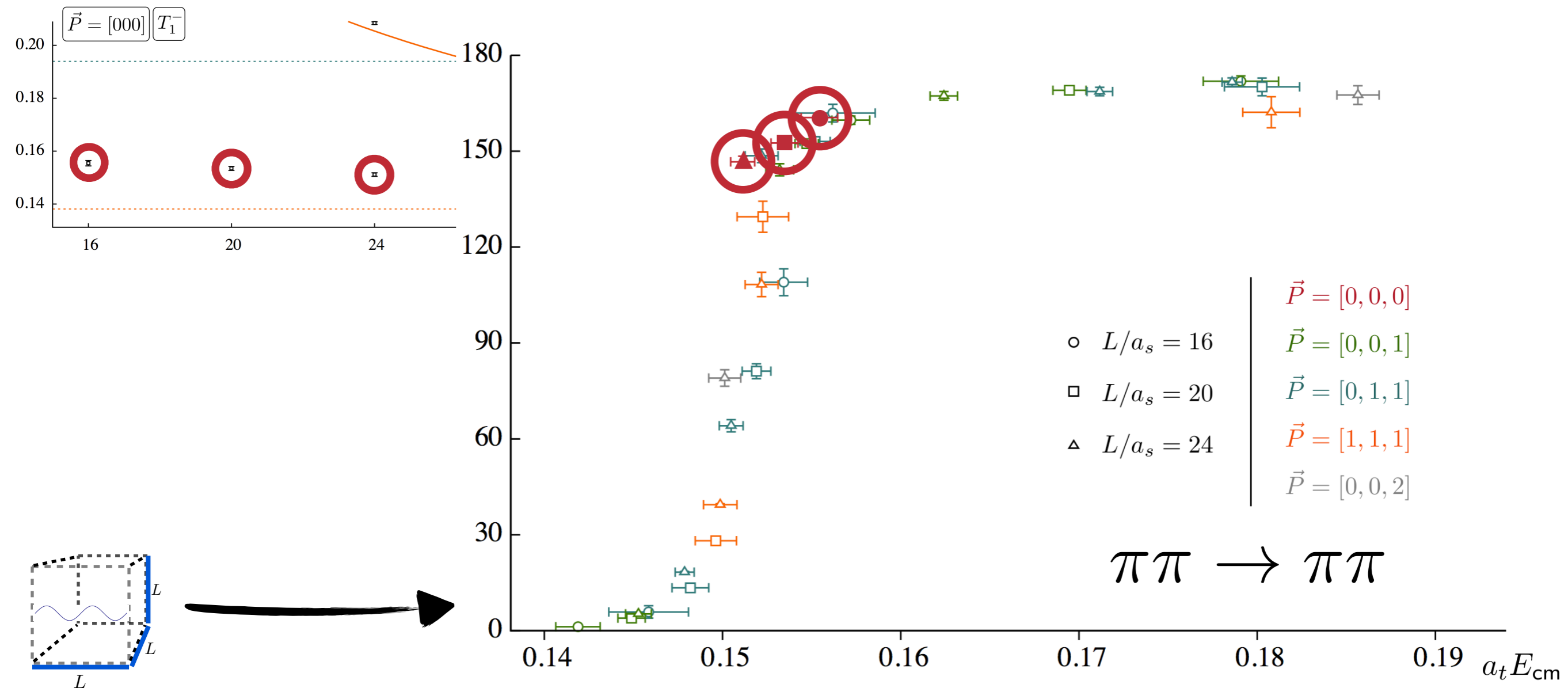
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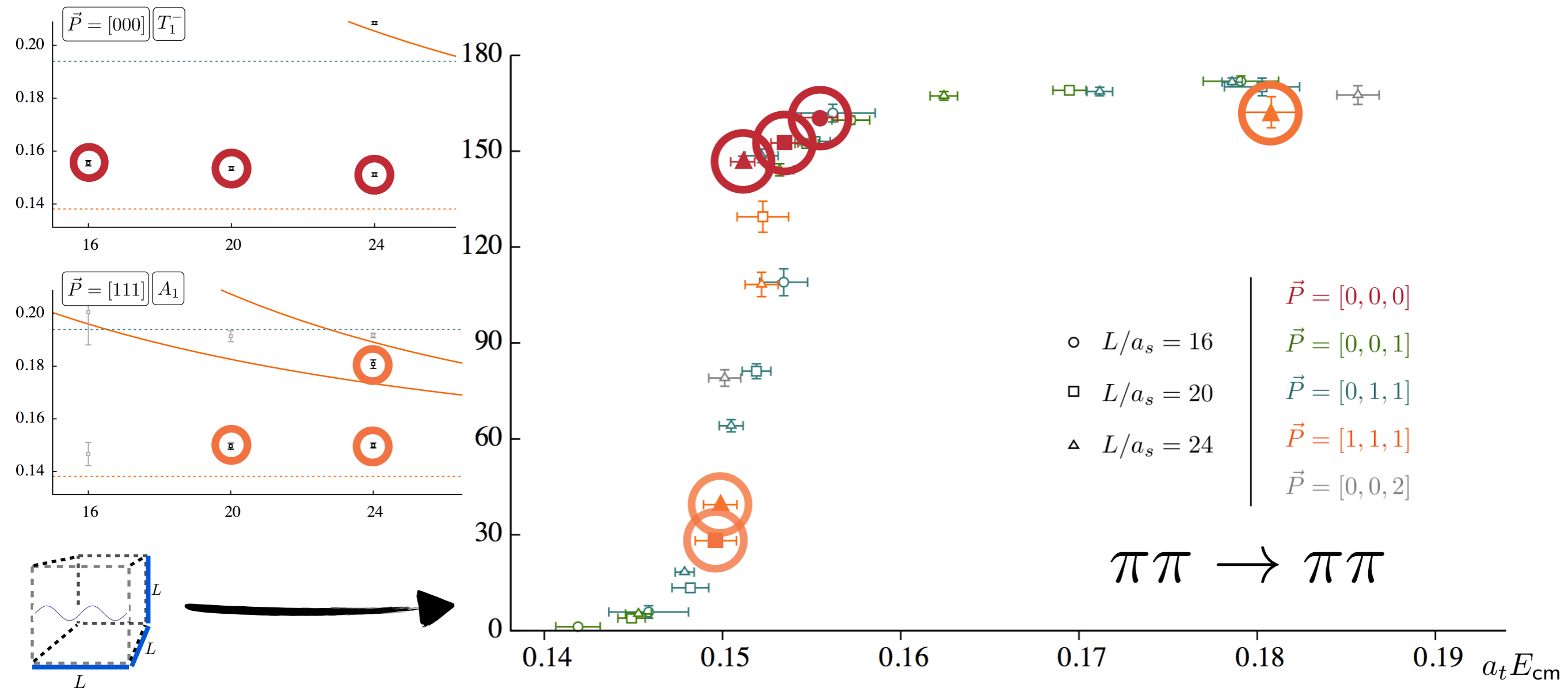
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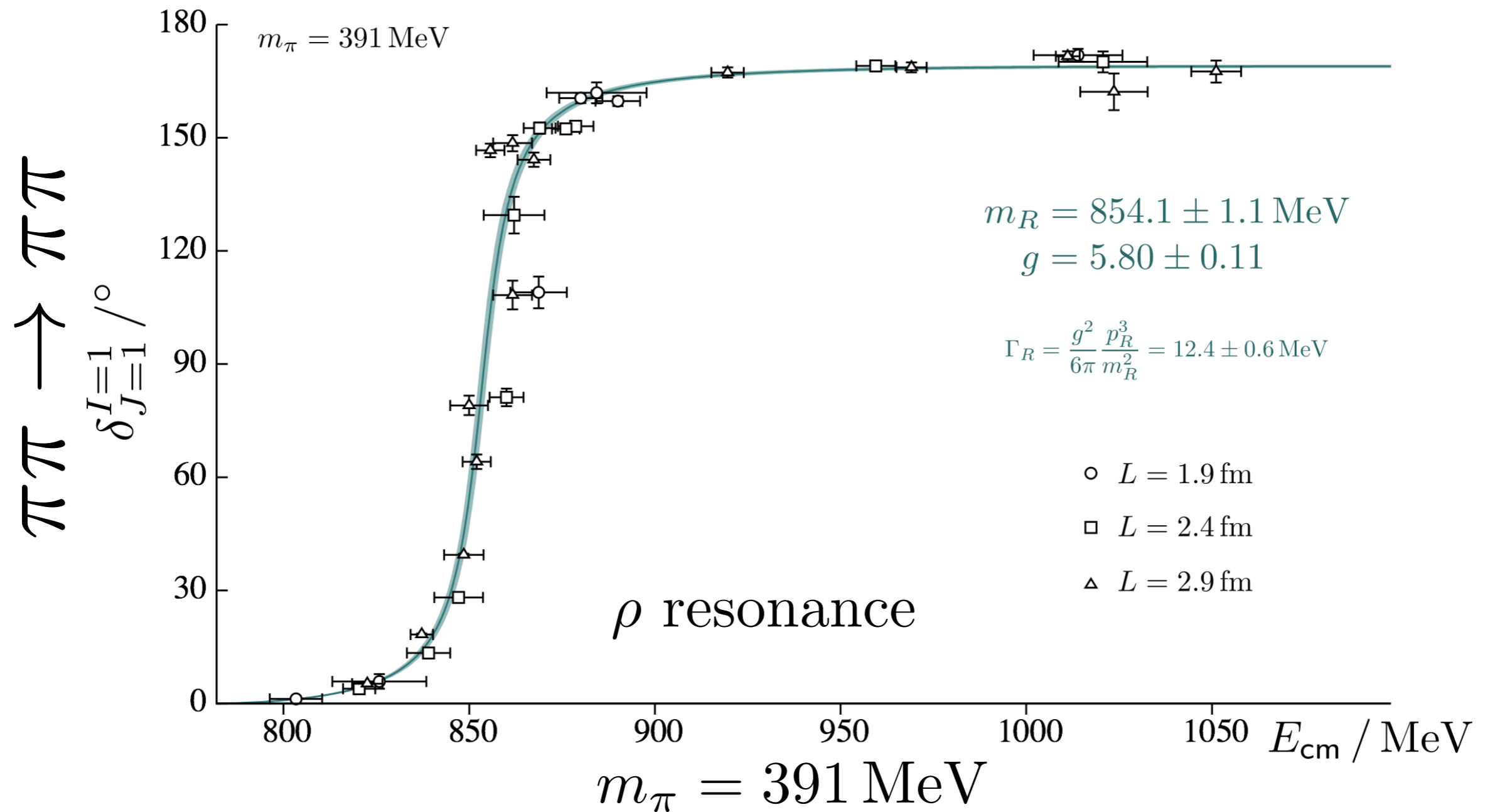
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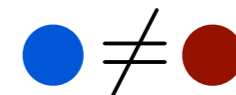
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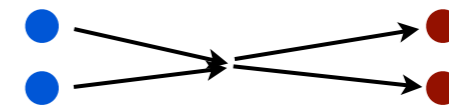
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Has since been generalized to include...

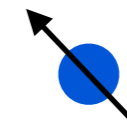
non-identical particles



multiple two-particle channels



particles with spin



MTH and Sharpe, *Phys.Rev. D86* (2012) 016007

Briceño and Davoudi, *Phys.Rev. D88* (2013) 094507

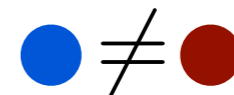
Briceño, *Phys. Rev. D 89*, 074507 (2014)

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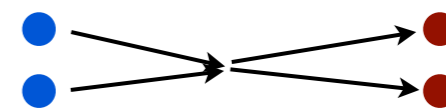
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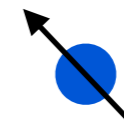
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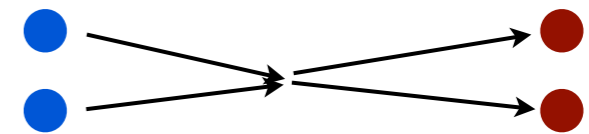
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The basic form of the equation stays the same,
but the **matrix space** and **definition of F** change

Multiple two-particle channels

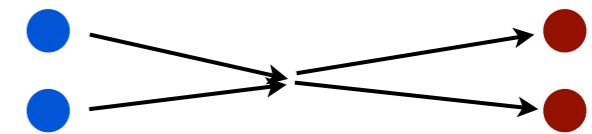


Must now include
a channel index

MTH and Sharpe/Briceño and Davoudi

$$\det \left[\begin{pmatrix} \mathcal{M}_{a \rightarrow a} & \mathcal{M}_{a \rightarrow b} \\ \mathcal{M}_{b \rightarrow a} & \mathcal{M}_{b \rightarrow b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0$$

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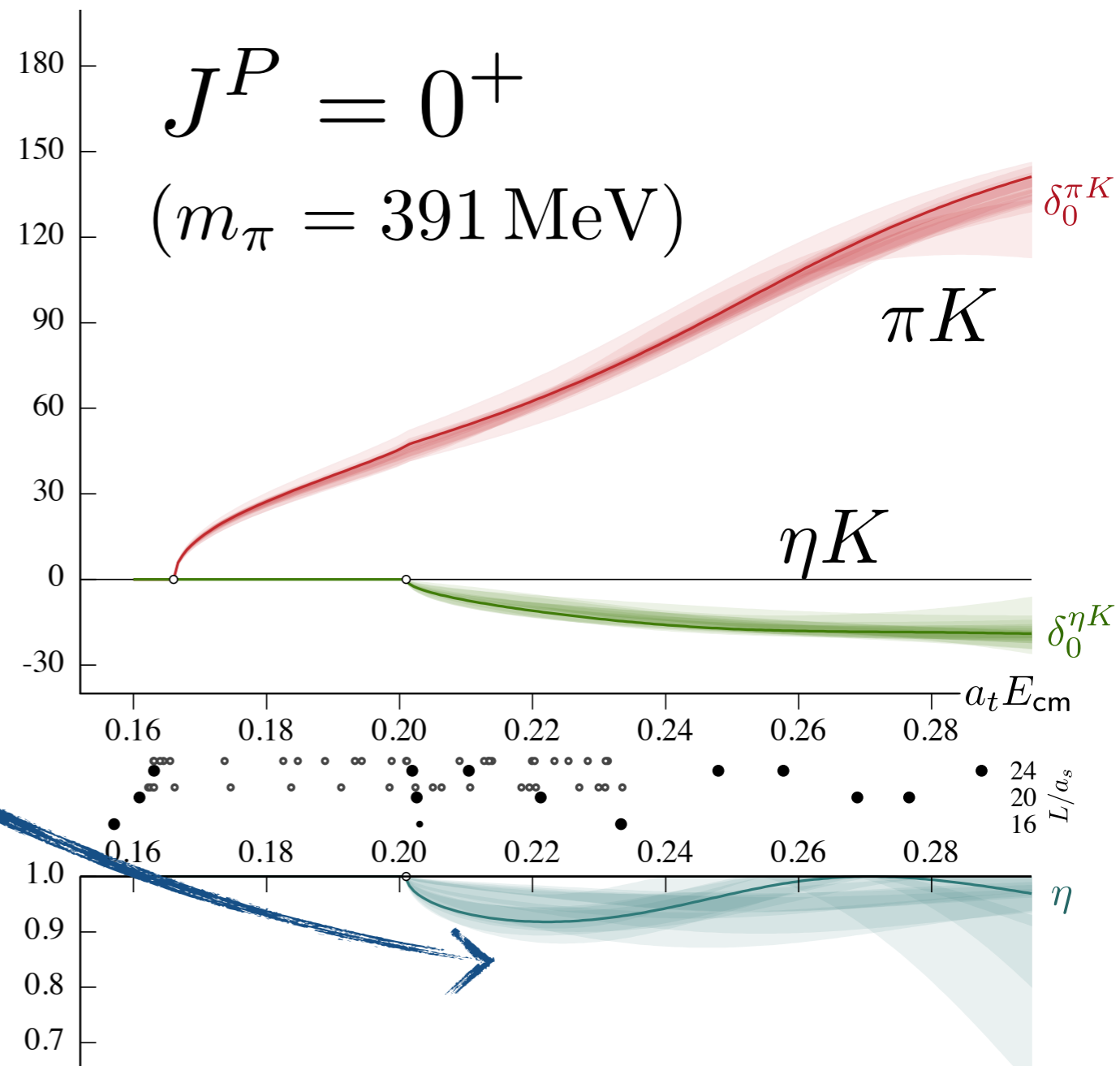
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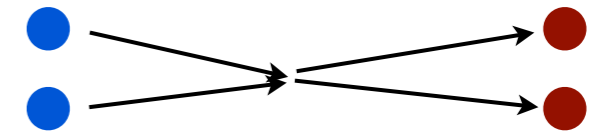
Already used in JLab study of
 $\pi K, \eta K$

$$\mathcal{M}(\pi K \rightarrow \eta K) \sim \sqrt{1 - \eta^2}$$

Wilson, Dudek, Edwards, Thomas,
Phys. Rev. D 91, 054008 (2015)
arXiv: 1411.2004



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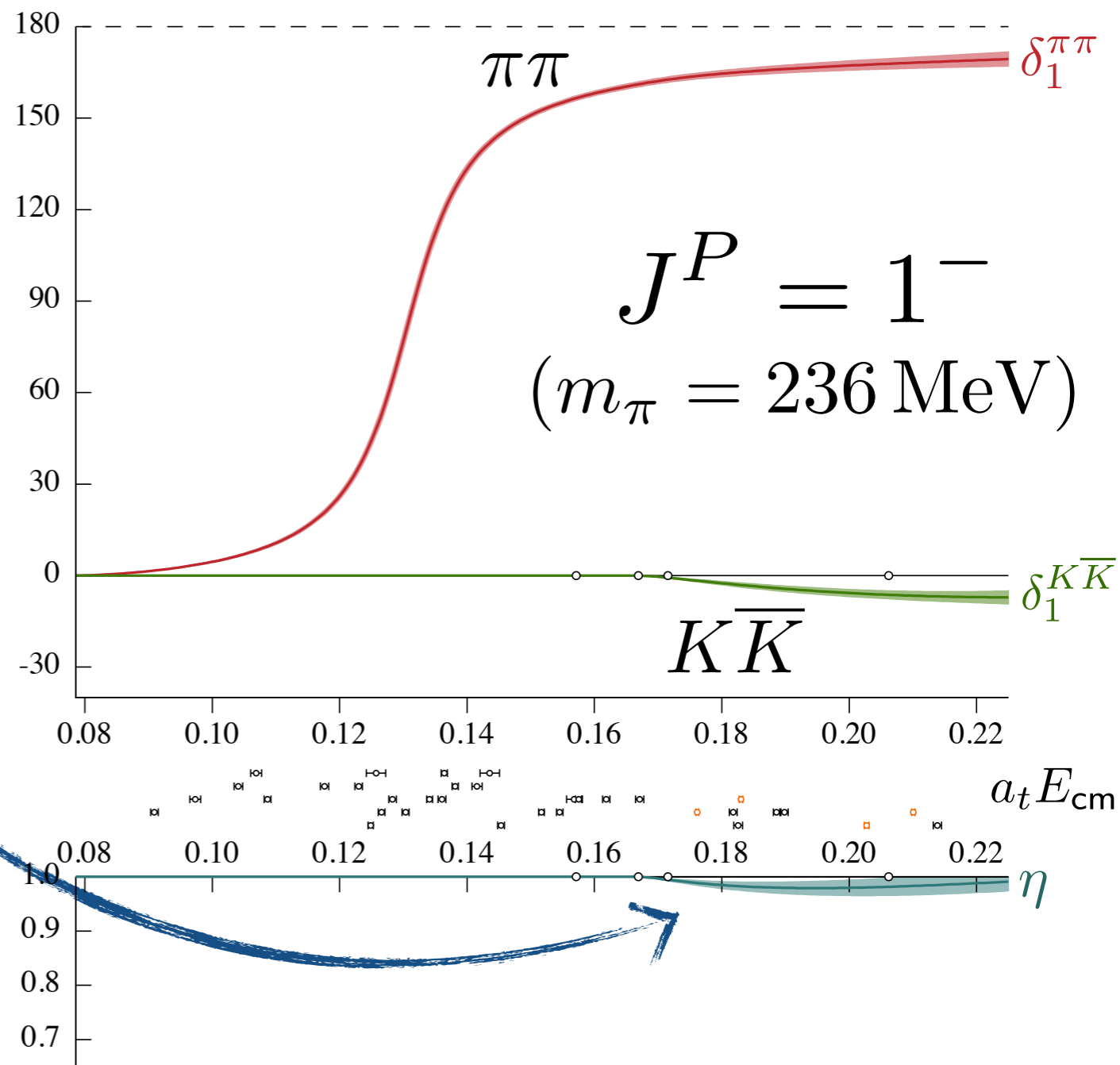
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As well as JLab rho study with
 $\pi\pi$, $K\bar{K}$

$$\mathcal{M}(\pi\pi \rightarrow K\bar{K}) \sim \sqrt{1 - \eta^2}$$

Wilson, Briceño, Dudek,
Edwards, Thomas,
arXiv:1507:02599



Two-particle scattering

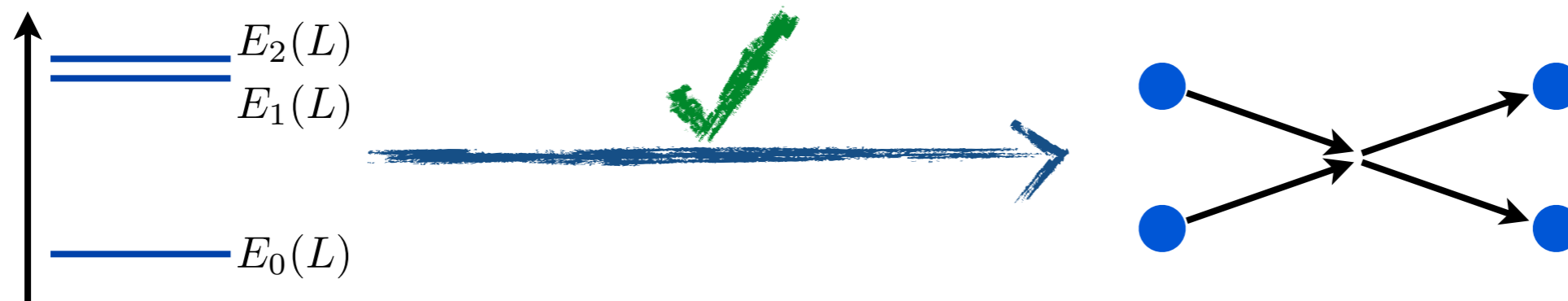
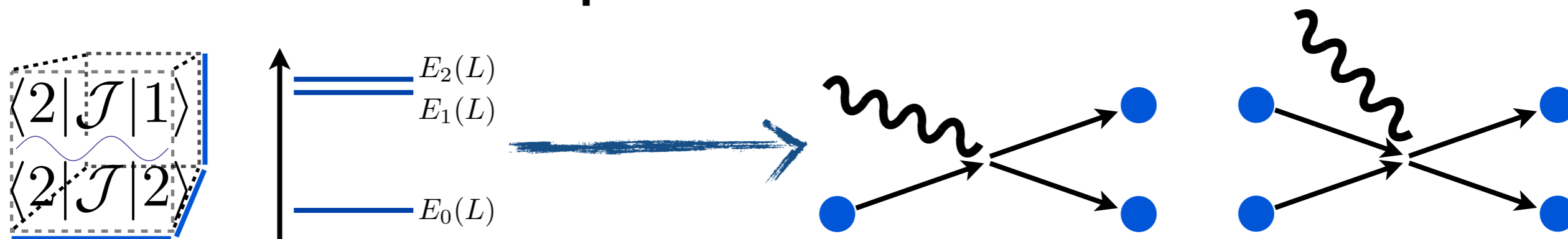
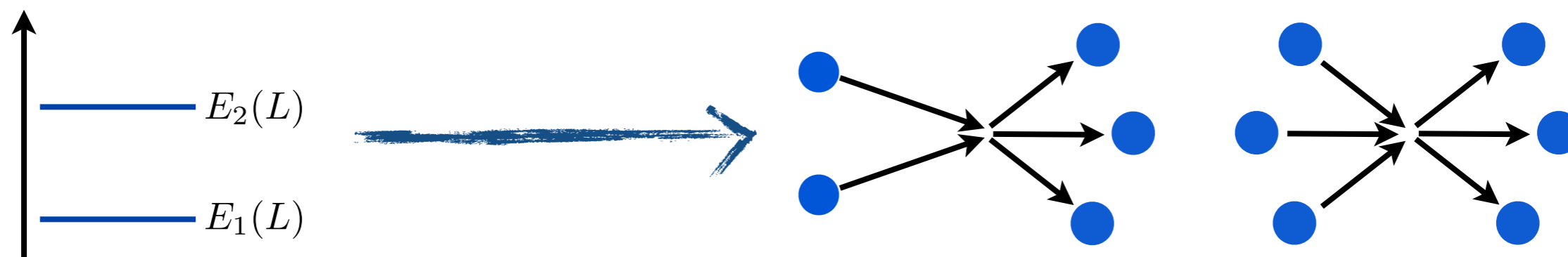


Photo- and electroproduction

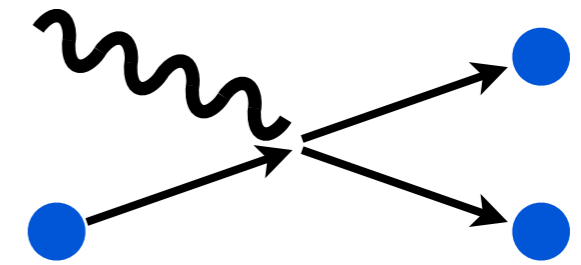


Three-particle scattering



Photoproduction

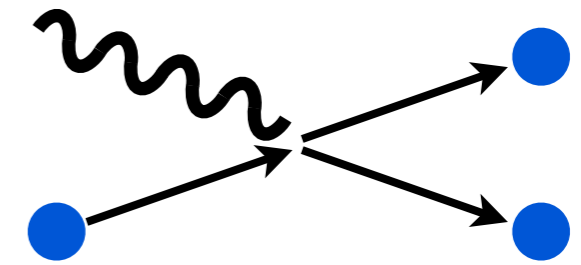
$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv$$



How can we get this from finite-volume observables?

Photoproduction

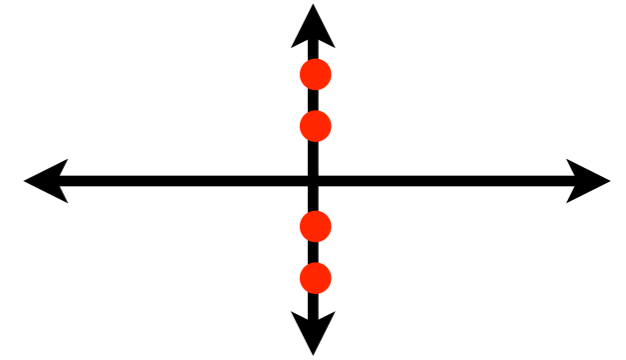
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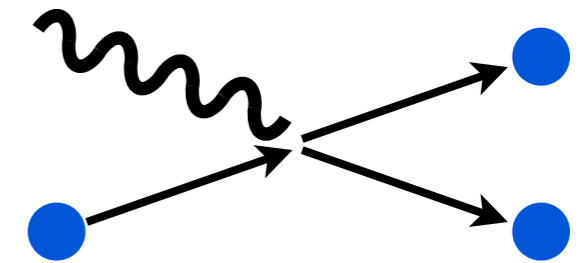
Why did we expect $C_L(P)$ to have poles?

$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$



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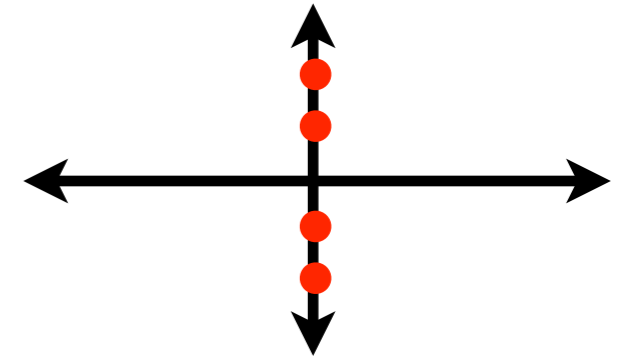
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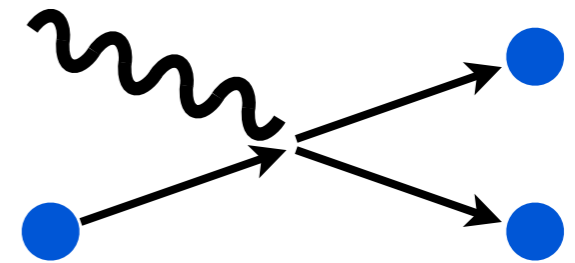
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Insert a complete set finite-volume of states

Photoproduction

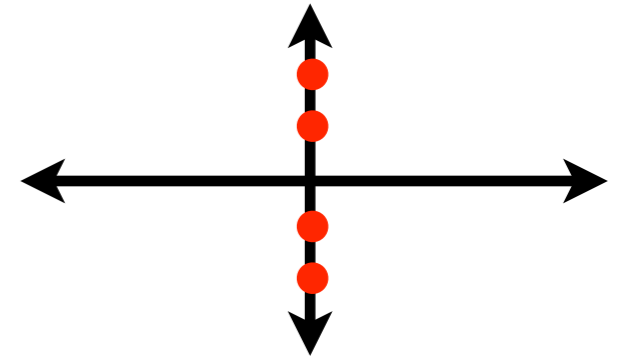
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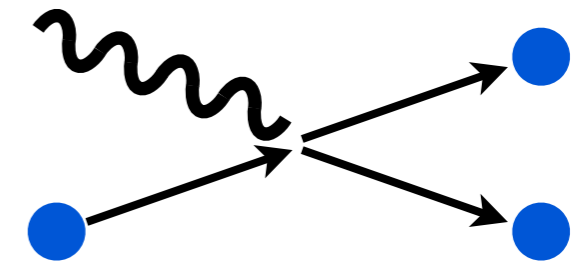


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$$C_L(P) \xrightarrow{P_4 \rightarrow iE_n} \frac{L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle}{(E_n + iP_4)}$$

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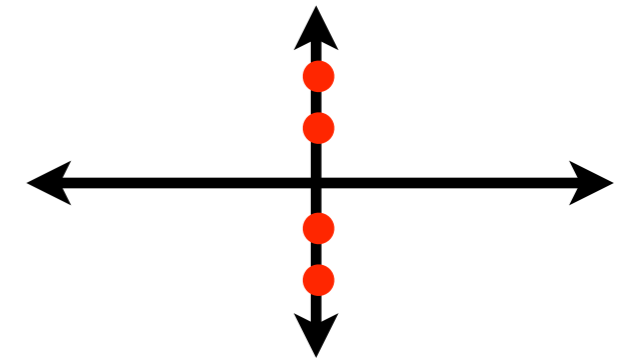
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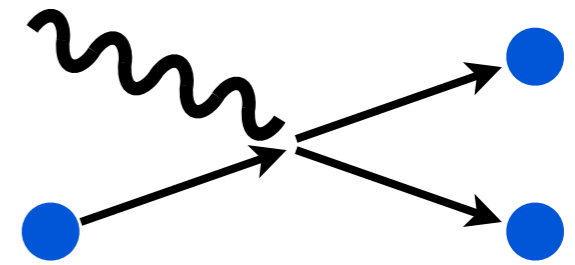
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Now compare this to our factorized result

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

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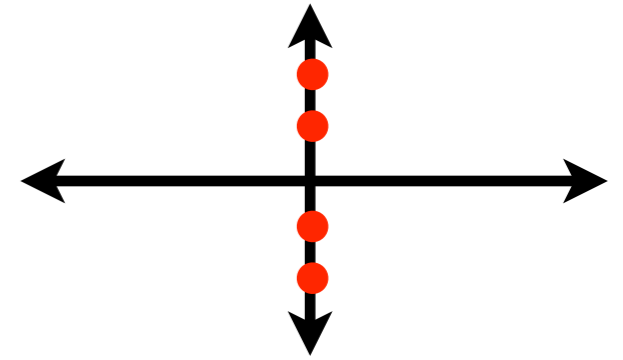
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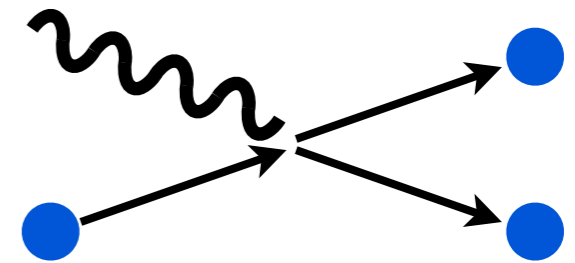
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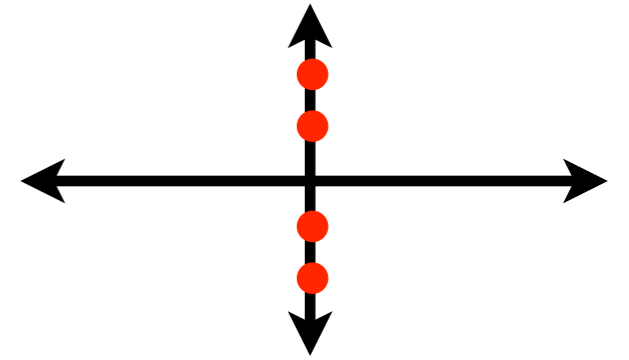
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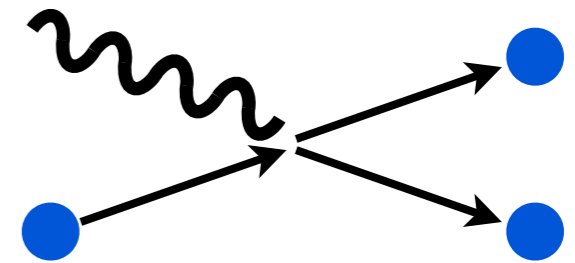
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\mathcal{R} is the residue of this matrix

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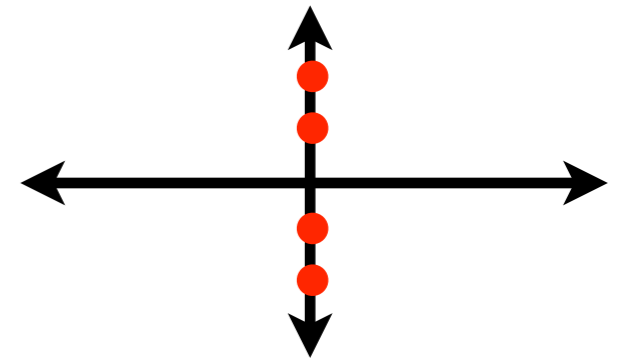
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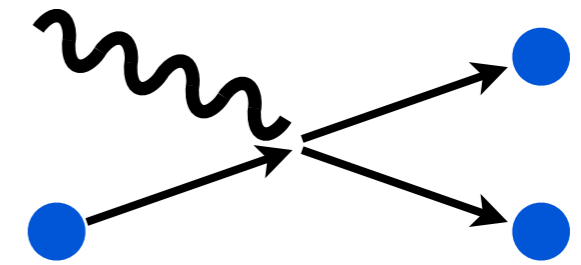
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Photoproduction

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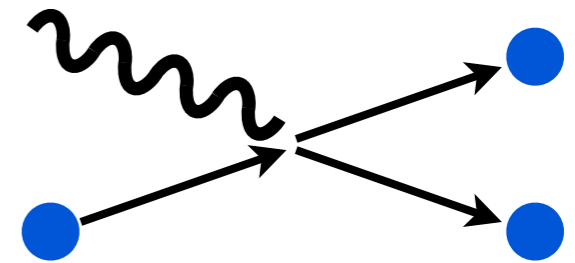
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$$L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle =$$

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Photoproduction

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How can we get this from finite-volume observables?

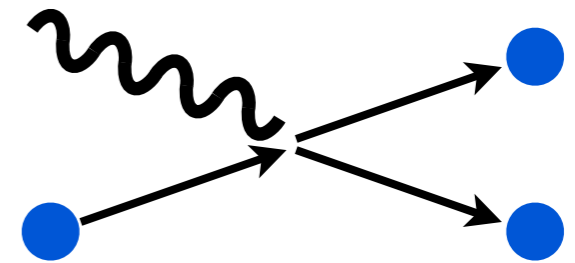
$$L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle =$$

$$\langle 0 | \mathcal{O}(0) | \pi\pi, \text{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \text{out} | \mathcal{O}^\dagger(0) | 0 \rangle$$

One has the freedom to choose \mathcal{O}^\dagger such that $\mathcal{O}^\dagger | 0 \rangle = \mathcal{J}_\mu | \pi \rangle$.
(Finite-volume effects are exponentially suppressed for single particles.)

Photoproduction

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$$2\omega_\pi L^6 |\langle n, \vec{P}, L | \mathcal{J}_\mu(0) | \pi, L \rangle|^2 =$$

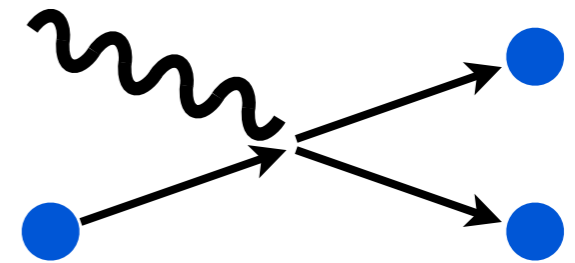
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R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev. D* **91**, 034501 (2015)

R. A. Briceño, MTH, *Phys. Rev. D* **92**, 074509 (2015)

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get this from the lattice

$$2\omega_\pi L^6 |\langle n, \vec{P}, L | \mathcal{J}_\mu(0) | \pi, L \rangle|^2 =$$

**experimental
observable**

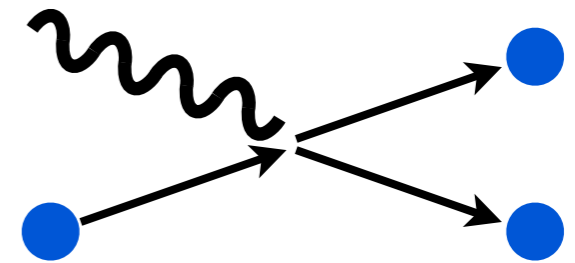
$$\langle \pi | \mathcal{J}_\mu(0) | \pi\pi, \text{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \text{out} | \mathcal{J}_\mu(0) | \pi \rangle$$

R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev. D* 91, 034501 (2015)

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Photoproduction

$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv$$



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get this from the lattice

$$2\omega_\pi L^6 |\langle n, \vec{P}, L | \mathcal{J}_\mu(0) | \pi, L \rangle|^2 =$$

**experimental
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$$\langle \pi | \mathcal{J}_\mu(0) | \pi\pi, \text{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \text{out} | \mathcal{J}_\mu(0) | \pi \rangle$$

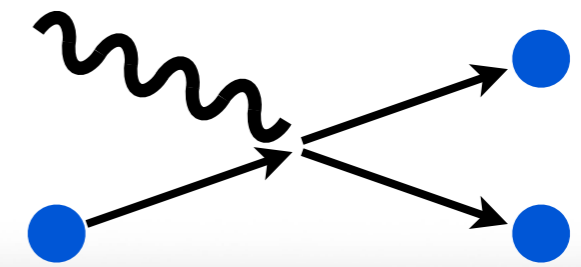
$$\mathcal{R}(E_n, \vec{P}, L) = -\text{Residue}_{E_n} \left[\frac{1}{F^{-1} + \mathcal{M}_{2 \rightarrow 2}} \right]$$

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Photoproduction

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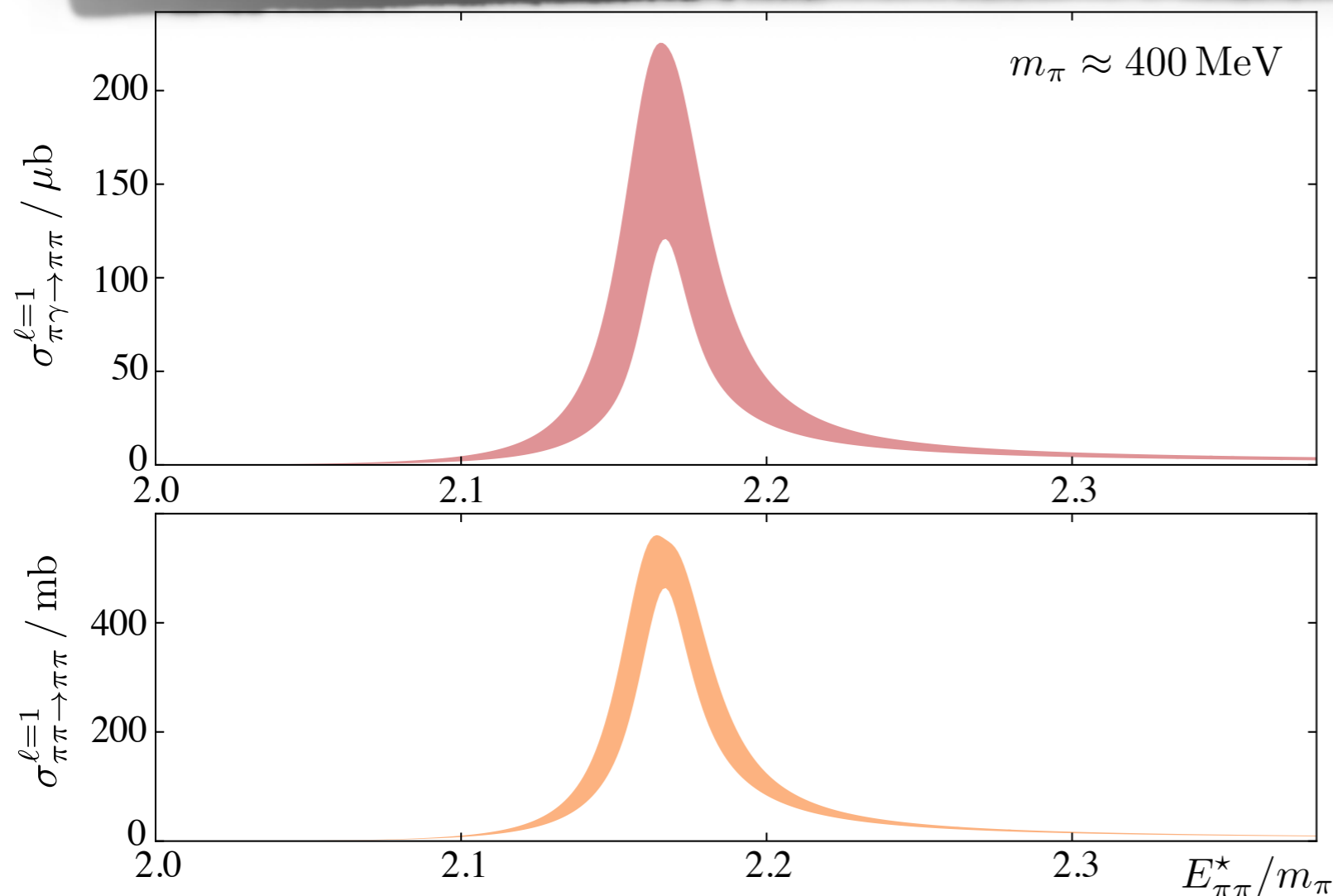
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Briceño, MTH, Walker-Loud/Briceño, MTH

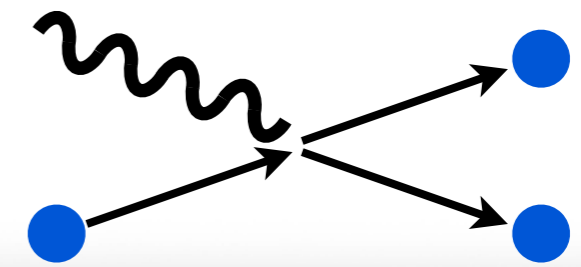


Photoproduction
in the rho channel

Briceño, Dudek, Edwards,
Schultz, Thomas, Wilson
arXiv: 1507.6622

Photoproduction

$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv$$



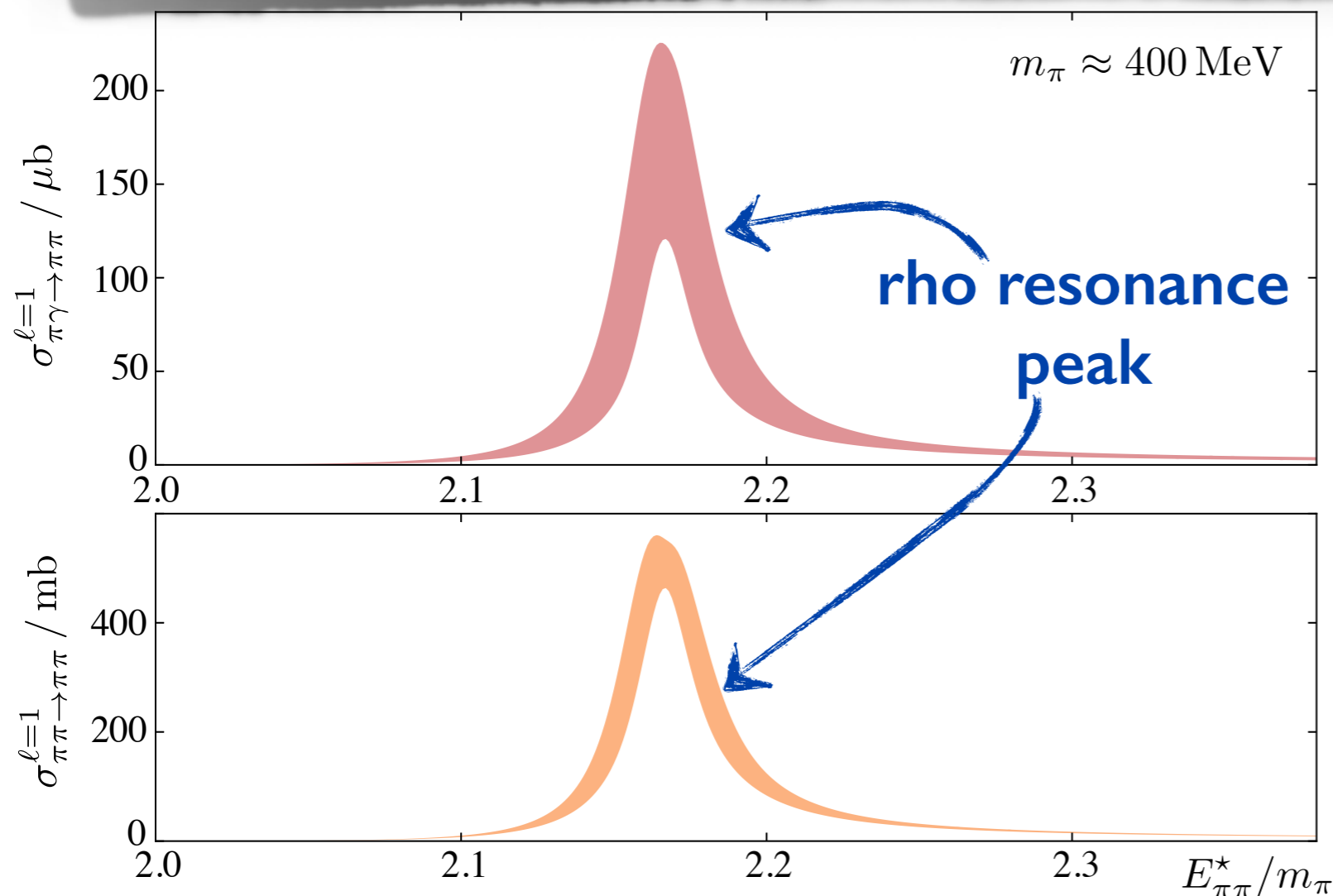
get this from the lattice

$$2\omega_\pi L^6 |\langle n, \vec{P}, L | \mathcal{J}_\mu(0) | \pi, L \rangle|^2 =$$

experimental
observable

$$\langle \pi | \mathcal{J}_\mu(0) | \pi\pi, \text{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \text{out} | \mathcal{J}_\mu(0) | \pi \rangle$$

Briceño, MTH, Walker-Loud/Briceño, MTH

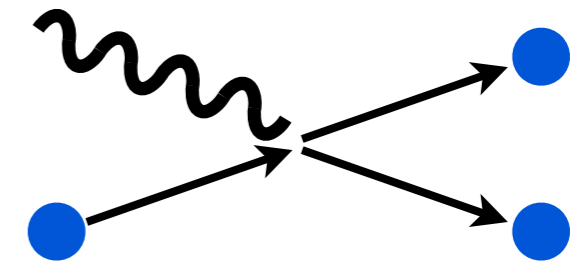


Photoproduction
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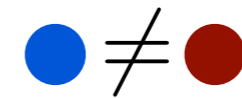
Photoproduction

$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv$$

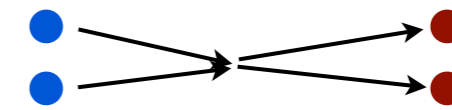


Result is very general

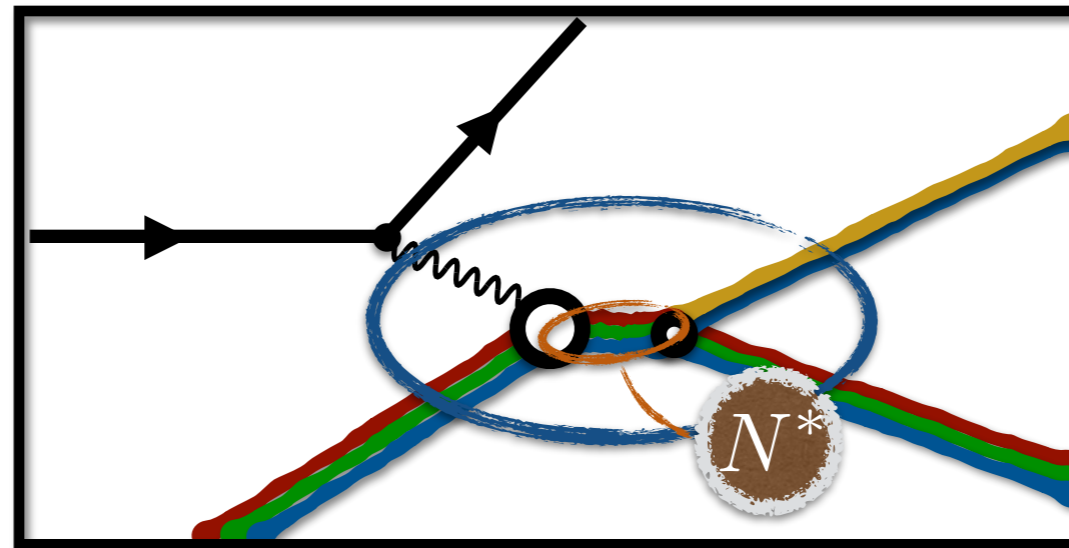
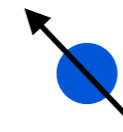
non-identical particles



multiple two-particle channels



particles with spin

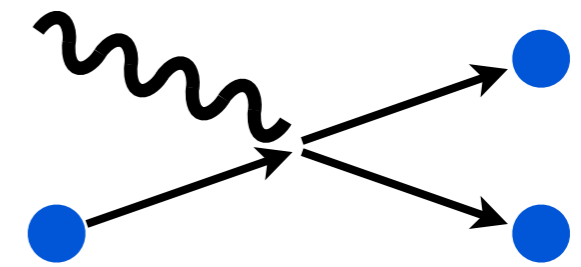


R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev. D* 91, 034501 (2015)

R. A. Briceño, MTH, *Phys. Rev. D* 92, 074509 (2015)

Photoproduction

$$\langle \pi\pi, \text{out} | \mathcal{T}_\mu | \pi \rangle \equiv$$



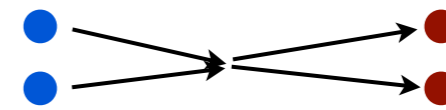
Result is very general

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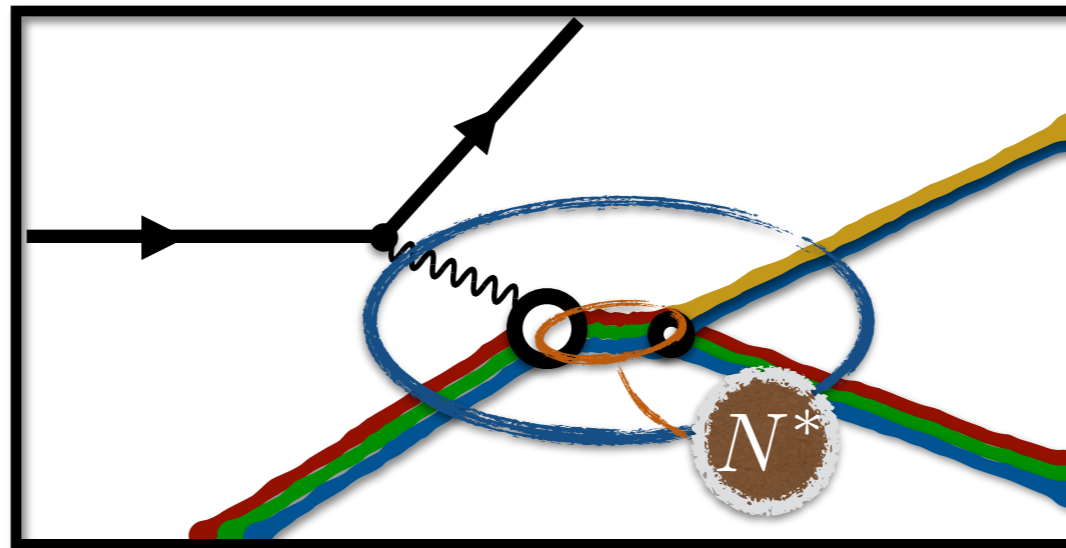
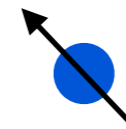
$$\bullet \neq \bullet$$



multiple two-particle channels



particles with spin



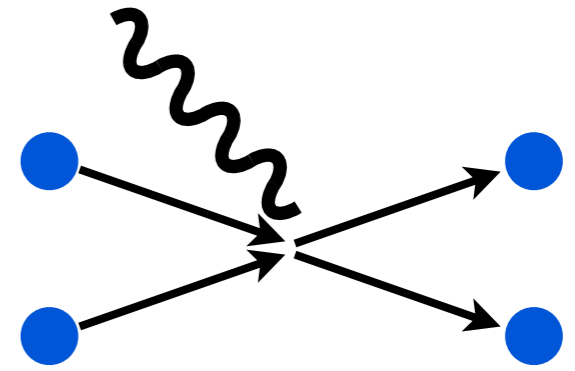
Formalism is in place to give Lattice QCD predictions of this process (ignoring three particles)

R. A. Briceño, MTH, A. Walker-Loud, *Phys. Rev. D* 91, 034501 (2015)

R. A. Briceño, MTH, *Phys. Rev. D* 92, 074509 (2015)

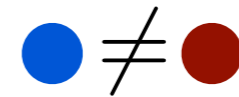
**Two-to-two
transitions**

$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi\pi, \text{in} \rangle \equiv$$

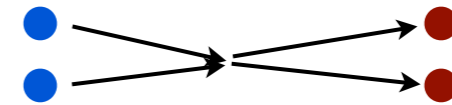


Formalism complete

non-identical particles



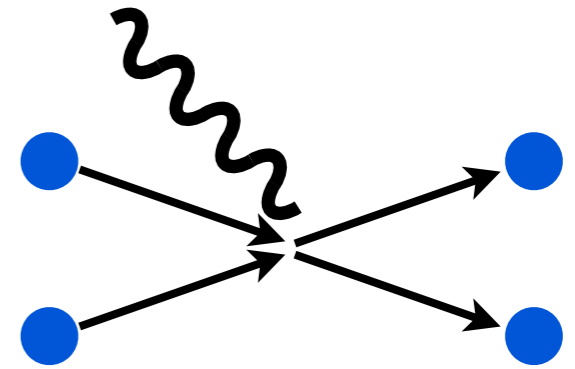
multiple two-particle channels



R. A. Briceño, MTH, arXiv: 1509.08507

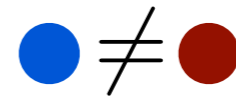
**Two-to-two
transitions**

$$\langle \pi\pi, \text{out} | \mathcal{J}_\mu | \pi\pi, \text{in} \rangle \equiv$$

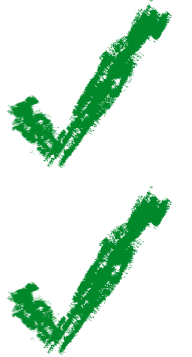
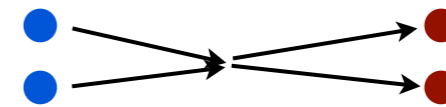


Formalism complete

non-identical particles

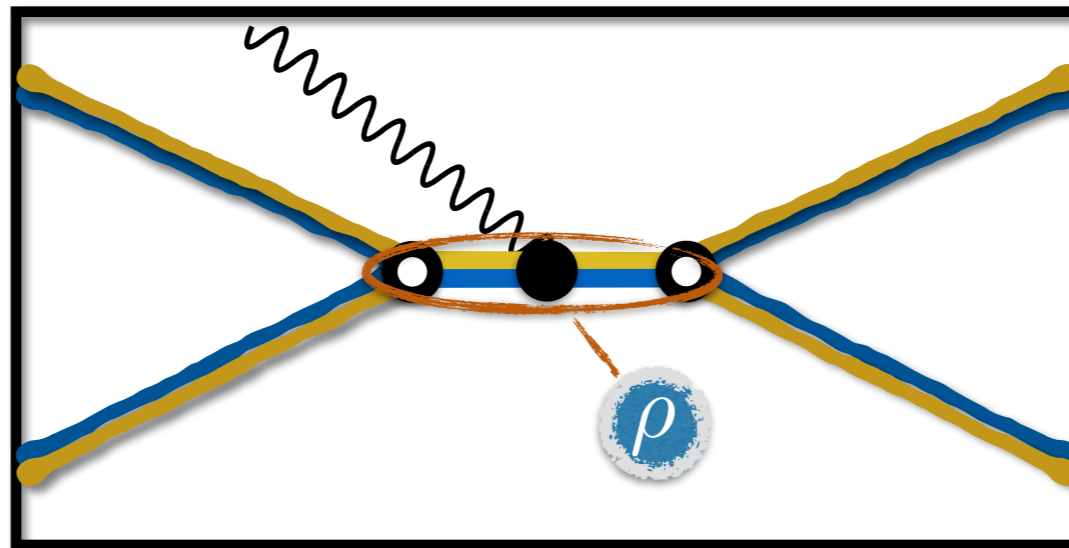


multiple two-particle channels



R. A. Briceño, MTH, arXiv: 1509.08507

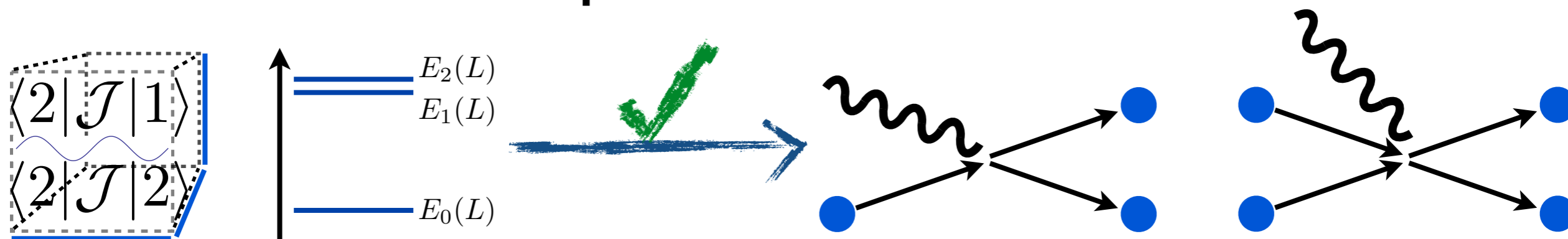
Required to extract resonance form factors



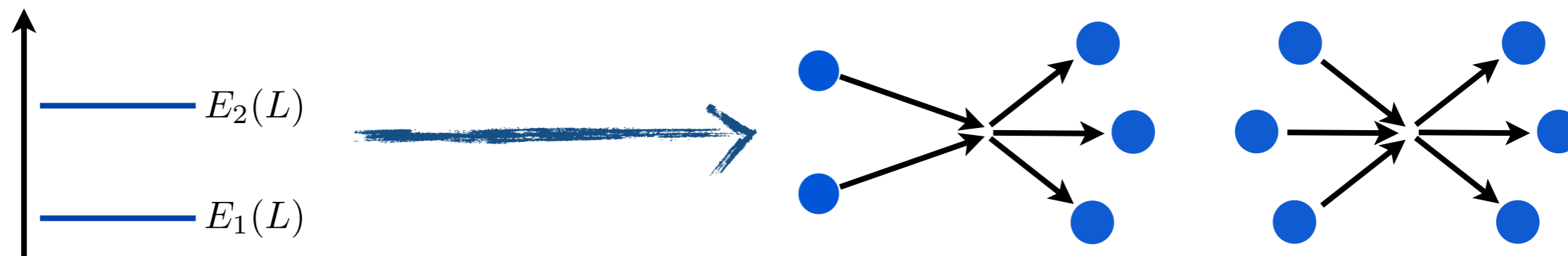
Two-particle scattering



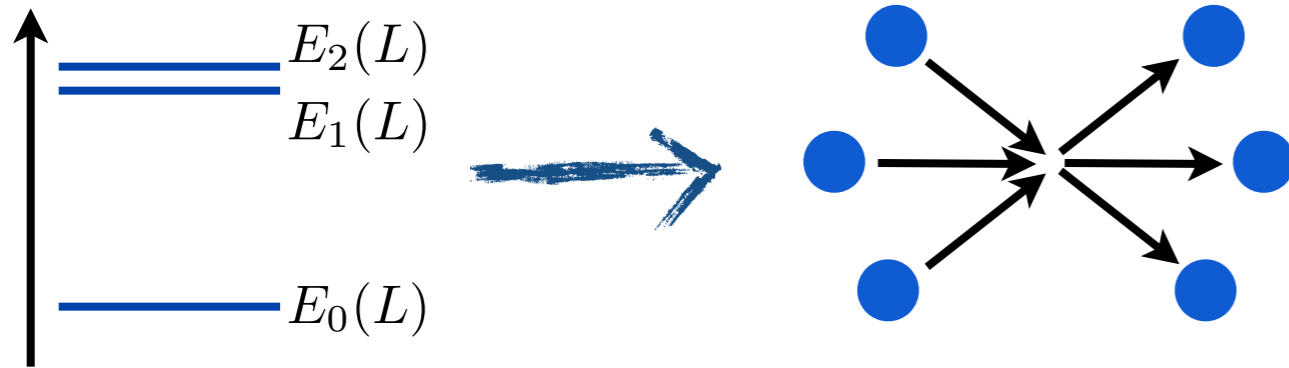
Photo- and electroproduction



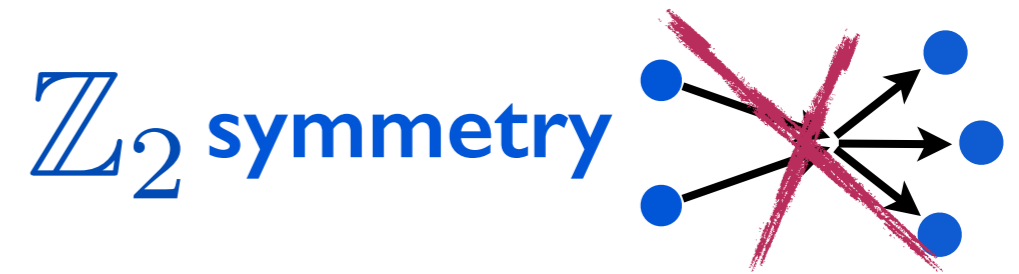
Three-particle scattering



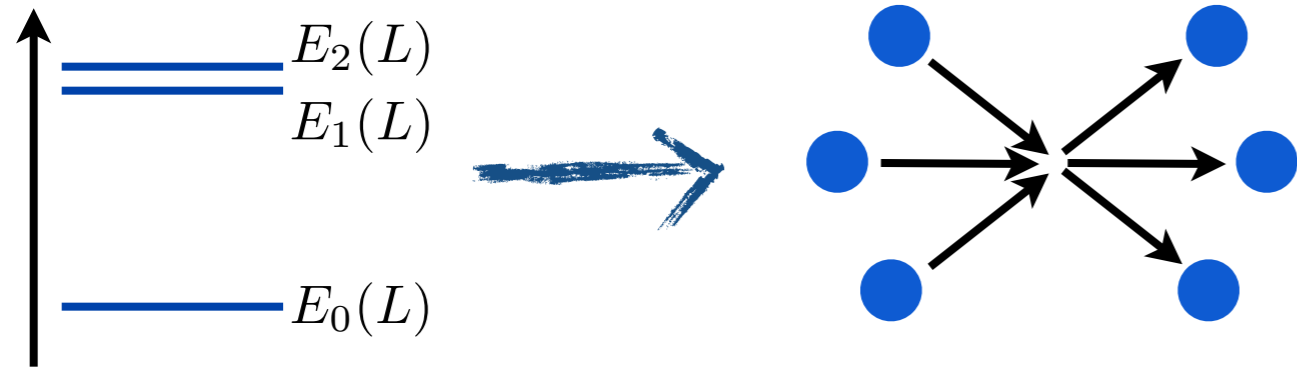
Three-to-three scattering



For now assume...
identical scalars, mass m

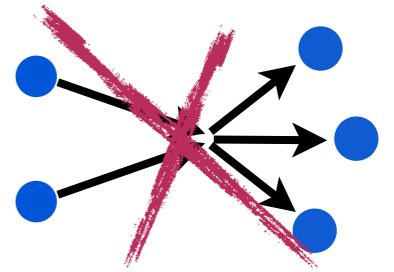


Three-to-three scattering



For now assume...
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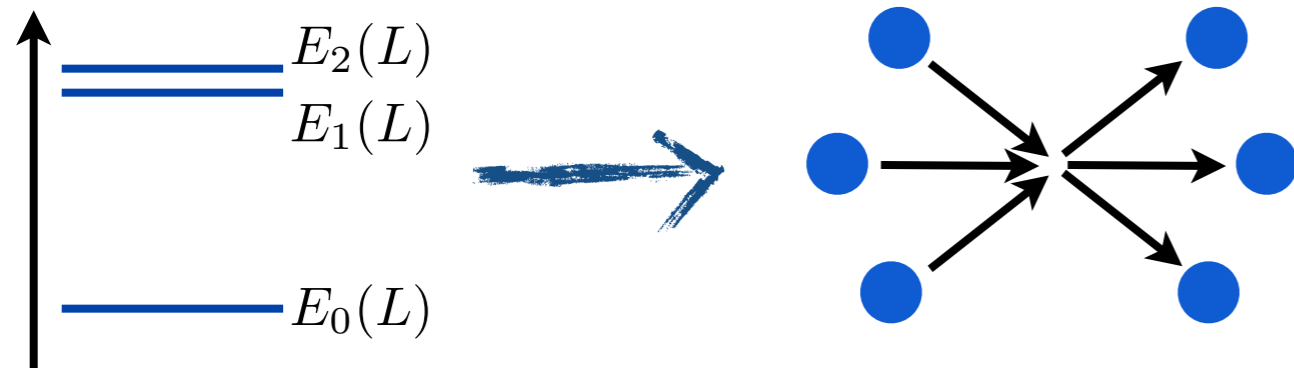
\mathbb{Z}_2 symmetry



$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

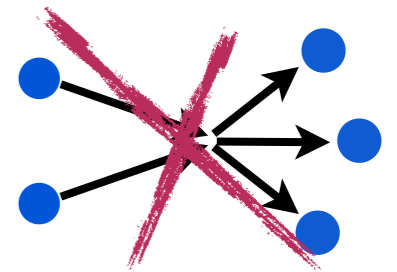
three-pion
interpolator

Three-to-three scattering



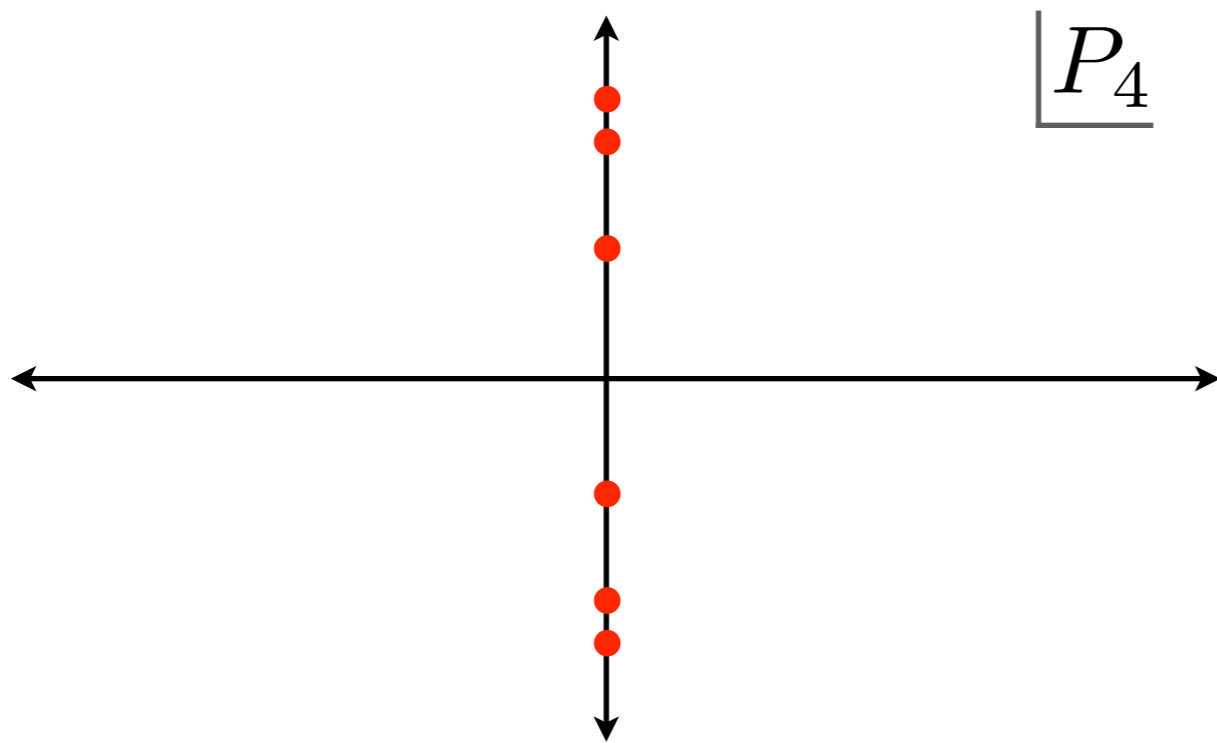
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\mathbb{Z}_2 symmetry



$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

three-pion
interpolator



Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.

Require $m < E^* < 5m$ to isolate three-particle states

Recall for two particles we started with a “skeleton expansion”

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

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$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

So now we need the same for three...

$$C_L(E, \vec{P}) \stackrel{?}{=} \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

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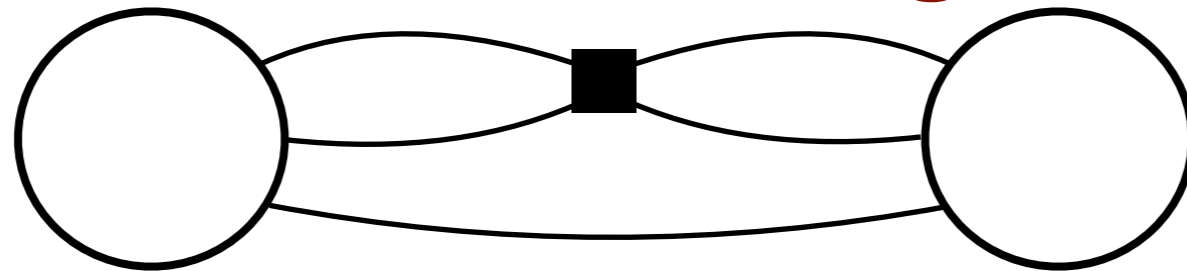
The diagram shows a series of terms in an expansion. Each term consists of two external circles labeled \mathcal{O}^\dagger and \mathcal{O} . The first term is a simple propagator with two vertices. The second term has a central circle labeled iK . The third term has two central circles labeled iK . Each vertex in these diagrams is connected to the external circles by two lines, and the internal vertices are connected to each other by two lines. Dashed boxes enclose the internal vertices in each term.

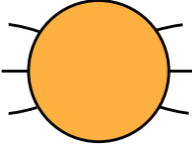
So now we need the same for three...

$$C_L(E, \vec{P}) \stackrel{?}{=} \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The diagram shows a series of terms in an expansion. Each term consists of three external circles. The first term is a simple propagator with two vertices. The second term has a central orange circle. The third term has two central orange circles. Each vertex in these diagrams is connected to the external circles by two lines, and the internal vertices are connected to each other by two lines. Dashed boxes enclose the internal vertices in each term.

No! We also need diagrams like



Disconnected diagrams in  lead to singularities that invalidate the derivation

New skeleton expansion

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

$$+ \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots$$

The diagrams in the expansion are:

- Diagram 1: Two white circles connected by two arcs, enclosed in a dashed box.
- Diagram 2: A white circle connected to an orange circle, which is connected to another white circle, all enclosed in a dashed box.
- Diagram 3: A white circle connected to two orange circles, which are connected to another white circle, all enclosed in a dashed box.
- Diagram 4: A white circle connected to a purple circle, which is connected to another white circle, all enclosed in a dashed box.
- Diagram 5: A white circle connected to two purple circles, which are connected to another white circle, all enclosed in a dashed box.
- Diagram 6: A white circle connected to three purple circles, which are connected to another white circle, all enclosed in a dashed box.

Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The diagrams in the kernel definition are:

- Diagram 1: A purple circle with four external lines (two on the left, two on the right).
- Diagram 2: A diagram with two vertices (black squares) connected by two arcs, with external lines at each vertex.
- Diagram 3: A diagram with two vertices connected by two arcs, with external lines at each vertex.

$$\text{Orange circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The diagrams in the kernel definition are:

- Diagram 1: An orange circle with four external lines (two on the left, two on the right).
- Diagram 2: A diagram with two vertices connected by a single line, with external lines at each vertex.
- Diagram 3: A diagram with two vertices connected by two arcs, with external lines at each vertex.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots
 \end{aligned}$$

The diagrams in the expansion are arranged in four rows. The first row contains three diagrams with orange nodes, the second row contains three diagrams with purple nodes, the third row contains three diagrams with purple nodes, and the fourth row contains two diagrams with purple nodes. Each diagram shows a sequence of nodes connected by lines, with dashed boxes highlighting specific sub-structures.

Kernel definitions:

$$\text{Purple Node} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The purple node is defined as the sum of three diagrams: a single vertex, a vertex with two arcs, and a vertex with two arcs and a central line segment.

$$\text{Orange Node} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The orange node is defined as the sum of three diagrams: a single vertex, a vertex with two vertices connected by a line, and a vertex with two arcs and a central line segment.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

The diagrams in the expansion represent various skeleton structures. The first row shows skeletons with 0, 1, and 2 orange nodes. The second and third rows show skeletons with 1, 2, and 3 purple nodes. The fourth row shows skeletons with 1, 2, and 3 purple nodes and 1 orange node. Dashed boxes in the diagrams indicate the skeleton structure.

Kernel definitions:

$$\text{Purple Node} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The purple node is defined as the sum of three diagrams: a vertex with four external lines, a vertex with four external lines and two internal arcs, and a vertex with four external lines and two internal arcs in a different configuration.

$$\text{Orange Node} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The orange node is defined as the sum of three diagrams: a vertex with four external lines, a vertex with four external lines and two internal arcs, and a vertex with four external lines and two internal arcs in a different configuration.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

The diagrams in the expansion are Feynman diagrams with two external legs. The first row shows diagrams with orange circles (kernels) on the internal lines. The second and third rows show diagrams with purple circles (kernels) on the internal lines. The fourth row shows diagrams with both orange and purple kernels. Dashed boxes in the diagrams indicate the skeleton structure.

Kernel definitions:

$$\text{Purple Kernel} \equiv \text{Diagram A} + \text{Diagram B} + \text{Diagram C} + \dots$$

Diagram A: A purple circle with four external lines (two on the left, two on the right).

Diagram B: A circle with four external lines, containing two internal lines forming a lens shape.

Diagram C: A circle with four external lines, containing two internal lines forming a lens shape, with a small black square at the top vertex.

$$\text{Orange Kernel} \equiv \text{Diagram D} + \text{Diagram E} + \text{Diagram F} + \dots$$

Diagram D: An orange circle with four external lines.

Diagram E: A circle with four external lines, containing a horizontal line with a small black square at its center.

Diagram F: A circle with four external lines, containing two internal lines forming a lens shape.

Significantly more complicated than two-particle story

Three-to-three scattering

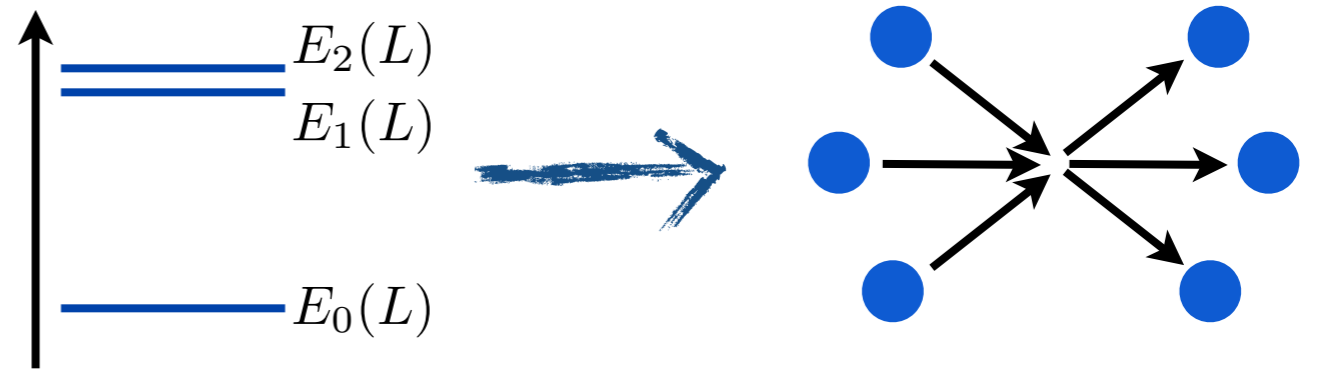


1. Work out the three particle skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots
 \end{aligned}$$

The expansion consists of three rows of diagrams. The first row contains three diagrams with orange circles: the first has one orange circle, the second has two orange circles, and the third has three orange circles. The second row contains three diagrams with purple circles: the first has one purple circle, the second has two purple circles, and the third has three purple circles. The third row contains three diagrams with purple circles: the first has two purple circles, the second has three purple circles, and the third has four purple circles. Each diagram shows a chain of white circles connected by lines, with the specified colored circles inserted into the chain. Dotted boxes highlight the internal structure of each diagram.

Three-to-three scattering



1. Work out the three particle skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots
 \end{aligned}$$

The diagrams in the expansion are:

- Diagram 1: Two white circles connected by a double line, with a dashed box around the connection.
- Diagram 2: A white circle, a double line, an orange circle, a double line, and another white circle, with dashed boxes around the connections.
- Diagram 3: A white circle, a double line, an orange circle, a double line, another orange circle, a double line, and a white circle, with dashed boxes around the connections.
- Diagram 4: A white circle, a double line, a purple circle, a double line, and another white circle, with a dashed box around the connection.
- Diagram 5: A white circle, a double line, two purple circles, a double line, and another white circle, with a dashed box around the connection.
- Diagram 6: A white circle, a double line, three purple circles, a double line, and another white circle, with a dashed box around the connection.
- Diagram 7: A white circle, a double line, two purple circles, a double line, another white circle, a double line, and a third white circle, with a dashed box around the first connection.
- Diagram 8: A white circle, a double line, three purple circles, a double line, another white circle, a double line, and a third white circle, with a dashed box around the first connection.
- Diagram 9: A white circle, a double line, four purple circles, a double line, another white circle, a double line, and a third white circle, with a dashed box around the first connection.

2. Break diagrams into finite- and infinite-volume parts

Three-to-three scattering



1. Work out the three particle skeleton expansion

$$C_L(E, \vec{P}) = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} + \dots$$

$$+ \text{[Diagram 4]} + \text{[Diagram 5]} + \text{[Diagram 6]} + \dots$$

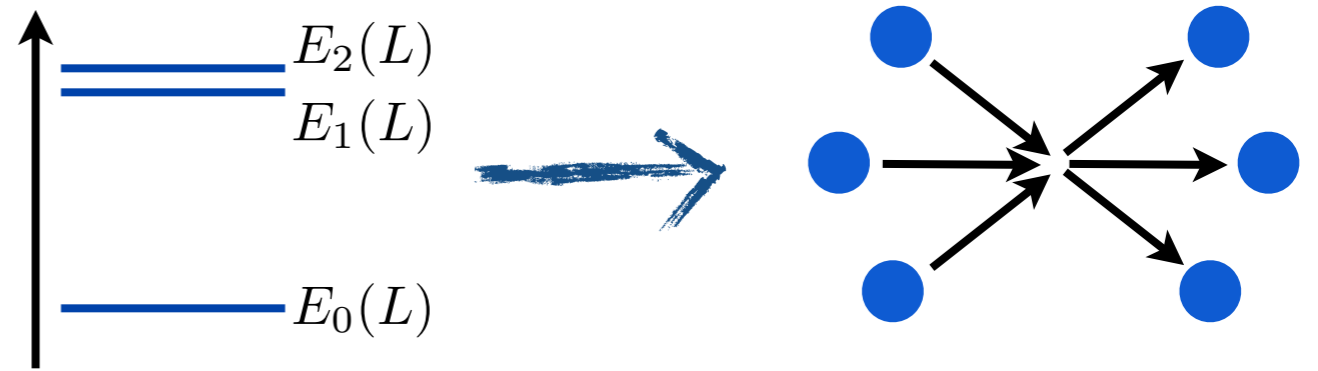
$$+ \text{[Diagram 7]} + \text{[Diagram 8]} + \text{[Diagram 9]} + \dots$$

The diagrams in the expansion represent various particle interactions. The first row shows diagrams with orange circles, the second row with purple circles, and the third row with purple circles. Each diagram consists of a chain of circles connected by lines, with some circles highlighted in color. Dotted lines indicate the boundaries of the diagrams.

2. Break diagrams into finite- and infinite-volume parts

3. Organize and sum terms to identify
infinite-volume observables

Three-to-three scattering



1. Work out the three particle skeleton expansion

$$C_L(E, \vec{P}) = \text{[Diagrammatic expansion of } C_L(E, \vec{P}) \text{ as a sum of skeleton diagrams with orange and purple particles]} + \dots$$

The diagrammatic expansion shows three rows of terms. The first row contains diagrams with orange particles (representing two-particle states) connected by lines, with dashed boxes indicating sub-diagrams. The second row contains diagrams with purple particles (representing three-particle states) connected by lines, also with dashed boxes. The third row contains more complex diagrams with multiple purple particles. Each term is separated by a plus sign and followed by an ellipsis.

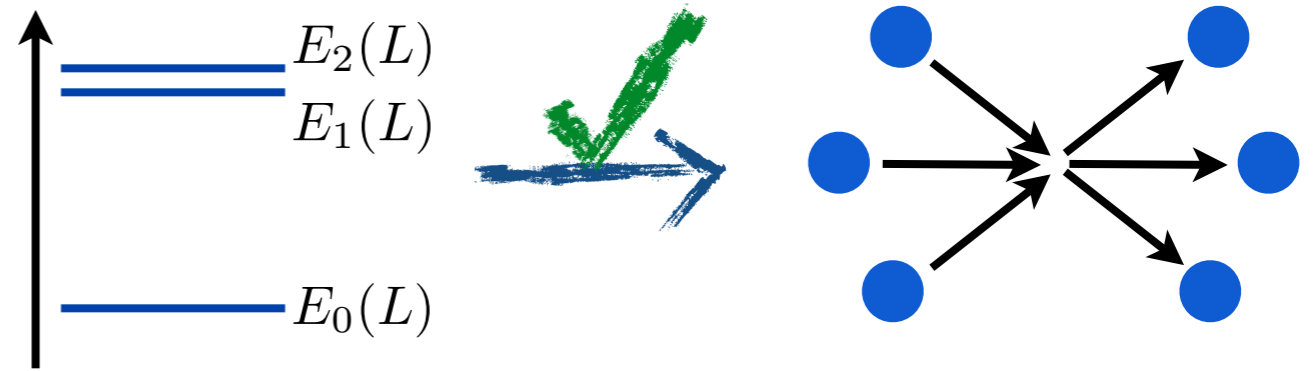
2. Break diagrams into finite- and infinite-volume parts

3. Organize and sum terms to identify
infinite-volume observables

Major complicating factors:

More diagram topologies, more degrees of freedom,
three-to-three amplitude contains **“long distance” kinematic poles**

Three-to-three scattering



Current status:

Formalism is complete for the simplest three-scalar system

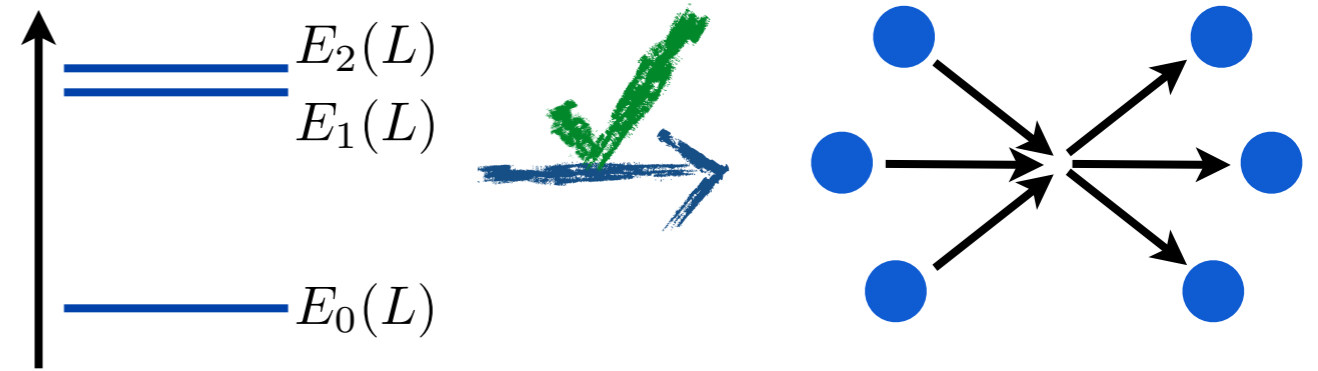
General, model-independent relation between finite-volume energies and three-to-three scattering amplitude

Derived using a generic relativistic field theory

MTH and Sharpe, *Phys. Rev. D* 90, 116003 (2014)

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Important caveats:

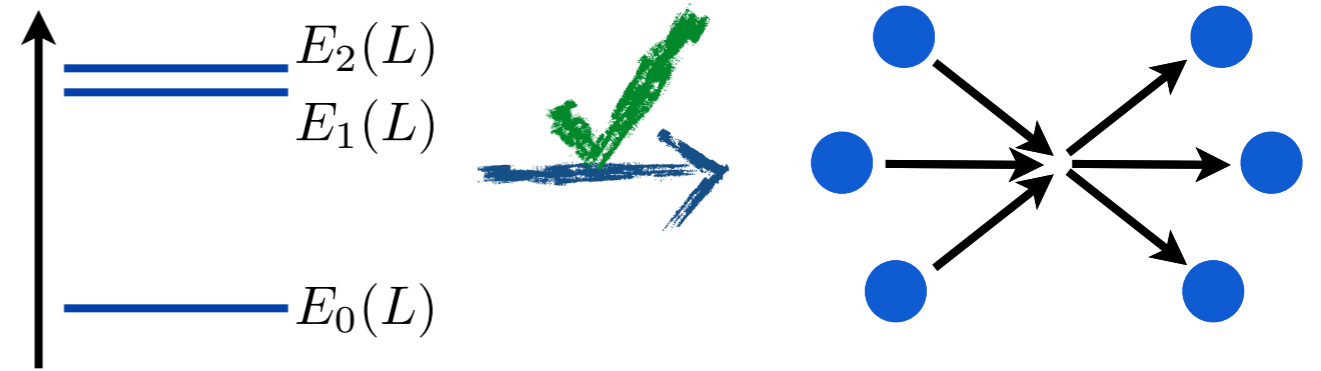
Identical particles with no two-to-three transitions

$$\pi\pi\pi \rightarrow \pi\pi\pi$$

Requires that two-particle scattering phase is bounded

$$|\delta_\ell(E)| < \pi/2$$

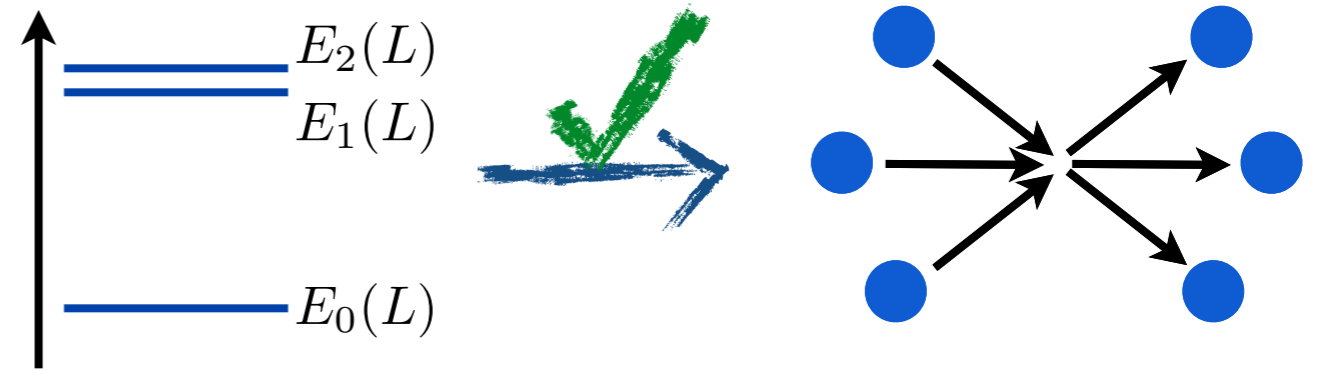
Three-to-three scattering



To check the result we have expanded the lowest three-particle energy in powers of $1/L$.

$$E = 3m + \frac{a_3}{L^3} + \frac{a_4}{L^4} + \frac{a_5}{L^5} + \frac{a_6}{L^6} + \mathcal{O}(1/L^7)$$

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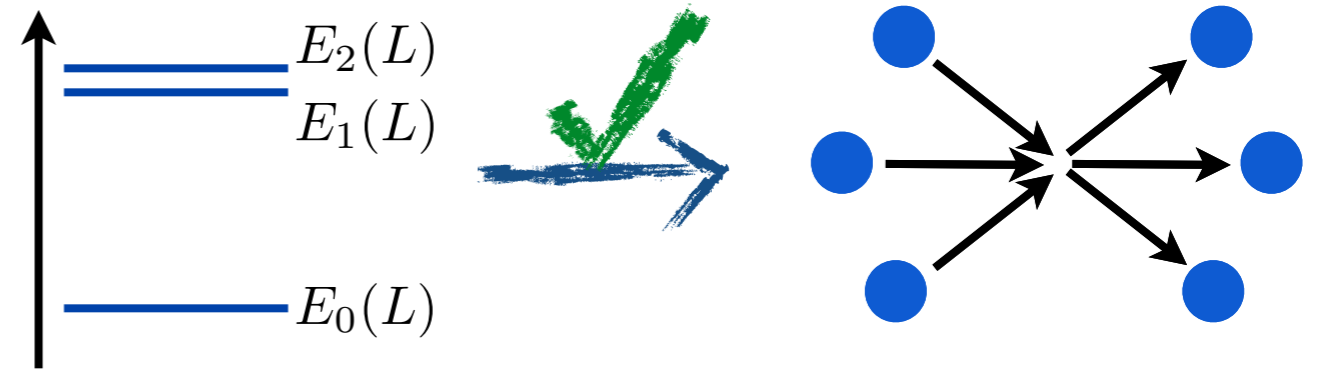
These terms were already known.

Our result agrees, providing a strong check

K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775

Beane, Detmold, Savage, *Phys. Rev.* D76 (2007) 074507

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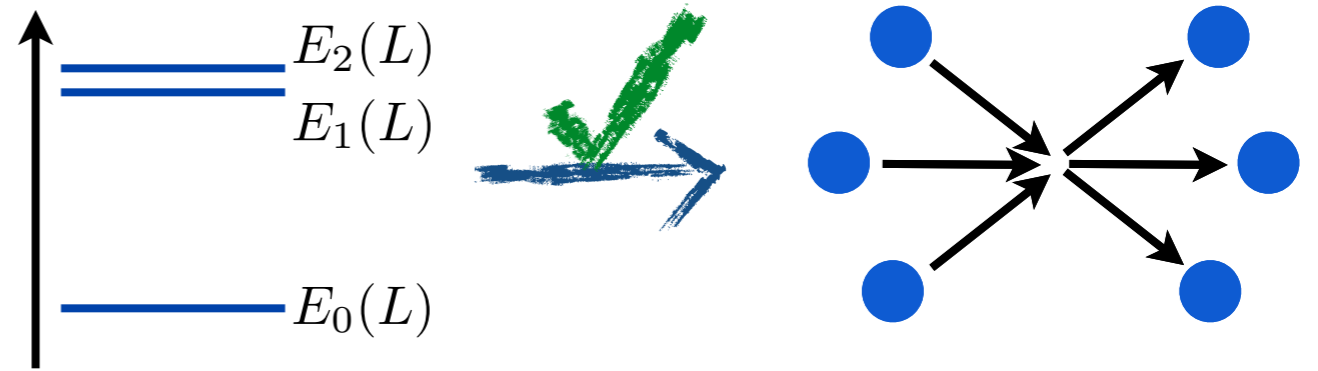
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This part is new... turns out that relativistic effects enter at this order...

$$\frac{a_6}{a_3} \equiv \left(\frac{a}{\pi} \right)^3 \left[2532.01 + \frac{16\pi^3}{3} (3\sqrt{3} - 4\pi) \log \left(\frac{mL}{2\pi} \right) \right] - 37.25 \frac{a^2}{m} + \frac{3\pi a}{m^2} + 6\pi r a^2 - \frac{\mathcal{M}_{3,\text{thr}}}{48m^3 a_3}$$

MTH and Sharpe, arXiv:1602.00324

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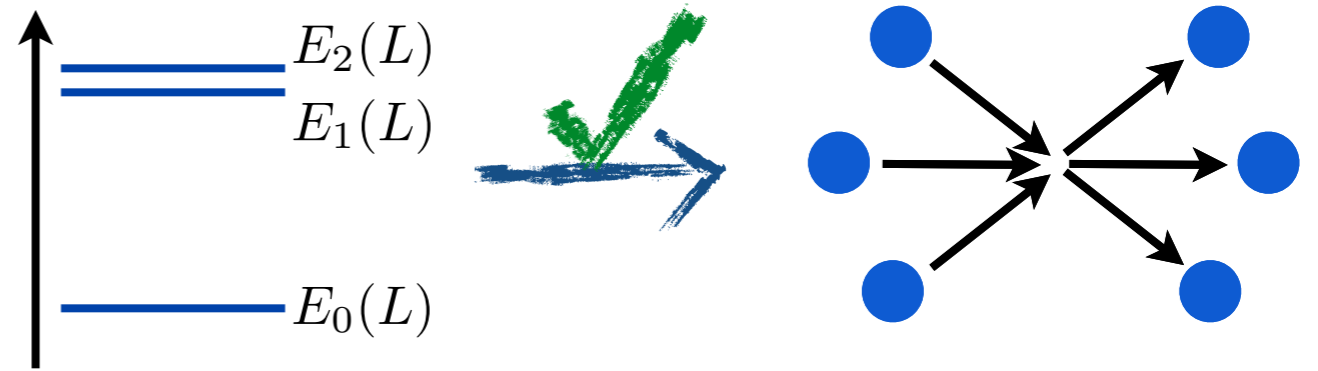
$$- 37.25 \frac{a^2}{m} + \frac{3\pi a}{m^2} + 6\pi r a^2 - \frac{\mathcal{M}_{3,\text{thr}}}{48m^3 a_3}$$

checked independently in $\lambda\phi^4$ theory through $\mathcal{O}(\lambda^3)$

MTH and Sharpe, *Phys. Rev. D* 93, 014506 (2016)

MTH and Sharpe, arXiv:1602.00324

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relativistic three-particle observable

Add in a known “long distance” piece to get the standard amplitude.

MTH and Sharpe, arXiv:1602.00324

Currently underway:

Relax all simplifying assumptions:

**Allow all particle types, allow two-to-three couplings,
remove bound on phase shift**

$$K\pi \rightarrow K\pi\pi \quad N\pi \rightarrow N\pi\pi \quad NNN \rightarrow NNN$$

Briceño, MTH, Sharpe, *in development*

Currently underway:

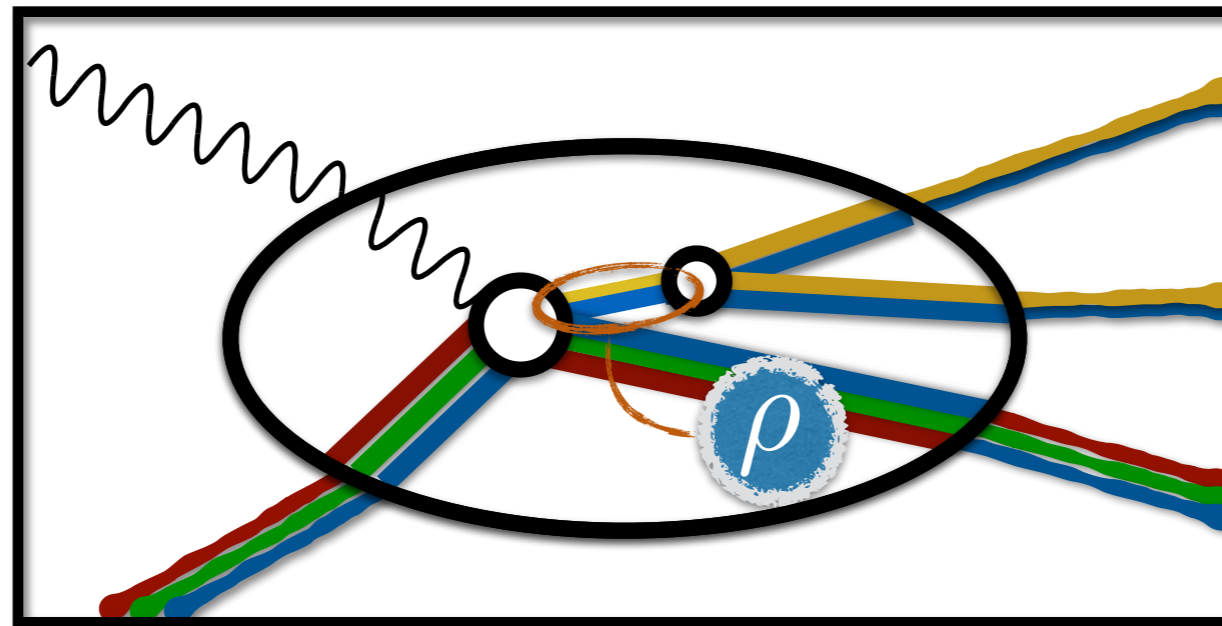
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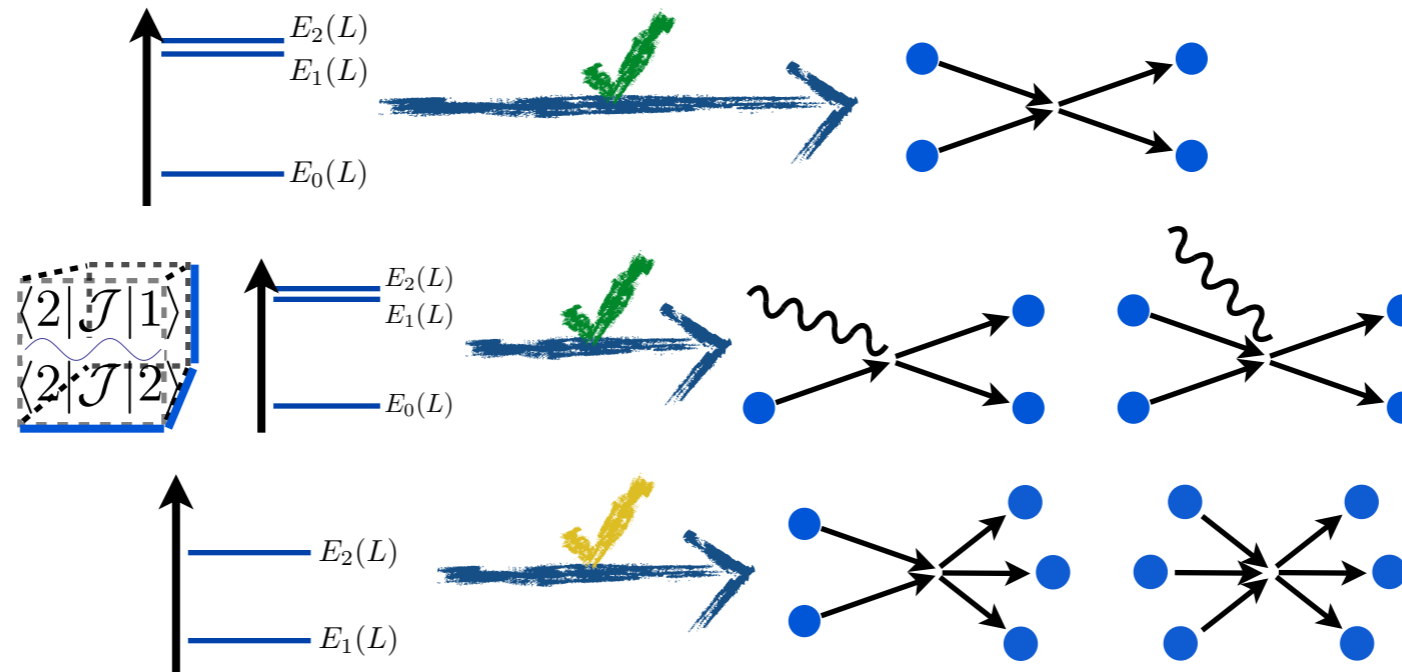
Use matching trick to recover transition amplitudes



$$p\gamma \rightarrow N\rho \rightarrow N\pi\pi$$

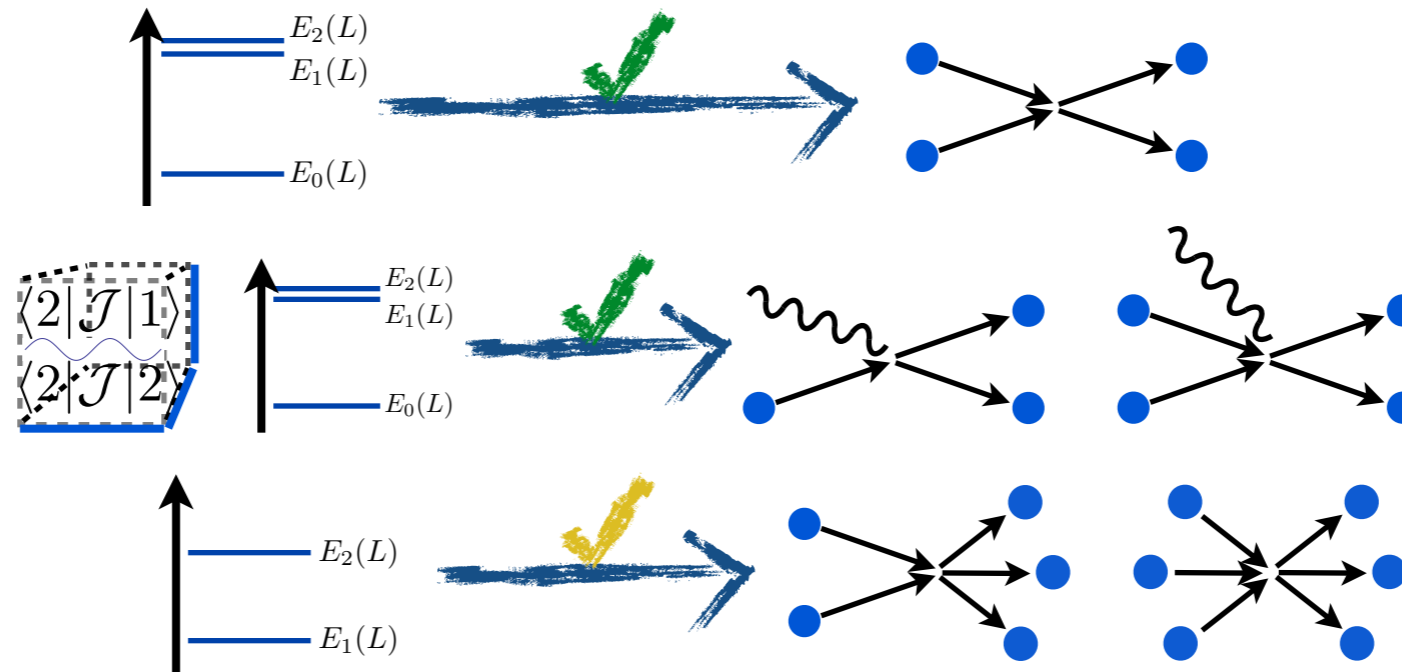
Summary

Reviewed methods to map finite-volume observables into physically observable scattering and transition amplitudes



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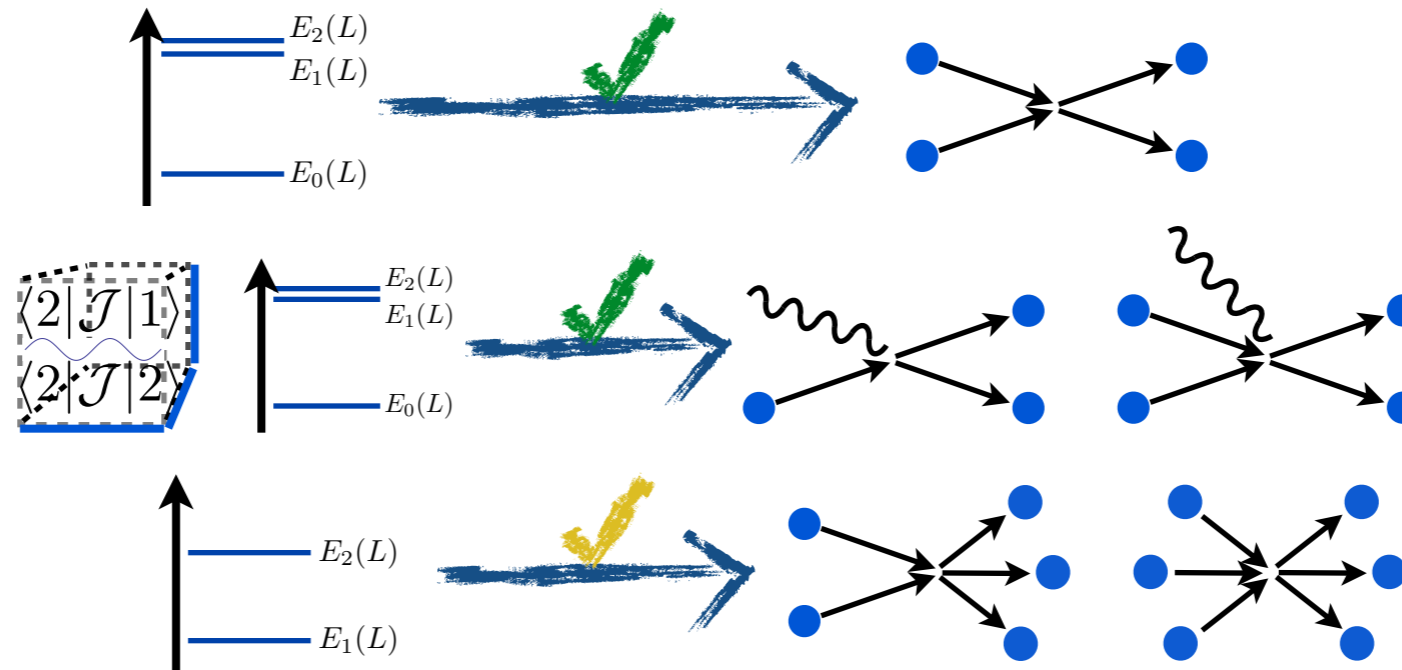


Results come from studying all-orders expansions in generic relativistic quantum field theory

The work is technical and requires developing new tools and methods for each new system

Summary

Reviewed methods to map finite-volume observables into physically observable scattering and transition amplitudes



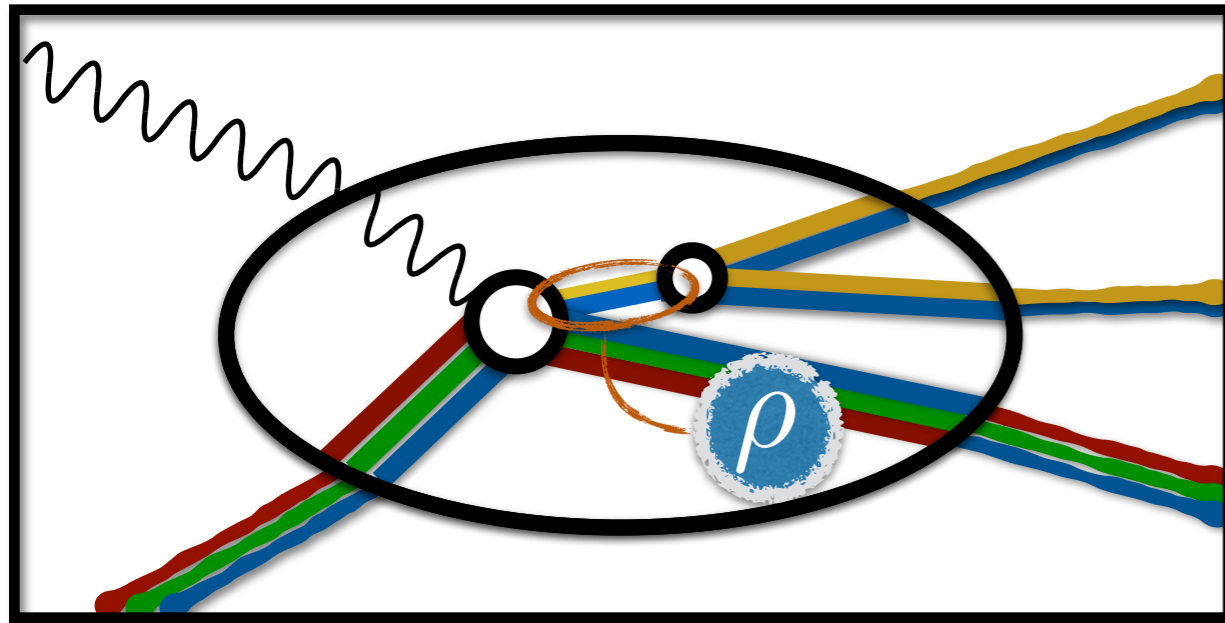
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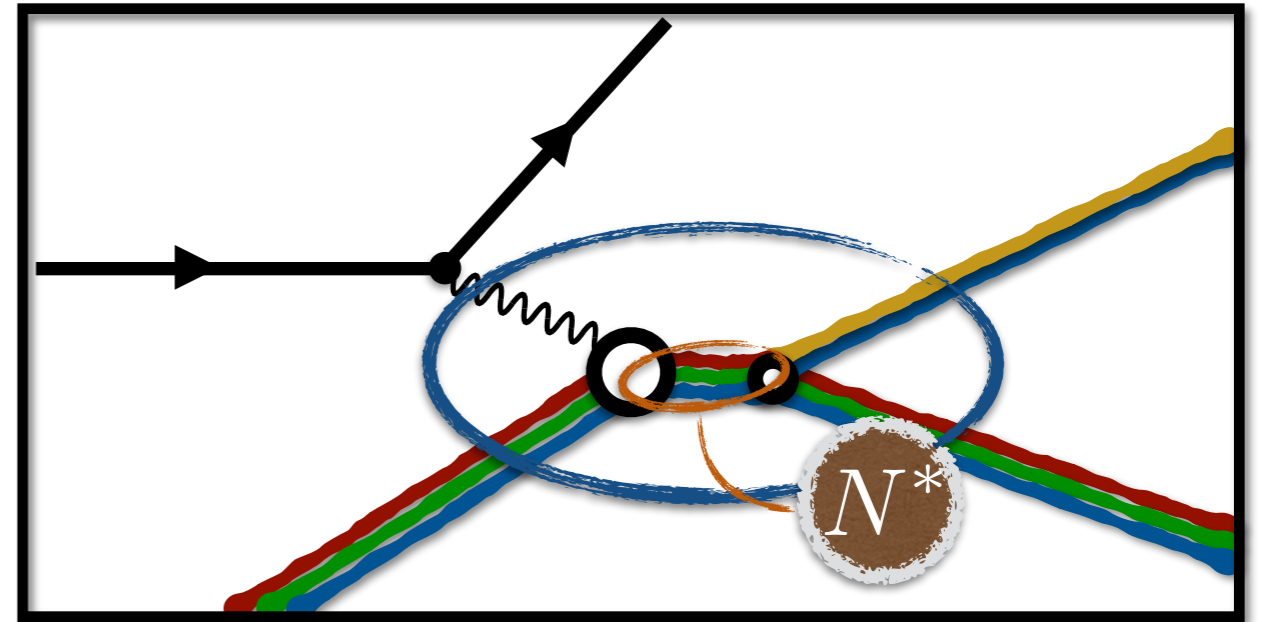
Can the scattering and transition amplitudes of QCD be extracted from Lattice QCD in a general, model independent way?

So far all signs point to **yes!**

My work at JLab:



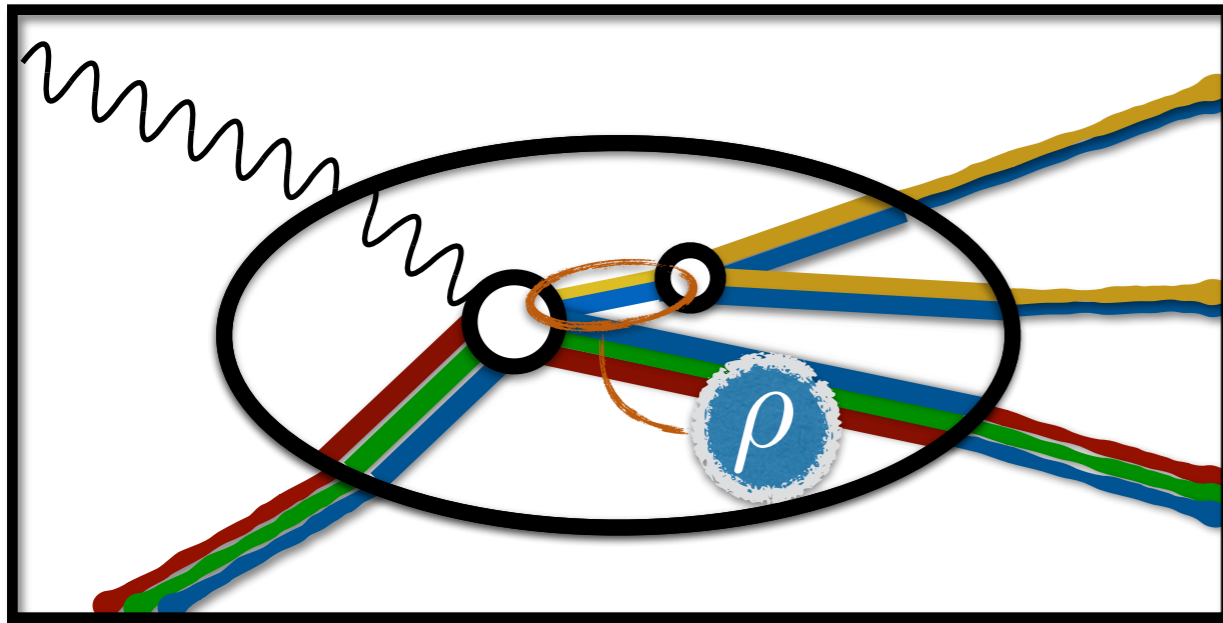
$$p\gamma \rightarrow N\rho \rightarrow N\pi\pi$$



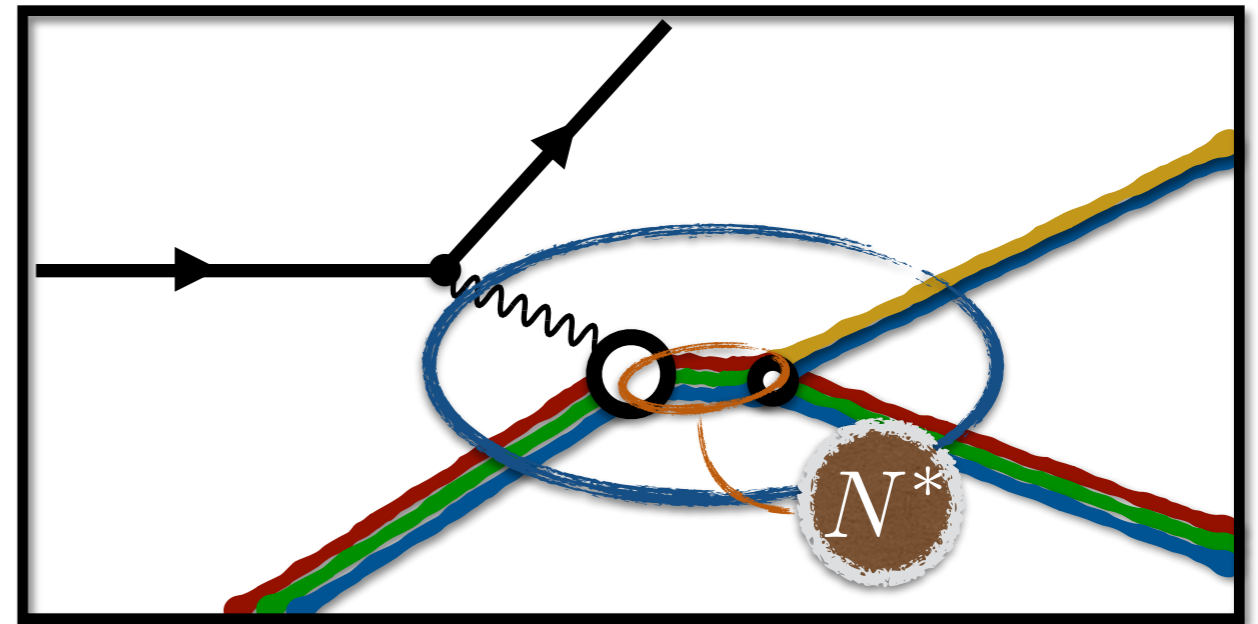
$$p\gamma^* \rightarrow N^* \rightarrow N\pi, N\eta$$

Experimental groups at JLab are measuring exactly the kinds of processes accommodated by this formalism.

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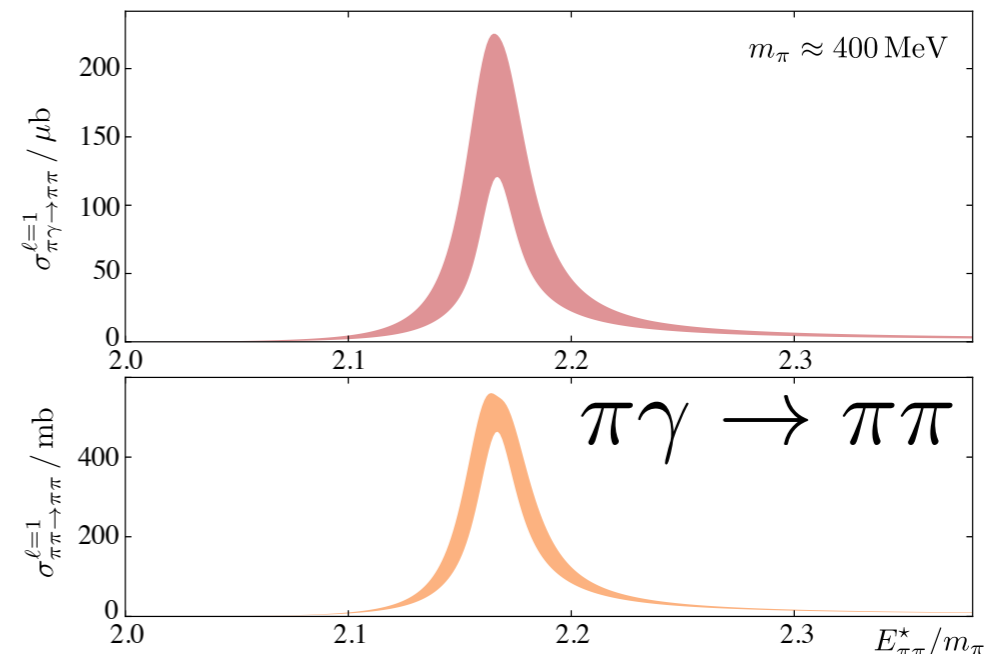
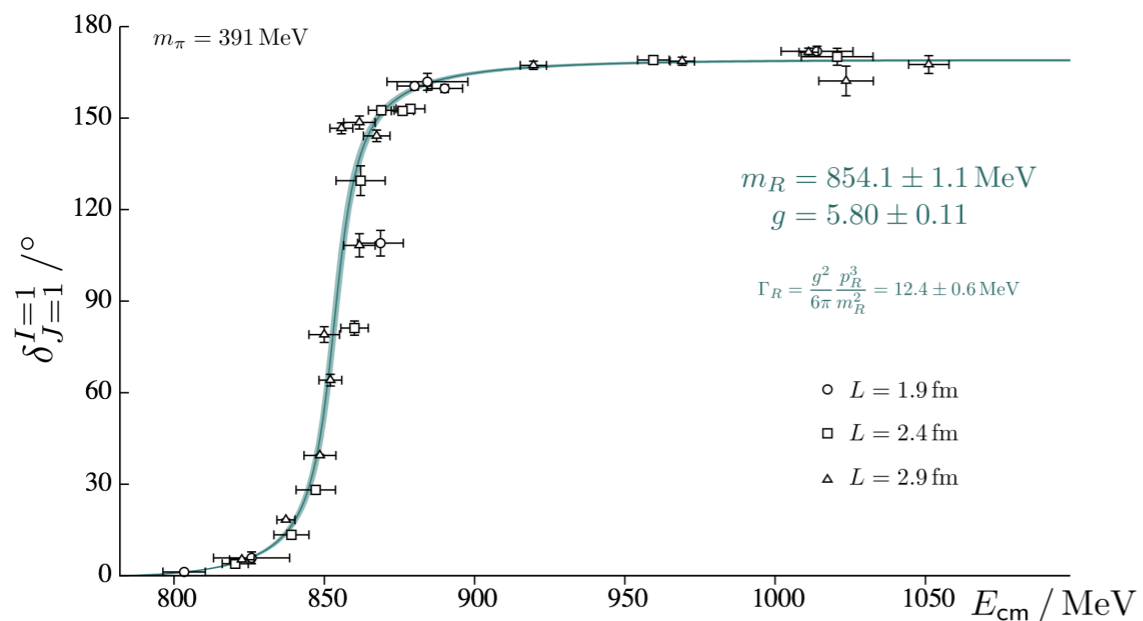


$$p\gamma \rightarrow N\rho \rightarrow N\pi\pi$$



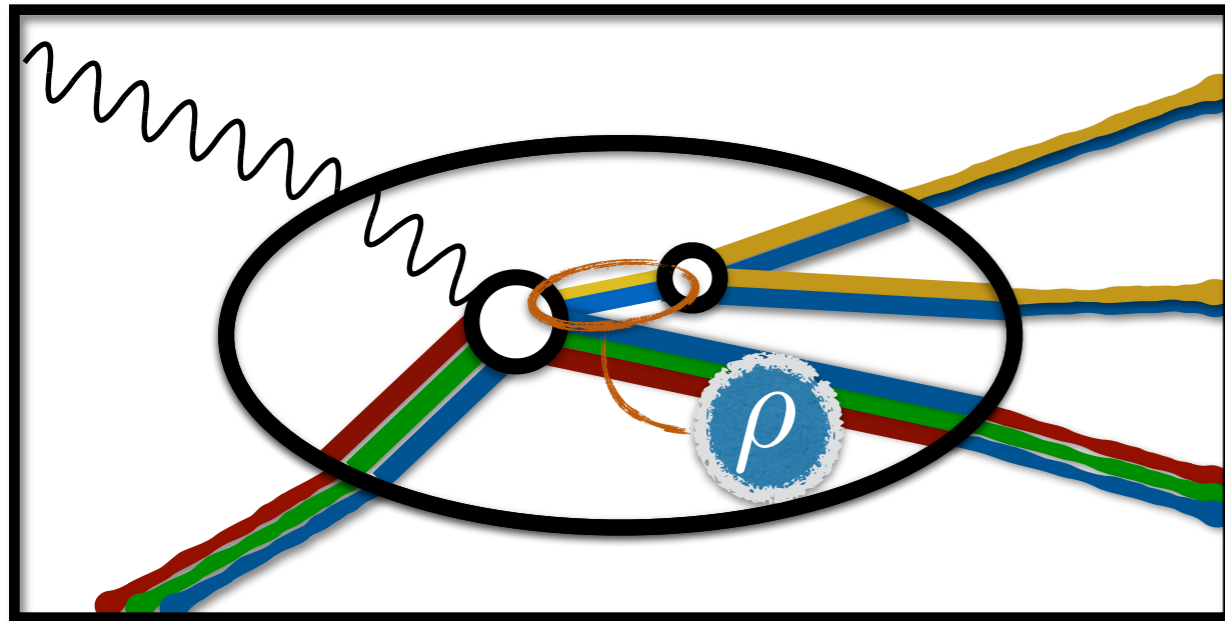
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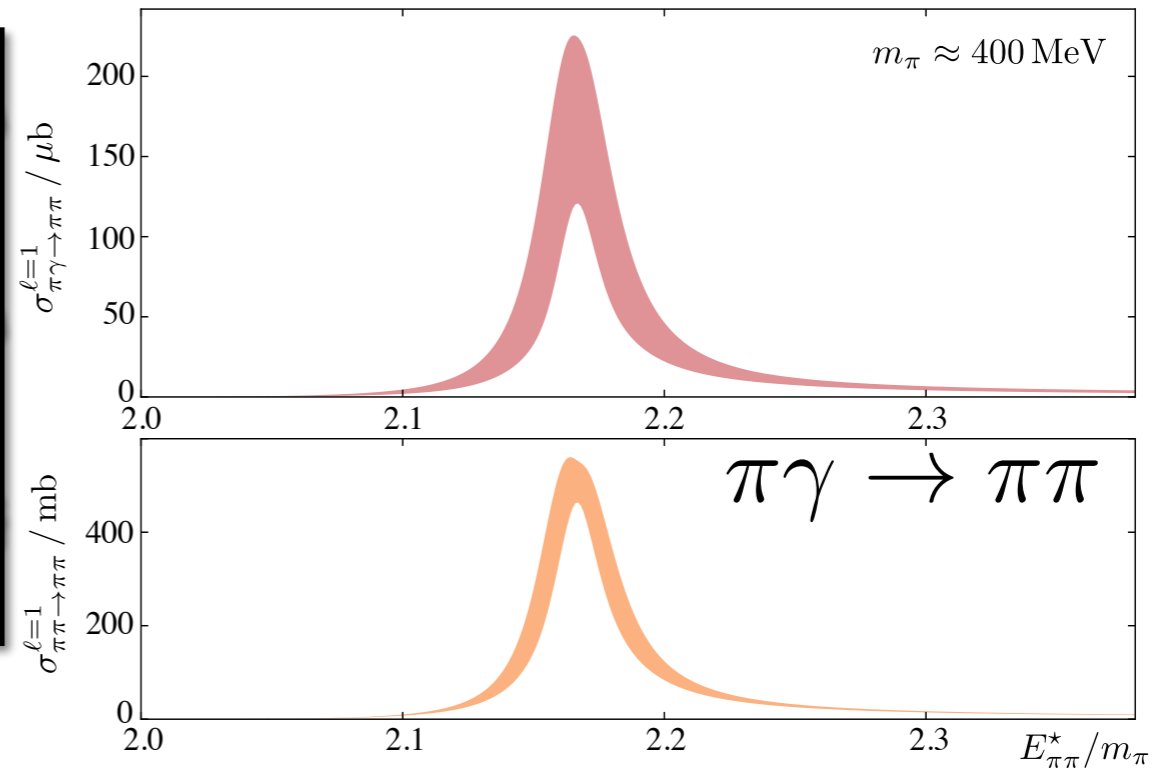


Lattice group at JLab leads the field in applying this kind of formalism

My work at JLab:

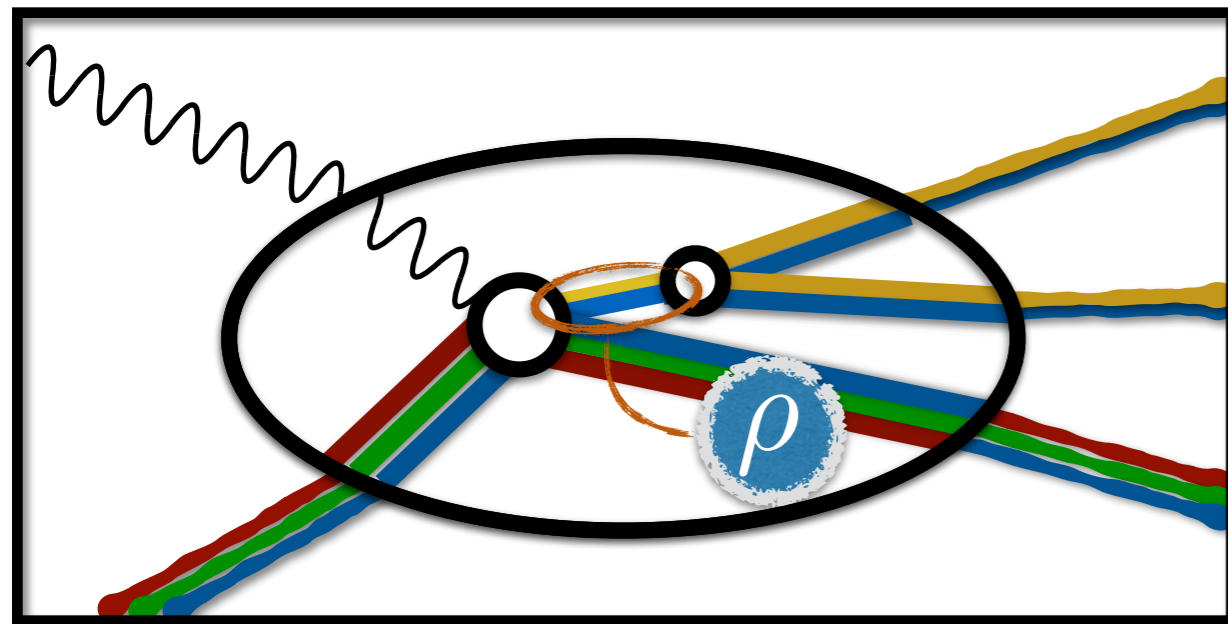


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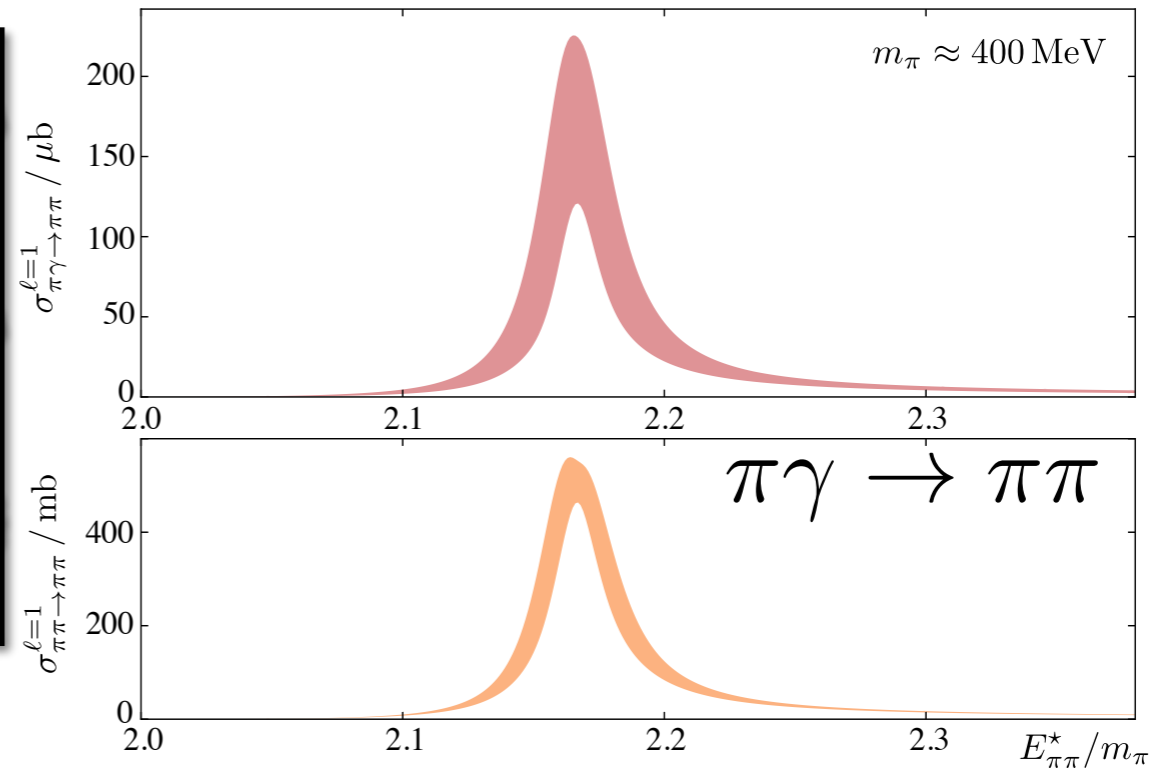


It would accelerate progress significantly if I had the opportunity to continue developing and also to apply this formalism here at Jlab.

My work at JLab:



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It would accelerate progress significantly if I had the opportunity to continue developing and also to apply this formalism here at Jlab.

The ideal scenario...

Regular interaction with experimental, lattice and theory groups:

Identifying the most relevant observables,

Developing formalism to extract these,

Performing the calculations

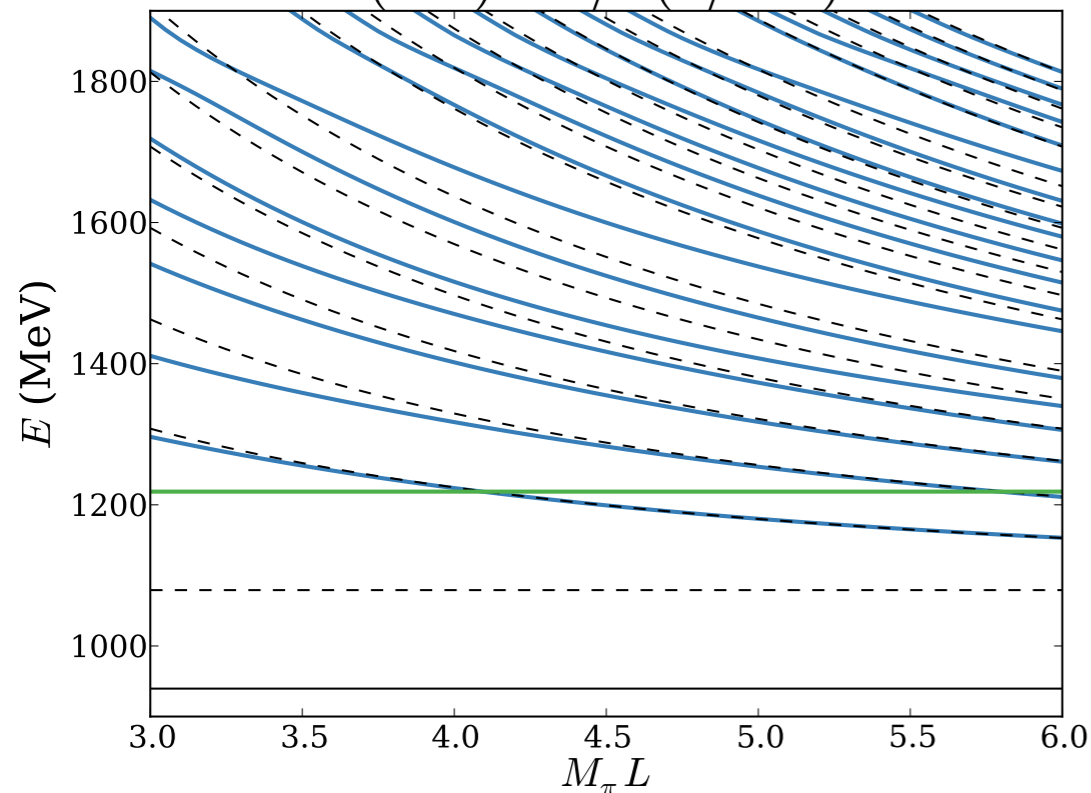
My work at JLab:

One example of symbiosis...

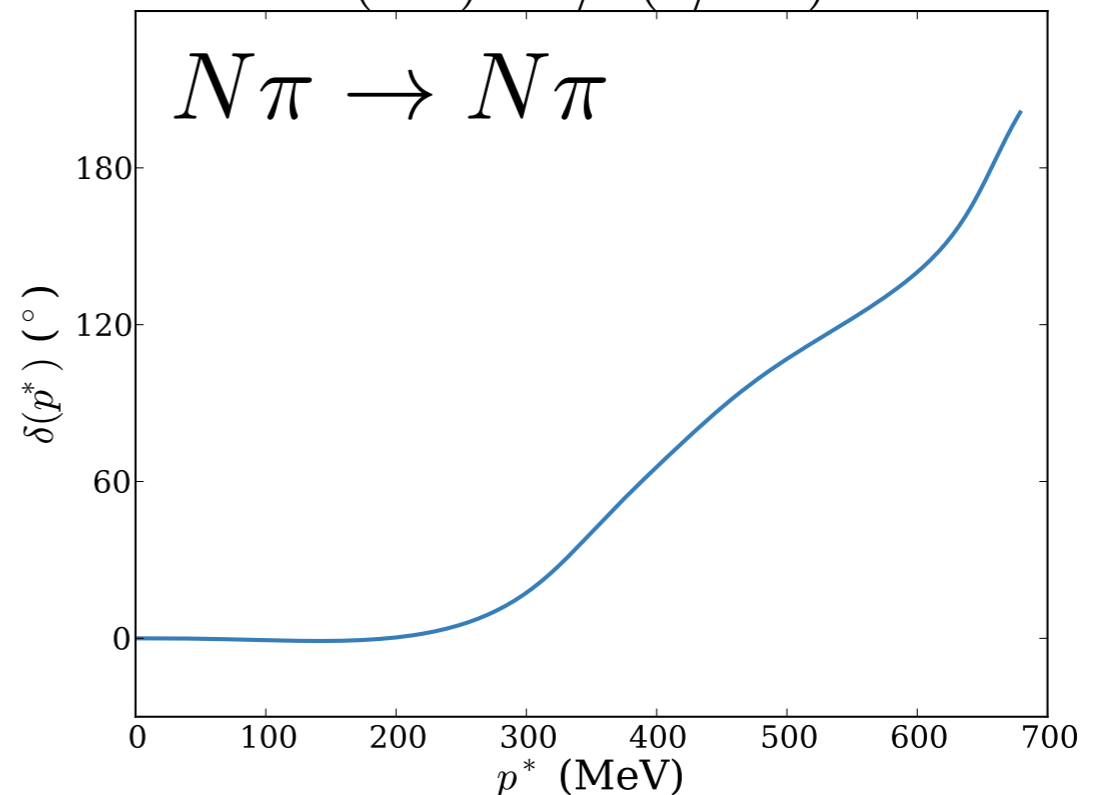
**Formalism can also be applied in the “other direction”
to gain insight on lattice observables**



$$I(J^P) = 1/2(1/2^+)$$



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**GW partial-wave data base
(solution WI08)**

MTH and Meyer, *to appear*

My work at JLab:

More concretely...

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More concretely...

In one to two years:

The formalism needed for $N\pi \rightarrow N\pi\pi$ and $N\gamma \rightarrow N^* \rightarrow N\pi\pi$ expected to be complete.

First lattice studies of three-particle systems

$$K\pi \rightarrow K\pi\pi \quad \omega \rightarrow \pi\pi\pi$$

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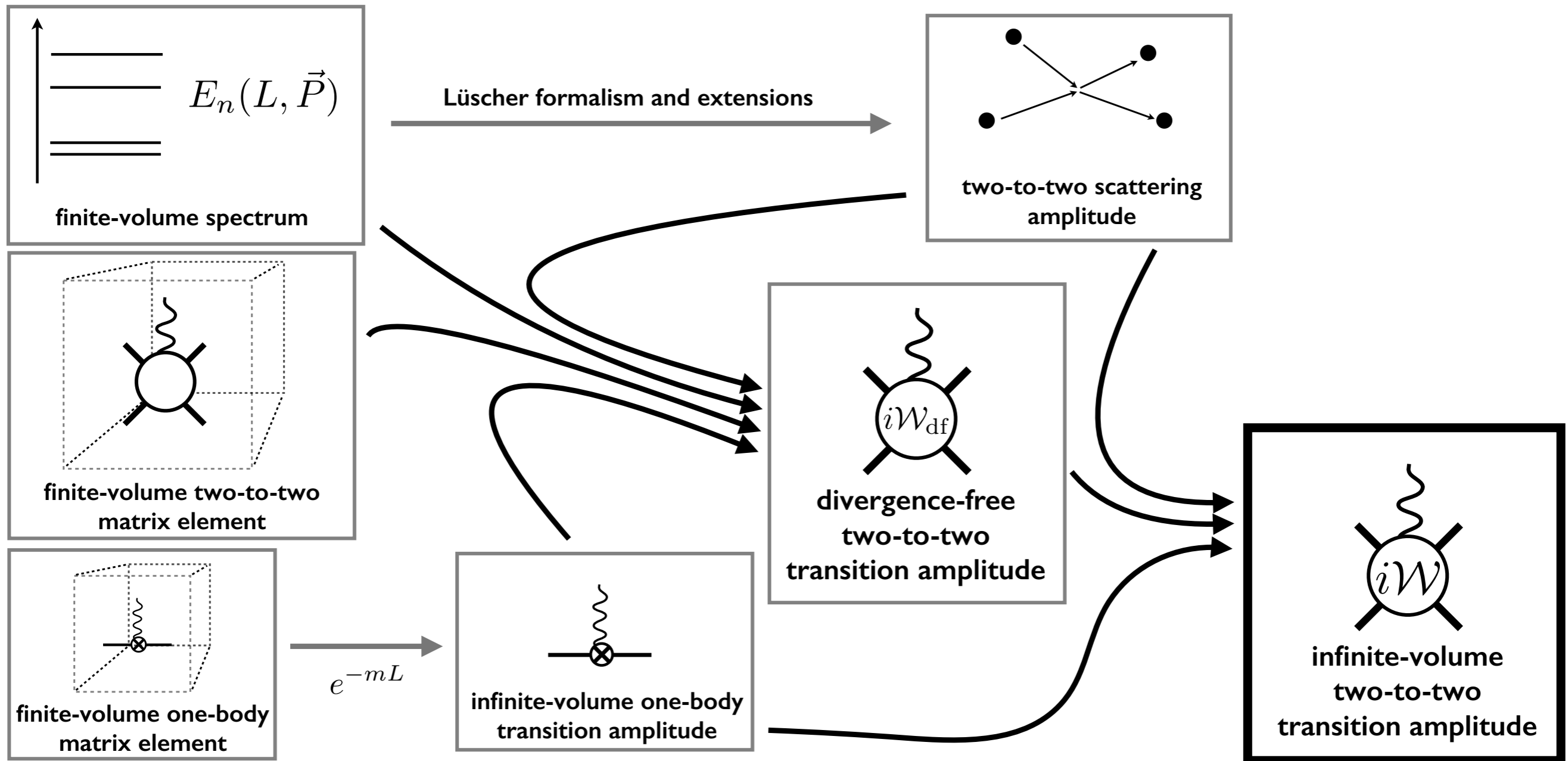
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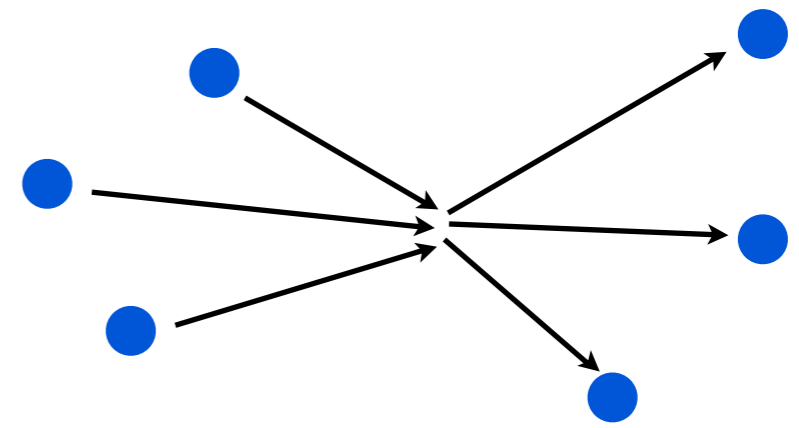
Thanks for listening!

Backup Slides

Two-to-two transition amplitudes

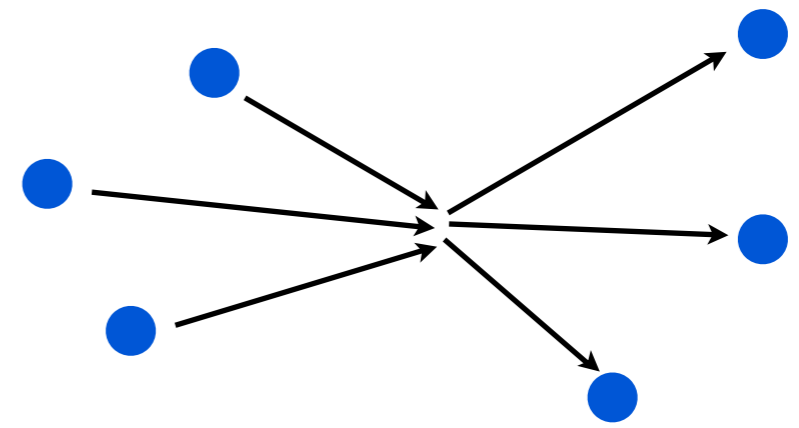


Infinite volume

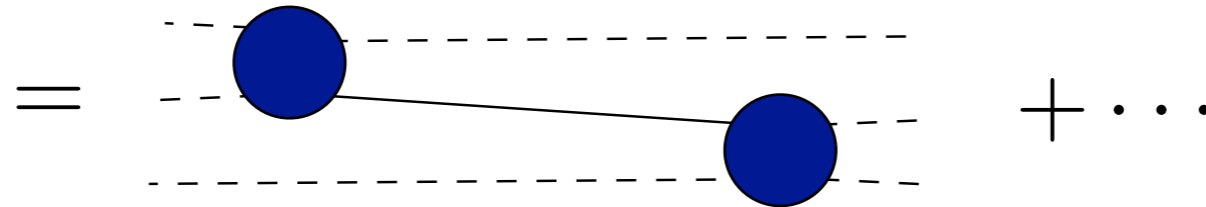


$i\mathcal{M}_{3\rightarrow 3} \equiv$ **Sum of all connected Feynman diagrams
with six external legs**

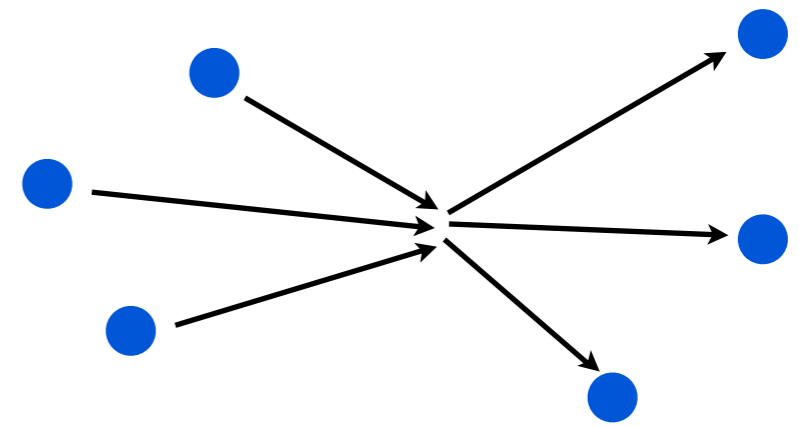
Infinite volume



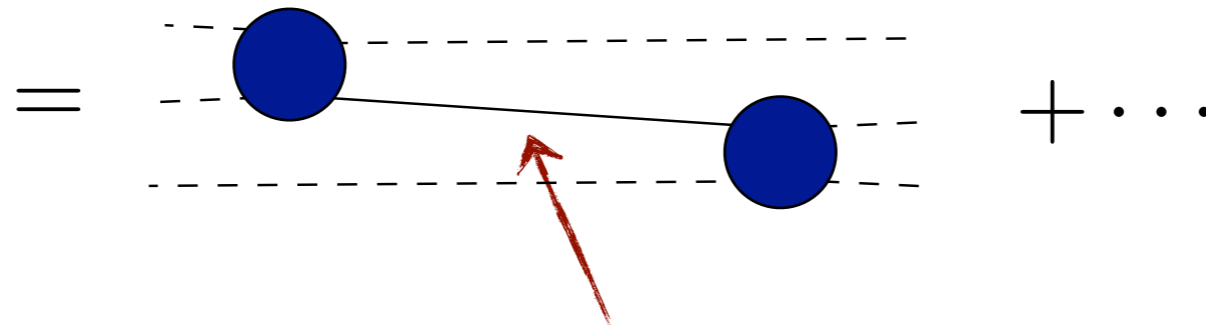
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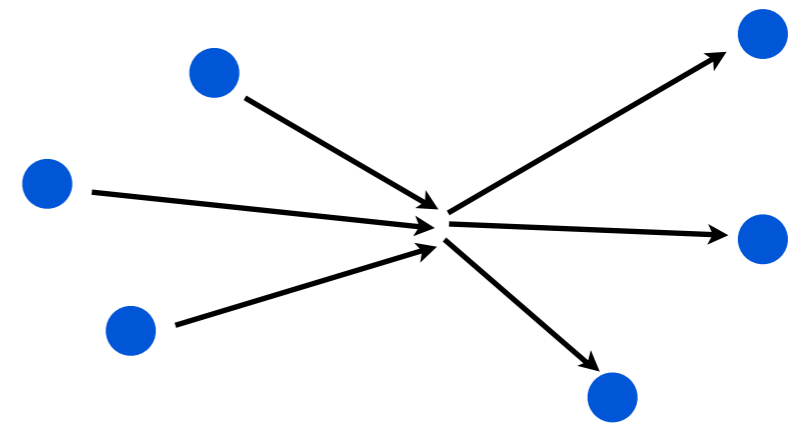


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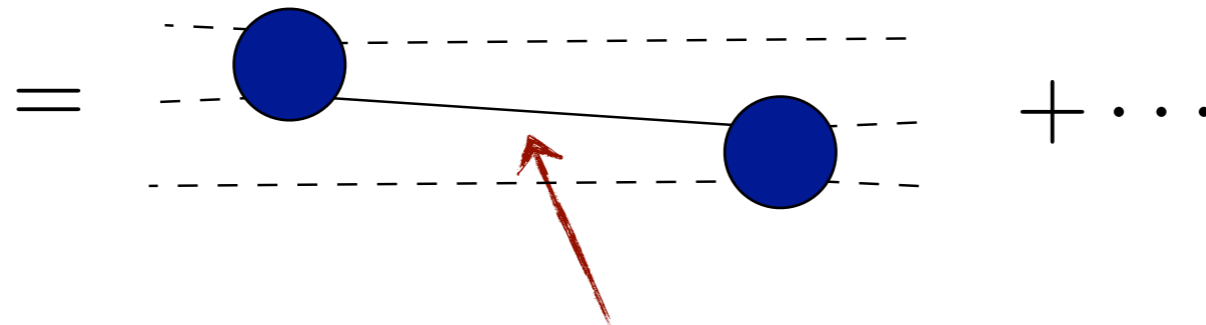


Certain external momenta put this on-shell!

Infinite volume



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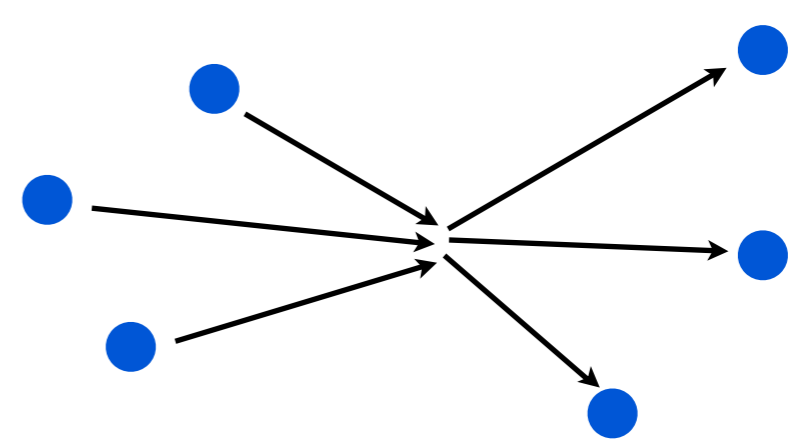


Certain external momenta put this on-shell!

$\mathcal{M}_{3\rightarrow 3}$ has kinematic singularities at certain momenta

No dominance of lowest partial waves

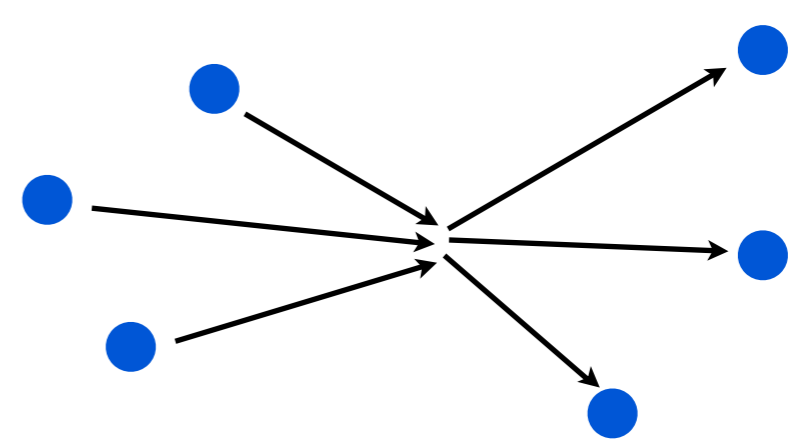
Infinite volume



Degrees of freedom for three on-shell particles with (E, \vec{P})



Infinite volume



Degrees of freedom for three on-shell particles with (E, \vec{P})



$$\vec{k}, l, m$$

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
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 & + \dots \\
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 \end{aligned}$$

The diagrams in the expansion are as follows:

- Row 1: Three diagrams showing two white circles connected by two lines. The first diagram has a dashed box around the lines. The second diagram has an orange circle on the top line between the dashed boxes. The third diagram has two orange circles on the top line between the dashed boxes.
- Row 2: Three diagrams showing two white circles connected by two lines. The first diagram has a purple circle on the top line between the dashed boxes. The second diagram has two purple circles on the top line between the dashed boxes. The third diagram has three purple circles on the top line between the dashed boxes.
- Row 3: Three diagrams showing two white circles connected by two lines. The first diagram has two purple circles on the top line between the dashed boxes. The second diagram has three purple circles on the top line between the dashed boxes. The third diagram has four purple circles on the top line between the dashed boxes.
- Row 4: Two diagrams showing two white circles connected by two lines. The first diagram has three purple circles on the top line between the dashed boxes. The second diagram has four purple circles on the top line between the dashed boxes.
- Row 5: Two diagrams showing two white circles connected by two lines. The first diagram has a purple circle on the top line between the dashed boxes and an orange circle on the top line between the dashed boxes. The second diagram has an orange circle on the top line between the dashed boxes and a purple circle on the top line between the dashed boxes.

Compare to two-particle skeleton expansion

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The diagrams in the two-particle skeleton expansion are as follows:

- Diagram 1: Two white circles connected by two lines, with a dashed box around the lines.
- Diagram 2: Two white circles connected by two lines, with a purple circle on the top line between the dashed boxes.
- Diagram 3: Two white circles connected by two lines, with two purple circles on the top line between the dashed boxes.

What is new here?

1. Degrees of freedom are different

two particles

two-particle angular momentum

three particles

\vec{k} + two-particle angular momentum



Our result only depends on finite-volume momentum

$$\vec{k} = \frac{2\pi}{L} \vec{n}$$

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Quantization condition expressed using matrices with indices

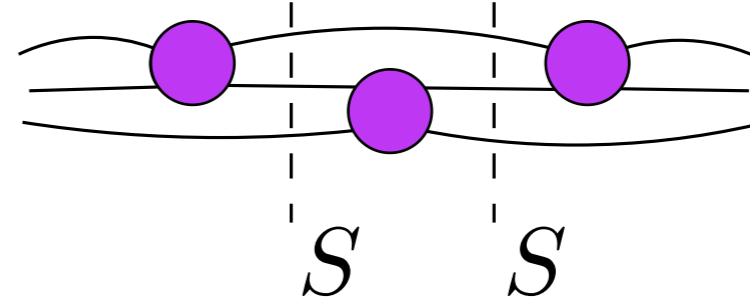
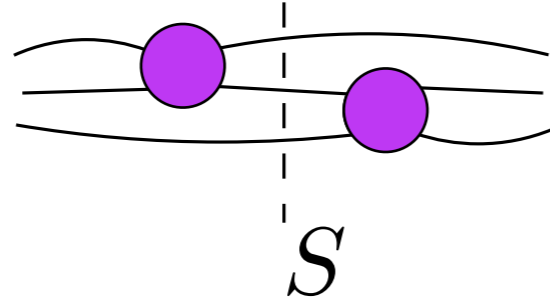
$$\vec{k}, l, m$$

What is new here?

2. Three particle divergences

Define $i\mathcal{M}_{\text{df},3\rightarrow 3}$

$$\equiv i\mathcal{M}_{3\rightarrow 3} - \left[i\mathcal{M}_{2\rightarrow 2} S i\mathcal{M}_{2\rightarrow 2} + \int i\mathcal{M}_{2\rightarrow 2} S i\mathcal{M}_{2\rightarrow 2} S i\mathcal{M}_{2\rightarrow 2} + \dots \right]$$



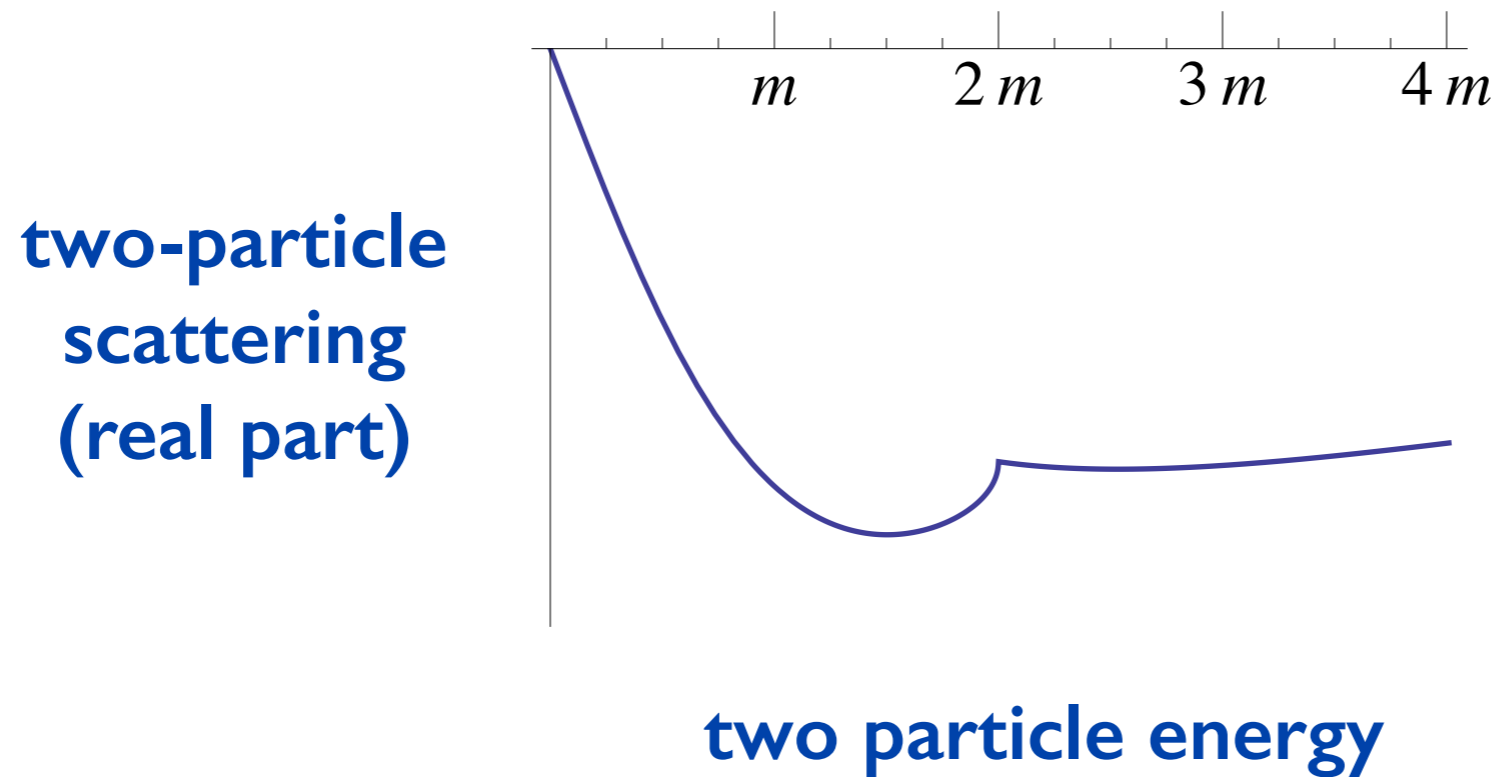
only on-shell
amplitudes here

infinite series
built with factors of $S i\mathcal{M}_{2\rightarrow 2}$

**This subtraction emerges naturally in our
finite-volume analysis**

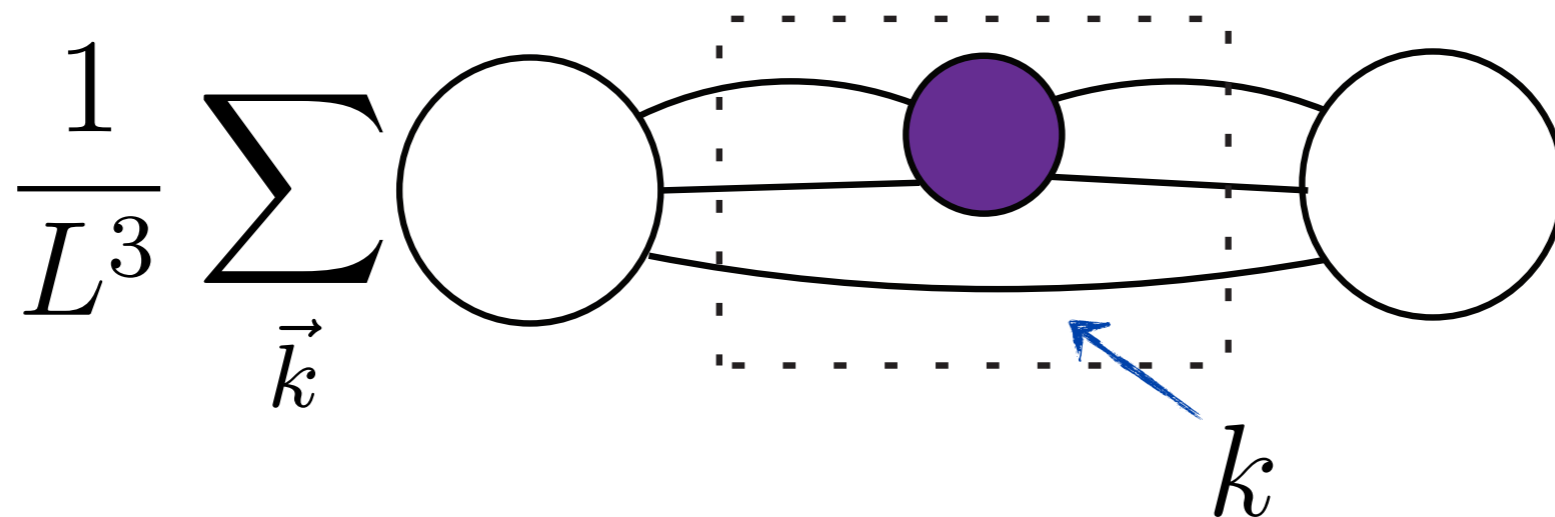
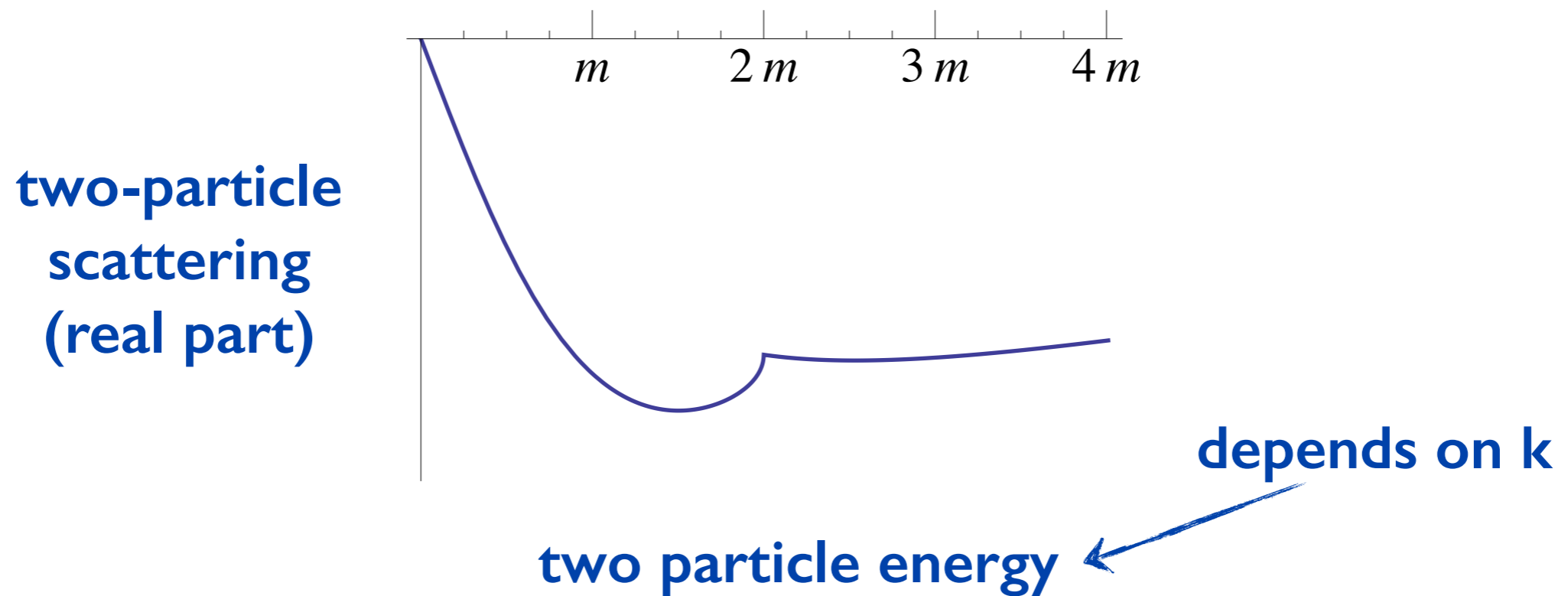
What is new here?

3. Must now worry about sum crossing
two-particle unitary cusp



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What is new here?

3. Must now worry about sum crossing
two-particle unitary cusp

To remove cusp

$i\epsilon$ prescription  principal value \widetilde{PV}

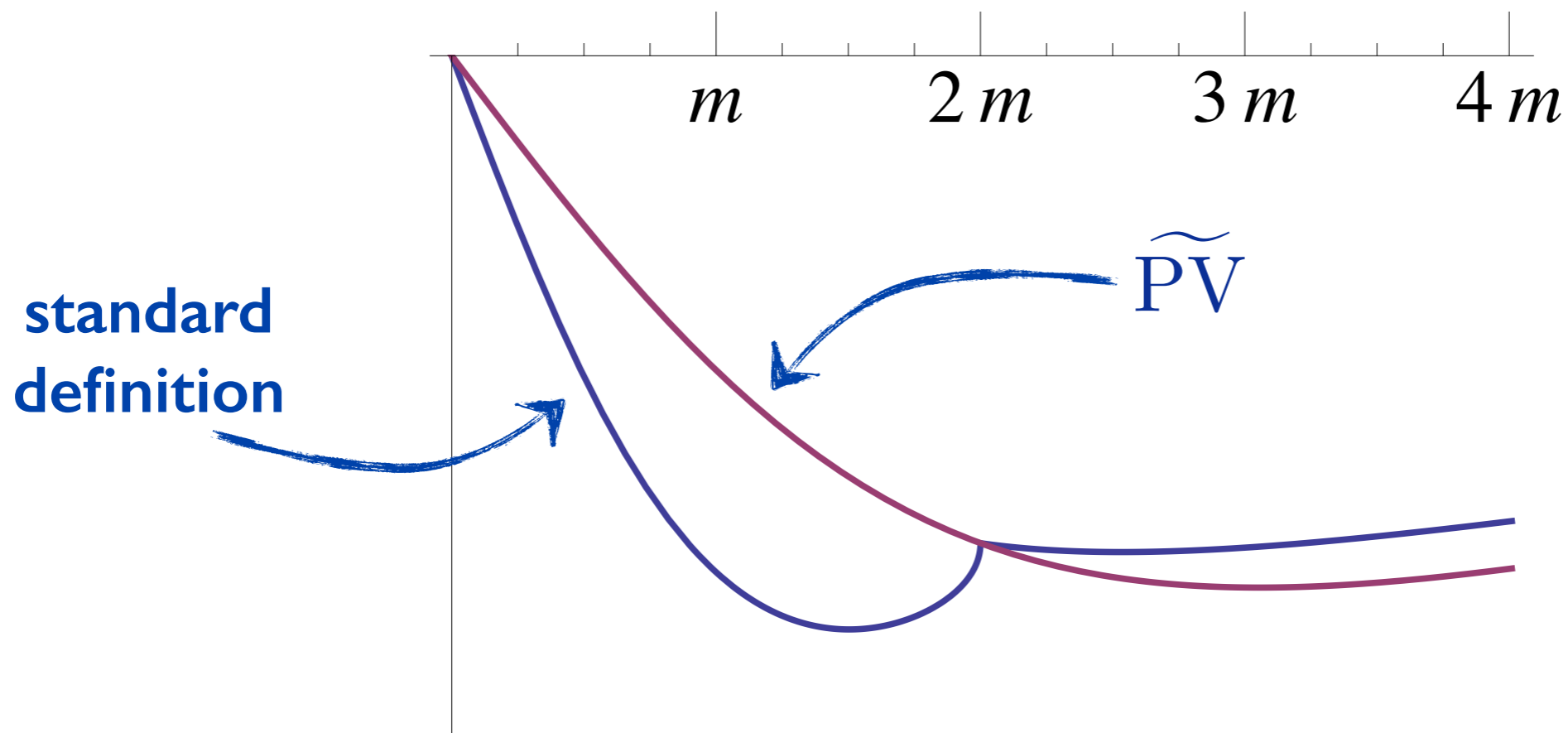
Analytically continue principal value below threshold
then interpolate to prescription-free subthreshold form

What is new here?

3. Must now worry about sum crossing
two-particle unitary cusp

To remove cusp

$i\epsilon$ prescription $\xrightarrow{\hspace{10em}}$ principal value \widetilde{PV}



What is new here?

3. Must now worry about sum crossing
two-particle unitary cusp

has a cusp

$$i\mathcal{M}_{2\rightarrow 2} = \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} + \dots$$

$$i\tilde{\mathcal{K}}_{2\rightarrow 2} = \text{---} \bullet \text{---} =$$

$$\text{---} \bullet \text{---} + \text{---} \bullet \overset{\sim}{\text{PV}} \bullet \text{---} + \text{---} \bullet \overset{\sim}{\text{PV}} \bullet \overset{\sim}{\text{PV}} \bullet \text{---} + \dots$$

What is new here?

3. Must now worry about sum crossing
two-particle unitary cusp

$$\begin{array}{ccc} i\mathcal{M}_{2\rightarrow 2} & \xrightarrow{\quad} & i\mathcal{K}_{2\rightarrow 2} \\ i\mathcal{M}_{\text{df},3\rightarrow 3} & & i\mathcal{K}_{\text{df},3\rightarrow 3} \end{array}$$

**We relate these infinite-volume quantities
to the finite-volume spectrum**

Three-particle result

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'_3 iF_3 \frac{1}{1 - i\mathcal{K}_{\text{df},3 \rightarrow 3} iF_3} A_3$$

$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} iG} i\mathcal{M}_{L,2 \rightarrow 2} iF \right]$$

$$i\mathcal{M}_{L,2 \rightarrow 2} \equiv i\mathcal{K}_{2 \rightarrow 2} \frac{1}{1 - iF i\mathcal{K}_{2 \rightarrow 2}}$$

All factors are matrices with indices \vec{k}, ℓ, m

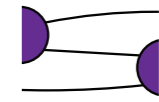
Three-particle result

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'_3 iF_3 \frac{1}{1 - i\mathcal{K}_{\text{df},3\rightarrow3} iF_3} A_3$$

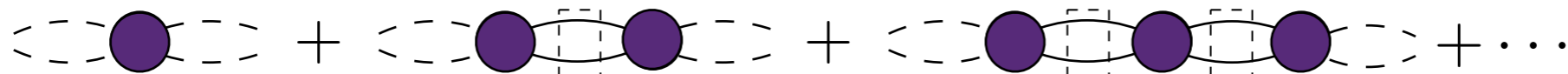
sum-integral difference 

$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\rightarrow2} iG} i\mathcal{M}_{L,2\rightarrow2} iF \right]$$

$$i\mathcal{M}_{L,2\rightarrow2} \equiv i\mathcal{K}_{2\rightarrow2} \frac{1}{1 - iF i\mathcal{K}_{2\rightarrow2}}$$

encodes switches 

sum of all two-particle loops (with summed momenta)



All factors are matrices with indices \vec{k}, ℓ, m

Three-particle result

At fixed (L, \vec{P}) , finite-volume spectrum is all solutions to

$$\det [1 - i\mathcal{K}_{\text{df},3\rightarrow 3} iF_3] = 0$$

$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\rightarrow 2} iG} i\mathcal{M}_{L,2\rightarrow 2} iF \right] \quad i\mathcal{M}_{L,2\rightarrow 2} \equiv i\mathcal{K}_{2\rightarrow 2} \frac{1}{1 - iF i\mathcal{K}_{2\rightarrow 2}}$$

MTH and Sharpe, *Phys. Rev. D*90, 116003 (2014)

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Model independent general result of relativistic scalar field theory

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Assumes two-particle phase shift is bounded by $\pi/2$

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MTH and Sharpe, *Phys. Rev. D* 90, 116003 (2014)

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Infinite matrices truncate if we truncate in angular momentum

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MTH and Sharpe, *Phys. Rev. D* 90, 116003 (2014)

Model independent general result of relativistic scalar field theory

Assumes two-particle phase shift is bounded by $\pi/2$

Infinite matrices truncate if we truncate in angular momentum

Strongest truncation is the isotropic limit, gives simple result

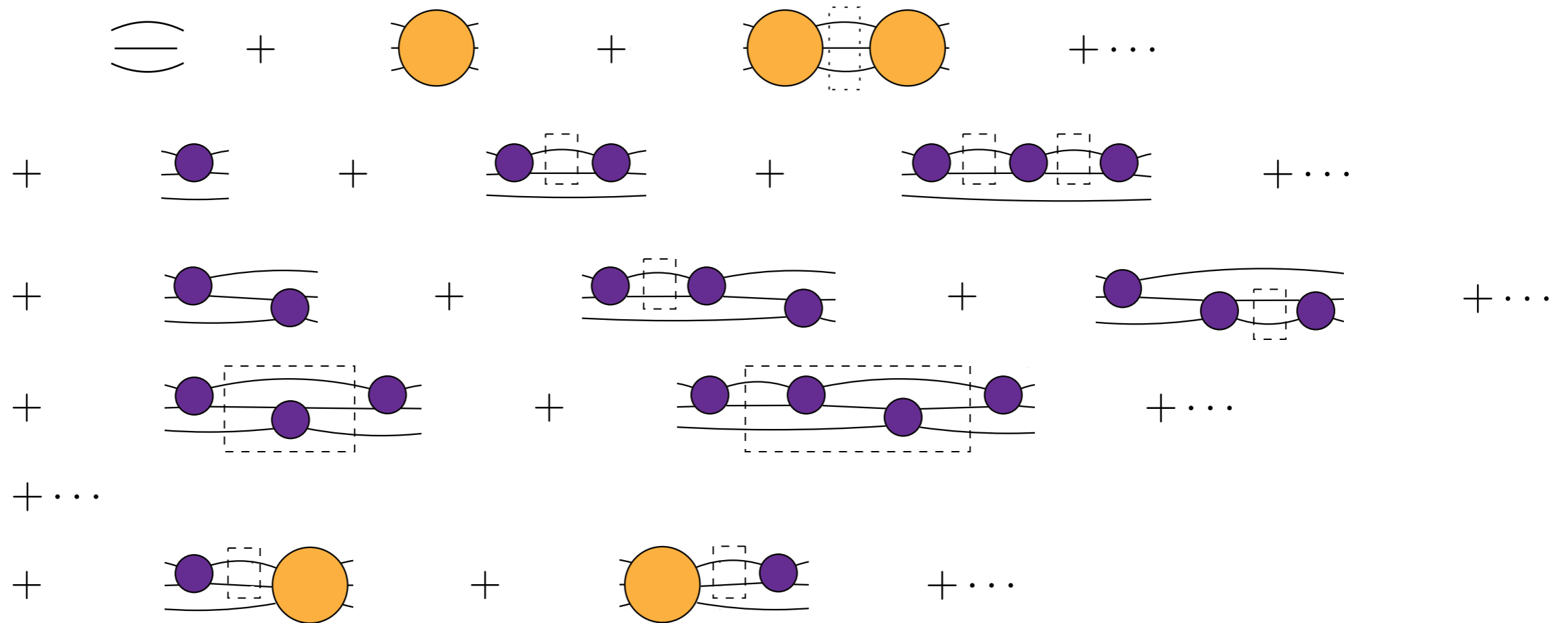
$$\mathcal{K}_{\text{df},3\rightarrow 3}(E_n^*) = -[F_{3,\text{iso}}(E_n, \vec{P}, L)]^{-1}$$

Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3\rightarrow 3}$

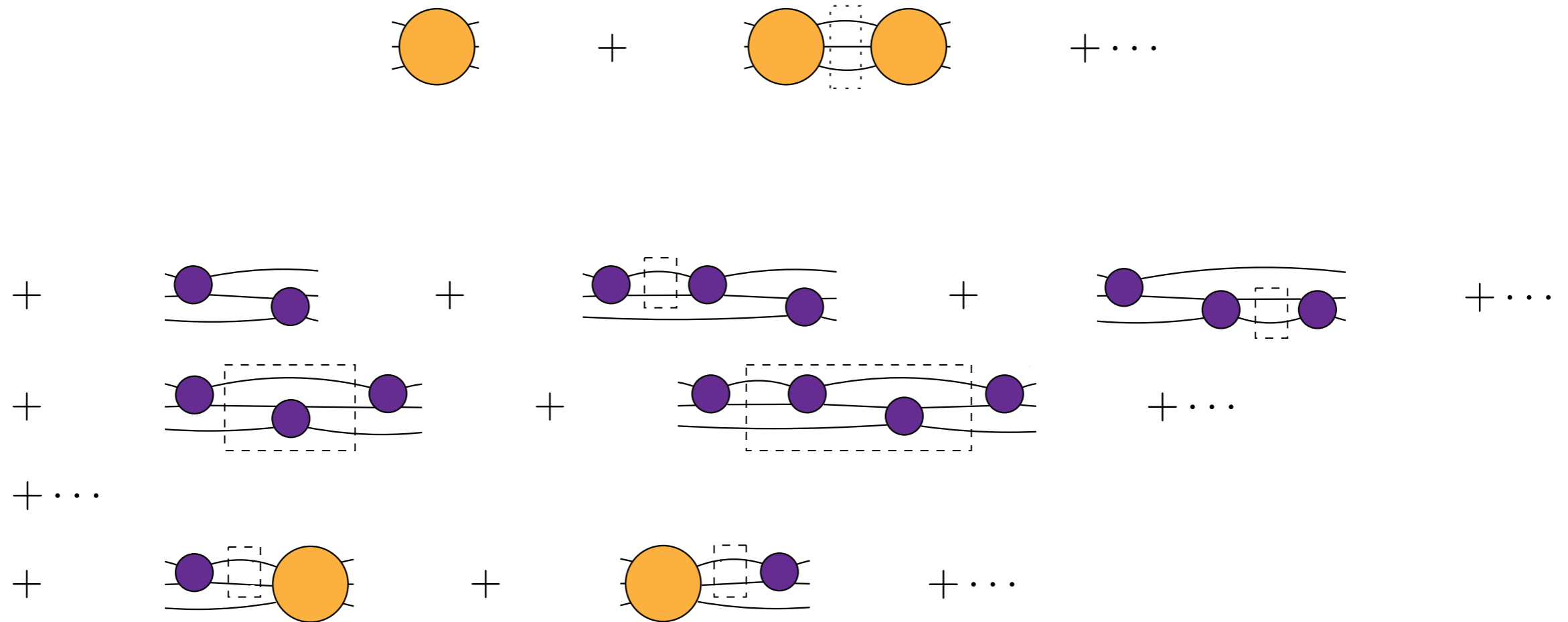
Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$



First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3\rightarrow 3}$

1. Amputate interpolating fields

Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$



First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3\rightarrow 3}$

2. Drop disconnected diagrams

Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

$$i\mathcal{M}_{L,3\rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{l} \text{Orange circle} + \text{Two orange circles connected by a dashed box} + \dots \\ + \text{Three purple circles on lines} + \text{Three purple circles on lines with a dashed box} + \text{Three purple circles on lines with a dashed box} + \dots \\ + \text{Three purple circles on lines with a dashed box} + \text{Three purple circles on lines with a dashed box} + \dots \\ + \dots \\ + \text{Purple circle and orange circle} + \text{Orange circle and purple circle} + \dots \end{array} \right\}$$

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3\rightarrow 3}$

3. Symmetrize

Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

$$i\mathcal{M}_{L,3\rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{l} \text{[Orange circle]} + \text{[Orange circle with dashed box]} + \dots \\ + \text{[Purple circles on lines]} + \text{[Purple circles on lines with dashed box]} + \text{[Purple circles on lines with dashed box]} + \dots \\ + \text{[Purple circles on lines with dashed box]} + \text{[Purple circles on lines with dashed box]} + \dots \\ + \dots \\ + \text{[Purple circle on line with dashed box and orange circle]} + \text{[Orange circle with dashed box and purple circle]} + \dots \end{array} \right\}$$

Replacing all loop momentum sums with i -epsilon prescription integrals gives physical three-to-three scattering amplitude

$$i\mathcal{M}_{3\rightarrow 3} = \lim_{L \rightarrow \infty} \Big|_{i\epsilon} i\mathcal{M}_{L,3\rightarrow 3}$$

Relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

$$i\mathcal{M}_{L,3\rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[\mathcal{L}_L \ i\mathcal{K}_{\text{df},3\rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3\rightarrow 3}} \ \mathcal{R}_L \right]$$

$$i\mathcal{M}_{3\rightarrow 3} = \lim_{L\rightarrow\infty} \left. i\mathcal{M}_{L,3\rightarrow 3} \right|_{i\epsilon}$$

MTH and Sharpe, *Phys. Rev. D* 92, 114509 (2015)

Gives integral equation relating $i\mathcal{K}_{\text{df},3\rightarrow 3}$ to $i\mathcal{M}_{3\rightarrow 3}$

Completes formal story (for the setup considered!)

Relation only depends on on-shell scattering quantities

$1/L$ expansions

In 1957, Huang and Yang determined energy shift for n identical bosons in a box

K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775

$$E_0(n, L) = \frac{4\pi a}{M L^3} \left\{ \binom{n}{2} - \left(\frac{a}{\pi L}\right) \binom{n}{2} \mathcal{I} + \left(\frac{a}{\pi L}\right)^2 \left\{ \binom{n}{2} \mathcal{I}^2 - \left[\binom{n}{2}^2 - 12 \binom{n}{3} - 6 \binom{n}{4} \right] \mathcal{J} \right\} \right\} + \mathcal{O}(L^{-6})$$

where a is the two-particle scattering length and

$$\mathcal{I} = \lim_{\Lambda \rightarrow \infty} \sum_{\substack{|\mathbf{i}| \leq \Lambda \\ \mathbf{i} \neq \mathbf{0}}} \frac{1}{|\mathbf{i}|^2} - 4\pi\Lambda = -8.91363291781$$

$$\mathcal{J} = \sum_{\mathbf{i} \neq \mathbf{0}} \frac{1}{|\mathbf{i}|^4} = 16.532315959$$

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In 2007 Beane, Detmold and Savage pushed the order to $1/L^6$ and the latter two calculated to $1/L^7$ the next year

Beane, S., Detmold, W. & Savage, M. *Phys. Rev.* D76 (2007) 074507

Detmold, W. & Savage, M. *Phys. Rev.* D77 (2008) 057502

At $1/L^6$ a three-particle contact term appears

$1/L$ expansions

Last year Detmold and Flynn performed a similar calculation for
matrix elements

Detmold and Flynn, *Phys. Rev. D* 91, 074509 (2015)

$$\begin{aligned} \langle n|J|n\rangle = & n\alpha_1 + \frac{n\alpha_1 a^2}{\pi^2 L^2} \binom{n}{2} \mathcal{J} + \frac{\alpha_2}{L^3} \binom{n}{2} \\ & + \frac{2n\alpha_1 a^3}{\pi^3 L^3} \binom{n}{2} \left\{ \mathcal{K} \binom{n}{2} - \left[\mathcal{I} \mathcal{J} + 4\mathcal{K} \binom{n-2}{1} + \mathcal{K} \binom{n-2}{2} \right] \right\} - \frac{2\alpha_2 a}{\pi L^4} \binom{n}{2} \mathcal{I} \\ & + \frac{n\alpha_1 a^4}{\pi^4 L^4} \left[3\mathcal{I}^2 \mathcal{J} + \mathcal{L} \left(186 - \frac{241n}{2} + \frac{29}{2} n^2 \right) + \mathcal{J}^2 \left(\frac{n^2}{4} + \frac{3n}{4} - \frac{7}{2} \right) \right. \\ & \left. + \mathcal{I} \mathcal{K}(4n - 14) + \mathcal{U}(32n - 64) + \mathcal{V}(16n - 32) \right] + \mathcal{O}(1/L^5). \end{aligned}$$

Here $\mathcal{I}, \mathcal{J}, \dots$ are known geometric constants
and α_1, α_2 are one- and two-boson current couplings

Nonperturbative and non-relativistic

Non-relativistic Faddeev analysis

In 2012, Polejaeva and Rusetzky derived a Lüscher-like result using
non-relativistic Faddeev equations

Polejaeva and Rusetzky, *Eur. Phys. J. A*48, 67 (2012)

Demonstrates that on-shell S-matrix determines spectrum

Difficult to extract scattering from the formalism

Nonperturbative and non-relativistic

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Dimer formalism

In 2013, Briceño and Davoudi studied three-particles in finite-volume
using the Dimer formalism

Briceño and Davoudi, *Phys. Rev. D*87, 094507 (2013)

Recovered Lüscher result when two of the three become bound

$$k \cot \delta = -k \cot \phi + \eta \frac{e^{-\gamma L}}{L}$$

**Final result involves an integral equation that one
needs to solve numerically**

Three-particle bound state

This year Meißner, Rios and Rusetsky determined the **finite-volume energy shift to a three-body bound state**

$$\Delta E = c \frac{\kappa^2}{m} \frac{|A|^2}{(kL)^{3/2}} \exp(-2\kappa L / \sqrt{3}) + \dots$$

Meißner, Rios and Rusektsky. *Phys. Rev. Lett.* 114, 091602 (2015)

Assumes the unitary limit for two-particle scattering

Result derived using non-relativistic quantum mechanics

Review..

Review...

1

$$C_L(P) = \begin{array}{c} \text{Diagram 1} + \text{Diagram 2} \\ + \text{Diagram 3} + \dots \end{array}$$

The diagrammatic equation for $C_L(P)$ is shown. The first row contains two terms: \mathcal{O}^\dagger connected to \mathcal{O} via two vertices (top and bottom) enclosed in a dashed box, and \mathcal{O}^\dagger connected to iK via two vertices (top and bottom) enclosed in a dashed box, which is then connected to \mathcal{O} via two vertices (top and bottom) enclosed in a dashed box. The second row contains a term where \mathcal{O}^\dagger is connected to iK via two vertices (top and bottom) enclosed in a dashed box, which is then connected to another iK via two vertices (top and bottom) enclosed in a dashed box, which is finally connected to \mathcal{O} via two vertices (top and bottom) enclosed in a dashed box. Ellipses follow. A blue callout box highlights a series of diagrams: a vertex connected to three vertices, followed by a vertex connected to two vertices, followed by a vertex connected to one vertex, followed by an ellipsis.

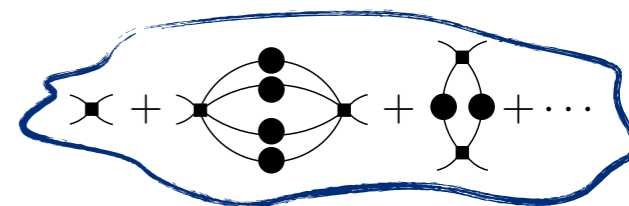
Review...

1

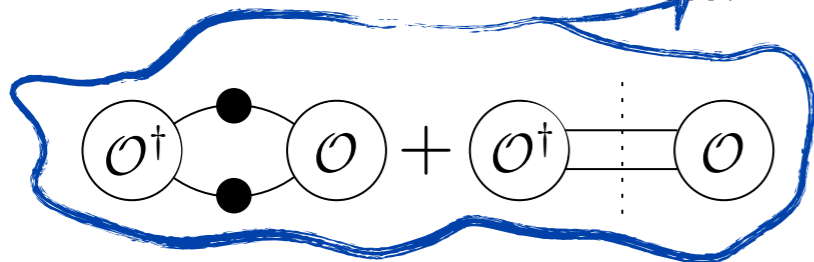
$$C_L(P) = \text{diagram 1} + \text{diagram 2}$$

The first diagram shows a circle labeled \mathcal{O}^\dagger on the left and a circle labeled \mathcal{O} on the right. Two black dots are positioned between them, enclosed in a vertical dashed rectangle. Two arcs connect the dots to the \mathcal{O}^\dagger circle, and two arcs connect the dots to the \mathcal{O} circle.

The second diagram is identical to the first, but the central circle is labeled iK .



2



$$+ \text{diagram 3} + \text{diagram 4} + \dots$$

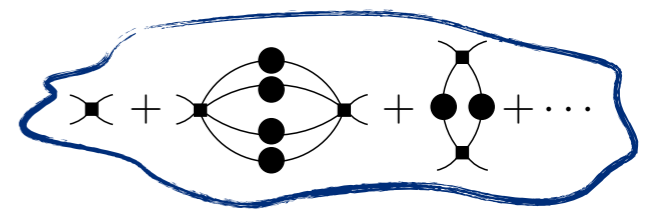
The third diagram shows a sequence of circles: \mathcal{O}^\dagger , iK , iK , and \mathcal{O} . Each of the iK circles has two dots between it and its neighbors, enclosed in a vertical dashed rectangle. Two arcs connect the dots to the \mathcal{O}^\dagger circle, and two arcs connect the dots to the \mathcal{O} circle.

The fourth diagram is identical to the third, but with an ellipsis following the \mathcal{O} circle.

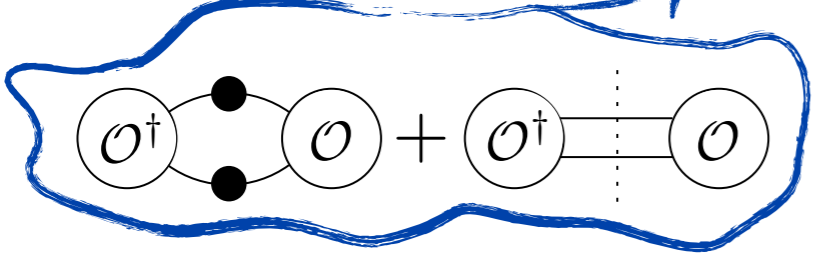
Review...

1

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O}$$

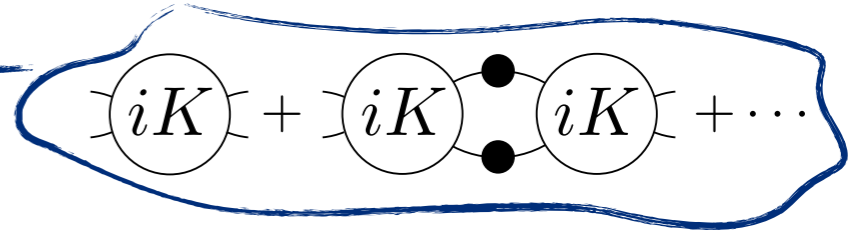


2



$$+ \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \dots$$

$$C_L(P) = C_\infty(P)$$



3

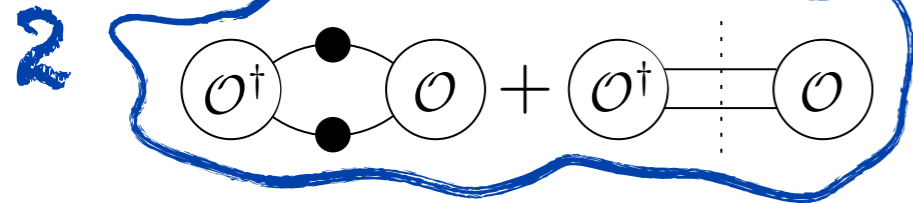
$\langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$

$$+ \begin{array}{c} A \\ \vdots \\ F \end{array} \begin{array}{c} A' \\ \vdots \\ F \end{array} + \begin{array}{c} A \\ \vdots \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \vdots \\ F \end{array} \begin{array}{c} A' \\ \vdots \\ F \end{array} + \begin{array}{c} A \\ \vdots \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \vdots \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \vdots \\ F \end{array} \begin{array}{c} A' \\ \vdots \\ F \end{array} + \dots$$

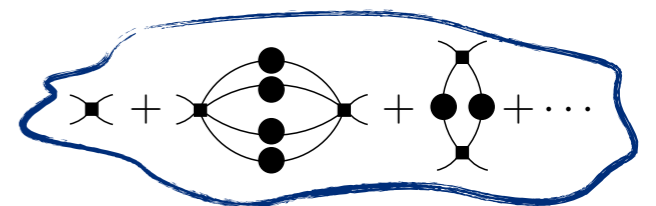
$\langle 0 | \mathcal{O} | \pi\pi, \text{in} \rangle$

Review...

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \mathcal{O}$$

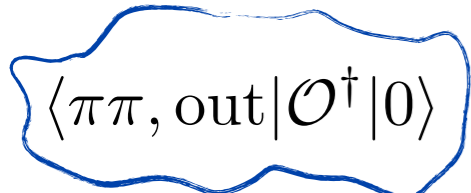


$$+ \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \mathcal{O} + \dots$$

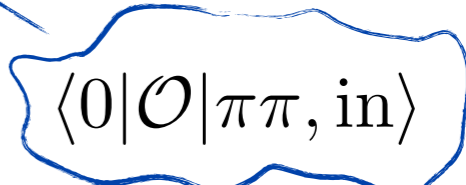
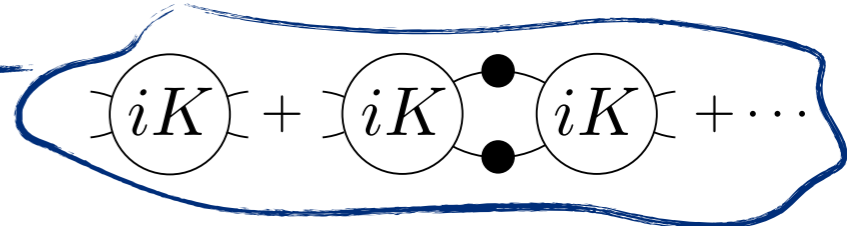


$$C_L(P) = C_\infty(P)$$

$$+ \begin{array}{c} A \\ \text{---} \\ F \end{array} \begin{array}{c} A' \\ \text{---} \\ F \end{array} + \begin{array}{c} A \\ \text{---} \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \text{---} \\ F \end{array} \begin{array}{c} A' \\ \text{---} \\ F \end{array}$$



$$+ \begin{array}{c} A \\ \text{---} \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \text{---} \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \text{---} \\ F \end{array} \begin{array}{c} A' \\ \text{---} \\ F \end{array} + \dots$$

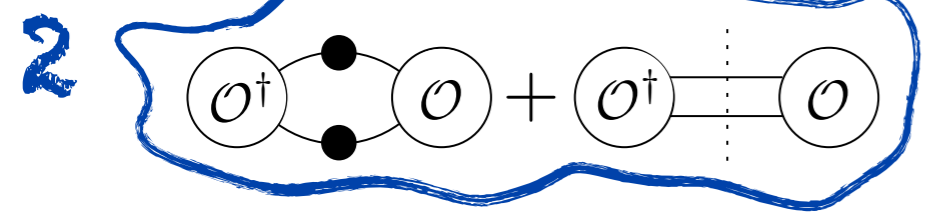


We deduce...

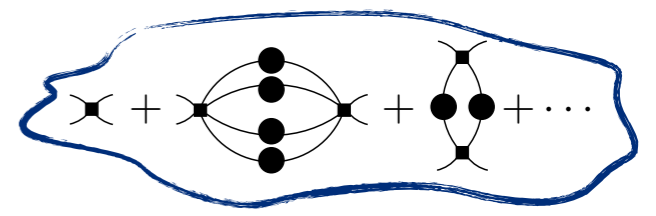
$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

Review...

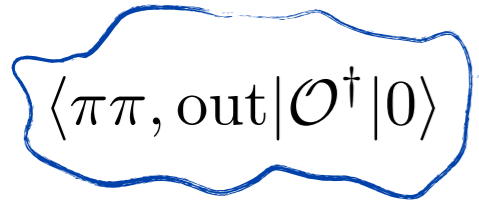
$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O}$$



$$+ \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \dots$$

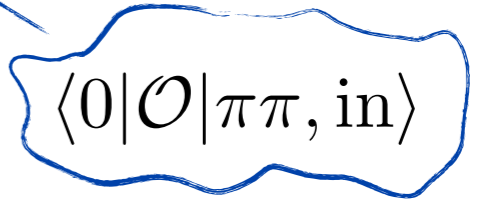
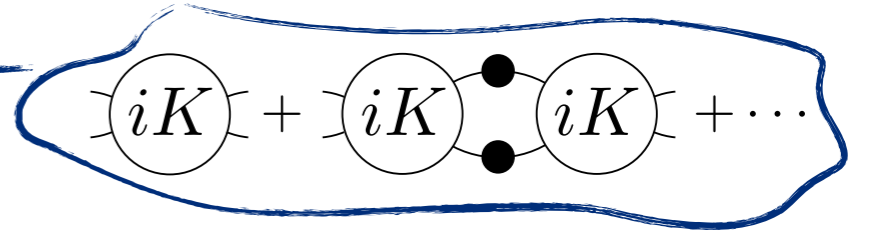


$$C_L(P) = C_\infty(P)$$



$$+ \begin{array}{c} A \\ \vdots \\ F \end{array} \begin{array}{c} A' \\ \vdots \\ F \end{array} + \begin{array}{c} A \\ \vdots \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \vdots \\ F \end{array} \begin{array}{c} A' \\ \vdots \\ F \end{array}$$

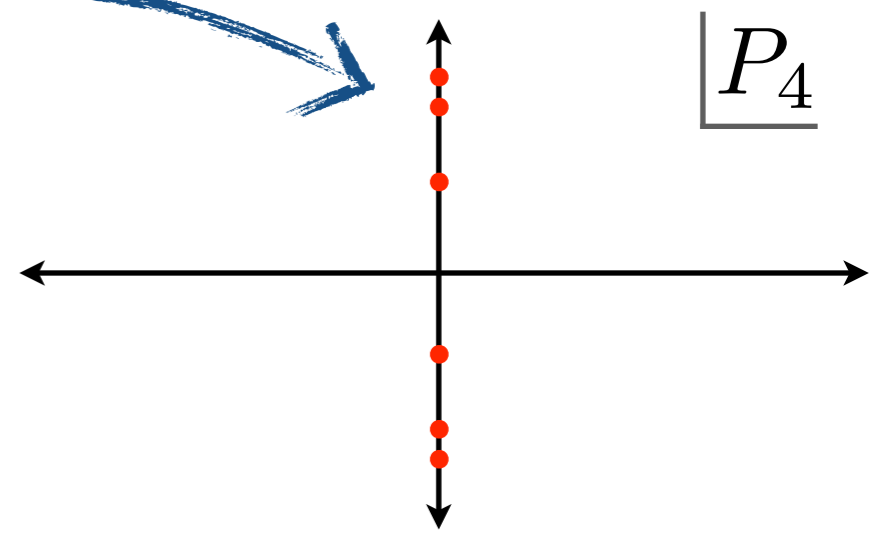
$$+ \begin{array}{c} A \\ \vdots \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \vdots \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \vdots \\ F \end{array} \begin{array}{c} A' \\ \vdots \\ F \end{array} + \dots$$



We deduce...

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

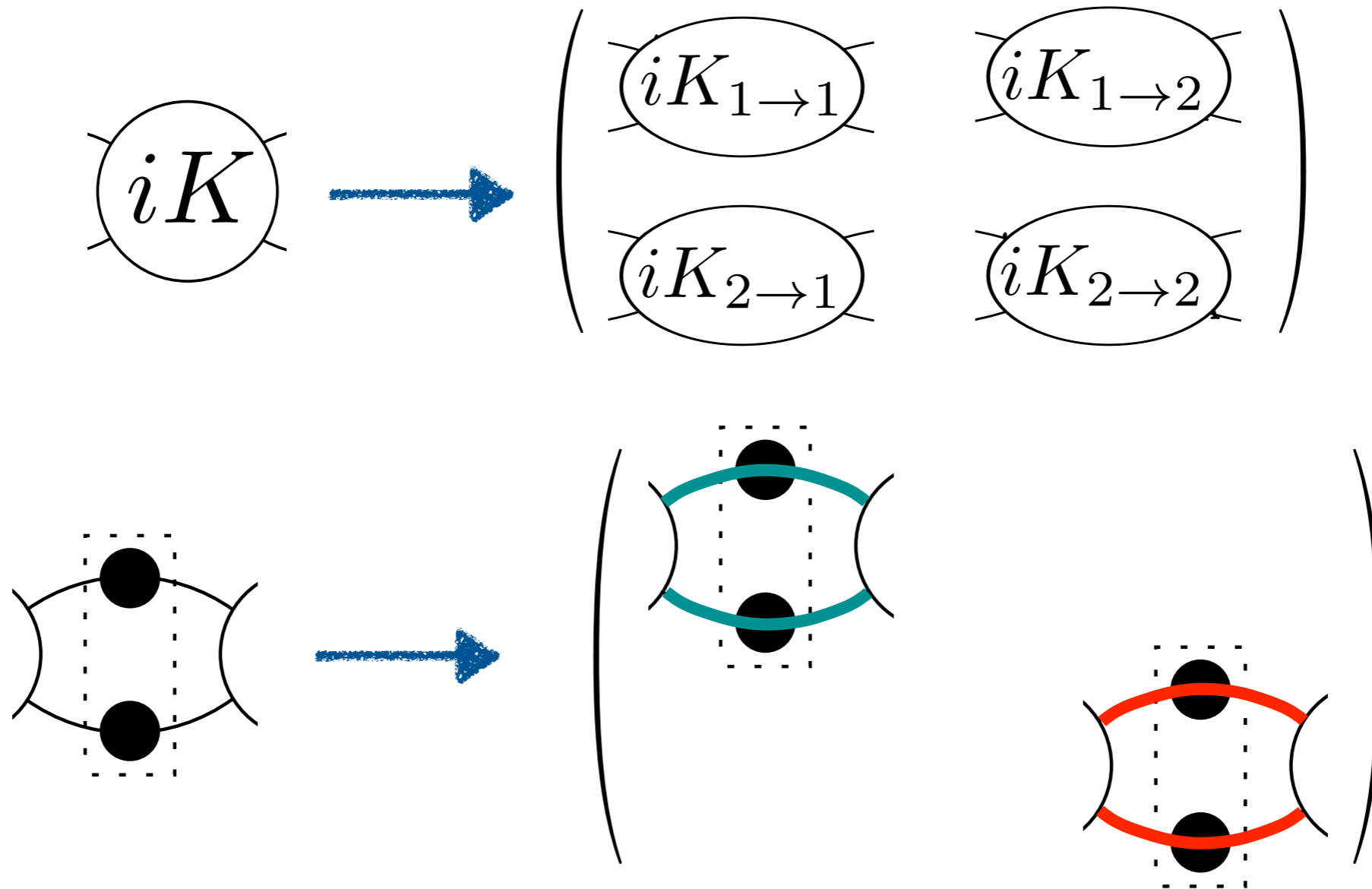
poles are in here



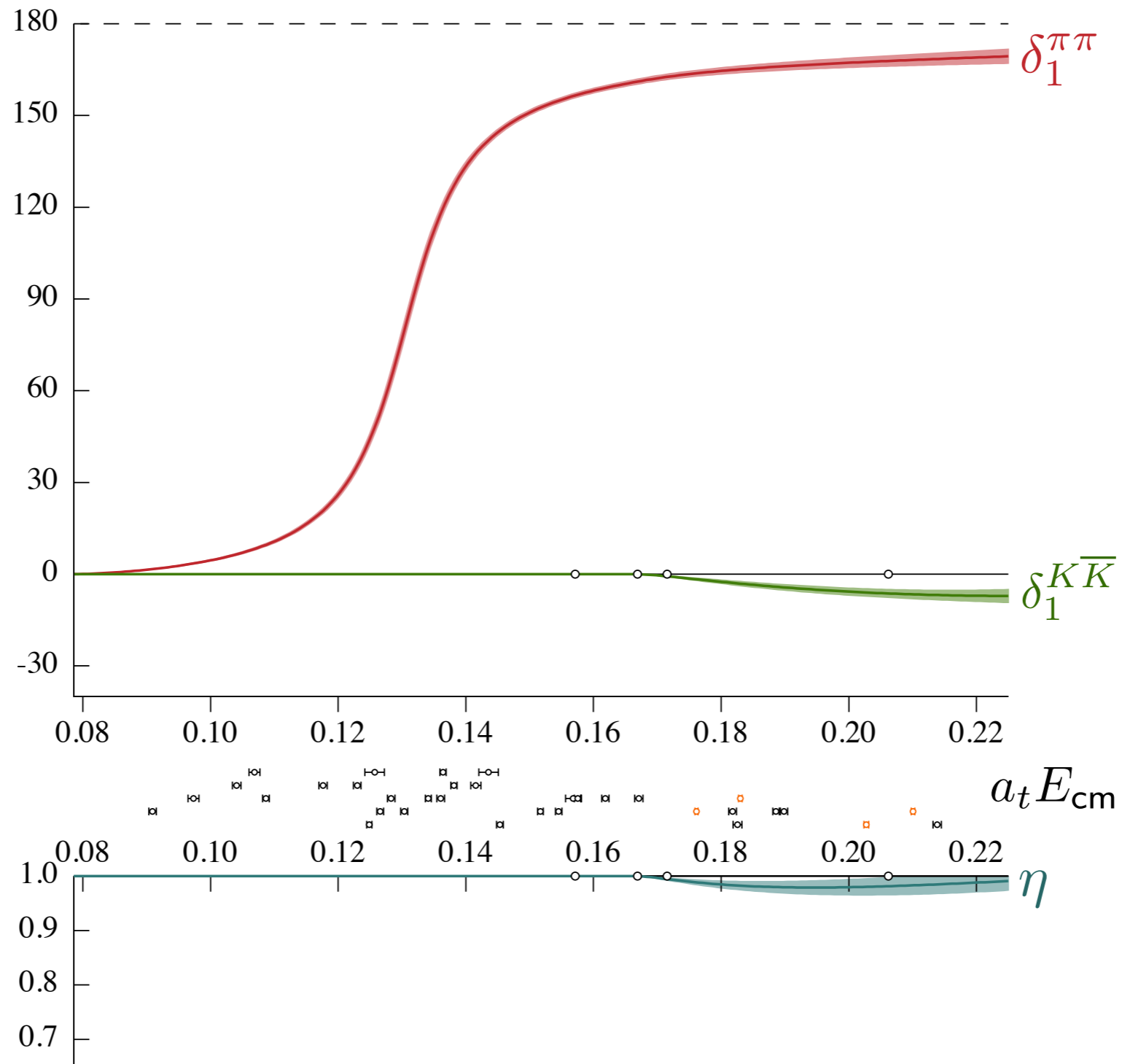
Scattering of multiple two-particle channels

$$\pi\pi \rightarrow \bar{K}K \quad \pi K \rightarrow \eta K$$

Make following replacements

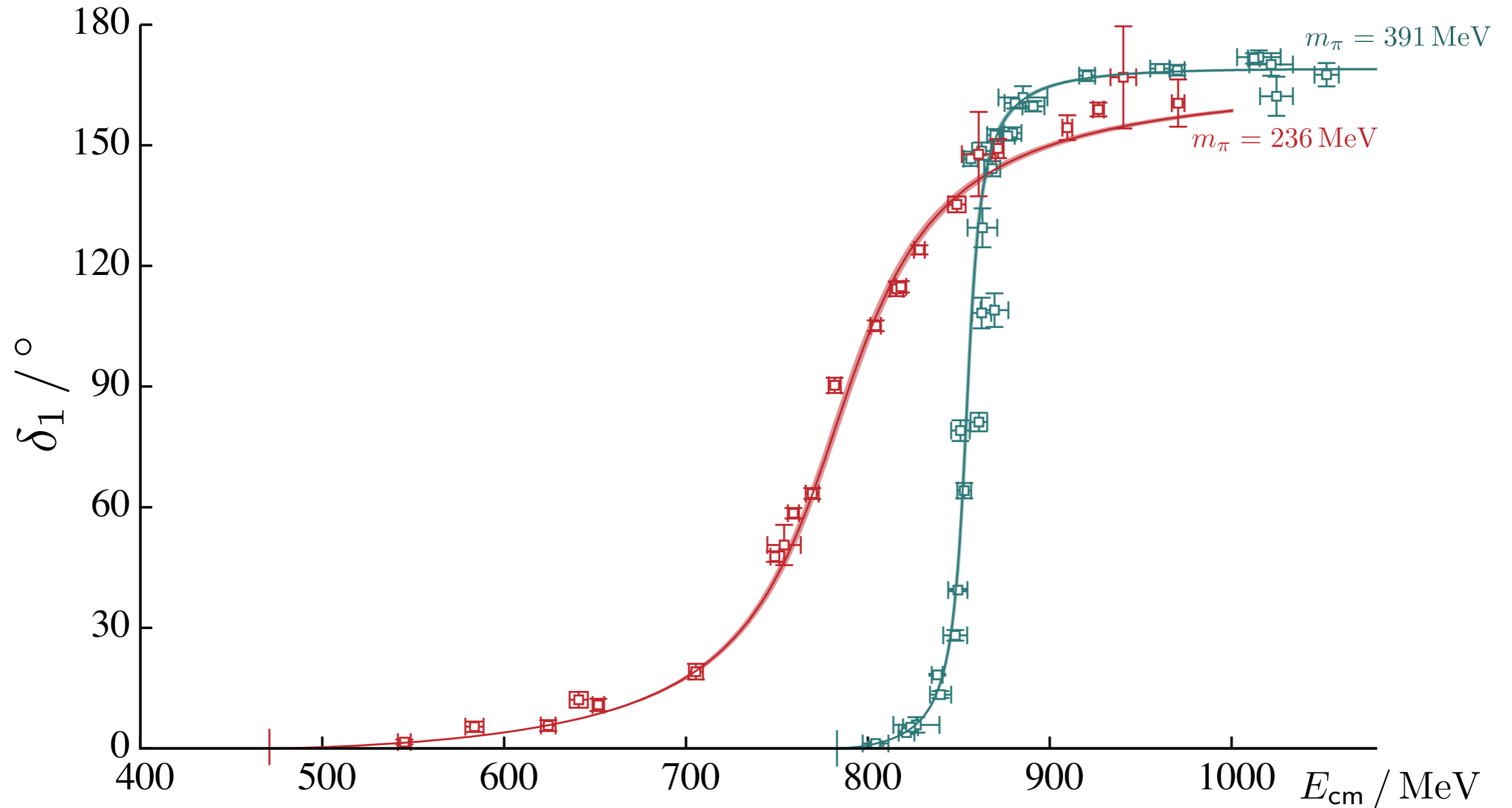


And also for the rho meson



Wilson, Briceño, Dudek, Edwards, Thomas, arXiv:1507:02599

And also for the rho meson



Wilson, Briceño, Dudek, Edwards, Thomas, arXiv:1507:02599