Extracting scattering and resonance properties from the lattice

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February 10th, 2016
Hall B

CLAS12 Torus
Magnet complete
CLAS12 Torus Magnet complete

Start taking data next year!
CLAS12 Torus Magnet complete
Start taking data next year!
Observed polarized rho photoproduction in 2015!

Add new hall
upgrade magnets and power supplies
double cryo capacity
5 new cryomodules

Hall D (GlueX)
Observed polarized rho photoproduction in 2015!

Hall B

CLAS12 Torus Magnet complete
Start taking data next year!
Observed polarized rho photoproduction in 2015!

JLab Physics
Hall D (GlueX)

Observed polarized rho photoproduction in 2015!

\( p\gamma \rightarrow N\rho \rightarrow N\pi\pi \)

Hall B

CLAS12 Torus Magnet complete
Start taking data next year!
\[ p\gamma \rightarrow N\rho \rightarrow N\pi\pi \]

\[ p\gamma^* \rightarrow N^* \rightarrow N\pi, N\eta \]
Resonances are not directly detected. Outgoing hadrons are used to reconstruct resonance properties. It is thus highly valuable to predict these transition amplitudes from the underlying theory of QCD.
Resonances are not directly detected. Outgoing hadrons are used to reconstruct resonance properties. It is thus highly valuable to predict these transition amplitudes from the underlying theory of QCD.

Combining accurate, model-independent predictions with experiment will lead to a deeper understanding of QCD’s rich resonance structure.
What can we extract from the lattice?

We are trying to evaluate a difficult integral numerically

\[
\text{observable} = \int \mathcal{D}\phi \ e^{iS} \left[ \begin{array}{c}
\text{interpolator} \\
\text{for observable}
\end{array} \right]
\]
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We are trying to evaluate a difficult integral numerically

\[
\text{observable} = \int \prod_i d\phi_i \; e^{-S} \quad \text{[interpolator for observable]}
\]

To do so we have to make four compromises
What can we extract from the lattice?

We are trying to evaluate a difficult integral numerically

\[
\text{observable} = \int \prod_i^N d\phi_i \ e^{-S} \left[ \text{interpolator for observable} \right]
\]

To do so we have to make four compromises

1. nonzero lattice spacing
2. finite volume, \( L \)
What can we extract from the lattice?

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1. nonzero lattice spacing
2. finite volume, \( L \)
3. Unphysical pion masses \( M_{\pi, \text{lattice}} > M_{\pi, \text{our universe}} \)
4. Euclidean correlators

But calculations at the physical pion mass do now exist.
What can we extract from the lattice?

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$$\left( \text{observable?} \right) = \int \prod_i d\phi_i \ e^{-S} \left[ \text{interpolator for observable} \right]$$

To do so we have to make four compromises:

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What can we extract from the lattice?

Not possible to directly calculate

$$\langle \pi \pi | \pi \pi \rangle$$

$$\langle N \pi | J_\mu | N \rangle$$

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\[ \langle \pi \pi | \pi \pi \rangle \]
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multi-particle in- and outstates
What can we extract from the lattice?

Not possible to directly calculate

multi-particle in- and outstates

\[
\langle \pi\pi, \text{out} | \pi\pi, \text{in} \rangle = \text{amputate and put on-shell} \\
\langle 0 | \tilde{\pi}(p^\prime) \tilde{\pi}(k^\prime) \tilde{\pi}(p) \tilde{\pi}(k) | 0 \rangle
\]
What can we extract from the lattice?

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multi-particle in- and outstates

\[
\langle \pi \pi, \text{out} | \pi \pi, \text{in} \rangle = \text{amputate and put on-shell} \]
\[
\langle 0 | \tilde{\pi}(p') \tilde{\pi}(k') \tilde{\pi}(p) \tilde{\pi}(k) | 0 \rangle
\]

\[
\langle N \pi \pi, \text{out} | J_\mu(x) | N \rangle = \text{amputate and put on-shell} \]
\[
\langle 0 | \tilde{N}(p'_1) \tilde{\pi}(p'_2) \tilde{\pi}(p'_3) J_\mu(x) \tilde{N}(P) | 0 \rangle
\]
What can we extract from the lattice?

Not possible to directly calculate

\[ \langle \pi \pi | \pi \pi \rangle \]
\[ \langle N\pi | J_\mu | N \rangle \]
\[ \langle N\pi\pi | J_\mu | N \rangle \]

multi-particle in- and outstates

\[ \langle \pi\pi, \text{out} | \pi\pi, \text{in} \rangle = \text{amputate and put on-shell} \]

\[ \langle 0 | \tilde{\pi}(p') \tilde{\pi}(k') \tilde{\pi}(p) \tilde{\pi}(k) | 0 \rangle \]

\[ \langle N\pi\pi, \text{out} | J_\mu(x) | N \rangle = \text{amputate and put on-shell} \]

\[ \langle 0 | \tilde{N}(p_1') \tilde{\pi}(p_2') \tilde{\pi}(p_3') J_\mu(x) \tilde{N}(P) | 0 \rangle \]

Requires Minkowski momenta and infinite volume
What can we extract from the lattice?

Instead we can only access finite-volume energies and matrix elements.

\[ H_{\text{QCD}} |n, L\rangle = |n, L\rangle E_n(L) \]
\[ \langle n, L, "N\pi\pi" | J_\mu(x) | "N", L\rangle \]

finite-volume energies and matrix elements
labels in quotes indicate quantum numbers
What can we extract from the lattice?

Instead we can only access

\[ H_{QCD} |n, L\rangle = |n, L\rangle E_n(L) \]

finite-volume energies and matrix elements

labels in quotes indicate quantum numbers

How can we determine

\[ \langle \pi\pi, \text{out} | \pi\pi, \text{in} \rangle \quad \text{and} \quad \langle N\pi\pi, \text{out} | J_\mu(x) | N \rangle \]

from

\[ E_n(L) \quad \text{and} \quad \langle n, L, \text{“}N\pi\pi\text{”} | J_\mu(x) | \text{“}N\text{”}, L \rangle \]
It is possible to derive relations between finite- and infinite-volume physics
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Two-particle scattering

\[ E_2(L) \]
\[ E_1(L) \]
\[ E_0(L) \]
It is possible to derive relations between finite- and infinite-volume physics.
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Two-particle scattering

\[ E_2(L) \quad E_1(L) \quad E_0(L) \]

Photo- and electroproduction

\[ \langle 2 | \mathcal{J} | 1 \rangle \quad \langle 2 | \mathcal{J} | 2 \rangle \]

Three-particle scattering

\[ E_2(L) \quad E_1(L) \]
Finite volume
Finite volume

**cubic**, spatial volume (extent $L$)

**periodic** boundary conditions

$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

**time direction** infinite
Finite volume

**cubic**, spatial volume (extent $L$)

**periodic** boundary conditions

$$\vec{p} \in (2\pi / L) \mathbb{Z}^3$$

time direction **infinite**

$L$ large enough to ignore $e^{-mL}$
Finite volume

**cubic**, spatial volume (extent $L$)

**periodic** boundary conditions

$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

time direction **infinite**

$L$ large enough to ignore $e^{-mL}$

Assume lattice effects are small and accommodated elsewhere

*Work in continuum field theory throughout*
Finite volume

quantum field theory

generic relativistic QFT

1. Include all interactions

2. no power-counting scheme

\textbf{cubic}, spatial volume (extent } L \text{)

\textbf{periodic} boundary conditions

\( \vec{p} \in (2\pi / L) \mathbb{Z}^3 \)

time direction \textbf{infinite}

\( L \) large enough to ignore \( e^{-mL} \)

Assume lattice effects are small and accommodated elsewhere

\textbf{Work in continuum field theory throughout}
Finite volume

quantum field theory

generic relativistic QFT

1. Include all interactions

2. no power-counting scheme

Not possible to directly calculate scattering observables to all orders

But it is possible to derive general, all-orders relations to finite-volume quantities

Assume lattice effects are small and accommodated elsewhere

Work in continuum field theory throughout

cubic, spatial volume (extent $L$)

periodic boundary conditions

$p \in (2\pi/L)\mathbb{Z}^3$

time direction infinite

$L$ large enough to ignore $e^{-mL}$
Two-to-two scattering

For now assume...

identical scalars, mass \( m \)

\[ \mathbb{Z}_2 \text{ symmetry} \]

---


Two-to-two scattering

\[ E_0(L) \rightarrow E_1(L) \rightarrow E_2(L) \]

For now assume...
identical scalars, mass \( m \)

\( \mathbb{Z}_2 \) symmetry

crossed diagram

two-particle interpolator

\[ C_L(P) \equiv \int_{L} d^4x \ e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle \]


Two-to-two scattering

\[ \begin{align*}
E_2(L) & \quad \rightarrow \quad E_1(L) \\
E_1(L) & \quad \rightarrow \quad E_2(L)
\end{align*} \]

For now assume...
identical scalars, mass \( m \)

\( \mathbb{Z}_2 \) symmetry

\[ C_L(P) \equiv \int \limits_L d^4x \ e^{-iPx} \langle 0 | T\mathcal{O}(x)\mathcal{O}^\dagger(0) | 0 \rangle \]

Euclidean convention
two-particle interpolator

\[ P = (P_4, \vec{P}) = (P_4, 2\pi n/L) \]
but allow \( P_4 \) to be real or imaginary

Two-to-two scattering

For now assume...
identical scalars, mass $m$

$\mathbb{Z}_2$ symmetry

Two-particle interpolator

Euclidean convention

$$C_L(P) \equiv \int_L d^4x \; e^{-iP \cdot x} \langle 0 | T O(x) O^\dagger(0) | 0 \rangle$$

two-particle interpolator

Pixels

CM frame energy is then

$$E^* \equiv \sqrt{-P_4^2 - \vec{P}^2}$$

Require $E^* < 4m$ to isolate two-to-two scattering

Two-to-two scattering

At fixed poles in $L$, $\mathbb{P}, \mathbb{C}$

analytic structure

Identical scalars, mass $m$

For now assume... $\mathbb{Z}_2$ symmetry

\[
C_L(P) \equiv \int_L d^4x \ e^{-iPx} \langle 0 | T\mathcal{O}(x)\mathcal{O}^\dagger(0) | 0 \rangle
\]

At fixed $L, \mathbb{P}$, poles in $C_L$ give finite-volume spectrum

$C_L$ analytic structure $C_\infty$ analytic structure
Two-to-two scattering

At fixed $L$, $\bar{P}$, poles in $C_L$ give finite-volume spectrum

$$C_L(P) \equiv \int_L d^4x \ e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

For now assume…

identical scalars, mass $m$

$\mathbb{Z}_2$ symmetry

Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.
\[ C_L(P) = o^\dagger \begin{array}{c} \circ \end{array} o + o^\dagger \begin{array}{c} iK \end{array} o \]

\[ + o^\dagger \begin{array}{c} iK \end{array} iK o + \cdots \]


\[ C_L(P) = \begin{array}{c} \text{ spatial loop momenta} \\
\text{are summed} \end{array} \]

\[
\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}
\]

\[ Lüscher, M. \textit{Nucl. Phys} B354, 531-578 (1991) \]

\[ C_L(P) = \langle \mathcal{O}^\dagger \mathcal{O} \rangle + \langle \mathcal{O}^\dagger iK \mathcal{O} \rangle + \cdots \]

Spatial loop momenta are summed

\[
\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}
\]


$$C_L(P) = \langle \bar{O}^\dagger \rangle \langle \bar{O} \rangle + \langle \bar{O}^\dagger \rangle \langle iK \bar{O} \rangle + \langle \bar{O}^\dagger \rangle \langle iK iK \bar{O} \rangle + \cdots$$

Spatial loop momenta are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)^3} \int \frac{dk^0}{2\pi}$$

\[ \Delta \equiv \text{fully dressed propagator} \]

Bethe Salpeter kernel

\[ C_L(P) = \langle \mathcal{O}^\dagger \mathcal{O} \rangle + \langle \mathcal{O}^\dagger iK \mathcal{O} \rangle + \langle \mathcal{O}^\dagger iK iK \mathcal{O} \rangle + \cdots \]

Spatial loop momenta are summed

\[
\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}
\]

\[ \Delta = \quad \text{fully dressed propagator} \]

\[ \Delta_L = \Delta_\infty + \mathcal{O}(e^{-mL}) \]

If \( E^* < 4m \) then

\[ K_L = K_\infty + \mathcal{O}(e^{-mL}) \]


Now we introduce an important identity.

\[ C_L(P) = o^\dagger \circ o + o^\dagger \circ iK \circ o + \cdots \]

\[ \frac{1}{L^3} \sum_{\vec{k}} \int_{\vec{k}} \]

contains all power-law corrections

Now we introduce an important identity.

Now we introduce an important identity.

\[
C_L(P) = O^\dagger \, \theta \, O + O^\dagger \, iK \, O + \cdots
\]

\[
\frac{1}{L^3} \sum_{\vec{k}} \int_{\vec{k}} F
\]

contains all power-law corrections

In all four-momenta are projected on shell.


Now we introduce an important identity.

\[ C_L(P) = \circledast o^\dagger \circledast o + \circledast o^\dagger iK \circledast o + \cdots \]

\[ \text{contains all power-law corrections} \]

In \[ \text{all four-momenta are projected on shell.} \]

Physical, propagating states give dominate finite-volume effects.

$$C_L(P) = \circlearrowleft_{\circ^\dagger} \circ \circlearrowleft_{\circ^\dagger} \circ \circlearrowleft_{\circ^\dagger} \circ K \circ\circlearrowright_{\circ^\dagger} \circ + \circlearrowleft_{\circ^\dagger} \circ K \circ K \circ \circlearrowright_{\circ^\dagger} \circ + \cdots$$
Now regroup by number of Fs

$$C_L(P) = \circ^{\dagger} \cdot \circ + \circ^{\dagger} \cdot iK \cdot \circ + \cdots$$

$$C_L(E, \tilde{P}) = C_\infty(E, \tilde{P}) + \cdots$$
Now regroup by number of Fs

\[ C_L(P) = C_L(E, \vec{P}) = C_{\infty}(E, \vec{P}) + (A \xrightarrow{F} A') + \]
Now regroup by number of Fs

\[ C_L(P) = C_{L}(E, \tilde{P}) = C_{\infty}(E, \tilde{P}) + A + \cdots \]

zero Fs
\[ C_L(E, \tilde{P}) = C_{\infty}(E, \tilde{P}) + A + \cdots \]

one F
\[ = \langle \pi \pi, \text{out} | O^\dagger | 0 \rangle \]
\[ C_L(P) = O^\dagger O + O^\dagger iK O \]

Now regroup by number of Fs

zero Fs
\[ C_L(E, \tilde{P}) = C_\infty(E, \tilde{P}) + \begin{array}{c}
A \end{array}_{F} A' \]

one F
\[ + \begin{array}{c}
iM \end{array}_{F'} A' \]

two Fs
\[ + \cdots \]

\[ = \langle \pi \pi, \text{out} | O^\dagger | 0 \rangle \]
\[ C_L(P) = \big( \mathcal{O}^\dagger \big) \circ O + \big( \mathcal{O}^\dagger \big) \circ iK \circ O \]

Now regroup by number of Fs

\[ C_L(E, \tilde{P}) = C_\infty(E, \tilde{P}) + A \bigg( A' \bigg) \bigg( F \bigg) + A \bigg( i\mathcal{M} \bigg) \bigg( A' \bigg) \bigg( F \bigg) + \cdots \]

zero Fs

\[ = \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle \]
$C_L(P) = \mathcal{O} + \mathcal{O} \mathcal{I} \mathcal{K} \mathcal{O}$

Now regroup by number of Fs

$C_L(E, \bar{P}) = C_\infty(E, \bar{P}) + A A' + A iM A' + \cdots$

$= \langle \pi \pi, \text{out} | \mathcal{O} | 0 \rangle$

When we factorize diagrams and group infinite-volume parts…

physical observables emerge!
Review...
Review...

\[ C_L(P) = O^\dagger \begin{array}{c} \bullet \end{array} O + O^\dagger \begin{array}{c} iK \end{array} O \]

\[ + O^\dagger \begin{array}{c} iK \end{array} iK O + \cdots \]
Review...

\[ C_L(P) = \sigma^\dagger \sigma + iK \sigma^\dagger \sigma + \sigma^\dagger iK \sigma^\dagger iK \sigma + \cdots \]
Review...

\[ C_L(P) = \Omega^\dagger \rightleftharpoons \Omega + \Omega^\dagger \rightleftharpoons iK \rightleftharpoons \Omega + \cdots \]

\[ C_L(P) = C_\infty(P) \]

\[ + A_F \rightleftharpoons A'_{F'} + A_F \rightleftharpoons iM \rightleftharpoons A'_{F'} \]

\[ + A_F \rightleftharpoons iM \rightleftharpoons iM \rightleftharpoons A'_{F'} + \cdots \]

\[ \langle \pi\pi, \text{out} | \Omega^\dagger | 0 \rangle \]

\[ \langle 0 | \Omega | \pi\pi, \text{in} \rangle \]
Review...

\[ C_L(P) = C_\infty(P) \]

\[ + \begin{array}{c}
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Review...

\[ C_L(P) = O^\dagger O + O^\dagger iK O + O^\dagger iK iK iK O + \cdots \]

\[ C_L(P) = C_\infty(P) + A_{FF} + A_{FF} iM + A_{FF} iM iM + A_{FF} + \cdots \]

\[ \langle \pi\pi, \text{out} | O^\dagger | 0 \rangle \]

\[ \langle 0 | O | \pi\pi, \text{in} \rangle \]

We deduce...

\[ C_L(P) = C_\infty(P) - A_{FF} \frac{1}{1 + M_{2\rightarrow2F}} A \]

poles are in here

\[ P_4 \]
Two-particle result

At fixed \((L, \vec{P})\), finite-volume energies are solutions to

\[
\det[\mathcal{M}_{2\rightarrow 2}^{-1} + F] = 0
\]


Matrices defined using angular-momentum states
Two-particle result

At fixed \((L, \vec{P})\), finite-volume energies are solutions to

\[
\det[\mathcal{M}^{-1}_{2\rightarrow2} + F] = 0
\]

Matrices defined using angular-momentum states

\[
\mathcal{M}_{2\rightarrow2} \equiv \begin{pmatrix}
& & \\
& & \\
& & 
\end{pmatrix}
\]
diagonal matrix, parametrized by \(\delta_\ell(E^*)\)

Two-particle result

At fixed \((L, \vec{P})\), finite-volume energies are solutions to

\[
\det[\mathcal{M}^{-1}_{2\rightarrow2} + F] = 0
\]

Matrices defined using angular-momentum states

\[
\mathcal{M}_{2\rightarrow2} \equiv \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\bullet & \rightarrow & \bullet \\
\end{array}
\]

diagonal matrix, parametrized by \(\delta_\ell(E^*)\)

\[
F \equiv \text{non-diagonal matrix of known geometric functions}
\]

Two-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to

$$\det \left[ M_{2 \rightarrow 2}^{-1} + F \right] = 0$$

Matrices defined using angular-momentum states

$M_{2 \rightarrow 2} \equiv \begin{array}{c}
\includegraphics{diagram1} \\
\end{array}$ diagonal matrix, parametrized by $\delta_l(E^*)$

$F \equiv \begin{array}{c}
\includegraphics{diagram2} \\
\end{array}$ non-diagonal matrix of known geometric functions

$\equiv \begin{array}{c}
\includegraphics{diagram3} \\
\end{array}$ difference of two-particle loops

in finite and infinite volume depends on $L, E, \vec{P}$


Two-particle result

At fixed \((L, \vec{P})\), finite-volume energies are solutions to

\[
\det [\mathcal{M}_{2\to 2}^{-1} + F] = 0
\]


Matrices defined using angular-momentum states

\[
\mathcal{M}_{2\to 2} \equiv \text{diagonal matrix, parametrized by } \delta_\ell(E^*)
\]

\[
F \equiv \text{non-diagonal matrix of known geometric functions}
\]

\[
\equiv \text{difference of two-particle loops in finite and infinite volume}
\]

At low energies, lowest partial waves dominate \(\mathcal{M}_{2\to 2}\)

\[
cot \delta(E^*_n) + \cot \phi(E_n, \vec{P}, L) = 0
\]

e.g. s-wave only
with some rearranging

scattering phase known function
Using the result (p-wave)

$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$

Using the result (p-wave)

\[ \cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0 \]
Using the result (p-wave)

\[
cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0
\]

Using the result (p-wave)

\[ \cot \delta_{\ell=1}(E^*_n) + \cot \phi(E_n, \vec{P}, L) = 0 \]

Using the result (p-wave)

\[ \cot \delta_{\ell=1}(E_n^*) + \cot \phi(\vec{E}_n, \vec{P}, L) = 0 \]
\[ \cot \delta_{\ell=1}(E^*_n) + \cot \phi(E_n, \vec{P}, L) = 0 \]

\[ m_\pi = 391 \text{ MeV} \]

\[ m_R = 854.1 \pm 1.1 \text{ MeV} \]
\[ g = 5.80 \pm 0.11 \]
\[ \Gamma_R = \frac{g^2}{6\pi} \frac{p^2_{\pi}}{m_R^2} = 12.4 \pm 0.6 \text{ MeV} \]

\( \delta_{J=1}^{\ell=1}/^0 \)

\( \rho \text{ resonance} \)

Two-particle result

At fixed \((L, \vec{P})\), finite-volume energies are solutions to
\[
\det[M^{-1}_{2 \rightarrow 2} + F] = 0
\]

Has since been generalized to include...

- non-indentical particles
  \(\bullet \neq \bullet\)
- multiple two-particle channels
- particles with spin

Two-particle result

At fixed \((L, \vec{P})\), finite-volume energies are solutions to

\[
\det[\mathcal{M}_{2\rightarrow 2}^{-1} + F] = 0
\]

Has since been generalized to include...

- non-indentetical particles
- multiple two-particle channels
- particles with spin


The basic form of the equation stays the same, but the **matrix space and definition of F** change
Multiple two-particle channels

Must now include a channel index

\[ \det \left[ \begin{pmatrix} M_{a \rightarrow a} & M_{a \rightarrow b} \\ M_{b \rightarrow a} & M_{b \rightarrow b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0 \]
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MTH and Sharpe/Briceño and Davoudi

Already used in JLab study of \( \pi K, \eta K \)

\[ \mathcal{M}(\pi K \to \eta K) \sim \sqrt{1 - \eta^2} \]


[arXiv: 1411.2004]
Multiple two-particle channels

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\text{det} \left[ \begin{pmatrix} \mathcal{M}_{a \to a} & \mathcal{M}_{a \to b} \\ \mathcal{M}_{b \to a} & \mathcal{M}_{b \to b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0
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As well as JLab rho study with \( \pi\pi, \ K\bar{K} \)

\[ \mathcal{M}(\pi\pi \to K\bar{K}) \sim \sqrt{1 - \eta^2} \]

Wilson, Briceño, Dudek, Edwards, Thomas, arXiv:1507:02599
Two-particle scattering

$E_0(L)$

$E_1(L)$

$E_2(L)$

Photo- and electroproduction

$E_0(L)$

$E_1(L)$

$E_2(L)$

Three-particle scattering

$E_0(L)$

$E_1(L)$

$E_2(L)$
Photoproduction \[ \langle \pi \pi, \text{out} | J_\mu | \pi \rangle \equiv \]

How can we get this from finite-volume observables?
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How can we get this from finite-volume observables?

Why did we expect \( C_L(P) \) to have poles?

\[
C_L(P) \equiv \int_L d^4x \, e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^+(0) | 0 \rangle
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Insert a complete set finite-volume of states
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\[
C_L(P) \xrightarrow{P_4 \to iE_n} \frac{L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle}{(E_n + iP_4)}
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Now compare this to our factorized result

\[ C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to2} F} A \]
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\[ \left\langle 0 | \mathcal{O}(0) | \pi\pi, \text{in} \right\rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \text{out} | \mathcal{O}^\dagger(0) | 0 \rangle \]

\[ \frac{1}{(E_n + iP_4)} \]
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\[
\xrightarrow{P_4 \to iE_n} \frac{1}{(E_n + iP_4)}
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\( \mathcal{R} \) is the residue of this matrix
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\[ C_L(P) \xrightarrow[\mathcal{P}_4 \rightarrow iE_n]{} L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle (E_n + iP_4) \]

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One has the freedom to choose \( \mathcal{O}^\dagger \) such that \( \mathcal{O}^\dagger | 0 \rangle = \mathcal{J}_\mu | \pi \rangle \).

(Finite-volume effects are exponentially suppressed for single particles.)
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\[ 2\omega_\pi L^6 \langle n, \vec{P}, L \mid \mathcal{J}_\mu(0) \mid \pi, L \rangle^2 = \]

\[ \langle \pi \mid \mathcal{J}_\mu(0) \mid \pi\pi, \text{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi\pi, \text{out} \mid \mathcal{J}_\mu(0) \mid \pi \rangle \]

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How can we get this from finite-volume observables?

\[ L^3 \langle 0 | O(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | O^\dagger(0) | 0 \rangle = \]

\[ \langle 0 | O(0) | \pi \pi, \text{in} \rangle R(E_n, \vec{P}, L) \langle \pi \pi, \text{out} | O^\dagger(0) | 0 \rangle \]

One has the freedom to choose \( O^\dagger \) such that \( O^\dagger | 0 \rangle = J_\mu | \pi \rangle \).
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get this from the lattice

\[ 2\omega_\pi L^6 | \langle n, \vec{P}, L | J_\mu(0) | \pi, L \rangle |^2 = \]

experimental observable

\[ \langle \pi | J_\mu(0) | \pi \pi, \text{in} \rangle R(E_n, \vec{P}, L) \langle \pi \pi, \text{out} | J_\mu(0) | \pi \rangle \]

Photoproduction \( \langle \pi \pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv \)

How can we get this from finite-volume observables?

\[
L^3 \langle 0 | \mathcal{O}(0) | n, \vec{P}, L \rangle \langle n, \vec{P}, L | \mathcal{O}^\dagger(0) | 0 \rangle = \\
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\]

\[
\mathcal{R}(E_n, \vec{P}, L) = -\text{Residue} \left[ \frac{1}{E_n F^{-1} + \mathcal{M}_{2\rightarrow2}} \right]
\]

Photoproduction $\langle \pi \pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv \langle \pi | \mathcal{J}_\mu (0) | \pi \pi, \text{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi \pi, \text{out} | \mathcal{J}_\mu (0) | \pi \rangle$

Briceño, MTH, Walker-Loud/Briceño, MTH

Experimental observable

$2\omega_\pi L^6 |\langle n, \vec{P}, L | \mathcal{J}_\mu (0) | \pi, L \rangle|^2 = \langle \pi | \mathcal{J}_\mu (0) | \pi \pi, \text{in} \rangle \mathcal{R}(E_n, \vec{P}, L) \langle \pi \pi, \text{out} | \mathcal{J}_\mu (0) | \pi \rangle$

$m_\pi \approx 400 \text{ MeV}$

Photoproduction in the rho channel

Briceño, Dudek, Edwards, Schultz, Thomas, Wilson

arXiv: 1507.6622
Photoproduction \[ \langle \pi \pi, \text{out} | \mathcal{J}_\mu | \pi \rangle \equiv \]

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- non-indentical particles
- multiple two-particle channels
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Formalism is in place to give Lattice QCD predictions of this process (ignoring three particles)

Two-to-two transitions

\[ \langle \pi \pi, \text{out} | \mathcal{J}_\mu | \pi \pi, \text{in} \rangle \equiv \]

Formalism complete

non-identical particles

multiple two-particle channels

R. A. Briceño, MTH, arXiv: 1509.08507
Two-to-two transitions

\[ \langle \pi \pi, \text{out} | J_\mu | \pi \pi, \text{in} \rangle \equiv \]

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R. A. Briceño, MTH, arXiv: 1509.08507

Required to extract resonance form factors
Two-particle scattering

Two-particle scattering

Photo- and electroproduction

Three-particle scattering
Three-to-three scattering

For now assume... identical scalars, mass $m$ 

$\mathbb{Z}_2$ symmetry
Three-to-three scattering

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identical scalars, mass $m$

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$$C_L(P) \equiv \int_L d^4 x \ e^{-iP \cdot x} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

three-pion interpolator
Three-to-three scattering

For now assume…

identical scalars, mass \( m \)

\[ \mathbb{Z}_2 \text{ symmetry} \]

\[
C_L(P) \equiv \int_L d^4x \ e^{-iP\cdot x} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle
\]

Calculate \( C_L(P) \) to all orders in perturbation theory and determine locations of poles.

Require \( m < E^* < 5m \) to isolate three-particle states
Recall for two particles we started with a “skeleton expansion”

\[ C_L(P) = O_\dagger \circ \circ + O_\dagger iK \circ O + O_\dagger iK iK O + \cdots \]
Recall for two particles we started with a “skeleton expansion”

\[ C_L(P) = \mathcal{O}^\dagger \mathcal{O} + \mathcal{O}^\dagger iK \mathcal{O} + \mathcal{O}^\dagger iK^2 \mathcal{O} + \cdots \]

So now we need the same for three…

\[ C_L(E, \vec{P}) = \mathcal{O} + \mathcal{O}^\dagger iK \mathcal{O} + \mathcal{O}^\dagger iK^2 \mathcal{O} + \cdots \]
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No! We also need diagrams like

Disconnected diagrams in \[ \mathcal{O} \] lead to singularities that invalidate the derivation
$C_L(E, \tilde{P}) = \text{Diagram 1} + \text{Diagram 2} + \cdots$

**Kernel definitions:**

- \[ \bullet \quad \equiv \quad \otimes + \bigotimes + \cdots \]
- \[ \otimes \quad \equiv \quad \otimes + \boxtimes + \cdots \]
New skeleton expansion

\[ C_L(E, \bar{P}) = \sum \int_{E} + \sum \int_{E} + \sum \int_{E} + \cdots \]

Kernel definitions:

\[ \begin{align*}
\text{Purple kernel} : & \equiv \bigoplus_{\text{bands}} + \bigoplus_{\text{bands}} + \cdots \\
\text{Orange kernel} : & \equiv \bigoplus_{\text{bands}} + \bigoplus_{\text{bands}} + \cdots
\end{align*} \]
New skeleton expansion

\[ C_L(E, \bar{P}) = \sum + \sum + \sum + \sum + \ldots \]

Kernel definitions:

\[ \begin{align*}
\begin{array}{c}
\text{Kernel 1} \equiv & x + \text{other kernel} + \ldots \\
\text{Kernel 2} \equiv & x + \text{other kernel} + \ldots
\end{array}
\end{align*} \]
New skeleton expansion

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Kernel definitions:

- \[ \begin{array}{c}
    \text{\purple} \\
    \equiv \mathcal{X} \oplus \mathcal{X} \oplus \mathcal{X} \oplus \cdots
    \end{array} \]

- \[ \begin{array}{c}
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Significantly more complicated than two-particle story
Three-to-three scattering

1. Work out the three particle skeleton expansion

\[ C_L(E, \vec{P}) = \ldots + \text{diagrams} + \ldots \]
Three-to-three scattering

1. Work out the three particle skeleton expansion

\[ C_{L}(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \cdots \]

2. Break diagrams into finite- and infinite-volume parts
Three-to-three scattering

1. Work out the three particle skeleton expansion

\[ C_L(E, \bar{P}) = C_L^{E_0(L)} + C_L^{E_1(L)} + C_L^{E_2(L)} + \cdots \]

2. Break diagrams into finite- and infinite-volume parts

3. Organize and sum terms to identify infinite-volume observables
Three-to-three scattering

1. Work out the three particle skeleton expansion

\[ C_L(E, \vec{P}) = \ldots + \begin{array}{c}
\includegraphics[scale=0.5]{diagram1.png}
\end{array} + \ldots + \begin{array}{c}
\includegraphics[scale=0.5]{diagram2.png}
\end{array} + \ldots \]

2. Break diagrams into finite- and infinite-volume parts

3. Organize and sum terms to identify infinite-volume observables

Major complicating factors:
More diagram topologies, more degrees of freedom, three-to-three amplitude contains “long distance” kinematic poles
Three-to-three scattering

Current status:
Formalism is complete for the simplest three-scalar system

General, model-independent relation between finite-volume energies and three-to-three scattering amplitude

Derived using a generic relativistic field theory

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Important caveats:
Identical particles with no two-to-three transitions
\[ \pi \pi \pi \rightarrow \pi \pi \pi \]

Requires that two-particle scattering phase is bounded
\[ \left| \delta_\ell(E) \right| < \pi/2 \]
Three-to-three scattering

To check the result we have expanded the lowest three-particle energy in powers of $1/L$.

$$E = 3m + \frac{a_3}{L^3} + \frac{a_4}{L^4} + \frac{a_5}{L^5} + \frac{a_6}{L^6} + \mathcal{O}(1/L^7)$$
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This part is new... turns out that relativistic effects enter at this order...

$$\frac{a_6}{a_3} \equiv \left(\frac{a}{\pi}\right)^3 \left[ 2532.01 + \frac{16\pi^3}{3} (3\sqrt{3} - 4\pi) \log \left( \frac{mL}{2\pi} \right) \right]$$

$$- 37.25 \frac{a^2}{m} + \frac{3\pi a}{m^2} + 6\pi ra^2 - \frac{M_{3,\text{thr}}}{48m^3a_3}$$

MTH and Sharpe, arXiv:1602.00324
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checked independently in

$\lambda \phi^4$ theory

through $\mathcal{O}(\lambda^3)$


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K. Huang and C. Yang, Phys. Rev. 105 (1957) 767-775


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Currently underway:

Relax all simplifying assumptions:

Allow all particle types, allow two-to-three couplings, remove bound on phase shift

\[ K \pi \rightarrow K \pi \pi \quad N \pi \rightarrow N \pi \pi \quad NNN \rightarrow NNN \]

Briceño, MTH, Sharpe, *in development*
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Briceño, MTH, Sharpe, *in development*

Use matching trick to recover transition amplitudes

\[ p\gamma \rightarrow N\rho \rightarrow N\pi\pi \]
Summary

Reviewed methods to map finite-volume observables into physically observable scattering and transition amplitudes.
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Summary

Reviewed methods to map finite-volume observables into physically observable scattering and transition amplitudes.

Results come from studying all-orders expansions in generic relativistic quantum field theory.

The work is technical and requires developing new tools and methods for each new system.

Can the scattering and transition amplitudes of QCD be extracted from Lattice QCD in a general, model independent way?

So far all signs point to yes!
Experimental groups at JLab are measuring exactly the kinds of processes accommodated by this formalism.
My work at JLab:

Experimental groups at JLab are measuring exactly the kinds of processes accommodated by this formalism.

Lattice group at JLab leads the field in applying this kind of formalism.
My work at JLab:

$p\gamma \rightarrow N\rho \rightarrow N\pi\pi$

It would accelerate progress significantly if I had the opportunity to continue developing and also to apply this formalism here at JLab.
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The ideal scenario...
Regular interaction with experimental, lattice and theory groups:
Identifying the most relevant observables,
Developing formalism to extract these,
Performing the calculations
My work at JLab:
One example of symbiosis...

Formalism can also be applied in the “other direction” to gain insight on lattice observables

\[ I(J^P) = 1/2(1/2^+) \]

GW partial-wave data base (solution WI08)

MTH and Meyer, to appear
My work at JLab:

More concretely...
My work at JLab:

More concretely...

In one to two years:

The formalism needed for $N\pi \rightarrow N\pi\pi$ and $N\gamma \rightarrow N^* \rightarrow N\pi\pi$ expected to be complete.

First lattice studies of three-particle systems

$K\pi \rightarrow K\pi\pi$ \quad $\omega \rightarrow \pi\pi\pi$
My work at JLab:

More concretely...

In one to two years:

The formalism needed for $N\pi \rightarrow N\pi\pi$ and $N\gamma \rightarrow N^* \rightarrow N\pi\pi$ expected to be complete.

First lattice studies of three-particle systems

$K\pi \rightarrow K\pi\pi$  \quad $\omega \rightarrow \pi\pi\pi$

In five years:

Automated result for n-body scattering and transitions implemented in a publicly available code library
My work at JLab:
More concretely...

In one to two years:
The formalism needed for $N\pi \rightarrow N\pi\pi$ and $N\gamma \rightarrow N^* \rightarrow N\pi\pi$ expected to be complete.

First lattice studies of three-particle systems

$K\pi \rightarrow K\pi\pi$  \hspace{1cm} $\omega \rightarrow \pi\pi\pi$

In five years:

Automated result for n-body scattering and transitions implemented in a publicly available code library

Thanks for listening!
Backup Slides
Two-to-two transition amplitudes

finite-volume spectrum

$L_\text{üscher}$ formalism and extensions

two-to-two scattering amplitude

divergence-free two-to-two transition amplitude

finite-volume two-to-two matrix element

finite-volume one-body matrix element

infinite-volume one-body transition amplitude

$E_n(L, \vec{P})$

$e^{-mL}$

infinite-volume two-to-two transition amplitude
Infinite volume

\[ iM_{3\rightarrow 3} \equiv \text{Sum of all connected Feynman diagrams with six external legs} \]
Infinite volume

\[ iM_{3 \to 3} \equiv \text{Sum of all connected Feynman diagrams with six external legs} \]

\[ = + \cdots \]
Infinite volume

\[ iM_{3 \rightarrow 3} \equiv \text{Sum of all connected Feynman diagrams with six external legs} \]

\[ = \ldots \]

Certain external momenta put this on-shell!
Infinite volume

\[ i M_{3 \to 3} \equiv \text{Sum of all connected Feynman diagrams with six external legs} \]

[Diagram showing a sum of connected Feynman diagrams]

Certain external momenta put this on-shell!

\[ M_{3 \to 3} \] has kinematic singularities at certain momenta

No dominance of lowest partial waves
Infinite volume

Degrees of freedom for three on-shell particles with $\left( E, \vec{P} \right)$

$\left( \omega_k, \vec{k} \right) \rightarrow \left( E - \omega_k, \vec{P} - \vec{k} \right)$

$\hat{a}^* \rightarrow \ell, m$
Infinite volume

Degrees of freedom for three on-shell particles with \((E, \vec{P})\)
New skeleton expansion

\[ C_L(E, \vec{P}) = \quad \square + \quad \square + \quad \square + \quad \square + \cdots + \cdots + \cdots \]

Compare to two-particle skeleton expansion

\[ C_L(E, \vec{P}) = \quad \square + \quad \square + \quad \square + \quad \square + \cdots + \cdots + \cdots \]
What is new here?

1. Degrees of freedom are different

two particles

two-particle angular momentum

\((E - \omega_k, \hat{P} - \vec{k})\)

\((\omega_k, \vec{k})\)

three particles

\(\vec{k} + \text{two-particle angular momentum}\)

\(\hat{a}^* \rightarrow \ell, m\)

Our result only depends on finite-volume momentum

\(\vec{k} = \frac{2\pi \vec{n}}{L}\)
What is new here?

1. Degrees of freedom are different

- two particles
  - two-particle angular momentum

- three particles
  - two-particle angular momentum

Our result only depends on finite-volume momentum

\[ \vec{k} = \frac{2\pi}{L} \vec{n} \]

Quantization condition expressed using matrices with indices

\[ \vec{k}, \ell, m \]
What is new here?

2. Three particle divergences

Define $i\mathcal{M}_{df, 3\rightarrow 3}$

$$\equiv i\mathcal{M}_{3\rightarrow 3} - \left[ i\mathcal{M}_{2\rightarrow 2} S i\mathcal{M}_{2\rightarrow 2} + \int i\mathcal{M}_{2\rightarrow 2} S i\mathcal{M}_{2\rightarrow 2} S i\mathcal{M}_{2\rightarrow 2} + \cdots \right]$$

This subtraction emerges naturally in our finite-volume analysis
What is new here?

3. Must now worry about sum crossing
two-particle unitary cusp

two-particle scattering
(real part)

two particle energy
What is new here?

3. Must now worry about sum crossing two-particle unitary cusp

\[ \frac{1}{L^3} \sum_{\vec{k}} \text{two-particle scattering (real part)} \]

depends on \( k \)

two particle energy

\[ m \quad 2m \quad 3m \quad 4m \]
What is new here?

3. Must now worry about sum crossing two-particle unitary cusp

To remove cusp

\[ i \epsilon \text{ prescription} \xrightarrow{\text{principal value}} \overline{\text{PV}} \]

Analytically continue principal value below threshold then interpolate to prescription-free subthreshold form

What is new here?

3. Must now worry about sum crossing two-particle unitary cusp

To remove cusp

\( i\epsilon \) prescription → principal value \( \widetilde{PV} \)

standard definition

\( m \quad 2m \quad 3m \quad 4m \)
What is new here?

3. Must now worry about sum crossing two-particle unitary cusp

\[ iM_{2 \rightarrow 2} = \text{Diagram} \]

has a cusp

\[ i\tilde{K}_{2 \rightarrow 2} = \text{Diagram} \]
What is new here?

3. Must now worry about sum crossing two-particle unitary cusp

\[ iM_{2\rightarrow 2} \]
\[ iM_{df,3\rightarrow 3} \]

We relate these infinite-volume quantities to the finite-volume spectrum
Three-particle result

\[
C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'_3 iF_3 \frac{1}{1 - iK_{df,3\to3} iF_3} A_3
\]

\[
iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - iM_{L,2\to2} iG} \right] iM_{L,2\to2} iF
\]

\[
iM_{L,2\to2} \equiv iK_{2\to2} \frac{1}{1 - iFiK_{2\to2} iF}
\]

All factors are matrices with indices $\vec{k}, \ell, m$
Three-particle result

\[
C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'_3 iF_3 \frac{1}{1 - iK_{df,3\to3}} iF_3 A_3
\]

\[iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - iM_{L,2\to2} iG} \right] iM_{L,2\to2} \equiv iK_{2\to2} \frac{1}{1 - iF iK_{2\to2}}\]

All factors are matrices with indices \( \vec{k}, \ell, m \).
Three-particle result

At fixed \((L, \vec{P})\), finite-volume spectrum is all solutions to

\[
\det \left[ 1 - i\mathcal{K}_{df,3\rightarrow3} iF_3 \right] = 0
\]

\[
iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\rightarrow2} iG} i\mathcal{M}_{L,2\rightarrow2} iF \right] \quad i\mathcal{M}_{L,2\rightarrow2} \equiv i\mathcal{K}_{2\rightarrow2} \frac{1}{1 - iF i\mathcal{K}_{2\rightarrow2}}
\]

Three-particle result

At fixed \((L, \vec{P})\), finite-volume spectrum is all solutions to

\[
\det \left[ 1 - i\mathcal{K}_{df,3\to3}iF_3 \right] = 0
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iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\to2}iG}i\mathcal{M}_{L,2\to2}iF \right]
\]

\[
i\mathcal{M}_{L,2\to2} \equiv i\mathcal{K}_{2\to2} \frac{1}{1 - iF_i\mathcal{K}_{2\to2}}
\]


Model independent general result of relativistic scalar field theory
Three-particle result

At fixed $(L, \vec{P})$, finite-volume spectrum is all solutions to

$$\det \left[ 1 - i\mathcal{K}_{df,3\rightarrow3}iF_3 \right] = 0$$

$$iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2\rightarrow2}iG} i\mathcal{M}_{L,2\rightarrow2}iF \right] \quad i\mathcal{M}_{L,2\rightarrow2} \equiv i\mathcal{K}_{2\rightarrow2} \frac{1}{1 - iF_i\mathcal{K}_{2\rightarrow2}}$$


Model independent general result of relativistic scalar field theory

Assumes two-particle phase shift is bounded by $\pi/2$
Three-particle result

At fixed \((L, \vec{P})\), finite-volume spectrum is all solutions to

\[
\det \left[ 1 - iK_{df,3 \rightarrow 3} iF_3 \right] = 0
\]

\[
iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - iM_{L,2 \rightarrow 2} iG} iM_{L,2 \rightarrow 2} iF \right] \quad iM_{L,2 \rightarrow 2} \equiv iK_{2 \rightarrow 2} \frac{1}{1 - iF iK_{2 \rightarrow 2}}
\]

MTH and Sharpe, Phys. Rev. D90, 116003 (2014)

Model independent general result of relativistic scalar field theory

Assumes two-particle phase shift is bounded by \(\pi/2\)

Infinite matrices truncate if we truncate in angular momentum
Three-particle result

At fixed \((L, \vec{P})\), finite-volume spectrum is all solutions to

\[
\det \left[ 1 - iK_{df,3\rightarrow 3} iF_3 \right] = 0
\]

\[
iF_3 \equiv \frac{iF}{2\omega L^3} \left[ \frac{1}{3} + \frac{1}{1 - iM_{L,2\rightarrow 2} iG} iM_{L,2\rightarrow 2} iF \right]
\]

\[
iM_{L,2\rightarrow 2} \equiv iK_{2\rightarrow 2} \frac{1}{1 - iF_2 iK_{2\rightarrow 2}}
\]


Model independent general result of relativistic scalar field theory

Assumes two-particle phase shift is bounded by \(\pi/2\)

Infinite matrices truncate if we truncate in angular momentum

Strongest truncation is the isotropic limit, gives simple result

\[
K_{df,3\rightarrow 3}(E_n^*) = - [F_{3,iso}(E_n, \vec{P}, L)]^{-1}
\]
Relating $i\mathcal{K}_{df,3\to 3}$ to $i\mathcal{M}_{3\to 3}$

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3\to 3}$
Relating $i\mathcal{K}_{df,3\rightarrow3}$ to $i\mathcal{M}_{3\rightarrow3}$

First we modify $C_L(E, \vec{P})$ to define $i\mathcal{M}_{L,3\rightarrow3}$

1. Amputate interpolating fields
Relating $iK_{df,3\rightarrow 3}$ to $iM_{3\rightarrow 3}$

First we modify $C_L(E, \vec{P})$ to define $iM_{L,3\rightarrow 3}$

2. Drop disconnected diagrams
Relating \( iK_{df,3\rightarrow 3} \) to \( iM_{3\rightarrow 3} \)

\[
iM_{L,3\rightarrow 3} \equiv S \left\{ \begin{array}{c}
\text{Diagram}
\end{array} \right\} + \text{...}
\]

First we modify \( C_L(E, \vec{P}) \) to define \( iM_{L,3\rightarrow 3} \)

3. Symmetrize
Relating $i\mathcal{K}_{df,3\to3}$ to $i\mathcal{M}_{3\to3}$

$i\mathcal{M}_{L,3\to3} \equiv S\left\{ \begin{array}{c}
\text{\textbullet} \quad + \quad \text{\textbullet} \quad + \quad \ldots \\
\text{\textbullet} \quad + \quad \text{\textbullet} \quad + \quad \ldots \\
\text{\textbullet} \quad + \quad \text{\textbullet} \quad + \quad \ldots \\
\ldots \\
\text{\textbullet} \quad + \quad \text{\textbullet} \quad + \quad \ldots \\
\end{array} \right\}$

Replacing all loop momentum sums with i-epsilon prescription integrals gives physical three-to-three scattering amplitude

$$i\mathcal{M}_{3\to3} = \lim_{L\to\infty} \bigg|_{i\epsilon} i\mathcal{M}_{L,3\to3}$$
Relating $i\mathcal{K}_{df,3\to 3}$ to $i\mathcal{M}_{3\to 3}$

\[
i\mathcal{M}_{L,3\to 3} = i\mathcal{D}_L + S \left[ \mathcal{L}_L \frac{1}{1 - iF_3} \frac{i\mathcal{K}_{df,3\to 3}}{i\mathcal{K}_{df,3\to 3}} \mathcal{R}_L \right]
\]

\[
i\mathcal{M}_{3\to 3} = \lim_{L \to \infty} \left. i\mathcal{M}_{L,3\to 3} \right|_{i\epsilon}
\]


Gives integral equation relating $i\mathcal{K}_{df,3\to 3}$ to $i\mathcal{M}_{3\to 3}$

Completes formal story (for the setup considered!)

Relation only depends on on-shell scattering quantities
1/L expansions

In 1957, Huang and Yang determined energy shift for n identical bosons in a box


\[ E_0(n, L) = \frac{4\pi a}{ML^3} \left\{ \left( \begin{array}{c} n \\ 2 \end{array} \right) - \left( \frac{a}{\pi L} \right) \left( \begin{array}{c} n \\ 2 \end{array} \right) I + \left( \frac{a}{\pi L} \right)^2 \left( \begin{array}{c} n \\ 2 \end{array} \right) I^2 - \left[ \left( \begin{array}{c} n \\ 2 \end{array} \right)^2 - 12 \left( \begin{array}{c} n \\ 3 \end{array} \right) - 6 \left( \begin{array}{c} n \\ 4 \end{array} \right) \right] J \right\} + O(L^{-6}) \]

where \( a \) is the two-particle scattering length and

\[ I = \lim_{\Lambda \to \infty} \sum_{i \neq 0}^{\mid i \mid \leq \Lambda} \frac{1}{|i|^2} - 4\pi \Lambda = -8.91363291781 \]
\[ J = \sum_{i \neq 0} \frac{1}{|i|^4} = 16.532315959 \]
$1/L$ expansions

In 1957, Huang and Yang determined energy shift for $n$ identical bosons in a box


$$E_0(n, L) = \frac{4\pi a}{M L^3} \left\{ \binom{n}{2} - \left( \frac{a}{\pi L} \right) \binom{n}{2} \mathcal{I} + \left( \frac{a}{\pi L} \right)^2 \left\{ \binom{n}{2} \mathcal{I}^2 - \left[ \binom{n}{2}^2 - 12 \binom{n}{3} - 6 \binom{n}{4} \right] \mathcal{J} \right\} \right\} + O(L^{-6})$$

where $a$ is the two-particle scattering length and

$$\mathcal{I} = \lim_{\Lambda \to \infty} \sum_{i \neq 0}^{||i|| \leq \Lambda} \frac{1}{||i||^2} - 4\pi \Lambda = -8.91363291781 \quad \mathcal{J} = \sum_{i \neq 0} \frac{1}{||i||^4} = 16.532315959$$

In 2007 Beane, Detmold and Savage pushed the order to $1/L^6$ and the latter two calculated to $1/L^7$ the next year


At $1/L^6$ a three-particle contact term appears
1/$L$ expansions

Last year Detmold and Flynn performed a similar calculation for matrix elements


$$
\langle n | J | n \rangle = n\alpha_1 + \frac{n\alpha_1 a^2}{\pi^2 L^2} \binom{n}{2} J + \frac{\alpha_2}{L^3} \binom{n}{2} 
+ \frac{2n\alpha_1 a^3}{\pi^3 L^3} \binom{n}{2} \left\{ K \binom{n}{2} - \left[ I J + 4K \binom{n-2}{1} + K \binom{n-2}{2} \right] \right\} - \frac{2\alpha_2 a}{\pi L^4} \binom{n}{2} I
+ \frac{n\alpha_1 a^4}{\pi^4 L^4} \left[ 3I^2 J + L \left( 186 - \frac{241n}{2} + \frac{29}{2} n^2 \right) + J^2 \left( \frac{n^2}{4} + \frac{3n}{4} - \frac{7}{2} \right) 
+ I K (4n - 14) + U (32n - 64) + V (16n - 32) \right] + \mathcal{O}(1/L^5).
$$

Here $\mathcal{I}, \mathcal{J}, \cdots$ are known geometric constants and $\alpha_1, \alpha_2$ are one- and two-boson current couplings.
Nonperturbative and non-relativistic

Non-relativistic Faddeev analysis

In 2012, Polejaeva and Rusetsky derived a Lüscher-like result using non-relativistic Faddeev equations


Demonstrates that on-shell S-matrix determines spectrum

Difficult to extract scattering from the formalism
Nonperturbative and non-relativistic

Non-relativistic Faddeev analysis

In 2012, Polejaeva and Rusetsky derived a Lüscher-like result using non-relativistic Faddeev equations


Demonstrates that on-shell S-matrix determines spectrum
Difficult to extract scattering from the formalism

Dimer formalism

In 2013, Briceño and Davoudi studied three-particles in finite-volume using the Dimer formalism


Recovered Lüscher result when two of the three become bound

\[ k \cot \delta = -k \cot \phi + \eta \frac{e^{-\gamma L}}{L} \]

Final result involves an integral equation that one needs to solve numerically
Three-particle bound state

This year Meißner, Rios and Rusetsky determined the finite-volume energy shift to a three-body bound state

\[ \Delta E = c \frac{\kappa^2}{m} \frac{|A|^2}{(kL)^{3/2}} \exp\left(-2\kappa L / \sqrt{3}\right) + \cdots \]


Assumes the unitary limit for two-particle scattering

Result derived using non-relativistic quantum mechanics
Review...
Review...

\[ C_L(P) = O^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} O + O^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} O + O^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} O + \cdots \]
Review...

\[ C_L(P) = o^\dagger o + o^\dagger iK o \]

\[ + o^\dagger iK iK o + \cdots \]
Review...

\[ C_L(P) = O^\dagger \circ \circ O + O^\dagger \circ iK \circ O + \ldots \]

\[ C_L(P) = C_\infty(P) \]

\[ + \begin{array}{ccc}
A & A' \\
F & F & F
\end{array} + \begin{array}{ccc}
A & iM & A' \\
F & F & F
\end{array} + \begin{array}{ccc}
iK & iK & iK \\
\end{array} + \ldots \]

\[ \langle \pi \pi, \text{out} | O^\dagger | 0 \rangle \]

\[ + \begin{array}{ccc}
A & iM & iM & A' \\
F & F & F & F
\end{array} + \ldots \]

\[ \langle 0 | O | \pi \pi, \text{in} \rangle \]
We deduce...

\[ C_L(P) = C_\infty(P) - A' F \frac{1}{1 + M_{2 \rightarrow 2 F} A} \]
Review...

\[ C_L(P) = \mathcal{O}^\dagger \mathcal{O} + \mathcal{O}^\dagger iK \mathcal{O} + \mathcal{O}^\dagger iK iK \mathcal{O} + \cdots \]

\[ C_L(P) = C_\infty(P) \]

\[ + \mathcal{A} \mathcal{A}' F F + \mathcal{A} \mathcal{M} \mathcal{A}' F F + \mathcal{A} \mathcal{M} \mathcal{M} \mathcal{A}' F F + \cdots \]

\[ \langle \pi \pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle \]

\[ \langle 0 | \mathcal{O} | \pi \pi, \text{in} \rangle \]

We deduce...

\[ C_L(P) = C_\infty(P) - A F \frac{1}{1 + \mathcal{M}_{2\rightarrow2} F} A \]

poles are in here

\[ P_4 \]
Scattering of multiple two-particle channels

\[ \pi\pi \rightarrow K \bar{K} \quad \pi K \rightarrow \eta K \]

Make following replacements

\[ \begin{pmatrix}
  iK_1 & iK_2 \\
  iK_1 & iK_2
\end{pmatrix} \]
And also for the rho meson
And also for the rho meson

Wilson, Briceño, Dudek, Edwards, Thomas, arXiv:1507:02599