Lorentz Covariance and Positivity Constraints in the Modeling of Generalized Parton Distributions

Theory Center Seminar  |  Hervé MOUTARDE

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Motivation.
Study nucleon structure to shed new light on nonperturbative QCD.

Perturbative QCD

Asymptotic freedom

Nonperturbative QCD

Perturbative AND nonperturbative QCD at work

- Define **universal** objects describing 3D nucleon structure: **Generalized Parton Distributions (GPD).**
- Relate GPDs to measurements using **factorization:** **Virtual Compton Scattering (DVCS, TCS),** **Deeply Virtual Meson production (DVMP).**
- Get **experimental knowledge** of nucleon structure.
Covariant and Positive GPD Models

Introduction

Correlation of the **longitudinal momentum** and the **transverse position** of a parton in a hadron.

DVCS recognized as the cleanest channel to access GPDs.

Deeply Virtual Compton Scattering (DVCS)

\[ R_\perp = \sum_i x_i r_{\perp i} \]

Transverse center of momentum
Anatomy of hadrons.
GPDs, 3D hadron imaging, and beyond.

- Correlation of the \textbf{longitudinal momentum} and the \textbf{transverse position} of a parton in a hadron.
- DVCS recognized as the cleanest channel to access GPDs.

Deeply Virtual Compton Scattering (DVCS)

\[ e^-, \gamma^*, Q^2, x + \xi, x - \xi, \gamma, F_i(x, t) \]

Transverse center of momentum \( R_\perp \)
\[ R_\perp = \sum_i x_i r_{\perp i} \]

Impact parameter \( b_\perp \)
Covariant and Positive GPD Models

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Status
- Experimental access
- Towards 3D images

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- Polynomiality
- Double Distributions
- Positivity
- Overlap

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- Radon transform
- Covariant extension
- Inverse Radon
- Example

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Anatomy of hadrons.
GPDs, 3D hadron imaging, and beyond.

- Correlation of the **longitudinal momentum** and the **transverse position** of a parton in a hadron.
- DVCS recognized as the cleanest channel to access GPDs.

**Deeply Virtual Compton Scattering (DVCS)**

**DVCS**
\[ e^-, \gamma^*, Q^2 \]
\[ \gamma \rightarrow e^- + xP^+ \]
\[ p + \xi \rightarrow x - \xi \]
\[ R_\perp = \sum_i x_i r_{\perp i} \]

Impact parameter \( b_\perp \)
Longitudinal momentum \( xP^+ \)
Anatomy of hadrons.
GPDs, 3D hadron imaging, and beyond.

- Correlation of the **longitudinal momentum** and the **transverse position** of a parton in a hadron.
- DVCS recognized as the cleanest channel to access GPDs.

### Deeply Virtual Compton Scattering (DVCS)

**Transverse center of momentum** \( R_\perp \)

\[
R_\perp = \sum_i x_i r_{\perp i}
\]

**Impact parameter** \( b_\perp \)

\[-1 < x < +1 \]

\[-1 < \xi < +1 \]

**Longitudinal momentum** \( xP^+ \)

- **24** GPDs \( F^i(x, \xi, t, \mu_F) \) for each parton type \( i = g, u, d, \ldots \) for leading and sub-leading twists.
Towards hadron tomography. GPDs as a scalpel-like probe of hadron structure.

1. **Phenomenology status**: relevance and need for parameterizations.

2. **Theoretical framework**: definition and existing constraints.

3. **GPDs from Light Front Wave Functions**: a promising computing strategy.

4. **The PARTONS platform**: a GPD toolkit.

How can we make this picture? What do we learn from it?

**Notes**

- **Phenomenology status**: relevance and need for parameterizations.
- **Theoretical framework**: definition and existing constraints.
- **GPDs from Light Front Wave Functions**: a promising computing strategy.
- **The PARTONS platform**: a GPD toolkit.
Phenomenology status
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Exclusive processes of current interest (1/2).
Factorization and universality.

DVCS
\[ e^-(p) + \gamma^*(Q^2) \rightarrow e^-(p_1) + \gamma(x) \]
\[ x + \xi \rightarrow x - \xi \]

Generalized Parton Distributions
\[ b_y \rightarrow b_x \]
nucleon

\[ Q^2 \]

\[ \gamma^* \]
Exclusive processes of current interest (1/2).

Factorization and universality.

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DVCS

\[ e^- e^- \gamma^* Q^2 \gamma \rightarrow x + \xi \rightarrow x - \xi \]

Perturbative

Nonperturbative

DVMP

\[ e^- \gamma^* Q^2 \rightarrow x + \xi \rightarrow x - \xi \]

Perturbative

Nonperturbative

Generalized Parton Distributions

\[ \gamma^* Q^2 \rightarrow b_y \rightarrow \text{nucleon} \]

TCS

\[ e^+ \gamma \rightarrow x + \xi \rightarrow x - \xi \]

Perturbative

Nonperturbative

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Exclusive processes of current interest (1/2).
Factorization and universality.

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DVCS
Perturbative
\( e^{-} \gamma^{*} Q^2 \gamma \)
factorization
\( x + \xi \)
\( x - \xi \)
\( p \)
\( p \)
t

Generalized Parton Distributions

\( Q^2 \)
\( \gamma^{*} \)

nucleon

DVMP
Perturbative
\( e^{-} \gamma^{*} Q^2 \pi, \rho, \ldots \)
factorization
\( x + \xi \)
\( x - \xi \)
\( p \)
\( p \)
t

TCS
Perturbative
\( e^{+} \gamma \gamma^{*} Q^2 e^{-} \)
factorization
\( x + \xi \)
\( x - \xi \)
\( p \)
\( p \)
t

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Bjorken regime : large $Q^2$ and fixed $x B \simeq 2 \xi / (1 + \xi)$

- Partonic interpretation relies on factorization theorems.
- All-order proofs for DVCS, TCS and some DVMP.
- GPDs depend on a (arbitrary) factorization scale $\mu_F$.
- Consistency requires the study of different channels.

- GPDs enter DVCS through Compton Form Factors:

$$ F(\xi, t, Q^2) = \int_{-1}^{1} dx \ C \left( x, \xi, \alpha_S(\mu_F), \frac{Q}{\mu_F} \right) F(x, \xi, t, \mu_F) $$

for a given GPD $F$.

- CFF $F$ is a complex function.
Summary of first extractions.
Feasibility of twist-2 analysis of existing data.

- **Dominance** of twist-2 and **validity** of a GPD analysis of DVCS data.

- $ImH$ **best determined**. Large uncertainties on $ReH$.

- However sizable **higher twist contamination** for DVCS measurements?

- Already some indications about the **invalidity** of the $H$-dominance hypothesis with **unpolarized data**.
Imaging the nucleon. How?
Extracting GPDs is not enough... Need to extrapolate!

1. Experimental data fits

\[ \Delta \sigma \ [\text{pb.GeV}^{-4}] \]
\[ \langle x_B \rangle = 0.5 \]
\[ \langle Q^2 \rangle = 6.3 \text{ GeV}^2 \]
\[ \langle -t \rangle = 0.735 \text{ GeV}^2 \]

2. GPD extraction

\[ f(x, t; z=0.2, Q^2=4) \]

3. Nucleon imaging

Images from Guidal et al., Rept. Prog. Phys. 76 (2013) 066202

Sidebar 2.2: The First 3D Pictures of the Nucleon

A computed tomography (CT) scan can help physicians pinpoint minute cancer tumors, diagnose tiny broken bones, and spot the early signs of osteoporosis.

Now physicists are using the principles behind the procedure to peer at the inner workings of the proton. This breakthrough is made possible by a relatively new concept in nuclear physics called generalized parton distributions.

An intense beam of high-energy electrons can be used...
Imaging the nucleon. How?
Extracting GPDs is not enough...Need to extrapolate!

1. **Extract** $H(x, \xi, t, \mu_F^{\text{ref}})$ from experimental data.

2. **Extrapolate** to vanishing skewness $H(x, 0, t, \mu_F^{\text{ref}})$.

3. **Extrapolate** $H(x, 0, t, \mu_F^{\text{ref}})$ up to infinite $t$.

4. **Compute** 2D Fourier transform in transverse plane:
   
   $$H(x, b_\perp) = \int_0^{+\infty} \frac{d\Delta_\perp}{2\pi} \Delta_\perp J_0(b_\perp \Delta_\perp) \ H(x, 0, -\Delta^2_\perp)$$

5. **Propagate** uncertainties.

6. **Control** extrapolations with an accuracy matching that of experimental data with **sound** GPD models.
Theoretical framework
Spin-0 Generalized Parton Distribution.
Definition and simple properties.

\[ H^q_{\pi}(x, \xi, t) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q} \left(-\frac{z}{2}\right) \gamma^+ q \left(\frac{z}{2}\right) \right| \pi, P - \frac{\Delta}{2} \right\rangle_{z^+ = 0, z_\perp = 0} \]

with \( t = \Delta^2 \) and \( \xi = -\Delta^+/(2P^+) \).

References

Spin-0 Generalized Parton Distribution.
Definition and simple properties.

\[
H^q_\pi(x, \xi, t) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q}\left(\frac{-z}{2}\right) \gamma^+ q\left(\frac{z}{2}\right) \right| \pi, P - \frac{\Delta}{2} \right\rangle_{z^+ = 0, z^- = 0}
\]

with \( t = \Delta^2 \) and \( \xi = -\Delta^+/(2P^+) \).

References


PDF forward limit
Form factor sum rule

\[
\int_{-1}^{+1} dx \, H^q(x, \xi, t) = F_1^q(t)
\]
Spin-0 Generalized Parton Distribution. Definition and simple properties.

\[ H^q_{\pi}(x, \xi, t) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q} \left( -\frac{z}{2} \right) \gamma^+ q \left( \frac{z}{2} \right) \right| \pi, P - \frac{\Delta}{2} \right\rangle \]

with \( t = \Delta^2 \) and \( \xi = -\Delta^+/(2P^+) \).

References


- PDF forward limit
- Form factor sum rule
- \( H^q \) is an even function of \( \xi \) from time-reversal invariance.
Spin-0 Generalized Parton Distribution.
Definition and simple properties.

\[ H^q_{\pi}(x, \xi, t) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q} \left( -\frac{z}{2} \right) \gamma^+ q \left( \frac{z}{2} \right) \right| \pi, P - \frac{\Delta}{2} \right\rangle \]

with \( t = \Delta^2 \) and \( \xi = -\Delta^+/(2P^+) \).

References


- PDF forward limit
- Form factor sum rule
- \( H^q \) is an even function of \( \xi \) from time-reversal invariance.
- \( H^q \) is real from hermiticity and time-reversal invariance.
Spin-0 Generalized Parton Distribution.
Not so simple properties.

- **Polynomiality**

\[ \int_{-1}^{+1} dx x^n H^q(x, \xi, t) = \text{polynomial in } \xi \]
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Not so simple properties.

Polynomiality

Lorentz covariance

Spin-0 Generalized Parton Distribution.
Not so simple properties.
Polynomiality

\[ H^q(x, \xi, t) \leq \sqrt{q \left( \frac{x + \xi}{1 + \xi} \right) q \left( \frac{x - \xi}{1 - \xi} \right)} \]

Positivity

Lorentz covariance
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Polynomiality

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Lorentz covariance

Positivity of Hilbert space norm

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Spin-0 Generalized Parton Distribution.
Not so simple properties.

- **Polynomiality**

- **Positivity**

$L^q$ has support $x \in [-1, +1]$. 

**Lorentz covariance**

**Positivity of Hilbert space norm**

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Lorentz covariance

Positivity of Hilbert space norm

Relativistic quantum mechanics

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- **Conclusion**
Spin-0 Generalized Parton Distribution.
Not so simple properties.

- **Polynomiality**
  - Lorentz covariance

- **Positivity**
  - Positivity of Hilbert space norm

- $H^q$ has support $x \in [-1, +1]$.

- **Soft pion theorem** (pion target)
  $$H^q(x, \xi = 1, t = 0) = \frac{1}{2} \phi_\pi^q \left( \frac{1 + x}{2} \right)$$

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- **Polynomiality**
  - Lorentz covariance
- **Positivity**
  - Positivity of Hilbert space norm
  - $H^q$ has support $x \in [-1, +1]$.
  - Relativistic quantum mechanics
- **Soft pion theorem** (pion target)
  - Dynamical chiral symmetry breaking
Spin-0 Generalized Parton Distribution.
Not so simple properties.

- **Polynomiality**
  - Lorentz covariance

- **Positivity**
  - Positivity of Hilbert space norm
  - Relativistic quantum mechanics
  - Soft pion theorem (pion target)
  - Dynamical chiral symmetry breaking

**How can we implement *a priori* these theoretical constraints?**

- There is no known GPD parameterization relying only on first principles.
- In the following, focus on **polynomiality** and **positivity**.
Polynomiality.
Mixed constraint from Lorentz invariance and discrete symmetries.

- Express Mellin moments of GPDs as matrix elements:
  \[
  \int_{-1}^{+1} dx x^m H^q(x, \xi, t) = \frac{1}{2(P^+)^{m+1}} \left\langle P + \frac{\Delta}{2} \bigg| \bar{q}(0) \gamma^+(i \not{D}^+)^m q(0) \bigg| P - \frac{\Delta}{2} \right\rangle
  \]

- Identify the Lorentz structure of the matrix element:
  linear combination of \((P^+)^{m+1-k}(\Delta^+)^k\) for \(0 \leq k \leq m+1\)

- Remember definition of skewness \(\Delta^+ = -2\xi P^+\).

- Select even powers to implement time reversal.

- Obtain polynomiality condition:
  \[
  \int_{-1}^{1} dx x^m H^q(x, \xi, t) = \sum_{i=0}^{m} (2\xi)^i C_{mi}^q(t) + (2\xi)^{m+1} C_{mm+1}^q(t) .
  \]
Double Distributions.
A convenient tool to encode GPD properties.

**Define Double Distributions** $F^q$ and $G^q$ as matrix elements of *twist-2 quark operators*:

\[
\left\langle P + \frac{\Delta}{2} \left| \bar{q}(0) \gamma^{\mu} i D^{\mu_1} \ldots i D^{\mu_m} q(0) \right| P - \frac{\Delta}{2} \right\rangle = \sum_{k=0}^{m} \binom{m}{k}
\]

\[
\left[ F^q_{mk}(t) 2 P^{\mu} - G^q_{mk}(t) \Delta^{\mu} \right] P^{\mu_1} \ldots P^{\mu_{m-k}} \left( -\frac{\Delta}{2} \right)^{\mu_{m-k+1}} \ldots \left( -\frac{\Delta}{2} \right)^{\mu_m}
\]

with

\[
F^q_{mk} = \int_{\Omega_{DD}} d\beta d\alpha \, \alpha^k \beta^{m-k} F^q(\beta, \alpha)
\]

\[
G^q_{mk} = \int_{\Omega_{DD}} d\beta d\alpha \, \alpha^k \beta^{m-k} G^q(\beta, \alpha)
\]


Double Distributions.
Relation to Generalized Parton Distributions.

- **Representation of GPD:**

\[ H^q(x, \xi, t) = \int_{\Omega_{DD}} d\beta d\alpha \, \delta(x - \beta - \alpha \xi) \left( F^q(\beta, \alpha, t) + \xi G^q(\beta, \alpha, t) \right) \]

- **Support property:** \( x \in [-1, +1] \).
- **Discrete symmetries:** \( F^q \) is \( \alpha \)-even and \( G^q \) is \( \alpha \)-odd.
- **Gauge:** any representation \((F^q, G^q)\) can be recast in one representation with a single DD \( f^q \):

\[ H^q(x, \xi, t) = x \int_{\Omega_{DD}} d\beta d\alpha \, f^q_{BMKS}(\beta, \alpha, t) \delta(x - \beta - \alpha \xi) \]


\[ H^q(x, \xi, t) = (1 - x) \int_{\Omega_{DD}} d\beta d\alpha \, f^q_{P}(\beta, \alpha, t) \delta(x - \beta - \alpha \xi) \]


Choose $F^q(\beta, \alpha) = 3\beta \theta(\beta)$ and $G^q(\beta, \alpha) = 3\alpha \theta(\beta)$:

$$H^q(x, \xi) = 3x \int_\Omega d\beta d\alpha \delta(x - \beta - \alpha \xi)$$

- Simple analytic expressions for the GPD:

$$H(x, \xi) = \begin{cases} 
6x(1 - x) & \text{if } 0 < |\xi| < x < 1, \\
\frac{x(1 - \xi^2)}{1 - \xi^2} & \text{if } -|\xi| < x < |\xi| < 1. 
\end{cases}$$
### Compute first Mellin moments.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\int_{-\xi}^{+\xi} dx x^n H(x, \xi)$</th>
<th>$\int_{+\xi}^{+1} dx x^n H(x, \xi)$</th>
<th>$\int_{-\xi}^{-1} dx x^n H(x, \xi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1+\xi-2\xi^2}{1+\xi}$</td>
<td>$\frac{2\xi^2}{1+\xi}$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1+\xi+\xi^2-3\xi^3}{2(1+\xi)}$</td>
<td>$\frac{2\xi^3}{1+\xi}$</td>
<td>$\frac{1+\xi^2}{2}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{3(1-\xi)(1+2\xi+3\xi^2+4\xi^3)}{10(1+\xi)}$</td>
<td>$\frac{6\xi^4}{5(1+\xi)}$</td>
<td>$\frac{3(1+\xi^2)}{10}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1+\xi+\xi^2+\xi^3+\xi^4-5\xi^5}{5(1+\xi)}$</td>
<td>$\frac{6\xi^5}{5(1+\xi)}$</td>
<td>$\frac{1+\xi^2+\xi^4}{5}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1+\xi+\xi^2+\xi^3+\xi^4+\xi^5-6\xi^6}{7(1+\xi)}$</td>
<td>$\frac{6\xi^6}{7(1+\xi)}$</td>
<td>$\frac{1+\xi^2+\xi^4}{7}$</td>
</tr>
</tbody>
</table>

- Expressions get more complicated as $n$ increases... But they always yield polynomials!

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Positivity. A consequence of the positivity of the norm in a Hilbert space.

- Identify the matrix element defining a GPD as an inner product of two different states.
- Apply Cauchy-Schwartz inequality, and identify PDFs at specific kinematic points, e.g.:

\[
|H^q(x, \xi, t)| \leq \sqrt{\frac{1}{1 - \xi^2}} q \left( \frac{x + \xi}{1 + \xi} \right) q \left( \frac{x - \xi}{1 - \xi} \right)
\]

- This procedures yields infinitely many inequalities stable under LO evolution.


- The overlap representation guarantees a priori the fulfillment of positivity constraints.
Overlap representation.
A first-principle connection with Light Front Wave Functions.

- Decompose an hadronic state \(|H; P, \lambda\rangle\) in a Fock basis:

  \[
  |H; P, \lambda\rangle = \sum_{N, \beta} \int [dx dk_\perp] N \psi_N^{(\beta, \lambda)}(x_1, k_\perp 1, \ldots, x_N, k_\perp N) |\beta, k_1, \ldots, k_N\rangle
  \]

- Derive an expression for the pion GPD in the DGLAP region \(\xi \leq x \leq 1\):

  \[
  H^q(x, \xi, t) \propto \sum_{\beta, j} \int [d\tilde{x} d\tilde{k}_\perp] \delta_j(q\delta(x-\tilde{x}_j)(\psi_N^{(\beta, \lambda)})^*(\tilde{x}', \tilde{k}_\perp)\psi_N^{(\beta, \lambda)}(\tilde{x}, \tilde{k}_\perp)
  \]

  with \(\tilde{x}, \tilde{k}_\perp \) (resp. \(\tilde{x}', \tilde{k}_\perp' \)) generically denoting incoming (resp. outgoing) parton kinematics.


- Similar expression in the ERBL region \(-\xi \leq x \leq \xi\), but with overlap of \(N\)- and \((N+2)\)-body LFWFs.
Overlap representation.
Advantages and drawbacks.

- Physical picture.
- Positivity relations are fulfilled by construction.
- Implementation of symmetries of $N$-body problems.

What is not obvious anymore

What is not obvious to see from the wave function representation is however the continuity of GPDs at $x = \pm \xi$ and the polynomiality condition. In these cases both the DGLAP and the ERBL regions must cooperate to lead to the required properties, and this implies nontrivial relations between the wave functions for the different Fock states relevant in the two regions. An ad hoc Ansatz for the wave functions would almost certainly lead to GPDs that violate the above requirements.

GPDs from Light Front Wave Functions
The Radon transform.
Definition and properties.

\[ \mathcal{R}f(s, \phi) = \int_{-\infty}^{+\infty} d\beta d\alpha \ f(\beta, \alpha) \delta(s - \beta \cos \phi - \alpha \sin \phi) \]

For \( s > 0 \) and \( \phi \in [0, 2\pi] \):

\[ \mathcal{R}f(-s, \phi) = \mathcal{R}f(s, \phi \pm \pi) \]

Relation to GPDs:

\[ x = \frac{s}{\cos \phi} \text{ and } \xi = \tan \phi \]

Relation between GPD and DD in Belistky et al. gauge

\[ \frac{\sqrt{1 + \xi^2}}{x} H(x, \xi) = \mathcal{R}f_{\text{BMKS}}(s, \phi), \]

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The Radon transform.
Definition and properties.

\[ \mathcal{R}f(s, \phi) = \int_{-\infty}^{+\infty} d\beta d\alpha f(\beta, \alpha) \delta(s - \beta \cos \phi - \alpha \sin \phi) \]

and:

\[ \mathcal{R}f(-s, \phi) = \mathcal{R}f(s, \phi \pm \pi) \]

Relation to GPDs:

\[ x = \frac{s}{\cos \phi} \text{ and } \xi = \tan \phi \]

Relation between GPD and DD in Pobylitsa gauge:

\[ \sqrt{1 + \xi^2} \frac{H(x, \xi)}{1 - x} = \mathcal{R}f_P(s, \phi) \]
The Mellin moments of a Radon transform are **homogeneous polynomials** in \( \omega = (\sin \phi, \cos \phi) \).

The converse is also true:

**Theorem (Hertle, 1983)**

Let \( g(s, \omega) \) an even compactly-supported distribution. Then \( g \) is itself the Radon transform of a compactly-supported distribution if and only if the **Ludwig-Helgason consistency condition** hold:

1. \( g \) is \( C^\infty \) in \( \omega \),
2. \( \int ds s^m g(s, \omega) \) is a homogeneous polynomial of degree \( m \) for all integer \( m \geq 0 \).

Double Distributions and the Radon transform are the **natural solution** of the polynomiality condition.
Implementing Lorentz covariance.
Extend an overlap in the DGLAP region to the whole GPD domain.

\[
\Omega_{DD}(|\alpha| + |\beta| \leq 1) \Rightarrow \begin{cases} 
(x, \xi) \in \text{DGLAP} & \iff |s| \geq |\sin \phi| , \\
(x, \xi) \in \text{ERBL} & \iff |s| \leq |\sin \phi| . 
\end{cases}
\]

\[
\beta = (x - \xi)/(1 + \xi) \\
\alpha = \frac{1}{\xi}(x - \beta) \\
\beta = (x - \xi)/(1 - \xi) \\
\beta = (x + \xi)/(1 + \xi)
\]

Each point \((\beta, \alpha)\) with \(\beta \neq 0\) contributes to both DGLAP and ERBL regions.
For any model of LFWF, one has to address the following three questions:

1. Does the extension exist?
2. If it exists, is it unique?
3. How can we compute this extension?
Ill-posedness in the sense of Hadamard.
A first glimpse at the inverse Radon transform.

- Numerical evaluation *almost unavoidable* (polar vs cartesian coordinates).
- Ill-posedness by *lack of continuity*.
- The *unlimited* Radon inverse problem is *midly* ill-posed while the *limited* one is *severely* ill-posed.
Theorem

Let $f$ be a compactly-supported locally summable function defined on $\mathbb{R}^2$ and $\mathcal{R}f$ its Radon transform. Let $(s_0, \omega_0) \in \mathbb{R} \times S^1$ and $U_0$ an open neighborhood of $\omega_0$ such that:

$$\text{for all } s > s_0 \text{ and } \omega \in U_0 \quad \mathcal{R}f(s, \omega) = 0.$$ 

Then $f(\mathcal{R}) = 0$ on the half-plane $\langle \mathcal{R} | \omega_0 \rangle > s_0$ of $\mathbb{R}^2$.

Consider a GPD $H$ being zero on the DGLAP region.

- Take $\phi_0$ and $s_0$ s.t. $\cos \phi_0 \neq 0$ and $|s_0| > |\sin \phi_0|$.
- Neighborhood $U_0$ of $\phi_0$ s.t. $\forall \phi \in U_0$ $|\sin \phi| < |s_0|$.
- The underlying DD $f$ has a zero Radon transform for all $\phi \in U_0$ and $s > s_0$ (DGLAP).
- Then $f(\beta, \alpha) = 0$ for all $(\beta, \alpha) \in \Omega_{\text{DD}}$ with $\beta \neq 0$.
- Extension unique up to adding a D-term: $\delta(\beta)D(\alpha)$. 
Computation of the extension.
Numerical evaluation of the inverse Radon transform (1/3).

A discretized problem

Consider \( N + 1 \) Hilbert spaces \( H, H_1, \ldots, H_N \), and a family of continuous surjective operators \( R_n : H \to H_n \) for \( 1 \leq n \leq N \). Being given \( g_1 \in H_1, \ldots, g_n \in H_n \), we search \( f \) solving the following system of equations:

\[
R_n f = g_n \quad \text{for } 1 \leq n \leq N
\]

Fully discrete case

Assume \( f \) piecewise-constant with values \( f_m \) for \( 1 \leq m \leq M \). For a collection of lines \( (L_n)_{1 \leq n \leq N} \) crossing \( \Omega_{\text{DD}} \), the Radon transform writes:

\[
g_n = R f = \int_{L_n} f = \sum_{m=1}^{M} f_m \times \text{Measure}(L_n \cap C_m) \quad \text{for } 1 \leq n \leq N
\]
Computation of the extension.
Numerical evaluation of the inverse Radon transform (2/3).

**Kaczmarz algorithm**

Denote $P_n$ the orthogonal projection on the affine subspace $R_n f = g_n$. Starting from $f^0 \in H$, the sequence defined iteratively by:

$$f^{k+1} = P_N P_{N-1} \ldots P_1 f^k$$

converges to the solution of the system.

The convergence is exponential if the projections are randomly ordered.

Computation of the extension.
Numerical evaluation of the inverse Radon transform (2/3).

\[ f^\lambda = P_4 P_3 P_2 P_1 \phi \]

\[ f = P_4 P_3 P_2 P_1 f^\lambda \]

\[ P_2 P_1 f^\lambda \]

\[ P_3 P_2 P_1 f^\lambda \]

\[ P_1 \phi \]

\[ K_1 \]

\[ K_2 \]

\[ K_3 \]

\[ K_4 \]
And if the input data are inconsistent?

- Instead of solving $g = \mathcal{R}f$, find $f$ such that $\|g - \mathcal{R}f\|_2$ is minimum.
- The solution always exists.
- The input data are inconsistent if $\|g - \mathcal{R}f\|_2 > 0$. 
Relaxed Kaczmarz algorithm

Let $\omega \in ]0, 2[$ and:

$$P_n^\omega = (1 - \omega) \text{Id}_H + \omega P_n \quad \text{for } 1 \leq n \leq N$$

Write:

$$RR^\dagger = (R_i R_j^\dagger)_{1 \leq i, j \leq N} = D + L + L^\dagger$$

where $D$ is diagonal, and $L$ is lower-triangular with zeros on the diagonal.
Theorem

Let $0 < \omega < 2$. For $f \in \text{Ran} \ R^\dagger$ (e.g. $f = 0$), the Kaczmarz method with relaxation converges to the unique solution $f^\omega \in \text{Ran} \ R^\dagger$.

\[ R^\dagger (D + \omega L)^{-1} (g - Rf^\omega) = 0, \]

where the matrix $D$ and $L$ appear in the decomposition of $RR^\dagger$.

If $g = Rf$ has a solution, then $f^\omega$ is its solution of minimal norm. Otherwise:

\[ f^\omega = f^{\text{MP}} + O(\omega), \]

where $f^{\text{MP}}$ is the minimizer in $H$ of:

\[ \langle g - Rf | g - Rf \rangle_D. \]

The inner product being defined by:

\[ \langle h | k \rangle_D = \langle D^{-1} h | k \rangle. \]
Test on a 1D example.
Recovering a PDF from the knowledge of its Mellin moments.

A pion valence PDF-like example

Aim: reconstruct the PDF \( q(x) = 30x^2(1 - x)^2 \) from the knowledge of its first 30 Mellin moments.

- Piecewise-constant PDF: 20 values.
- Input: 30 Mellin moments.
- Unrelaxed method \( \omega = 1 \).
- 10000 iterations.

Extensive testing in progress
- Various inputs: PDFs and LFWFs.
- Numerical noise and numerical techniques.
PARTONS Project

PARtonic Tomography Of Nucleon Software
Computing chain design.
Differential studies: physical models and numerical methods.

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First principles and fundamental parameters

Evolution

GPD at $\mu \neq \mu_F^{\text{ref}}$

GPD at $\mu_F^{\text{ref}}$

DVCS

TCS

DVMP

DVCS

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GPD at $\mu \neq \mu_F^{\text{ref}}$

Evolution

- Many observables.
- Kinematic reach.
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**Experimental data and phenomenology**
Need for modularity

Computation of amplitudes

- **DVCS**
- **TCS**
- **DVMP**

**First principles and fundamental parameters**

- **GPD at** $\mu \neq \mu_F^{\text{ref}}$
- **Evolution**

- **GPD at** $\mu_F^\text{ref}$

---

- Many observables.
- Kinematic reach.

- Perturbative approximations.
- Physical models.
- Fits.
- Numerical methods.
- Accuracy and speed.
Computing chain design.
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- Many observables.
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**Computation of amplitudes**

First principles and fundamental parameters

- Many observables.
- Kinematic reach.

- Perturbative approximations.
- Physical models.
- Fits.
- Numerical methods.
- Accuracy and speed.

GPD at \( \mu \neq \mu_F^{\text{ref}} \)

Evolution

GPD at \( \mu_F^{\text{ref}} \)

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- Accuracy and speed.

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GPD at $\mu_F^{\text{ref}}$

Evolution
Status.
Currently: integration, tests, validation.

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3 stages:

1. Design.
2. Integration and validation.
3. Production.

Flexible software architecture.


1 new physical development = 1 new module.

Aggregate knowledge and know-how. Do not reinvent the wheel!

What can be automated will be automated.

Get ready for 12 GeV!
Modularity.
Inheritance, standardized inputs and outputs.

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- Steps of logic sequence in parent class.
- Model description and related mathematical methods in daughter class.
GPD computing made simple.
Each line of code corresponds to a physical hypothesis.

```cpp
// Lots of includes
#include <src/Partons.h>
...

// Retrieve GPD service
GPDService* pGPDService = ServiceObjectRegistry::getGPDService();
// Load GPD module with the BaseModuleFactory
GPDModule* pGK11Model = ModuleObjectFactory::newGPDModule(GK11Model::classId);
// Create a GPDKinematic(x, xi, t, MuF, MuR)
GPDKinematic gpdKinematic(0.1, xBToXi(0.001), 0.3, 8., 8.);
// Compute data and store results
GPDResult gpdResult = pGPDService->computeGPDModelRestrictedByGPDType(gpdKinematic, pGK11Model, GPDType::ALL);
// Print results
std::cout << gpdResult.toString() << std::endl;

delete pGK11Model;
pGK11Model = 0;
```

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GPD computing automated.
Each line of code corresponds to a physical hypothesis.

```xml
<?xml version="1.0" encoding="UTF-8" standalone="yes" ?>
<scenario id="01" date="" description="Example: computation of one GPD model (GK11) without evolution">
    <!-- Select type of computation -->
    <task service="GPDService" method="computeGPDModel" >
        <!-- Specify kinematics -->
        <GPDKinematic>
            <param name="x" value="0.1" />
            <param name="xi" value="1.00050025" />
            <param name="t" value="-0.3" />
            <param name="MuF2" value="8" />
            <param name="MuR2" value="8" />
        </GPDKinematic>
        <!-- Choose GPD model and set parameters -->
        <GPDModule>
            <param name="id" value="GK11Model" />
        </GPDModule>
    </task>
</scenario>
```

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GPD computing automated.
Each line of code corresponds to a physical hypothesis.

computeOneGPD.xml

```xml
<?xml version="1.0" encoding="UTF-8" standalone="yes"?>
<scenario id="01" date="" description="Example: computation of one GPD model (GK11) without evolution">
  <!-- Select type of computation -->
  <task service="GPDService" method="computeGPDModel">
    <!-- Specify kinematics -->
    <GPDKinematic>
      <param name="x" value="0.1"/>
      <param name="xi" value="1.00050025"/>
      <param name="t" value="-0.3"/>
      <param name="MuF2" value="8"/>
      <param name="MuR2" value="8"/>
    </GPDKinematic>
    <!-- Choose GPD model and set parameters -->
    <GPDModule>
      <param name="id" value="GK11Model"/>
    </GPDModule>
  </task>
</scenario>
```

\[
\begin{align*}
H^u &= 0.822557 \\
H^u(+) &= 0.165636 \\
H^u(-) &= 1.47948 \\
H^d &= 0.421431 \\
H^d(+) &= 0.0805182 \\
H^d(-) &= 0.762344 \\
H^s &= 0.00883408 \\
H^s(+) &= 0.0176682 \\
H^s(-) &= 0 \\
H^g &= 0.385611 \\
\end{align*}
\]

and \( E, \tilde{H}, \tilde{E} \).
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computeOneCFF.xml

```xml
<?xml version="1.0" encoding="UTF-8" standalone="yes" ?>
<scenario id="03" date="" description="Example: computation of one convol coeff function model (DVCSCFF) with GPD model (GK11)"
<task service="DVCSConvolCoeffFunctionService" method="computeWithGPDModel"
    <DVCSConvolCoeffFunctionKinematic>
        <param name="xi" value="0.5" />
        <param name="t" value="-0.1346" />
        <param name="Q2" value="1.5557" />
        <param name="MuF2" value="4" />
        <param name="MuR2" value="4" />
    </DVCSConvolCoeffFunctionKinematic>
    <GPDModule>
        <param name="id" value="GK11Model" />
    </GPDModule>
    <DVCSConvolCoeffFunctionModule>
        <param name="id" value="DVSCSFFModel" />
        <param name="qcd_order_type" value="LO" />
    </DVCSConvolCoeffFunctionModule>
</task>
</scenario>
```

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CFF computing automated.
Each line of code corresponds to a physical hypothesis.

```xml
<?xml version="1.0" encoding="UTF-8" standalone="yes" ?>
<scenario id="03" date="" description="Example: computation of one convol coeff function model (DVCSCFF) with GPD model (GK11)"
<task service="DVCSConvolCoeffFunctionService" method="computeWithGPDModel"
<DVCSConvolCoeffFunctionKinematic>
<param name="xi" value="0.5" />
<param name="t" value="-0.1346" />
<param name="Q2" value="1.5557" />
<param name="MuF2" value="4" />
<param name="MuR2" value="4" />
</DVCSConvolCoeffFunctionKinematic>
<GPDModule>
<param name="id" value="GK11Model" />
</GPDModule>
</DVCSConvolCoeffFunctionService>

\[ \mathcal{H} = 1.47722 + 1.76698 i \]
\[ \mathcal{E} = 0.12279 + 0.512312 i \]
\[ \tilde{\mathcal{H}} = 1.54911 + 0.953728 i \]
\[ \tilde{\mathcal{E}} = 18.8776 + 3.75275 i \]
```

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Multidisciplinary development team

Berthou (Irfu)  Binosi (ECT*)  Chouika (Irfu)  Guidal (IPNO)  Mezrag (ANL)  Moutarde (Irfu)  Sabatié (Irfu)  Sznajder (IPNO)  Wagner (NCBJ)

IPN et LPT (Orsay), Irfu (Saclay) and CPhT (Polytechnique)
Experimental data analysis  Perturbative QCD
World data fits  GPD modeling
Conclusion
Conclusions and prospects.
Positivity and polynomiality constraints consistently implemented.

- Last decade demonstrated maturity of GPD phenomenology.
- **Challenging constraints** expected from Jefferson Lab in the valence region.
- **Systematic** procedure to construct GPD models from any "reasonable" Ansatz of LFWFs.
- Characterization of the existence and **uniqueness** of the extension from the DGLAP to the ERBL region.
- Development of the platform PARTONS for **phenemonology** and **theory** purposes.
- Numerical tests *in progress.*