Towards a first principles picture of the proton

Christopher Monahan
New High Energy Theory Center
Rutgers, The State University of New Jersey
EIC @ JLab

12 GeV
Early universe evolution

Neutron stars

Nuclear landscape

12 GeV

EIC @ JLab

LHCb

LUX

Nuclear landscape

Neutron stars

12 GeV

EIC @ JLab

LHCb

LUX
\[ x = \frac{Q^2}{2P \cdot q} \]
\[ q^2 = -Q^2 \]
Inclusive and semi-inclusive deep inelastic scattering

PDFs in deep valence region $x > 0.5$

Nucleon polarisation asymmetries

Spin structure function $g_2$ and higher twists

Pion structure functions at high $x$

Transversity distributions and GPDs

Reduced PDF uncertainties $10^{-3} < x < 1$

Kinematic reach to $x \sim 10^{-4}$

Resolve origin of proton spin via gluon and sea quark helicities

Explore gluon and sea quark TMDs

3D imaging via GPDs
First principles picture:

Derived from QCD action

Independent of phenomenological modelling

Uncertainties systematically improvable
First principles picture:

Derived from QCD action

Independent of phenomenological modelling

Uncertainties systematically improvable
Direct determination of parton distributions

1. Nucleon structure on the lattice

2. PDFs from lattice QCD: an unsolved challenge

3. A solution: the gradient flow

4. No free lunch: the gradient flow scheme

5. Looking forward
Nucleon structure:

- Nucleon mass
- Axial charge
- Form factors
- Parton distributions
- Transverse momentum and generalised parton distributions
- Wigner functions
Nucleon mass
Axial charge
Form factors
Parton distributions
Transverse momentum and generalised parton distributions
Wigner functions

Borsanyi et al., Science 347 (2015) 1452
Nucleon mass
Axial charge
Form factors
Parton distributions
Transverse momentum and generalised parton distributions
Wigner functions
Nucleon mass
Axial charge
Form factors
Parton distributions
Transverse momentum and generalised parton distributions
Wigner functions
Nucleon mass
Axial charge
Form factors
Parton distributions
Transverse momentum and generalised parton distributions
Wigner functions

Capture longitudinal momentum structure of constituents of fast-moving nucleons
Capture longitudinal momentum structure of constituents of fast-moving nucleons

- Nucleon mass
- Axial charge
- Form factors
- Parton distributions
- Transverse momentum and generalised parton distributions
- Wigner functions
2. PDFs from lattice QCD: An unsolved challenge
Deep inelastic scattering

\[ x = \frac{Q^2}{2P \cdot q} \]

\[ q^2 = -Q^2 \]

Decompose cross-section

\[ \frac{d\sigma}{d\Omega dE'} = \frac{e^4}{16\pi^2 Q^4} \ell^{\mu \nu} W_{\mu \nu} \]

Hadronic tensor

\[ W_{\mu \nu}(p, q) = \frac{1}{4\pi} \int d^4x \ e^{i q \cdot x} \langle p, \lambda' \ | [j_\mu(x), j_\nu(x)] | p, \lambda \rangle \]

Express in terms of structure functions \( F_1, F_2, g_1, g_2 \)

\[ F(x, Q^2) = \int dy \ C \left( \frac{x}{y}, \frac{Q^2}{\mu^2} \right) f_{q/N}(x, \mu^2) \]

(Light front) parton distributions universal

\[ f_{q/N}(x, Q^2) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dy^- e^{-i y^- p^+} \langle N \ | \bar{\psi}(0^+, y^-, 0_T) \gamma_+ U(y^-, 0) \psi(0) | N \rangle \]
Relate hadronic tensor to forward Compton amplitude

\[ W_{\mu\nu} = \frac{1}{2\pi} \text{Im}\{T_{\mu\nu}\} \]

Operator product expansion generates “twist” (dimension - spin) expansion

 Twist-2 operators dominate in Bjorken limit

\[ \overline{\psi} \gamma_{\{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n}\} \psi \quad \text{traces} \]

Mellin moments

\[ \langle x^n \rangle_{f_{q/N}} = \int_{-1}^{1} dx \ x^n \ f_{q/N}(x) \]

\[ 2 \langle x^n \rangle_{f_{q/N}} P_{\mu_1} \cdots P_{\mu_n} = \frac{1}{2} \langle N(P) | \overline{\psi} \gamma_{\{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n}\} \psi | N(P) \rangle \]

Wick rotation of moments is trivial

... however...
Rotational symmetry broken

Mixing patterns under renormalisation more complicated on the lattice
Rotational symmetry broken

Mixing patterns under renormalisation more complicated on the lattice

Mixing between operators of different mass dimension: power-divergent mixing

\[
\overline{\psi}\gamma_4 \gamma_5 \overleftrightarrow{D}_4 \overleftrightarrow{D}_4 \psi \sim \frac{1}{a^2} \overline{\psi}\gamma_4 \gamma_5 \psi
\]
Power-divergent mixing restricts lattice calculations to first four moments

3. A solution: The gradient flow
“Smearing” partially restores rotational symmetry: 

**suppresses operator mixing**

Construct operators with improved continuum limits
Gradient flow: deterministic evolution in new parameter - flow time

Drives fields to minimise action - removes UV fluctuations

Renormalised boundary theory remains finite

Lüscher & Weisz, JHEP 1102 (2011) 51
Luscher, JHEP 04 (2013) 123
Some technical comments:

Four-dimensional smearing in Euclidean spacetime

Status unclear for nonrenormalisable theories

Perhaps best to think of this as just a tool
Some technical comments:

Four-dimensional smearing in Euclidean spacetime

Status unclear for nonrenormalisable theories

Perhaps best to think of this as just a tool

Up for debate?
Scalar field theory

\[ \frac{\partial}{\partial \tau} \bar{\phi}(\tau, x) = \partial^2 \bar{\phi}(\tau, x) \quad \bar{\phi}(\tau = 0, x) = \phi(x) \quad \tilde{\phi}(\tau, p) = e^{-\tau p^2} \tilde{\phi}(p) \]

Exact solution possible with Dirichlet boundary conditions

\[ \bar{\phi}(\tau, x) = \int d^4y \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x-y)} e^{-\tau p^2} \phi(y) = \frac{1}{16\pi^2\tau^2} \int d^4y e^{-\frac{(x-y)^2}{4\tau}} \phi(y) \]

Smearing radius \( s_{\text{rms}} = \sqrt{8\tau} \)

Interactions occur at zero flow time (i.e. in the original “boundary” theory)
Flow-time
Gradient flow in QCD

\[
\frac{\partial}{\partial \tau} B_\mu(\tau, x) = D_\nu \left( \partial_\nu B_\mu - \partial_\mu B_\nu + [B_\nu, B_\mu] \right) \quad D_\mu = \partial_\mu + [B_\mu, \cdot]
\]

\[
\frac{\partial}{\partial \tau} \chi(\tau, x) = D^F_\mu D^F_\mu \chi(\tau, x) \quad D^F_\mu = \partial_\mu + B_\mu
\]

Dirichlet boundary conditions

\[
B_\mu(\tau = 0, x) = A_\mu(x) \quad \chi(\tau = 0, x) = \psi(x)
\]

Tree-level expansion

\[
B_\mu(\tau, x) = \int d^4 y \left\{ K_\tau(x - y)_{\mu\nu} A_\nu(y) + \int_0^\tau d\sigma K_{\tau-\sigma}(x - y)_{\mu\nu} R_\nu(\sigma, y) \right\}
\]

“Flow propagator”

\[
K_\tau(x)_{\mu\nu} = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ipx}}{p^2} \left\{ (\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-\tau p^2} + p_\mu p_\nu \right\}
\]

\[
R_\mu(\tau, x) = 2[B_\nu, \partial_\nu B_\mu] - [B_\nu, \partial_\mu B_\nu] - [B_\mu, \partial_\nu B_\nu] + [B_\nu, [B_\nu, B_\mu]]
\]

Lüscher & Weisz, JHEP 1102 (2011) 51
Luscher, JHEP 04 (2013) 123
Directed "tree" graphs

\[ B_\mu(\tau, x) \]
\[ K_\tau(x)_{\mu\nu} \]
\[ A_\mu(x) \]

Two-point function

\[ \langle B^a_\mu(\tau, x) B^b_\nu(\sigma, y) \rangle = \]

[Diagram showing directed graph with nodes and arrows]
Directed "tree" graphs

\[ B_\mu(\tau, x) \]

\[ K_\tau(x)_{\mu\nu} \]

\[ A_\mu(x) \]

Two-point function

\[ \langle B^a_\mu(\tau, x) B^b_\nu(\sigma, y) \rangle = \]

\[ + \]

\[ + \]
Directed "tree" graphs

\[ B_\mu(\tau, x) \]

\[ K_\tau(x)_{\mu\nu} \]

\[ A_\mu(x) \]

Two-point function

\[ \left\langle B_\mu^a(\tau, x) B_\nu^b(\sigma, y) \right\rangle = \right\langle A_\mu^a(x) A_\nu^b(y) \right\rangle + \]
Directed “tree” graphs

\[ B_\mu(\tau, x) \quad \text{and} \quad A_\mu(x) \quad K_\tau(x)_{\mu\nu} \]

Two-point function

\[ \langle B^a_\mu(\tau, x) B^b_\nu(\sigma, y) \rangle = \quad + \quad \langle A^a_\mu(x) A^b_\nu(y) \rangle \]

\[ \langle B^a_\mu(\tau, x) B^b_\nu(\sigma, y) \rangle = \quad + \]

\[ \langle \tilde{B}^a_\mu(t, p) \tilde{B}^b_\nu(s, q) \rangle = (2\pi)^4 \delta(p + q) \delta^{ab} \delta_{\mu\nu} \frac{e^{-(s+t)p^2}}{p^2} + O(g^2) \]
Smearing removes power-divergent mixing in the continuum, at the expense of introducing a new scale.
4. No free lunch: The gradient flow scheme
Consider twist-2 operators

\[ \mathcal{T}_{\mu_1 \ldots \mu_n}(x) = \phi(x) \partial_{\mu_1} \ldots \partial_{\mu_n} \phi(x) - \text{traces} \]

Example: continuum matrix element

\[ \langle \Omega | \phi^2(0) \cdot \phi(0) \partial_{\mu} \partial_{\nu} \phi(0) | \Omega \rangle = 0 \]

On the lattice

\[ \langle \Omega | \phi^2(0) \cdot \phi(0) \nabla_{\mu} \nabla_{\nu} \phi(0) | \Omega \rangle = -\frac{\delta_{\mu\nu}}{32a^2} + \mathcal{O}(a^0, \lambda) \]

With smeared degrees of freedom

\[ \langle \Omega | \overline{\phi}(\tau, 0) \cdot \phi(\tau, 0) \nabla_{\mu} \nabla_{\nu} \overline{\phi}(\tau, 0) | \Omega \rangle = -\frac{\delta_{\mu\nu}}{256\pi^2} + \mathcal{O}(a^0, \lambda) \]
Two approaches to our new scale:

“Smeared Operator Product Expansion” (sOPE)

Small flow-time expansion
sOPE

Replace local operators in the OPE

\[ \mathcal{O}(x) \xrightarrow{x \to 0} \sum_{k} c_k(x, \mu) \mathcal{O}_{R}^{(k)}(0, \mu) + \ldots \]

with locally-smeared operators

\[ \mathcal{O}(x) \xrightarrow{x \to 0} \sum_{k} d_k(x, \mu, \tau) \overline{\mathcal{O}}^{(k)}(0, \mu, \tau) + \ldots \]
Small flow-time expansion

Relate locally-smeared operators to local operators

$$\bar{O}(\tau) = C(\mu, \tau) O_R(\mu) + \ldots$$
Case-study: determine heavy-light quark bilinear renormalisation parameters

\[ \Sigma^{\text{latt}}_\Gamma (a, M_Q) \equiv \langle \Omega | \overline{Q} \Gamma q \cdot \overline{Q} \Gamma q | \Omega \rangle^{\text{latt}} \]

\[ \Sigma^{\text{smear}}_\Gamma (\tau, a, M_Q) \equiv \langle \Omega | \overline{Q} \Gamma q(\tau) \cdot \overline{Q} \Gamma q(\tau) | \Omega \rangle^{\text{latt}} \]

Continuum matrix element \[ \Sigma^{R}_\Gamma (\mu, M_Q) = \lim_{a \to 0} Z^2_\Gamma (\mu, a, M_Q) \Sigma^{\text{latt}}_\Gamma (a, M_Q) \]
Case-study: determine heavy-light quark bilinear **renormalisation parameters**

\[
\Sigma_{\Gamma}^{\text{latt}}(a, M_Q) \equiv \langle \Omega | \bar{Q} \Gamma q \cdot \bar{Q} \Gamma q | \Omega \rangle^{\text{latt}}
\]

\[
\Sigma_{\Gamma}^{\text{smear}}(\tau, a, M_Q) \equiv \langle \Omega | \bar{Q} \Gamma q(\tau) \cdot \bar{Q} \Gamma q(\tau) | \Omega \rangle^{\text{latt}}
\]

Continuum matrix element

\[
\Sigma_{\Gamma}^{\text{R}}(\mu, M_Q) = \lim_{a \to 0} Z_{\Gamma}^2(\mu, a, M_Q) \Sigma_{\Gamma}^{\text{latt}}(a, M_Q)
\]
Case-study: determine heavy-light quark bilinear renormalisation parameters

\[ \Sigma_{\Gamma}^{\text{latt}}(a, M_Q) \equiv \langle \Omega | \overline{Q} \Gamma q \cdot \overline{Q} \Gamma q | \Omega \rangle^{\text{latt}} \]

\[ \Sigma_{\Gamma}^{\text{smear}}(\tau, a, M_Q) \equiv \langle \Omega | \overline{Q} \Gamma q(\tau) \cdot \overline{Q} \Gamma q(\tau) | \Omega \rangle^{\text{latt}} \]

Continuum matrix element

\[ \Sigma_{\Gamma}^{R}(\mu, M_Q) = \lim_{a \to 0} Z_{\Gamma}^2(\mu, a, M_Q) \Sigma_{\Gamma}^{\text{latt}}(a, M_Q) \]

\[ \Sigma_{\Gamma}(\tau, M_Q) = C(\mu, \tau) \Sigma_{\Gamma}^{R}(\mu, M_Q) \]

Small flow-time expansion
Case-study: determine heavy-light quark bilinear renormalisation parameters

\[ \Sigma_{\Gamma}^{\text{latt}} (a, M_Q) \equiv \langle \Omega | \overline{Q} \Gamma q \cdot \overline{Q} \Gamma q | \Omega \rangle^{\text{latt}} \]

\[ \Sigma_{\Gamma}^{\text{smear}} (\tau, a, M_Q) \equiv \langle \Omega | \overline{Q} \Gamma q(\tau) \cdot \overline{Q} \Gamma q(\tau) | \Omega \rangle^{\text{latt}} \]

Continuum matrix element

\[ \Sigma_{\Gamma}^{R} (\mu, M_Q) = \lim_{a \to 0} Z_{\Gamma}^2 (\mu, a, M_Q) \Sigma_{\Gamma}^{\text{latt}} (a, M_Q) \]

\[ \Sigma_{\Gamma} (\tau, M_Q) = C(\mu, \tau) \Sigma_{\Gamma}^{R} (\mu, M_Q) \]

\[ \Sigma_{\Gamma} (\tau, M_Q) = \lim_{a \to 0} \Sigma_{\Gamma}^{\text{smear}} (\tau, a, M_Q) \]
Case-study: matrix elements of heavy-light quark bilinears

\[ \Sigma^{\text{latt}}_\Gamma(a, M_Q) \equiv \langle \Omega | \overline{Q} \Gamma q \cdot \overline{Q} \Gamma q | \Omega \rangle^{\text{latt}} \]

\[ \Sigma^{\text{smear}}_\Gamma(\tau, a, M_Q) \equiv \langle \Omega | \overline{Q} \Gamma q(\tau) \cdot \overline{Q} \Gamma q(\tau) | \Omega \rangle^{\text{latt}} \]

Continuum matrix element

\[ \Sigma^R_\Gamma(\mu, M_Q) = \lim_{a \to 0} Z^2_\Gamma(\mu, a, M_Q) \Sigma^{\text{latt}}_\Gamma(a, M_Q) \]

Small flow-time expansion

\[ \Sigma_\Gamma(\tau, M_Q) = C(\mu, \tau) \Sigma^R_\Gamma(\mu, M_Q) \]

\[ \Sigma_\Gamma(\tau, M_Q) = \lim_{a \to 0} \Sigma^{\text{smear}}_\Gamma(\tau, a, M_Q) \]

Defines renormalisation scheme: gradient flow scheme

\[ Z^\text{GF}_\Gamma(\mu, a, M_Q) = \sqrt{\lim_{a \to 0} \frac{\Sigma^{\text{smear}}_\Gamma(\tau, a, M_Q)}{C(\mu, \tau)} \frac{1}{\Sigma^{\text{latt}}_\Gamma(a, M_Q)}} \]
Gradient flow scheme

\[ Z_{\Gamma}^{GF}(\mu, a, M_Q) = \sqrt{\frac{\lim_{a \to 0} \sum_{\Gamma}^{\text{smear}}(\tau, a, M_Q)}{C(\mu, \tau)} \frac{1}{\Sigma_{\Gamma}^{\text{latt}}(a, M_Q)}} \]
Gradient flow scheme

\[ Z_{\Gamma}^{\text{GF}}(\mu, a, M_Q) = \sqrt{\lim_{a \to 0} \frac{\sum_{\Gamma}^{\text{smea}}(\tau, a, M_Q)}{C(\mu, \tau)} \cdot \frac{1}{\sum_{\Gamma}^{\text{latt}}(a, M_Q)}} \]
Calculate with smeared fermions

Controls all divergences

But...

Requires fermion renormalisation

Complicates perturbation theory

Calculate with smeared gauge fields

Controls all gluonic divergences

But...

Does not remove tree-level divergence
Calculate with smeared fermions: [JLab Theory Seminar 12/9/2016]

\[ Z_{\Gamma}^{GF}(\mu, a, M_Q) = \sqrt{\frac{\lim_{a \to 0} \Sigma_{\Gamma}^{\text{smear}}(\tau, a, M_Q)}{C(\mu, \tau) \Sigma_{\Gamma}^{\text{latt}}(a, M_Q)}} \]
Update: calculate with smeared gauge fields

Remove tree-level divergence by taking continuum limit of

\[
\frac{\sum_{\Gamma}^{\text{smear}}(\tau, a, M_Q)}{\sum_{\Gamma}^{\text{tree}}(\tau, a, M_Q)}
\]

1. Fix physical volume

2. Tune bare coupling at different lattice sizes

3. Fix flow time in physical units
Update: calculate with smeared gauge fields

Remove tree-level divergence by taking continuum limit of

$$\frac{\Sigma_{\Gamma}^{\text{smear}}(\tau, a, M_Q)}{\Sigma_{\Gamma}^{\text{tree}}(\tau, a, M_Q)}$$

1. Fix physical volume
2. Tune bare coupling at different lattice sizes
3. Fix flow time in physical units
Update: calculate with smeared gauge fields

\[ Z_T^{GF}(\mu, a, M_Q) = \sqrt{\lim_{a \to 0} \frac{\sum_{\Gamma}^\text{smear}(\tau, a, M_Q)}{C(\mu, \tau)} \frac{1}{\Sigma_{\Gamma}^{\text{latt}}(a, M_Q)}} \]
Update: calculate with smeared gauge fields

\[ Z_T^{GF}(\mu, a, M_Q) = \lim_{a \to 0} \frac{\Sigma_T^{\text{smear}}(\tau, a, M_Q)}{C(\mu, \tau)} \frac{1}{\Sigma_T^{\text{latt}}(a, M_Q)} \]
Gradient flow scheme

\[ Z^\text{GF}_\Gamma (\mu, a, M_Q) = \sqrt{ \lim_{a \to 0} \frac{\sum_\Gamma^{\text{smear}} (\tau, a, M_Q)}{C(\mu, \tau)} \frac{1}{\sum_\Gamma^{\text{latt}} (a, M_Q)} } \]
5. Looking forward
Direct determination of PDFs

Via gradient flow procedure

- Hypercubic symmetry = power-divergent mixing

- Restricts us to lowest moments of PDFs or GPDs

- Gradient flow removes power-divergent mixing

- Gradient flow scheme relates matrix elements at non-zero and zero flow times

Determine moments

- Will allow determination of higher moments
  - involves only local operators
  - analysis a well-established procedure
Direct determination of PDFs

Via gradient flow procedure

Determine moments

LaMET

with David Richards, Kostas Orginos, Carl Carlson

X. Ji, Sc. China (2014)
X. Ji, PRL 110 (2013) 262002
Direct determination of PDFs

Via gradient flow procedure

Determine moments

LaMET

Renormalisation (K. Orginos)

Numerics (D. Richards & K. Orginos)
Direct determination of PDFs

Via gradient flow procedure

Determine moments

LaMET

Renormalisation (K. Orginos)

Lattice perturbation theory (C. Carlson)

Numerics (D. Richards & K. Orginos)
Direct determination of PDFs

Via gradient flow procedure

Determine moments

LaMET

Renormalisation (K. Orginos)

Lattice perturbation theory (C. Carlson)

Nonperturbative scheme

Numerics (D. Richards & K. Orginos)
Direct determination of PDFs

Via gradient flow procedure

LaMET

Renormalisation (K. Orginos)

Numerics (D. Richards & K. Orginos)

Lattice perturbation theory (C. Carlson)

Nonperturbative scheme

Determine moments
Direct determination of PDFs

Via gradient flow procedure

Determine moments

LaMET

Renormalisation (K. Orginos)

Lattice perturbation theory (C. Carlson)

Nonperturbative scheme

Numerics (D. Richards & K. Orginos)
Direct determination of PDFs

Via gradient flow (GF) procedure

Determine moments

Now  Summer `16  End of `16  2017  2018  2019  2020

GF: bilinears
Class C
GF: twist-2 ops.
Class B
GF: mixing

Estimated!
Direct determination of PDFs

Via gradient flow (GF) procedure

Determine moments

Now  Summer `16  End of `16  2017  2018  2019  2020

GF: bilinears

GF: twist-2 ops.

GF: mixing

Pion PDAs/PDFs

Proton PDA/PDFs

Class A

Systematic uncertainties?

No. of moments?

Range of $Q^2$?

Estimated!
Direct determination of PDFs

via gradient flow (GF) procedure

determine moments

Now  Summer `16  End of `16  2017  2018  2019  2020

GF: bilinears
GF: twist-2 ops.
GF: mixing
Pion PDAs/PDFs
Proton PDA/PDFs
Spin physics...
Sea quarks...
Tomography...

Estimated!
Calculations

1. Low moments of spin-(in)dependent structure functions
   - operators that do not mix
   - operators that mix

Example tests:
   - operators in the same lattice irrep have the same renormalisation
   - operators in the same continuum irrep and different lattice irreps give the same result

Physics

1. Basic tests
   - verification of basic implementation via comparison with previous results
   - verification that gradient flow removes mixing
Calculations

1. Low moments of spin-(in)dependent structure functions
   - operators that do not mix
   - operators that mix

2. Higher moments

Questions:
- how many moments?
- can full PDF be reconstructed?
- can high moments constrain global fits?

Physics

1. Basic tests
   - verification of basic implementation via comparison with previous results
   - verification that gradient flow removes mixing

2. First ever such calculation
   a) comparison with theory
      - nonrelativistic quark models at \( m_q \)
      - tests of “Borromean” picture of baryonic di-quark correlations in Dyson-Schwinger formalism
      - predictions from AdS/QCD
   b) experiment
      - global fits
      - JLab 12 GeV deep valence region
   c) constrain high-x regimes of PDFs
Calculations

3. Low moments of spin-(in)dependent generalised structure functions on unquenched configurations

4. Higher moments

Physics

3. Comparisons with:
- quenched lattice calculations for first three moments of GPDs
- experimental data
- Ji sum rule for total quark angular momentum

4. First ever such calculation...
Corollaries

Calculations

a) Gluon operators and mixing
b) Twist-3 contributions (e.g. to $g_2$)
c) Nonperturbative improvement
d) Nonperturbative Wilson coefficients

Physics

Largely unstudied in lattice QCD

Improvement in systematic uncertainties.
Summary

Gradient flow - tool to remove power-divergent mixing

Gradient flow scheme - operator renormalisation

Enable lattice calculations of moments of PDFs and GPDs
Thank you

chris.monahan@rutgers.edu
Update: calculate with smeared gauge fields

1. Choose some bare coupling and lattice size $L/a$

2. Measure finite volume renormalised coupling

3. Determine the matrix element at some flow time $t$

4. Fix renormalised coupling, which fixes the physical box size, and tune bare coupling to match at new lattice spacing $L'/a$

5. Determine matrix element at fixed physical flow time by choosing flow time in lattice units by fixing product $t' = (m_{\text{crit}})^2/(m_{\text{crit}}')^2$, where the critical mass is calculated at two loops in “cactus-improved” lattice perturbation theory

6. Repeat steps 3-5 to take continuum limit
Lattice determinations: nucleon structure

Meson distribution amplitudes
- quenched
  - Martinelli & Sachrajda, PLB 1 (1987) 184
  - Martinelli & Sachrajda, NPB 306 (1988) 805
- unquenched
  - Best et al, PRD 56 (1997) 2743

Nucleon
- axial charge
  - Capitani et al, PRD 86 (2012) 074502
  - Horsley et al, PLB 732 (2014) 41
- unpolarised
- polarised
  - Gockeler et al, PRD 53 (1996) 2317

higher twist contributions
- transverse momentum distributions
- generalised parton distributions
  - Y. Zhao, arXiv/1506.08832
  - Musch et al, PRD 83 (2011) 094507

W. Bietenholz et al, PoS LATTICE(2009) 138
Nucleon axial charge

\[ \langle x^0 \rangle_{\Delta q} = \int_0^1 dx \left[ \Delta q(x) + \Delta \bar{q}(x) \right] \]

\[ \Delta q(x) = q_\uparrow(x) - q_\downarrow(x) \]


Direct determination of PDFs: LaMET

Relate PDFs

$$q(x, \mu^2) = \int \frac{d\xi^-}{4\pi} e^{-ix\xi^-} P^+ \langle P | \overline{\psi}(\xi^-) \gamma^+ e^{-ig\int_{0}^{\xi^-} d\eta^- A^+(\eta^-)} \psi(0) | P \rangle$$

to “quasi”-distributions

$$\overline{q}(x, \mu^2, P^z) = \int \frac{dz}{4\pi} e^{izk^z} \langle P | \overline{\psi}(z) \gamma^z e^{-ig\int_{0}^{z} dz' A^z(z')} \psi(0) | P \rangle + \mathcal{O} \left( \Lambda_{QCD}^2 / (P^z)^2, M^2 / (P^z)^2 \right)$$

via a factorisation formula

$$\overline{q}(x, \mu^2, P^z) = \int_{x}^{1} \frac{dy}{y} Z \left( \frac{x}{y}, \frac{\mu}{P^z} \right) q(y, \mu^2) + \mathcal{O} \left( \Lambda_{QCD}^2 / (P^z)^2, M^2 / (P^z)^2 \right)$$

Requires renormalisation of nonlocal operators

Some progress towards this via HQET at NLO - relation to OPE-based approaches?

Initial lattice studies at a single lattice spacing

X. Ji et al, PRD 91 (2015) 074009
X. Ji, Sc. China (2014)
X. Ji, PRL 110 (2013) 262002

C. Alexandrou et al, PRD 92 (2015) 014502
H.-W. Lin et al, PRD 91 (2014) 054510
Moments of quark density

\[ \langle x^n \rangle_q = \int_0^1 dx \, x^n (q(x) + (-1)^{n+1} \bar{q}(x)) \quad q = q_\uparrow + q_\downarrow \]

helicity

\[ \langle x^n \rangle_{\Delta q} = \int_0^1 dx \, x^n (\Delta q(x) + (-1)^n \Delta \bar{q}(x)) \quad \Delta q = q_\uparrow - q_\downarrow \]

and transversity

\[ \langle x^n \rangle_{\delta q} = \int_0^1 dx \, x^n (\delta q(x) + (-1)^{n+1} \delta \bar{q}(x)) \quad \delta q = q_\uparrow - q_\downarrow \]

Odd moments related to spin-independent structure functions

\[ \int_0^1 dx \, x^{n-1} F_1(x, Q^2) = \frac{1}{2} c_n^{(q)} (Q^2/\mu^2) \sum_f e_f^2 \langle x^{n-1} \rangle_{q_f(\mu)} \]

\[ \int_0^1 dx \, x^{n-2} F_2(x, Q^2) = c_n^{(q)} (Q^2/\mu^2) \sum_f e_f^2 \langle x^{n-1} \rangle_{q_f(\mu)} \]

Even moments related to spin-dependent structure function

\[ \int_0^1 dx \, x^n g_1(x, Q^2) = \frac{1}{2} c_n^{(\Delta q)} (Q^2/\mu^2) \sum_f e_f^2 \langle x^n \rangle_{\Delta q_f(\mu)} \]
Moments are related to matrix elements of local operators

\[ O^{(q_f)}_{\{\mu_1 \ldots \mu_n\}} = \left( \frac{i}{2} \right)^{n-1} \bar{\psi}^f \gamma_{\{\mu_1} D_{\mu_2} \cdots D_{\mu_n\}} \psi^f \]

\[ O^{(\Delta q_f)}_{\{\sigma \mu_1 \ldots \mu_n\}} = \left( \frac{i}{2} \right)^n \bar{\psi}^f \gamma_5 \gamma_{\{\sigma} D_{\mu_1} \cdots D_{\mu_n\}} \psi^f \]

\[ O^{(\delta q_f)}_{\mu\{\nu \mu_1 \ldots \mu_n\}} = \left( \frac{i}{2} \right)^n \bar{\psi}^f \gamma_5 \sigma_{\mu\{\nu} D_{\mu_1} \cdots D_{\mu_n\}} \psi^f \]

Via

\[ 2 \langle x^{n-1} \rangle_{q_f} P_{\mu_1} \cdots P_{\mu_n} = \frac{1}{2} \sum_S \langle P, S | O^{(q_f)}_{\{\mu_1 \ldots \mu_n\}} | P, S \rangle \]

\[ \frac{2}{n+1} \langle x^n \rangle_{\Delta q_f} S_{\{\sigma} P_{\mu_1} \cdots P_{\mu_n\}} = - \langle P, S | O^{(\Delta q_f)}_{\{\sigma \mu_1 \ldots \mu_n\}} | P, S \rangle \]

\[ \frac{2}{m_N} \langle x^n \rangle_{\delta q_f} S_{[\mu} P_{\{\nu]} P_{\mu_1} \cdots P_{\mu_n} = \langle P, S | O^{(\delta q_f)}_{\mu\{\nu \mu_1 \ldots \mu_n\}} | P, S \rangle \]
For Euclidean lattice operators

\[ O^{(q_f)}_{\{\mu_1 \ldots \mu_n\}} = \bar{\psi}^f \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_n} \psi^f \]

\[ O^{(5)}_{\{\sigma \mu_1 \ldots \mu_n\}} = \bar{\psi}^f \gamma_{\sigma \gamma_5} D_{\mu_1} \cdots D_{\mu_n} \psi^f \]

Lie in same \( O(4) \) irrep, but inequivalent reps of \( H(4) \)

\[ O^{(q_f)}_{\{14\}} \quad O^{(q_f)}_{\{44\}} - \frac{1}{3} \sum_{i=1}^{3} O^{(q_f)}_{\{ii\}} \]

Lie in same \( H(4) \) irrep

\[ O^{(5)}_{\{14\}} \quad O^{(5)}_{\{24\}} \]

Second moment operator

\[ O^{(q_f)}_{\{114\}} - \frac{1}{2} \left( O^{(q_f)}_{\{224\}} + O^{(q_f)}_{\{334\}} \right) \]

Third moment operator

\[ O^{(q_f)}_{\{1144\}} + O^{(q_f)}_{\{2233\}} - O^{(q_f)}_{\{1133\}} - O^{(q_f)}_{\{2244\}} \]

which mixes with

\[ \bar{\psi}^f \sigma_{\mu \nu} \gamma_5 \tilde{D}_{\mu_1} \tilde{D}_{\mu_2} \psi^f \]

See, for example, Gockeler et al, PRD 54 (1996) 5705
JLab 12GeV physics

Study deep valence region of PDFs, $x > 0.5$. Proton well constrained for $x < 0.8$, but no free neutron targets limits precision for neutrons above $x \sim 0.5$, due to ignorance of nuclear modification effects. Answer question: why is $d$ quark distribution softer than expected from flavour symmetry?

Polarisation asymmetry asymmetry unknown in neutrons for $x > 0.6$. Though there exist rigorous QCD predictions.

Large-x distributions relevant to high energy collider backgrounds: high-x uncertainties at lower $Q^2$ feed into low-x region at higher $Q^2$ via perturbative evolution.

Proton and neutron polarisation asymmetries. Both at $x > 0.4$ for $Q^2 \sim 8-9$ GeV and for $x < 0.95$ in resonance region, $Q^2 \sim 2-7$ GeV.
JLab 12GeV physics

Ji sum rule - vector GPDs yield total contribution of quark OAM to nucleon spin. Cannot measure sum directly, but constrain models of GPDs that predict sum rule values.

Expect strong correlation between transverse size and longitudinal momentum: large $x$ -> soft $t$ dependence (small in transverse direction) & small $x$ -> stiff $t$ dependence

Test factorisation in DVCS and high $P_T$ meson production
MEIC @ JLab

JLab 12 GeV

Brady et al., JHEP 1206 (2012) 019