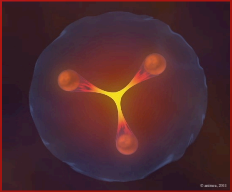
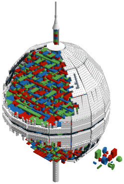
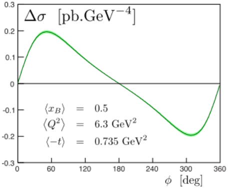
DE LA RECHERCHE À L'INDUSTRIE

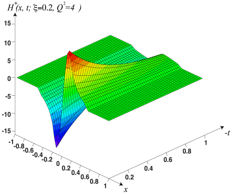
AGENCE NATIONALE DE LA RECHERCHE

Lorentz Covariance and Positivity Constraints in the Modeling of Generalized Parton Distributions









Theory Center Seminar | Hervé MOUTARDE

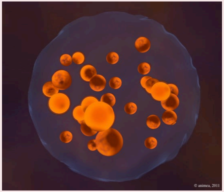
Feb. 22nd, 2016

www.cea.fr

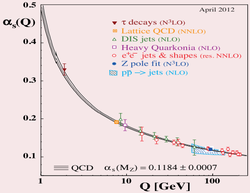
Motivation.

Study nucleon structure to shed new light on nonperturbative QCD.

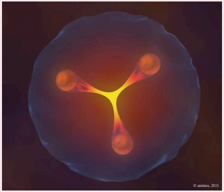
Perturbative QCD



Asymptotic freedom



Nonperturbative QCD

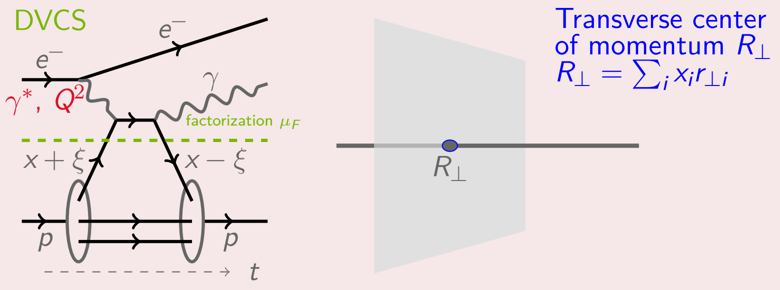


Perturbative AND nonperturbative QCD at work

- Define **universal** objects describing 3D nucleon structure: **Generalized Parton Distributions** (GPD).
- Relate GPDs to measurements using **factorization**: **Virtual Compton Scattering** (DVCS, TCS), **Deeply Virtual Meson production** (DVMP).
- Get **experimental knowledge** of nucleon structure.

- Correlation of the **longitudinal momentum** and the **transverse position** of a parton in a hadron.
- DVCS recognized as the cleanest channel to access GPDs.

Deeply Virtual Compton Scattering (DVCS)



Anatomy of hadrons.
GPDs, 3D hadron imaging, and beyond.

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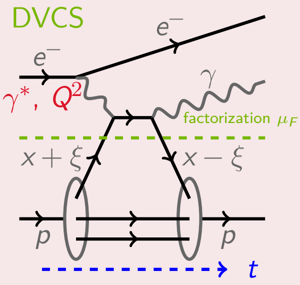
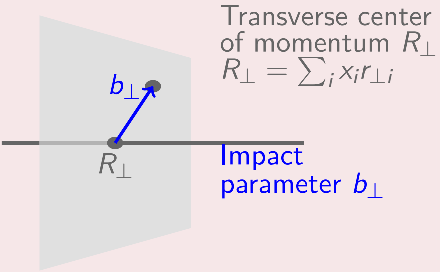
- DVCS recognized as the cleanest channel to access GPDs.

Deeply Virtual Compton Scattering (DVCS)





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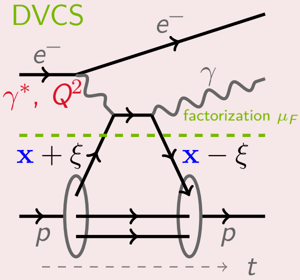
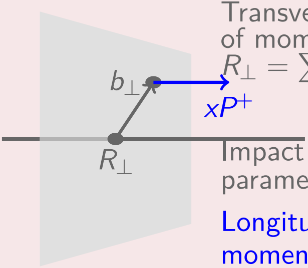
Deeply Virtual Compton Scattering (DVCS)





Transverse center of momentum $R_\perp$
$R_\perp = \sum_i x_i r_{\perp i}$

Impact parameter $b_\perp$

Longitudinal momentum xP^+

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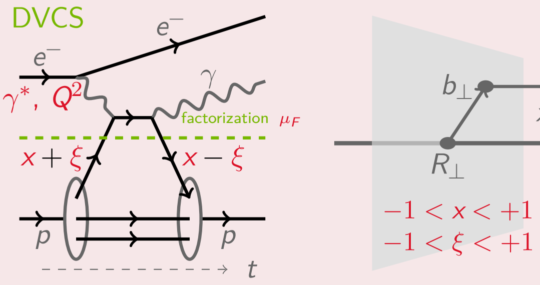
- DVCS recognized as the cleanest channel to access GPDs.

Deeply Virtual Compton Scattering (DVCS)



Transverse center of momentum $R_\perp$
$R_\perp = \sum_i x_i r_{\perp i}$

Impact parameter $b_\perp$

Longitudinal momentum xP^+

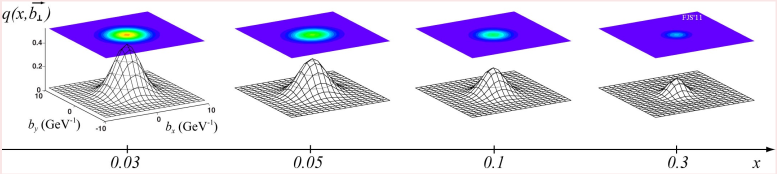
- **24 GPDs** $F^i(x, \xi, t, \mu_F)$ for each parton type $i = g, u, d, \ldots$ for leading and sub-leading twists.

Towards hadron tomography.

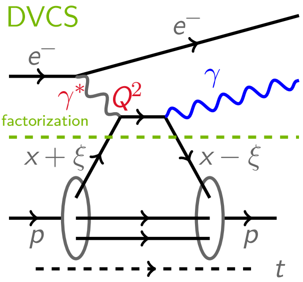
GPDs as a scalpel-like probe of hadron structure.

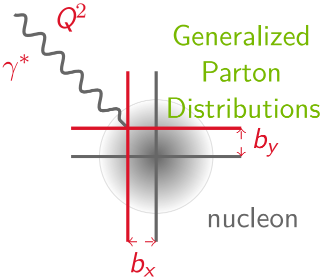
1. **Phenomenology status:** relevance and need for parameterizations.
2. **Theoretical framework:** definition and existing constraints.
3. **GPDs from Light Front Wave Functions:** a promising computing strategy.
4. **The PARTONS platform:** a GPD toolkit.

How can we make this picture? What do we learn from it?



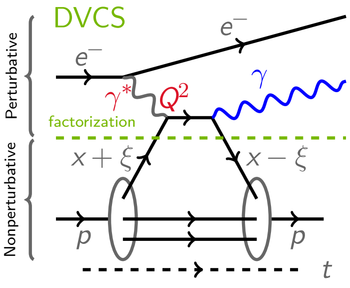
Phenomenology status

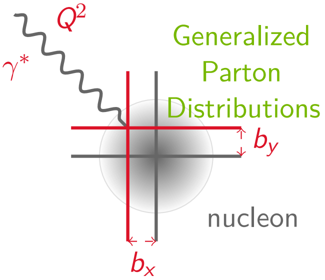

DVCS
e^-
e^-
γ^*
Q^2
γ
factorization
$x+\xi$
$x-\xi$
p
p
t

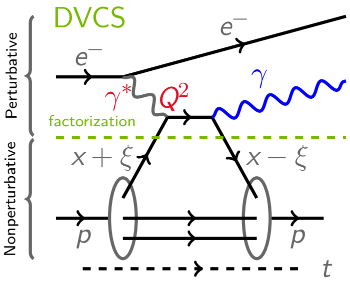

Q^2
γ^*
Generalized Parton Distributions
b_y
nucleon
b_x

Exclusive processes of current interest (1/2).

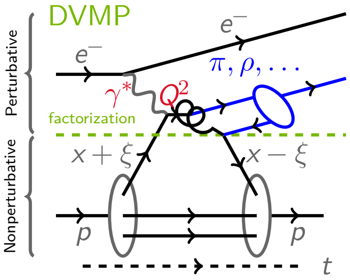
Factorization and universality.



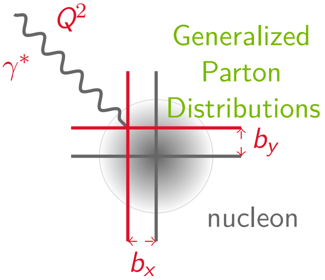




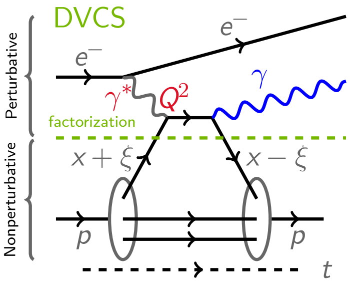
DVCS
Perturbative
Nonperturbative
e^-
e^-
γ^*
Q^2
γ
factorization
$x+\xi$
$x-\xi$
p
p
t



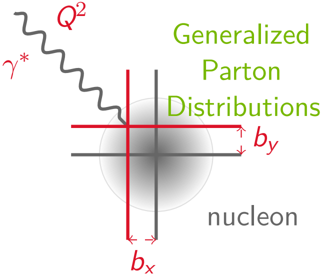
DVMP
Perturbative
Nonperturbative
e^-
e^-
$\pi, \rho, \ldots$
γ^*
Q^2
factorization
$x+\xi$
$x-\xi$
p
p
t



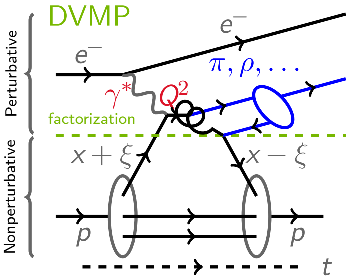
Q^2
γ^*
Generalized
Parton
Distributions
b_y
nucleon
b_x



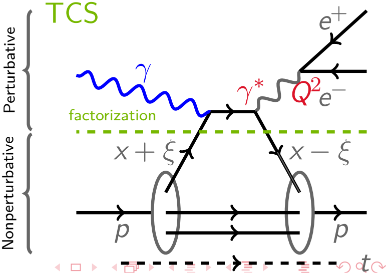
DVCS
Perturbative
Nonperturbative
e^-
e^-
γ^*
Q^2
γ
factorization
$x+\xi$
$x-\xi$
p
p
t



γ^*
Q^2
Generalized
Parton
Distributions
b_y
b_x
nucleon



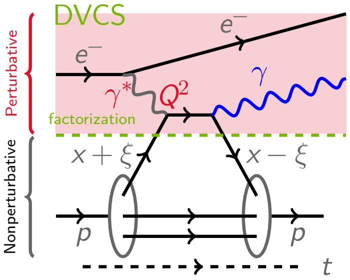
DVMP
Perturbative
Nonperturbative
e^-
e^-
$\pi, \rho, \ldots$
γ^*
Q^2
factorization
$x+\xi$
$x-\xi$
p
p
t

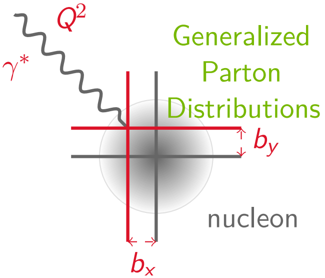


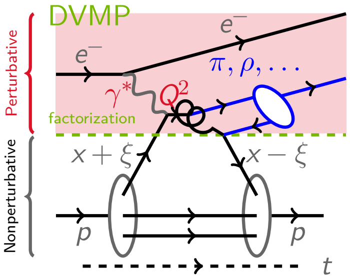
TCS
Perturbative
Nonperturbative
e^+
γ
γ^*
Q^2
e^-
factorization
$x+\xi$
$x-\xi$
p
p
t

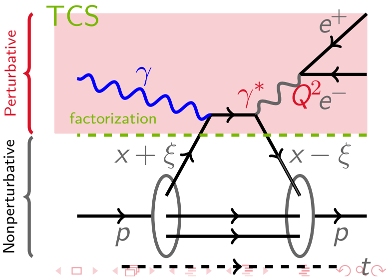
Exclusive processes of current interest (1/2).

Factorization and universality.



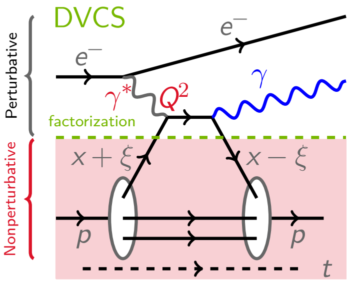


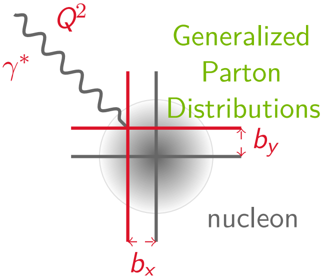


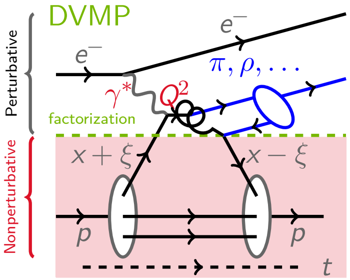


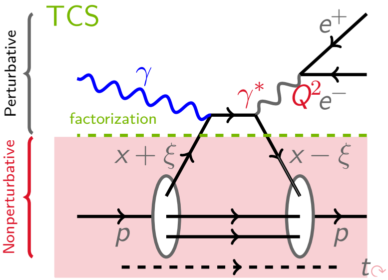
Exclusive processes of current interest (1/2).

Factorization and universality.









Exclusive processes of present interest (2/2).

Factorization and universality.

Bjorken regime : large Q^2 and fixed $xB \simeq 2\xi/(1+\xi)$

- Partonic interpretation relies on **factorization theorems**.
- All-order proofs for DVCS, TCS and some DVMP.
- GPDs depend on a (arbitrary) factorization scale μ_F.
- **Consistency** requires the study of **different channels**.

- GPDs enter DVCS through **Compton Form Factors** :

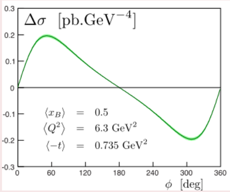
$$\mathcal{F}(\xi, t, Q^2) = \int_{-1}^{1} dx\, C\left(x, \xi, \alpha_S(\mu_F), \frac{Q}{\mu_F}\right) F(x, \xi, t, \mu_F)$$

for a given GPD F.

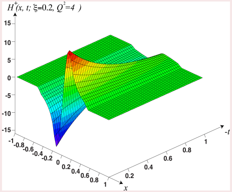
- CFF $\mathcal{F}$ is a **complex function**.

- **Dominance** of twist-2 and **validity** of a GPD analysis of DVCS data.
- $Im\mathcal{H}$ **best determined**. Large uncertainties on $Re\mathcal{H}$.
- However sizable **higher twist contamination** for DVCS measurements?
- Already some indications about the **invalidity** of the H-dominance hypothesis with **unpolarized data**.

1. Experimental data fits



2. GPD extraction



3. Nucleon imaging

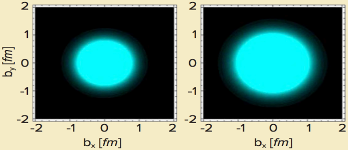
Images from Guidal et al.,
Rept. Prog. Phys. 76 (2013) 066202

Reaching for the Horizon

The 2015 Long Range Plan for Nuclear Science

Sidebar 2.2: The First 3D Pictures of the Nucleon

A computed tomography (CT) scan can help physicians pinpoint minute cancer tumors, diagnose tiny broken bones, and spot the early signs of osteoporosis. Now physicists are using the principles behind the procedure to peer at the inner workings of the proton. This breakthrough is made possible by a relatively new concept in nuclear physics called generalized parton distributions.



An intense beam of high-energy electrons can be used

1. **Extract** $H(x, \xi, t, \mu_F^{\mathrm{ref}})$ from experimental data.
2. **Extrapolate** to vanishing skewness $H(x, 0, t, \mu_F^{\mathrm{ref}})$.
3. **Extrapolate** $H(x, 0, t, \mu_F^{\mathrm{ref}})$ up to infinite t.
4. **Compute** 2D Fourier transform in transverse plane:

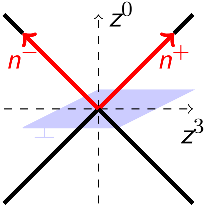
$$H(x, b_\perp) = \int_0^{+\infty} \frac{\mathrm{d}\Delta_\perp}{2\pi} \Delta_\perp \, J_0(b_\perp \Delta_\perp) \, H(x, 0, -\Delta_\perp^2)$$

5. **Propagate** uncertainties.
6. **Control** extrapolations with an accuracy matching that of experimental data with **sound** GPD models.

Theoretical framework

$$H_\pi^q(x,\xi,t) \quad = \quad \frac{1}{2}\int \frac{dz^-}{2\pi}\, e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0 \\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



References

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**

$$H^q(x,0,0) = q(x)$$

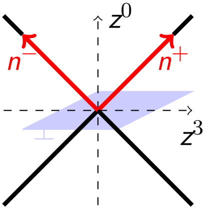
Spin-0 Generalized Parton Distribution.

Definition and simple properties.

$$H^q_\pi(x, \xi, t) \quad =$$

$$\frac{1}{2} \int \frac{\mathrm{d}z^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right) \gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P - \frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



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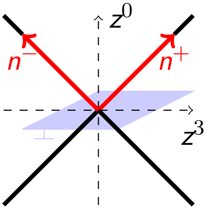
- PDF **forward limit**
- Form factor **sum rule**

$$\int_{-1}^{+1} dx\, H^q(x, \xi, t) = F^q_1(t)$$

$$H^q_\pi(x, \xi, t) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P + \frac{\Delta}{2} \right| \bar{q}\left(-\frac{z}{2}\right) \gamma^+ q\left(\frac{z}{2}\right) \left| \pi, P - \frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



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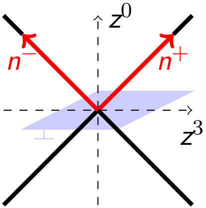
- PDF **forward limit**
- Form factor **sum rule**
- H^q is an **even function** of ξ from time-reversal invariance.

$$H^q_\pi(x, \xi, t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



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Radyushkin, Phys. Lett. **B380**, 417 (1996)

- PDF **forward limit**
- Form factor **sum rule**
- H^q is an **even function** of ξ from time-reversal invariance.
- H^q is **real** from hermiticity and time-reversal invariance.

Polynomiality

$$\int_{-1}^{+1} dx\, x^n H^q(x, \xi, t) = \text{polynomial in } \xi$$

Polynomiality

Lorentz covariance

- **Polynomiality**

Lorentz covariance

- **Positivity**

$$H^q(x, \xi, t) \leq \sqrt{q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

Spin-0 Generalized Parton Distribution.

Not so simple properties.

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

$$H^q(x, \xi = 1, t = 0) = \frac{1}{2}\phi^q_\pi\left(\frac{1+x}{2}\right)$$

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

  Dynamical chiral symmetry breaking

- **Polynomiality**

  Lorentz covariance

- **Positivity**

  Positivity of Hilbert space norm

- H^q has support $x \in [-1, +1]$.

  Relativistic quantum mechanics

- **Soft pion theorem** (pion target)

  Dynamical chiral symmetry breaking

How can we implement *a priori* these theoretical constraints?

- There is no known GPD parameterization **relying only on first principles**.
- In the following, focus on **polynomiality** and **positivity**.

Polynomiality.

Mixed constraint from Lorentz invariance and discrete symmetries.

- Express Mellin moments of GPDs as **matrix elements**:

$$\int_{-1}^{+1} \mathrm{d}x\, x^m H^q(x, \xi, t)$$

$$= \frac{1}{2(P^+)^{m+1}} \left\langle P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i \overleftrightarrow{D}^+)^m q(0) \middle| P - \frac{\Delta}{2} \right\rangle$$

- Identify the **Lorentz structure** of the matrix element:

linear combination of $(P^+)^{m+1-k}(\Delta^+)^k$ for $0 \leq k \leq m+1$

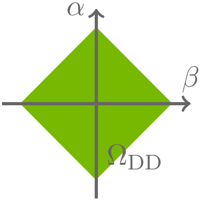
- Remember definition of **skewness** $\Delta^+ = -2\xi P^+$.
- Select **even powers** to implement time reversal.
- Obtain **polynomiality condition**:

$$\int_{-1}^{1} \mathrm{d}x\, x^m H^q(x, \xi, t) = \sum_{\substack{i=0 \\ \text{even}}}^{m} (2\xi)^i C^q_{mi}(t) + (2\xi)^{m+1} C^q_{mm+1}(t) \;.$$

- Define Double Distributions F^q and G^q as matrix elements of **twist-2 quark operators**:

$$\left\langle P+\frac{\Delta}{2}\middle|\bar{q}(0)\gamma^{\{\mu}i\overleftrightarrow{D}^{\mu_1}\dots i\overleftrightarrow{D}^{\mu_m\}}q(0)\middle|P-\frac{\Delta}{2}\right\rangle = \sum_{k=0}^{m}\binom{m}{k}$$

$$\left[F^q_{mk}(t)2P^{\{\mu}-G^q_{mk}(t)\Delta^{\{\mu}\right]P^{\mu_1}\dots P^{\mu_{m-k}}\left(-\frac{\Delta}{2}\right)^{\mu_{m-k+1}}\dots\left(-\frac{\Delta}{2}\right)^{\mu_m\}}$$



with

$$F^q_{mk} = \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta\mathrm{d}\alpha\,\alpha^k\beta^{m-k}F^q(\beta,\alpha)$$

$$G^q_{mk} = \int_{\Omega_{\mathrm{DD}}} \mathrm{d}\beta\mathrm{d}\alpha\,\alpha^k\beta^{m-k}G^q(\beta,\alpha)$$

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Radyushkin, Phys. Rev. **D59**, 014030 (1999)
Radyushkin, Phys. Lett. **B449**, 81 (1999)

Double Distributions.

Relation to Generalized Parton Distributions.

- Representation of GPD:

$$H^q(x,\xi,t) = \int_{\Omega_{\rm DD}} \mathrm{d}\beta\mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi)\big(F^q(\beta,\alpha,t)+\xi G^q(\beta,\alpha,t)\big)$$

- Support property: $x \in [-1,+1]$.
- Discrete symmetries: F^q is α-even and G^q is α-odd.
- **Gauge**: any representation (F^q, G^q) can be recast in one representation with a single DD f^q:

$$H^q(x,\xi,t) = x\int_{\Omega_{\rm DD}} \mathrm{d}\beta\mathrm{d}\alpha\, f^q_{\rm BMKS}(\beta,\alpha,t)\delta(x-\beta-\alpha\xi)$$

Belitsky *et al.*, Phys. Rev. **D64**, 116002 (2001)

$$H^q(x,\xi,t) = (1-x)\int_{\Omega_{\rm DD}} \mathrm{d}\beta\mathrm{d}\alpha\, f^q_{\rm P}(\beta,\alpha,t)\delta(x-\beta-\alpha\xi)$$

Pobylitsa, Phys. Rev. **D67**, 034009 (2003)
Müller, Few Body Syst. **55**, 317 (2014)

- Choose $F^q(\beta, \alpha) = 3\beta\theta(\beta)$ ad $G^q(\beta, \alpha) = 3\alpha\theta(\beta)$:

$$H^q(x, \xi) = 3x \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \, \delta(x - \beta - \alpha\xi)$$

- Simple analytic expressions for the GPD:

$$\begin{aligned} H(x, \xi) &= \frac{6x(1-x)}{1-\xi^2} \text{ if } 0 < |\xi| < x < 1, \\ H(x, \xi) &= \frac{3x(x+|\xi|)}{|\xi|(1+|\xi|)} \text{ if } -|\xi| < x < |\xi| < 1. \end{aligned}$$

- Compute first Mellin moments.

n	$\int_{-\xi}^{+\xi} \mathrm{d}x\, x^n H(x,\xi)$	$\int_{+\xi}^{+1} \mathrm{d}x\, x^n H(x,\xi)$	$\int_{-\xi}^{+1} \mathrm{d}x\, x^n H(x,\xi)$
0	$\frac{1+\xi-2\xi^2}{1+\xi}$	$\frac{2\xi^2}{1+\xi}$	1
1	$\frac{1+\xi+\xi^2-3\xi^3}{2(1+\xi)}$	$\frac{2\xi^3}{1+\xi}$	$\frac{1+\xi^2}{2}$
2	$\frac{3(1-\xi)(1+2\xi+3\xi^2+4\xi^3)}{10(1+\xi)}$	$\frac{6\xi^4}{5(1+\xi)}$	$\frac{3(1+\xi^2)}{10}$
3	$\frac{1+\xi+\xi^2+\xi^3+\xi^4-5\xi^5}{5(1+\xi)}$	$\frac{6\xi^5}{5(1+\xi)}$	$\frac{1+\xi^2+\xi^4}{5}$
4	$\frac{1+\xi+\xi^2+\xi^3+\xi^4+\xi^5-6\xi^6}{7(1+\xi)}$	$\frac{6\xi^6}{7(1+\xi)}$	$\frac{1+\xi^2+\xi^4}{7}$

- Expressions get more complicated as n increases... But they always yield polynomials!

- Identify the matrix element defining a GPD as an **inner product** of two different states.
- Apply Cauchy-Schwartz inequality, and identify PDFs at specific kinematic points, *e.g.*:

$$|H^q(x,\xi,t)| \leq \sqrt{\frac{1}{1-\xi^2} q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

- This procedures yields **infinitely many inequalities** stable under LO evolution.

Pobylitsa, Phys. Rev. **D66**, 094002 (2002)

- The **overlap representation** guarantees *a priori* the fulfillment of positivity constraints.

- Decompose an hadronic state $|H; P, \lambda\rangle$ in a Fock basis:

$$|H; P, \lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x\mathrm{d}\mathbf{k}_\perp]_N \psi_N^{(\beta,\lambda)}(x_1, \mathbf{k}_{\perp 1}, \ldots, x_N, \mathbf{k}_{\perp N}) \, |\beta, k_1, \ldots, k_N\rangle$$

- Derive an expression for the pion GPD in the DGLAP region $\xi \leq x \leq 1$:

$$H^q(x, \xi, t) \propto \sum_{\beta,j} \int [\mathrm{d}\bar{x}\mathrm{d}\bar{\mathbf{k}}_\perp]_N \delta_{j,q} \delta(x - \bar{x}_j) \left(\psi_N^{(\beta,\lambda)}\right)^* (\hat{x}', \hat{\mathbf{k}}'_\perp) \psi_N^{(\beta,\lambda)}(\tilde{x}, \tilde{\mathbf{k}}_\perp)$$

  with $\tilde{x}, \tilde{\mathbf{k}}_\perp$ (resp. $\hat{x}', \hat{\mathbf{k}}'_\perp$) generically denoting incoming (resp. outgoing) parton kinematics.

  Diehl *et al.*, Nucl. Phys. **B596**, 33 (2001)

- Similar expression in the ERBL region $-\xi \leq x \leq \xi$, but with overlap of N- and $(N+2)$-body LFWFs.

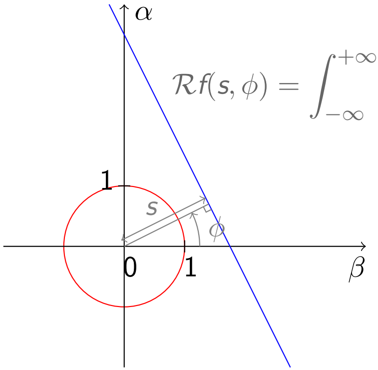
- Physical picture.
- Positivity relations are fulfilled **by construction**.
- Implementation of **symmetries of** N**-body problems**.

What is not obvious anymore

What is *not* obvious to see from the wave function representation is however the **continuity of GPDs at** $x = \pm\xi$ and the **polynomiality** condition. In these cases both the DGLAP and the ERBL regions must cooperate to lead to the required properties, and this implies **nontrivial relations between the wave functions** for the different Fock states relevant in the two regions. An *ad hoc* Ansatz for the wave functions would **almost certainly lead** to GPDs that **violate the above requirements**.

Diehl, Phys. Rept. **388**, 41 (2003)

GPDs from Light Front Wave Functions



For $s > 0$ and $\phi \in [0, 2\pi]$:

$$\mathcal{R}f(s, \phi) = \int_{-\infty}^{+\infty} \mathrm{d}\beta \mathrm{d}\alpha \, f(\beta, \alpha) \delta(s - \beta \cos\phi - \alpha \sin\phi)$$

and:

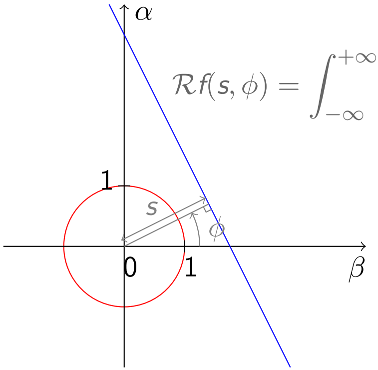
$$\mathcal{R}f(-s, \phi) = \mathcal{R}f(s, \phi \pm \pi)$$

Relation to GPDs:

$$x = \frac{s}{\cos\phi} \text{ and } \xi = \tan\phi$$

Relation between GPD and DD in Belistky *et al.* gauge

$$\frac{\sqrt{1+\xi^2}}{x} H(x, \xi) = \mathcal{R}f_{\mathrm{BMKS}}(s, \phi) \; ,$$



For $s > 0$ and $\phi \in [0, 2\pi]$:

$$\mathcal{R}f(s,\phi) = \int_{-\infty}^{+\infty} \mathrm{d}\beta \mathrm{d}\alpha\, f(\beta,\alpha)\delta(s - \beta\cos\phi - \alpha\sin\phi)$$

and:

$$\mathcal{R}f(-s,\phi) = \mathcal{R}f(s,\phi\pm\pi)$$

Relation to GPDs:

$$x = \frac{s}{\cos\phi} \text{ and } \xi = \tan\phi$$

Relation between GPD and DD in Pobylitsa gauge

$$\frac{\sqrt{1+\xi^2}}{1-x} H(x,\xi) = \mathcal{R}f_{\mathrm{P}}(s,\phi) \ ,$$

- The Mellin moments of a Radon transform are **homogeneous polynomials** in $\omega = (\sin\phi, \cos\phi)$.
- The converse is also true:

Theorem (Hertle, 1983)

Let $g(s, \omega)$ an even compactly-supported distribution. Then g is itself the Radon transform of a compactly-supported distribution if and only if the ***Ludwig-Helgason consistency condition*** *hold:*

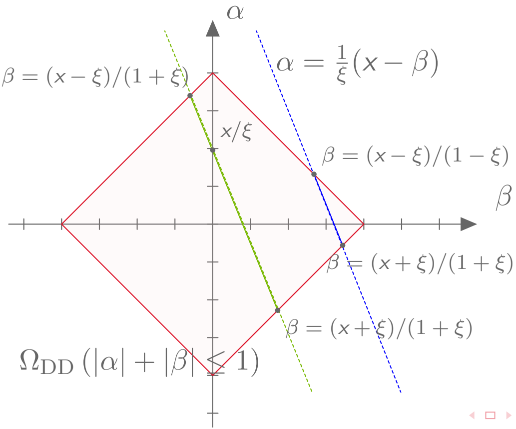
(i) *g is C^∞ in ω,*

(ii) *$\int \mathrm{d}s\, s^m g(s, \omega)$ is a homogeneous polynomial of degree m for all integer $m \geq 0$.*

- Double Distributions and the Radon transform are the **natural solution** of the polynomiality condition.

DGLAP and ERBL regions

$$(x, \xi) \in \text{DGLAP} \quad \Leftrightarrow \quad |s| \geq |\sin\phi| \,,$$
$$(x, \xi) \in \text{ERBL} \quad \Leftrightarrow \quad |s| \leq |\sin\phi| \,.$$



Each point (β, α) with $\beta \neq 0$ contributes to **both** DGLAP and ERBL regions.

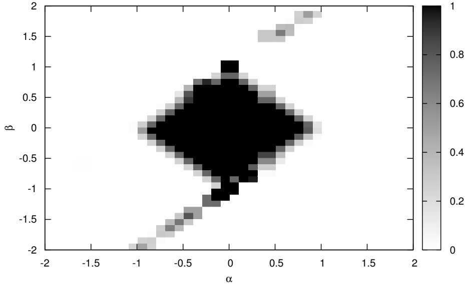
For **any model of LFWF**, one has to address the following three questions:

1. Does the extension exist?
2. If it exists, is it unique?
3. How can we compute this extension?

Ill-posedness in the sense of Hadamard.

A first glimpse at the inverse Radon transform.

- Numerical evaluation ***almost unavoidable*** (polar vs cartesian coordinates).
- Ill-posedness by **lack of continuity**.
- The **unlimited** Radon inverse problem is **midly** ill-posed while the **limited** one is **severely** ill-posed.



Theorem

Let f be a compactly-supported locally summable function defined on $\mathbb{R}^2$ and $\mathcal{R}f$ its Radon transform.
Let $(s_0, \omega_0) \in \mathbb{R} \times S^1$ and U_0 an open neighborhood of ω_0 such that:

$$\textit{for all } s > s_0 \textit{ and } \omega \in U_0 \quad \mathcal{R}f(s, \omega) = 0 \,.$$

Then $f(\aleph) = 0$ on the half-plane $\langle \aleph \,|\, \omega_0 \rangle > s_0$ of $\mathbb{R}^2$.

Consider a GPD H being zero on the DGLAP region.

- Take ϕ_0 and s_0 *s.t.* $\cos\phi_0 \neq 0$ and $|s_0| > |\sin\phi_0|$.
- Neighborhood U_0 of ϕ_0 *s.t.* $\forall \phi \in U_0 \quad |\sin \ \phi| < |s_0|$.
- The underlying DD f has a zero Radon transform for all $\phi \in U_0$ and $s > s_0$ (DGLAP).
- Then $f(\beta, \alpha) = 0$ for all $(\beta, \alpha) \in \Omega_{\mathrm{DD}}$ with $\beta \neq 0$.
- Extension **unique** up to adding a **D-term**: $\delta(\beta)D(\alpha)$.

A discretized problem

Consider $N+1$ Hilbert spaces H, H_1, ..., H_N, and a family of continuous surjective operators $R_n : H \to H_n$ for $1 \leq n \leq N$. Being given $g_1 \in H_1$, ..., $g_n \in H_n$, we search f solving the following system of equations:

$$R_n f = g_n \quad \text{for } 1 \leq n \leq N$$

Fully discrete case

Assume f piecewise-constant with values f_m for $1 \leq m \leq M$. For a collection of lines $(L_n)_{1 \leq n \leq N}$ crossing Ω_{DD}, the Radon transform writes:

$$g_n = \mathcal{R}f = \int_{L_n} f = \sum_{m=1}^{M} f_m \times \mathrm{Measure}(L_n \cap C_m) \quad \text{for } 1 \leq n \leq N$$

Kaczmarz algorithm

Denote P_n the orthogonal projection on the *affine* subspace $R_n f = g_n$. Starting from $f^0 \in H$, the sequence defined iteratively by:

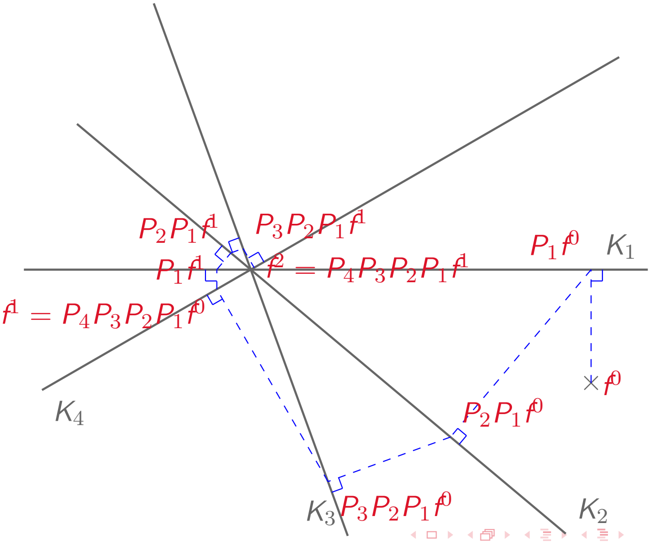
$$f^{k+1} = P_N P_{N-1} \ldots P_1 f^k$$

converges to the solution of the system.
The convergence is exponential if the projections are randomly ordered.

Strohmer and Vershynin, Jour. Four. Analysis and Appl. **15**, 437 (2009)

Computation of the extension.

Numerical evaluation of the inverse Radon transform (2/3).



And if the input data are inconsistent?

- Instead of solving $g = \mathcal{R}f$, find f such that $\|g - \mathcal{R}f\|_2$ is **minimum**.
- The solution **always exists**.
- The input data are **inconsistent** if $\|g - \mathcal{R}f\|_2 > 0$.

Relaxed Kaczmarz algorithm

Let $\omega \in]0, 2[$ and:

$$P_n^\omega = (1 - \omega)\,\mathrm{Id}_H + \omega P_n \quad \text{for } 1 \leq n \leq N$$

Write:

$$RR^\dagger = (R_i R_j^\dagger)_{1 \leq i,j \leq N} = D + L + L^\dagger$$

where D is diagonal, and L is lower-triangular with zeros on the diagonal.

Theorem

Let $0 < \omega < 2$. For $f^0 \in \mathrm{Ran}\ R^\dagger$ (e.g. $f^0 = 0$), the Kaczmarz method with relaxation converges to the unique solution $f^\omega \in \mathrm{Ran}\ R^\dagger$ of:

$$R^\dagger (D + \omega L)^{-1} (g - Rf^\omega) = 0 \ ,$$

where the matrix D and L appear in the decomposition of $RR^\dagger$. If $g = \mathcal{R}f$ has a solution, then f^ω is its solution of minimal norm. Otherwise:

$$f^\omega = f_{MP} + \mathcal{O}(\omega) \ ,$$

where f_{MP} is the minimizer in H of:

$$\langle g - \mathcal{R}f | g - \mathcal{R}f \rangle_D \ ,$$

the inner product being defined by:

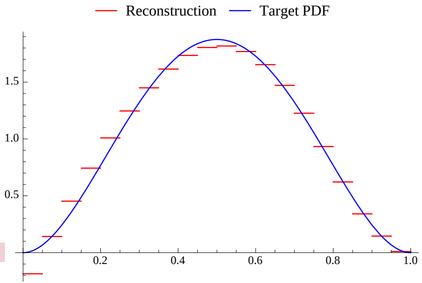
$$\langle h | k \rangle_D = \langle D^{-1} h | k \rangle \ .$$

Test on a 1D example.

Recovering a PDF from the knowledge of its Mellin moments.

A pion valence PDF-like example

Aim: reconstruct the PDF $q(x) = 30x^2(1-x)^2$ from the knowledge of its first 30 Mellin moments.



- Piecewise-constant PDF: 20 values.
- Input: 30 Mellin moments.
- Unrelaxed method $\omega = 1$.
- 10000 iterations.

- Extensive testing *in progress*
  - Various inputs: PDFs and LFWFs.
  - Numerical noise and numerical techniques.

PARTONS Project

PARtonic
Tomography
Of
Nucleon
Software

Computing chain design.

Differential studies: physical models and numerical methods.

Experimental data and phenomenology	Full processes
Computation of amplitudes	Small distance contributions
First principles and fundamental parameters	Large distance contributions

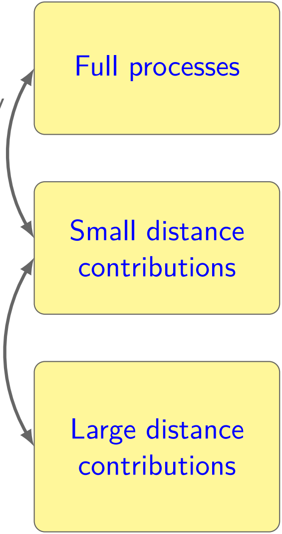
Computing chain design.

Differential studies: physical models and numerical methods.

Experimental data and phenomenology

Computation of amplitudes

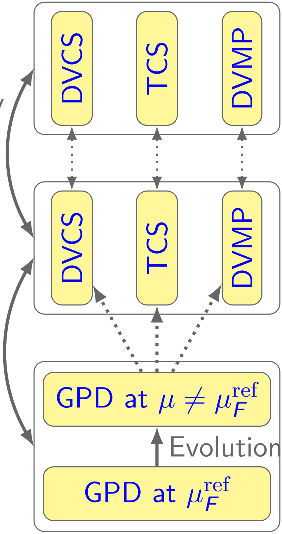
First principles and fundamental parameters

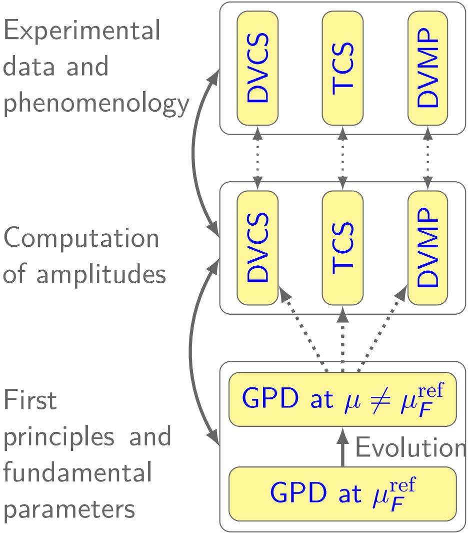


Experimental data and phenomenology

Computation of amplitudes

First principles and fundamental parameters





- Many observables.
- Kinematic reach.

Computing chain design.

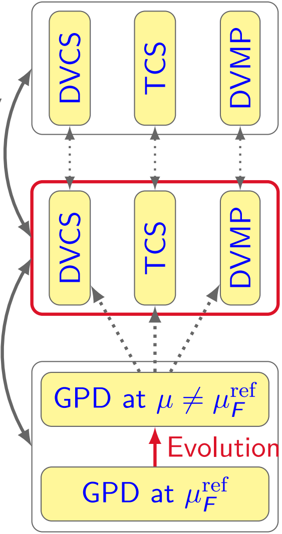
Differential studies: physical models and numerical methods.

Experimental data and phenomenology

Need for modularity

Computation of amplitudes

First principles and fundamental parameters



- Many observables.
- Kinematic reach.

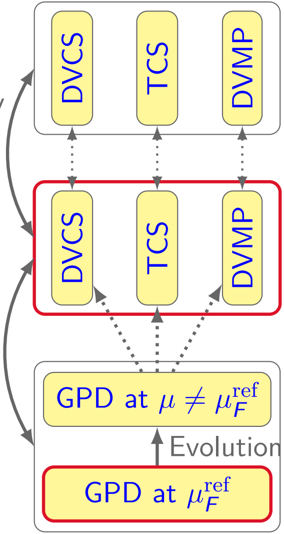
- Perturbative approximations.
- Physical models.
- Fits.
- Numerical methods.
- Accuracy and speed.

Experimental data and phenomenology

Need for modularity

Computation of amplitudes

First principles and fundamental parameters



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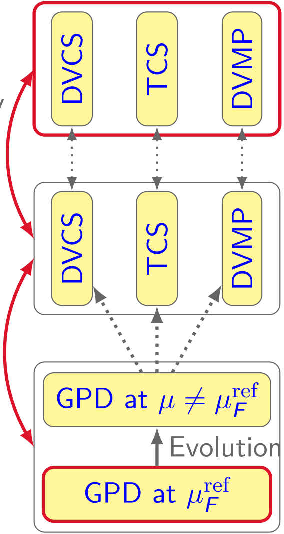

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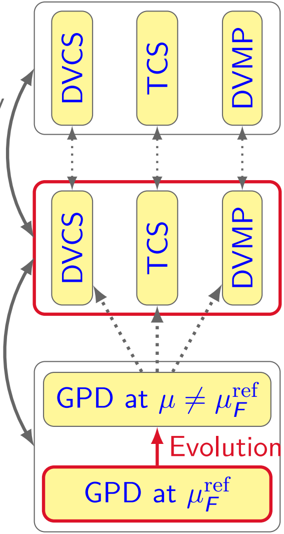

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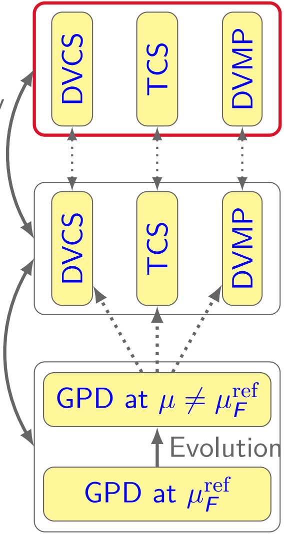


- Many observables.
- Kinematic reach.

- Perturbative approximations.
- Physical models.
- Fits.
- Numerical methods.
- Accuracy and speed.

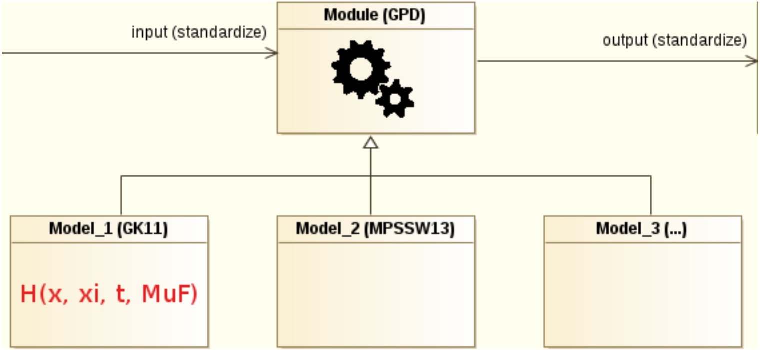
- 3 stages:
  1. Design.
  2. Integration and validation.
  3. Production.
- Flexible software architecture.

  B. Berthou *et al.*, *PARTONS: a computing platform for the phenomenology of Generalized Parton Distributions*

  to appear in Eur. Phys. J. C.
- 1 new physical development = 1 new module.
- *Aggregate* knowledge and know-how. *Do not* reinvent the wheel!
- What *can* be automated *will be* automated.
- Get ready for 12 GeV!

Modularity.

Inheritance, standardized inputs and outputs.



- Steps of logic sequence in parent class.
- Model description and related mathematical methods in daughter class.

gpdExample()

```
// Lots of includes
#include <src/Partons.h>
...

// Retrieve GPD service
GPDService* pGPDService = ServiceObjectRegistry::getGPDService();
// Load GPD module with the BaseModuleFactory
GPDModule* pGK11Model = ModuleObjectFactory::newGPDModule(
    GK11Model::classId);
// Create a GPDKinematic(x, xi, t, MuF, MuR)
GPDKinematic gpdKinematic(0.1, xBToXi(0.001), −0.3, 8., 8.);
// Compute data and store results
GPDResult gpdResult = pGPDService−>
computeGPDModelRestrictedByGPDType(gpdKinematic, pGK11Model,
     GPDType::ALL);
// Print results
std :: cout << gpdResult.toString() << std::endl;

delete pGK11Model;
pGK11Model = 0;
^^I^^I
```

GPD computing automated.

Each line of code corresponds to a physical hypothesis.

computeOneGPD.xml

```xml
<?xml version="1.0" encoding="UTF-8" standalone="yes" ?>
<scenario id="01" date="" description="Example␣:␣computation␣of␣one␣GPD
␣model␣(GK11)␣without␣evolution">
        <!-- Select type of computation -->
        <task service="GPDService" method="computeGPDModel" >
                <!-- Specify kinematics -->
                <GPDKinematic>
                        <param name="x" value="0.1" />
                        <param name="xi" value="1.00050025" />
                        <param name="t" value="-0.3" />
                        <param name="MuF2" value="8" />
                        <param name="MuR2" value="8" />
                </GPDKinematic>
                <!-- Choose GPD model and set parameters -->
                <GPDModule>
                        <param name="id" value="GK11Model" />
                </GPDModule>
        </task>
</scenario>
^^I^^I
```

GPD computing automated.

Each line of code corresponds to a physical hypothesis.

computeOneGPD.xml

```
<?xml version="1.0" encoding="UTF-8" stand
<scenario id="01" date="" description="Exam
␣model␣(GK11)␣without␣evolution">
        <!-- Select type of computation -->
        <task service="GPDService" method=
                <!-- Specify kinematics -->
                <GPDKinematic>
                        <param name="x" valu
                        <param name="xi" va
                        <param name="t" valu
                        <param name="MuF2"
                        <param name="MuR2"
                </GPDKinematic>
                <!-- Choose GPD model and
                <GPDModule>
                        <param name="id" va
                </GPDModule>
        </task>
</scenario>
^^I^^I
```

$H^u = 0.822557$
$H^{u(+)} = 0.165636$
$H^{u(-)} = 1.47948$

$H^d = 0.421431$
$H^{d(+)} = 0.0805182$
$H^{d(-)} = 0.762344$

$H^s = 0.00883408$
$H^{s(+)} = 0.0176682$
$H^{s(-)} = 0$

$H^g = 0.385611$
and E, $\tilde{H}$, $\tilde{E}$, ...

CFF computing automated.

Each line of code corresponds to a physical hypothesis.

computeOneCFF.xml

```xml
<?xml version="1.0" encoding="UTF-8" standalone="yes" ?>
<scenario id="03" date="" description="Example␣:␣computation␣of␣one␣convol␣coeff␣function␣model␣(DVCSCFF)␣with␣GPD␣model␣(GK11)">
        <task service="DVCSConvolCoeffFunctionService" method="computeWithGPDModel">
                <DVCSConvolCoeffFunctionKinematic>
                        <param name="xi" value="0.5" />
                        <param name="t" value="-0.1346" />
                        <param name="Q2" value="1.5557" />
                        <param name="MuF2" value="4" />
                        <param name="MuR2" value="4" />
                </DVCSConvolCoeffFunctionKinematic>
                <GPDModule>
                        <param name="id" value="GK11Model" />
                </GPDModule>
                <DVCSConvolCoeffFunctionModule>
                        <param name="id" value="DVCSCFFModel" />
                        <param name="qcd_order_type" value="LO" />
                </DVCSConvolCoeffFunctionModule>
        </task>
</scenario>
```

CFF computing automated.

Each line of code corresponds to a physical hypothesis.

computeOneCFF.xml

```
<?xml version="1.0" encoding="UTF-8" standalone="yes" ?>
<scenario id="03" date="" description="Example␣:␣computation␣of␣one␣convol␣coeff␣function␣model␣(DVCSCFF)␣with␣GPD␣model␣(GK11)">
        <task service="DVCSConvolCoeffFunctionService" method="computeWithGPDModel">
                <DVCSConvolCoeffFunctionKinematic>
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                        <param name="t" value="-0.1346" />
                        <param name="Q2" value="1.5557" />
                        <param name="MuF2" value="4" />
                        <param name="MuR2" value="4" />
                </DVCSConvolCoeffFunctionKinematic>
                <GPDModule>
                        <param name="id" value="GK11Model" />
                </GPDModule>
                <DVCSConvolC
                        <para                                            >
                        <para                                             />
                </DVCSConvol
        </task>
</scenario>
```

$$\mathcal{H} = 1.47722 + 1.76698\ i$$
$$\mathcal{E} = 0.12279 + 0.512312\ i$$
$$\widetilde{\mathcal{H}} = 1.54911 + 0.953728\ i$$
$$\widetilde{\mathcal{E}} = 18.8776 + 3.75275\ i$$

Members and areas of expertise

Network of developers, upstream contributors and users

Multidisciplinary development team

Berthou	Binosi	Chouika	Guidal	Mezrag	Moutarde	Sabatié	Sznajder	Wagner
(Irfu)	(ECT*)	(Irfu)	(IPNO)	(ANL)	(Irfu)	(Irfu)	(IPNO)	(NCBJ)

IPN et LPT (Orsay), Irfu (Saclay) and CPhT (Polytechnique)

AGENCE NATIONALE DE LA RECHERCHE ANR

Experimental data analysis	Perturbative QCD
World data fits	GPD modeling

Conclusion

- Last decade demonstrated **maturity of GPD phenomenology**.
- **Challenging constraints** expected from Jefferson Lab in the valence region.
- **Systematic** procedure to construct GPD models from any "reasonable" Ansatz of LFWFs.
- **Characterization** of the **existence** and **uniqueness** of the extension from the DGLAP to the ERBL region.
- Development of the platform PARTONS for **phenemonology** and **theory** purposes.
- Numerical tests *in progress*.

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