Three-loop QCD corrections from massive quarks to deep-inelastic structure functions

J. Ablinger[1], **A. Behring**[2], J. Blümlein[2], A. De Freitas[2], A. Hasselhuhn[3], A. von Manteuffel[4], M. Round[1], C. Schneider[1], F. Wißbrock[5]

[1] Johannes Kepler University, Linz
[2] DESY, Zeuthen
[3] KIT, Karlsruhe
[4] J. Gutenberg University, Mainz
[5] IHES, Bures-Sur-Yvette

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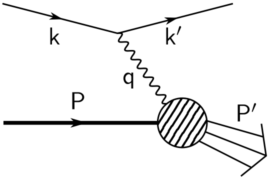
Outline

- Introduction
- Calculation of operator matrix elements
- Selected results
- Conclusion

Based on the following publications:

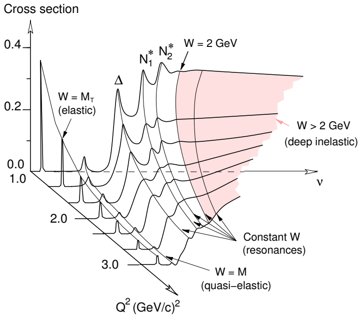
- A. Behring, I. Bierenbaum, J. Blümlein, A. De Freitas, S. Klein and F. Wißbrock, Eur.Phys.J. **C74** (2014), 3033
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider, F. Wißbrock, Nucl. Phys. **B886** (2014) 733
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider, Nucl. Phys. **B890** (2014) 48
- A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider, Nucl. Phys. **B897** (2015) 612
- A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, C. Schneider, Phys.Rev. **D92** (2015), 114005
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider, Comput. Phys. Commun. **202** (2016) 33

Deep-inelastic scattering (DIS)



$$Q^2 = -q^2 \qquad x = \frac{-q^2}{2P.q}$$

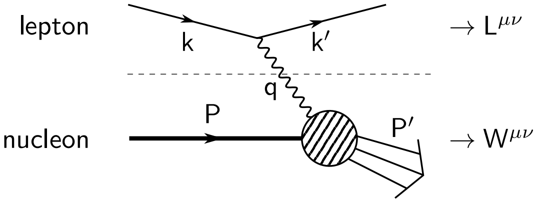
$$W^2 = P'^2 \qquad \nu = \frac{P.q}{M}$$



[Used with kind permission from Xiaochao Zheng]

- Lepton-nucleon scattering: measure scattered electron
- Deep-inelastic scattering: Sufficiently large W^2 and Q^2.
- Compare: experimental measurements $\leftrightarrow$ theoretical predictions
  - $\Rightarrow$ Extraction of important quantities: parton distributions (PDFs), strong coupling constant (α_s), mass of heavy quarks, . . .

Theoretical description



Cross section: $$\frac{d\sigma}{dx\, dQ^2} = \frac{2\pi\alpha^2}{xQ^4} L_{\mu\nu} W^{\mu\nu}$$

Leptonic tensor: $$L_{\mu\nu} = 2k_\mu k'_\nu + 2k_\nu k'_\mu - g_{\mu\nu} Q^2$$

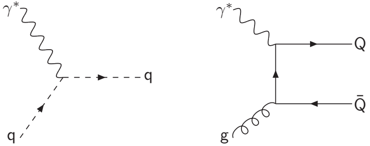
Hadronic tensor: $$W_{\mu\nu} = (\dots)_{\mu\nu} F_L(x, Q^2) + (\dots)_{\mu\nu} F_2(x, Q^2)$$

Structure functions receive contributions from

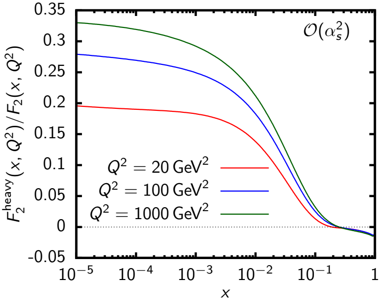
- Gluons and light quarks (up, down, strange; $m^2 < \Lambda^2_{\mathrm{QCD}}$)
- Heavy quarks (charm, bottom; $m^2 \gg \Lambda^2_{\mathrm{QCD}}$)

Heavy quarks in deep-inelastic scattering

- Structure functions: Contributions from light and heavy quarks



- Contributions from heavy quarks start at $\mathcal{O}(\alpha_s)$
- Gluon-photon fusion
  $\Rightarrow$ Sensitive to gluon PDF
- Heavy quarks yield significant contributions to structure functions
  (e.g. $20 - 30\%$ to $F_2(x, Q^2)$), esp. at low x



[Behring et al. '14]

Motivation for 3-loop calculations

- Comparison of theory and experiment requires sufficiently precise theory predictions
- Precision of experimental data for DIS: $\sim 1\%$ for $F_2(x, Q^2)$
  $\rightarrow$ requires theoretical description at $\mathcal{O}(\alpha_s^3)$

$\Rightarrow$ NNLO contributions of heavy quarks are important for the precise measurement of the strong coupling constant [d'Enterria et al. '15]

$$\delta\alpha_s(M_Z) \simeq 1\%$$

and the masses of the heavy quarks [Alekhin et al. '12 and updates]

$$m_c(m_c) = 1.25 \pm 0.02(\text{exp})^{+0.03}_{-0.02}(\text{scale})^{+0.00}_{-0.07}(\text{thy})\text{GeV}$$

$$m_b(m_b) = 3.91 \pm 0.14(\text{exp})^{+0.00}_{-0.11}(\text{thy})\text{GeV} \quad \text{(preliminary)}$$

($\overline{\text{MS}}$ scheme)

Theoretical description of heavy quarks

Hadronic tensor: $W_{\mu\nu} = (\dots)_{\mu\nu}\, F_L(x, Q^2) + (\dots)_{\mu\nu}\, F_2(x, Q^2)$

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Wilson coefficients (perturbative) — PDFs (non-perturbative)

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Wilson coefficients (perturbative) — PDFs (non-perturbative)

x- and N-space are connected by a Mellin transformation:

$$M[f(x)](N) = \int_0^1 \mathrm{d}x\, x^{N-1} f(x)$$

Representation simplifies in Mellin space.

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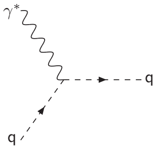
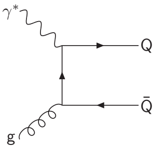
Wilson coefficients: $\mathbb{C}_{2,j}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = C_{2,j}\left(N,\frac{Q^2}{\mu^2}\right) + H_{2,j}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)$

massless Wilson coefficients
NNLO: [Moch, Vermaseren, Vogt '05]

heavy-flavor Wilson coefficients





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For F_2 and $Q^2/m^2 \gtrsim 10$ the heavy flavor Wilson coefficients factorise:
[Buza, Matiounine, Smith, Migneron, van Neerven '96]

Heavy flavor Wilson coefficients: $\boxed{H_{2,j}(N) = \sum_i A_{ij}(N) C_{2,i}(N)}$

massive operator matrix elements (OMEs)

LO: [Witten '76; Babcock, Sievers '78; Shifman, Vainshtein, Zakharov '78; Leveille, Weiler '79; Glück, Reya '79; Glück, Hoffmann, Reya '82]
NLO: [Laenen, van Neerven, Riemersma, Smith '93; Buza, Matiounine, Smith, Migneron, van Neerven '96; Bierenbaum, Blümlein, Klein '07a, '07b, '08, '09a]

massless Wilson coefficients

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Heavy flavor Wilson coefficients: $H_{2,j}(N) = \sum_i A_{ij}(N) C_{2,i}(N)$

OMEs A_{ij} also essential to define the variable flavor number scheme
→ describe transition $N_F \to N_F + 1$ massless quarks
→ transitions relevant for the PDFs at the LHC

Massive operator matrix elements (OME)

Definition of the operator matrix elements

massless, on-shell parton states

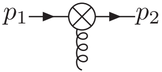
$$A_{ij} := \langle j | O_i | j \rangle$$

Local operators from the light-cone expansion

e.g. $O^{\mathrm{NS}}_{q,a;\mu_1,\ldots,\mu_N} = i^{N-1} S[\overline{\Psi}\gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_N} \frac{\lambda^a}{2}\Psi]$ − trace terms

Feynman rules for operators

$\propto (\Delta.p)^{N-1}$



$\propto \sum_{j=0}^{N-2} (\Delta.p_1)^j (\Delta.p_2)^{N-2-j}$

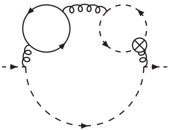
⋮

Depend on integer variable N (Mellin variable)

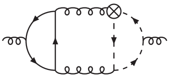
Massive operator matrix elements at NNLO

Fixed moments for OMEs: $N = 2 \ldots 10(14)$ ✓ [Bierenbaum, Blümlein, Klein, '09b]

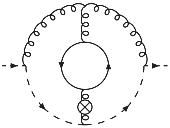
All logarithmic terms from renormalisation ✓ [Behring et al. '14]



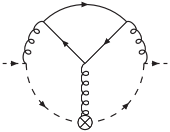
$A_{qq,Q}^{\mathsf{PS}}$
8 diagrams
✓ [Ablinger et al. '10]



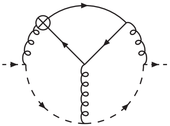
$A_{qg,Q}$
132 diagrams
✓ [Ablinger et al. '10]



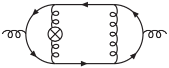
$A_{gq,Q}$
89 diagrams
✓ [Ablinger et al. '14a]



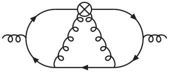
$A_{qq,Q}^{\mathsf{NS}}$ & $A_{qq,Q}^{\mathsf{TR}}$
112 diagrams
✓ [Ablinger et al. '14b]



A_{Qq}^{PS}
125 diagrams
✓ [Ablinger et al. '14c]



$A_{gg,Q}$
642 diagrams
✓

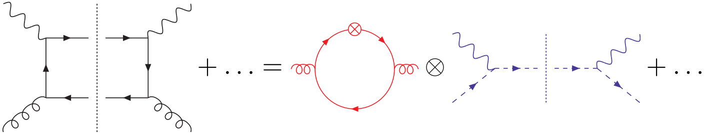


A_{Qg}
1233 diagrams
in progress
(1003 diags. done)

Feynman diagrams drawn using `axodraw` [Vermaseren '94]

Factorisation of Wilson coefficients for $Q^2 \gg m^2$

Factorisation into massive OMEs and massless Wilson coefficients



Example: [Buza, Matiounine, Smith, Migneron, van Neerven '96] [Bierenbaum, Blümlein, Klein, '09b]

$$
\begin{aligned}
H_{q,2}^{\mathrm{PS}}(N_F+1) = a_s^2 \Bigg[& A_{Qq}^{\mathrm{PS},(2)}(N_F+1) + \frac{C_{q,2}^{\mathrm{PS},(2)}(N_F+1)}{N_F+1} \Bigg] \\
+ a_s^3 \Bigg[& A_{Qq}^{\mathrm{PS},(3)}(N_F+1) + \frac{C_{q,2}^{\mathrm{PS},(3)}(N_F+1)}{N_F+1} \\
& + A_{gq,Q}^{(2)}(N_F+1)\frac{C_{g,2}^{(1)}(N_F+1)}{N_F+1} \\
& + A_{Qq}^{\mathrm{PS},(2)}(N_F+1)\, C_{q,2}^{\mathrm{NS},(1)}(N_F+1) \Bigg]
\end{aligned}
$$

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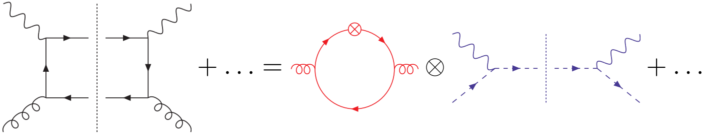
Factorisation into massive OMEs and massless Wilson coefficients



Status of heavy flavour Wilson coefficients at NNLO

$L_{q,2}^{\mathrm{PS}}$ $(\propto A_{qq,Q}^{\mathrm{PS},(3)})$	✓	[Ablinger et al. '10] [Behring et al. '14]
$L_{g,2}^{\mathrm{S}}$ $(\propto A_{qg,Q}^{(3)})$	✓	[Ablinger et al. '10] [Behring et al. '14]
$L_{q,2}^{\mathrm{NS}}$ $(\propto A_{qq,Q}^{\mathrm{NS},(3)})$	✓	[Ablinger et al. '14b]
$H_{q,2}^{\mathrm{PS}}$ $(\propto A_{Qq}^{\mathrm{PS},(3)})$	✓	[Ablinger et al. '14c]
$H_{g,2}^{\mathrm{S}}$ $(\propto A_{Qg}^{(3)})$		in progres

Variable flavour number scheme (VFNS)

- Transition from scheme with N_F massless and 1 massive flavour to scheme with N_F+1 effectively massless flavours
- Massive OMEs appear in the matching conditions of the PDFs

NNLO matching relations:
[Buza et al. '96] [Bierenbaum, Blümlein, Klein, '09a, '09b]

$$f_k(N_F+1,\mu^2)+\bar{f}_k(N_F+1,\mu^2)=A_{qq,Q}^{\mathrm{NS}}\otimes[f_k(N_F,\mu^2)+\bar{f}_k(N_F,\mu^2)] + \frac{1}{N_F}\left[A_{qq,Q}^{\mathrm{PS}}\otimes\Sigma(N_F,\mu^2)+A_{qg,Q}\otimes G(N_F,\mu^2)\right]$$

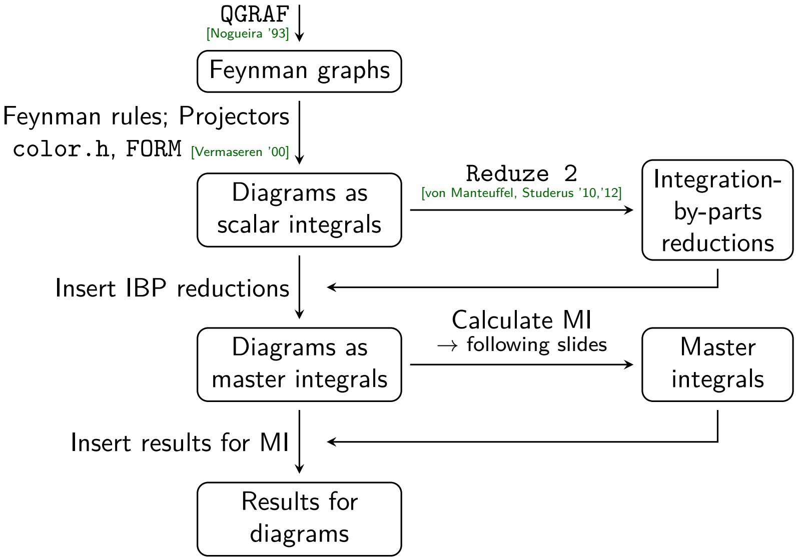
$$f_{Q+\bar{Q}}(N_F+1,\mu^2)=A_{Qq}^{\mathrm{PS}}\otimes\Sigma(N_F,\mu^2)+A_{Qg}\otimes G(N_F,\mu^2)$$

$$G(N_F+1,\mu^2)=A_{gq,Q}\otimes\Sigma(N_F,\mu^2)+A_{gg,Q}\otimes G(N_f,\mu^2)$$

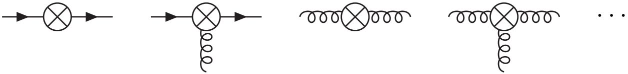
$$\Sigma(N_F+1,\mu^2)=\left[A_{qq,Q}^{\mathrm{NS}}+A_{qq,Q}^{\mathrm{PS}}+A_{Qq}^{\mathrm{PS}}\right]\otimes\Sigma(N_F,\mu^2) + \left[A_{qg,Q}+A_{Qg}\right]\otimes G(N_F,\mu^2)$$

with the singlet combination $\Sigma(N_F,\mu^2)=\sum_{k=1}^{N_F}[f_k(N_F,\mu^2)+f_k(N_F,\mu^2)]$

Outline of the calculation



Dealing with operator insertions



- Large number of scalar integrals ($\sim 10^5$) requires using integration-by-parts reductions to master integrals (474)
- Problem: Operators prevent straightforward application of Laporta's algorithm (N in exponents of scalar products)
- Solution: Introduce generating functions for operators

$$\sum_{N=0}^{\infty} t^N (\Delta.k)^N = \frac{1}{1 - t(\Delta.k)}$$

and similar expressions for more complex operators

$\Rightarrow$ treat them as linear propagators

- Allows to use `Reduze 2` to obtain IBP reductions
- Additional advantage: Allows to derive differential equations in t
- Result in N is recovered as Nth coefficient of expansion in t at the end of the calculation

Calculation of master integrals

Master integrals are calculated using a range of techniques:

- Hypergeometric function techniques
- Mellin-Barnes representations

⇒ Yields multi-sum representations

⇒ Simplify using summation algorithms based on $\Sigma\Pi$ fields/rings implemented in `Sigma` [Schneider '01-], `EvaluateMultiSums` and `SumProduction` [Ablinger, Blümlein, Hasselhuhn, Schneider'10-] and special function tools from `HarmonicSums` [Ablinger, Blümlein, Schneider '10,'13]

Moreover, we use

- Coupled systems of differential equations/difference equations [Ablinger et al. '15]
  `SolveCoupledSystem`
- Almkvist-Zeilberger algorithm [Almkvist, Zeilberger '90; Apagodu, Zeilberger '06]
  $\rightarrow$ `MultiIntegrate` [Ablinger '12]

⇒ Yields scalar recurrences for the integrals

⇒ Solve using the packages listed above

Nested sums and iterated integrals

Results require mathematical objects of increasing complexity:

$A_{qq,Q}^{\mathrm{PS}}$, $A_{qg,Q}$, $A_{qq,Q}^{\mathrm{NS}}$, $A_{gq,Q}$

Harmonic sums
[Vermaseren '98] [Blümlein, Kurth '98]

$$\sum_{i=1}^{N} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}$$

HPLs
[Remiddi, Vermaseren '99]

$$\int_0^x \frac{\mathrm{d}y}{y} \int_0^y \frac{\mathrm{d}z}{1-z}$$

A_{Qq}^{PS}

Generalised harmonic sums
[Moch, Uwer, Weinzierl '01]
[Ablinger, Blümlein, Schneider '13]

$$\sum_{i=1}^{N} \frac{2^{-i}}{i^2} \sum_{j=1}^{i} \frac{2^j}{j}$$

(Here:) HPLs at $1-2x$

$$\int_0^{1-2x} \frac{\mathrm{d}y}{y} \int_0^y \frac{\mathrm{d}z}{1-z}$$

$A_{gg,Q}$, A_{Qg} (so far)

Cyclotomic & binomial sums
[Ablinger, Blümlein, Schneider '11]
[Ablinger, Blümlein, Raab, Schneider '14]

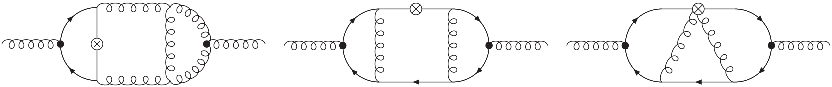
$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \binom{2j}{j} \frac{(-1)^j}{j^3}}{\binom{2i}{i}(2i+1)}$$

Cyclotomic HPLs
[Ablinger, Blümlein, Schneider '11]

& iterated integrals over root-valued letters
[Ablinger, Blümlein, Raab, Schneider '14]

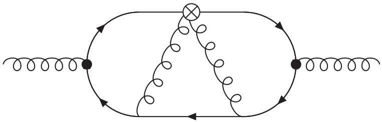
$$\int_0^x \frac{\mathrm{d}y}{y\sqrt{y+\frac{1}{4}}} \int_0^y \frac{\mathrm{d}z}{z} \int_0^z \frac{\mathrm{d}w}{w}$$

$A_{Qg}^{(3)}$: Ladder- and V-diagrams



- Ladder topologies enter $A_{Qg}^{(3)}$, $A_{gg,Q}^{(3)}$ and $A_{qg,Q}^{(3)}$
- Scalar prototypes (without numerators) of $A_{Qg}^{(3)}$ ladder diagrams were calculated in 2012 [Ablinger et al. '12]
- Here: Calculation of 12 physical $A_{Qg}^{(3)}$ diagrams (with numerators) via master integrals [Ablinger et al. '15]
- Benefits beyond the diagrams discussed here: Methods developed for this allow to calculate all $A_{gg,Q}^{(3)}$ and many $A_{Qg}^{(3)}$ diagrams

Example: V diagram 12



- Particularly difficult diagram: V topology with 5 massive lines
- Operator introduces “non-planarity” into planar diagram
- Reduction using integration-by-parts identities requires 92 master integrals
- Calculation of master integral is very demanding ($\mathcal{O}$(*weeks*) of computation time)
- Required new computer algebra tools for the solution of coupled systems of difference equations and handling of binomially weighted sums [Ablinger et al. '15]

Example: V diagram 12 – Result

[Ablinger et al. '15]

$$D_{12,b} = T_F \left(\frac{C_A}{2} - C_F\right)(C_A - C_F)\Bigg\{\frac{1}{\varepsilon^3}\Bigg[-\frac{128\big(N^2+N+1\big)}{3N(N+1)^2(N+2)} + \frac{128(N+3)}{3(N+1)^2(N+2)}S_1$$

$$- \frac{64}{3(N+1)(N+2)}\left[3S_2 + S_1^2 + 4S_{-2}\right] + (-1)^N \frac{128}{3N(N+1)^2(N+2)}\Bigg]$$

$$+ \frac{1}{\varepsilon^2}\Bigg[-\frac{64P_{324}}{3N(N+1)^3(N+2)^2} - \frac{32(2N+1)\big(4N^3+10N^2+17N+20\big)}{3N(N+1)^2(N+2)^2}S_2 + \dots\Bigg]$$

$$+ \frac{1}{\varepsilon}\Bigg[+\frac{128\big(N^2-5N+2\big)}{3N(N+1)(N+2)}S_{-2,1,1} + \frac{16\big(10N^3+62N^2+111N+60\big)}{9N(N+1)^2(N+2)^2}S_1^3 + \dots\Bigg]$$

$$- \frac{P_{328}}{N^3(N+1)(N+2)(2N+1)(2N+3)\binom{2N}{N}}\Bigg[16\left(\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}\right)\zeta_3$$

$$+ 16\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},1,i_1\right) + 48\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},-1,i_1\right)\Bigg]$$

$$+ \frac{N^2+4N+2}{N(N+1)(N+2)}\Bigg[-192\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}\left(\sum_{i_2=1}^{i_1}\frac{1}{\binom{2i_2}{i_2}i_2^2}\right)S_{1,2}\left(\frac{1}{2},1,i_1\right)$$

$$- 576\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}\left(\sum_{i_2=1}^{i_1}\frac{1}{\binom{2i_2}{i_2}i_2^2}\right)S_{1,2}\left(\frac{1}{2},-1,i_1\right) - 32\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}}{i_2^3}}{\binom{2i_1}{i_1}(1+i_1)}$$

$$+ \left(192\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},1,i_1\right) + 576\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},-1,i_1\right)\right)$$

$$\times \sum_{i_1=1}^{N}\frac{1}{\binom{2i_1}{i_1}i_1^2} - 64\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\binom{2i_2}{i_2}\frac{S_1(i_2)}{i_2^2}}{\binom{2i_1}{i_1}(1+i_1)} + 64\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}S_2(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+i_1)}$$

$$+ 96\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_2(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+i_1)} + 96\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_{-2}(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+i_1)} + 96\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_{1,1}(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+i_1)}$$

$$+ 192\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}S_{-2}(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+i_1)} - 64S_{2,1,2}\left(-2,\frac{1}{2},1\right) - 192S_{2,1,2}\left(-2,\frac{1}{2},-1\right)$$

$$+ \left(192\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}(-2)^{i_2}\binom{2i_2}{i_2}}{\binom{2i_1}{i_1}i_1^2} - 256S_2(-2)\right)\zeta_3\Bigg]$$

$$+ \frac{\big(3N^2+16\big)}{N(N+1)(N+2)}\Bigg[\frac{64}{3}\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}}{i_2^3}}{\binom{2i_1}{i_1}(1+2i_1)} - 128\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}S_{-2}(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+2i_1)}$$

$$+ \frac{128}{3}\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_1(i_2)}{i_2^2}}{\binom{2i_1}{i_1}(1+2i_1)} - 64\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_2(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+2i_1)} - 64\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_{-2}(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+2i_1)}$$

$$- \frac{128}{3}\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}S_2(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+2i_1)} - 64\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_{1,1}(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+2i_1)}\Bigg]$$

$$+ \frac{6N-5}{N(N+1)(N+2)}\Bigg[\Bigg(-256\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},1,i_1\right)$$

$$- 768\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},-1,i_1\right)\Bigg)\sum_{i_1=1}^{N}\frac{1}{\binom{2i_1}{i_1}i_1}$$

$$+ 768\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}\left(\sum_{i_2=1}^{i_1}\frac{1}{\binom{2i_2}{i_2}i_2}\right)S_{1,2}\left(\frac{1}{2},-1,i_1\right)$$

$$+ 256\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}\left(\sum_{i_2=1}^{i_1}\frac{1}{\binom{2i_2}{i_2}i_2}\right)S_{1,2}\left(\frac{1}{2},1,i_1\right)$$

$$+ \left(256S_1(-2) - 256\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}(-2)^{i_2}\binom{2i_2}{i_2}}{\binom{2i_1}{i_1}i_1}\right)\zeta_3\Bigg]$$

$$+ \frac{P_{343}}{N^3(N+1)^2(N+2)(2N+1)(2N+3)\binom{2N}{N}}\Bigg[\frac{32}{3}\sum_{i_1=1}^{N}\frac{(-1)^{i_1}\binom{2i_1}{i_1}S_2(i_1)}{i_1}$$

$$+ 32\sum_{i_1=1}^{N}\frac{(-1)^{i_1}\binom{2i_1}{i_1}S_{-2}(i_1)}{i_1} - \frac{16}{3}\sum_{i_1=1}^{N}\frac{(-1)^{i_1}\binom{2i_1}{i_1}}{i_1^3} - \frac{32}{3}\sum_{i_1=1}^{N}\frac{\binom{2i_1}{i_1}S_1(i_1)}{i_1^2}$$

$$+ 16\sum_{i_1=1}^{N}\frac{\binom{2i_1}{i_1}S_2(i_1)}{i_1} + 16\sum_{i_1=1}^{N}\frac{\binom{2i_1}{i_1}S_{-2}(i_1)}{i_1} + 16\sum_{i_1=1}^{N}\frac{\binom{2i_1}{i_1}S_{1,1}(i_1)}{i_1}\Bigg]$$

$$+ \frac{64\big(5N^2-N+10\big)}{3N(N+1)(N+2)}S_{-2,2,1} + \frac{2\big(34N^3+34N^2-119N-108\big)}{9N(N+1)^2(N+2)^2}S_1^4$$

$$- \frac{64\big(5N^2+19N+10\big)}{3N(N+1)(N+2)}S_{-4,1} - \frac{64\big(25N^3+4N^2+58N+20\big)}{3N^2(N+1)(N+2)}S_{-2,1,1}$$

$$+ \frac{32\big(5N^2+57N+10\big)}{3N(N+1)(N+2)}S_{2,-3} - \frac{64\big(7N^2-11N+14\big)}{3N(N+1)(N+2)}S_{-2,1,1,1}$$

$$+ \frac{64\big(9N^2-7N+18\big)}{3N(N+1)(N+2)}S_{2,1,-2} + \frac{32\big(19N^3-20N^2+62N+28\big)}{3N^2(N+1)(N+2)}S_{-3,1}$$

$$+ \frac{64\big(9N^2+4N+18\big)}{3N(N+1)(N+2)}S_{-2,3} + \frac{64\big(11N^3-4N^2+30N+12\big)}{3N^2(N+1)(N+2)}S_{-2,2}$$

$$+ \frac{32\big(13N^2+7N+26\big)}{3N(N+1)(N+2)}S_{-3,1,1} - \frac{64\big(13N^2+30N+26\big)}{3N(N+1)(N+2)}S_{-5}$$

$$+ \frac{128\big(15N^2+10N-6\big)}{3N(N+1)(N+2)}S_{-2,1,-2} - \frac{32\big(15N^2+412N+530\big)}{15N(N+1)(N+2)}S_5$$

$$+ \frac{64\big(16N^2+43N+16\big)}{3N(N+1)(N+2)}S_{2,2,1} + \frac{128\big(21N^2+58N+18\big)}{3N(N+1)(N+2)}S_{-2,-3} + \dots$$

Example: V diagram 12 – Result

[Ablinger et al. '15]

$$D_{12,b} = T_F\left(\frac{C_A}{2} - C_F\right)(C_A - C_F)\Bigg\{\frac{1}{\varepsilon^3}\left[-\frac{128(N^2+N+1)}{3N(N+1)^2(N+2)} + \frac{128(N+3)}{3(N+1)^2(N+2)}S_1 - \frac{64}{3(N+1)(N+2)}\left[3S_2 + S_1^2 + 4S_{-2}\right] + (-1)^N\frac{128}{3N(N+1)^2(N+2)}\right]$$

$$+\frac{1}{\varepsilon^2}\left[-\frac{64P_{324}}{3N(N+1)^3(N+2)^2} - \frac{32(2N+\ldots}{\ldots} \ldots\right]$$

$$+\frac{1}{\varepsilon}\left[+\frac{128(N^2-5N+2)}{3N(N+1)(N+2)}S_{-2,1,1} + \frac{16(10\ldots}{9\ldots} \ldots\right]$$

$$-\frac{P_{328}}{N^3(N+1)(N+2)(2N+1)(2N+3)\ldots} \ldots$$

$$+16\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},1,i_1\right) + 48\sum \ldots$$

$$+\frac{N^2+4N+2}{N(N+1)(N+2)}\left[-192\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}\left(\sum_{i_2=1}^{i_1}\frac{1}{\binom{2i_2}{i_2}i_2^2}\right)S_{1,2}\left(\frac{1}{2},1,i_1\right) \ldots\right.$$

$$+\frac{(3N^2+16)}{N(N+1)(N+2)}\left[\frac{64}{3}\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}}{i_2^3}}{\binom{2i_1}{i_1}(1+2i_1)} - 128\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}S_{-2}(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+2i_1)}\right.$$

$$+\frac{128}{3}\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_1(i_2)}{i_2^2}}{\binom{2i_1}{i_1}(1+2i_1)} - 64\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_2(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+2i_1)} - 64\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_{-2}(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+2i_1)}$$

$$-\frac{128}{3}\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{(-1)^{i_2}\binom{2i_2}{i_2}S_2(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+2i_1)} - 64\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_{1,1}(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+2i_1)}$$

$$+\frac{6N-5}{N(N+1)(N+2)}\left[\left(-256\sum_{i_1=1}^{N}(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},1,i_1\right) \ldots\right.\right.$$

$$\ldots(-2)^{i_1}\binom{2i_1}{i_1}S_{1,2}\left(\frac{1}{2},-1,i_1\right)\Bigg)\sum_{i_1=1}^{N}\frac{1}{\binom{2i_1}{i_1}i_1}$$

$$\ldots(-2)^{i_1}\binom{2i_1}{i_1}\left(\sum_{i_2=1}^{i_1}\frac{1}{\binom{2i_2}{i_2}i_2}\right)S_{1,2}\left(\frac{1}{2},-1,i_1\right)$$

$$\ldots(-2)^{i_1}\binom{2i_1}{i_1}\left(\sum_{i_2=1}^{i_1}\frac{1}{\binom{2i_2}{i_2}i_2}\right)S_{1,2}\left(\frac{1}{2},1,i_1\right)$$

$$+\left(256S_1(-2) - 256\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}(-2)^{i_2}\binom{2i_2}{i_2}}{\binom{2i_1}{i_1}i_1}\right)\zeta_3\Bigg]$$

$$+\frac{P_{343}}{N^3(N+1)^2(N+2)(2N+1)(2N+3)\binom{2N}{N}}\left[\frac{32}{3}\sum_{i_1=1}^{N}\frac{(-1)^{i_1}\binom{2i_1}{i_1}S_2(i_1)}{i_1} \ldots\right.$$

$$+\frac{32(13N^2+7N+26)}{3N(N+1)(N+2)}S_{-3,1,1} - \frac{64(13N^2+30N+26)}{3N(N+1)(N+2)}S_{-5}$$

$$+\frac{128(15N^2+10N-6)}{3N(N+1)(N+2)}S_{-2,1,-2} - \frac{32(15N^2+412N+530)}{15N(N+1)(N+2)}S_5$$

$$+\frac{64(16N^2+43N+16)}{3N(N+1)(N+2)}S_{2,2,1} + \frac{128(21N^2+58N+18)}{3N(N+1)(N+2)}S_{-2,-3} + \ldots$$

$$\sum_{i_1=1}^{N}\frac{\sum_{i_2=1}^{i_1}\frac{\binom{2i_2}{i_2}S_{1,1}(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+2i_1)}$$

Binomially weighted sums

- Nested sums with binomial coefficients $\binom{2i}{i}$ as weights
- Translate to iterated integrals over root-valued letters in x-space

Anomalous dimensions

- Renormalisation of the OMEs [Bierenbaum, Blümlein, Klein, '09b]
  involves the NNLO anomalous dimensions [Moch, Vermaseren, Vogt '04a, '04b]
  Example: $(\hat{\gamma}_{ij} = \gamma_{ij}(N_F + 1) - \gamma_{ij}(N_F))$

$$\hat{\hat{A}}_{qq,Q}^{\mathrm{NS},(3)} = \frac{1}{\varepsilon^3} \cdots + \frac{1}{\varepsilon^2} \cdots + \frac{1}{\varepsilon} \Bigg[\frac{\hat{\gamma}_{qq}^{\mathrm{NS},(2)}}{3} - 4 a_{qq,Q}^{\mathrm{NS},(2)} \left[\beta_0 + \beta_{0,Q}\right] + \beta_{1,Q}^{(1)} \gamma_{qq}^{(0)} + \frac{\gamma_{qq}^{(0)} \beta_0 \beta_{0,Q} \zeta_2}{2} - 2 \delta m_1^{(0)} \beta_{0,Q} \gamma_{qq}^{(0)} - \delta m_1^{(-1)} \hat{\gamma}_{qq}^{\mathrm{NS},(1)} \Bigg] + \mathcal{O}\left(\varepsilon^0\right)$$

$\Rightarrow$ $\mathcal{O}(N_F)$ contributions to anomalous dimensions

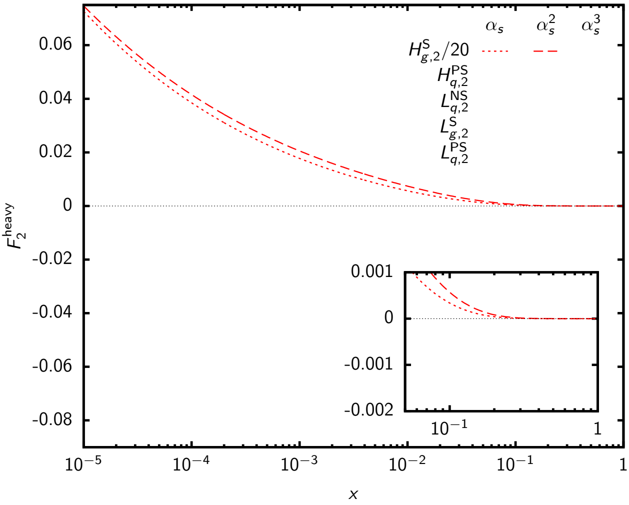
$A_{gq,Q} \to \gamma_{gq}^{(2)}$ [Ablinger et al. '14a] $A_{gg,Q} \to \gamma_{gg}^{(2)}$

$A_{qq,Q}^{\mathrm{NS}} \to \gamma_{qq}^{\mathrm{NS},(2)}$ [Ablinger et al. '14b] $A_{Qg} \to \gamma_{qg}^{(2)}$

$A_{Qq}^{\mathrm{PS}} \to \gamma_{qq}^{\mathrm{PS},(2)}$ [Ablinger et al. '14c] complete PS anom. dim.

- First independent calculation in a massive setting

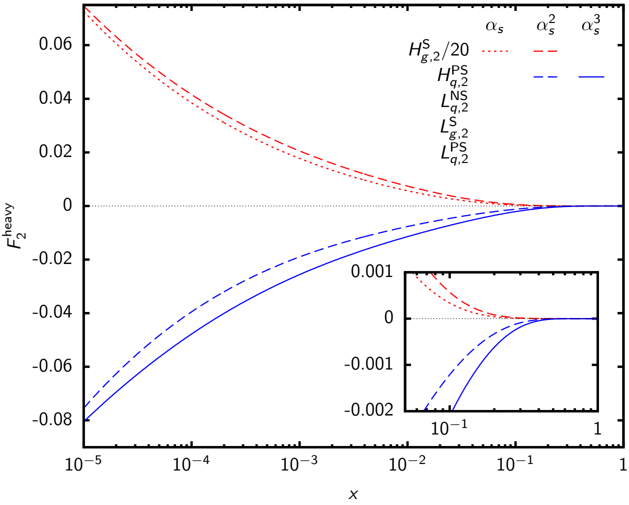
Contributions to the structure function F_2



$H^{S}_{g,2}$ not yet known at $\mathcal{O}(\alpha_s^3)$
$\rightarrow$ in progress

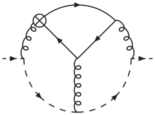
$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs
Renormalisation: α_s in $\overline{\text{MS}}$ scheme, m_c in on-shell scheme

Contributions to the structure function F_2



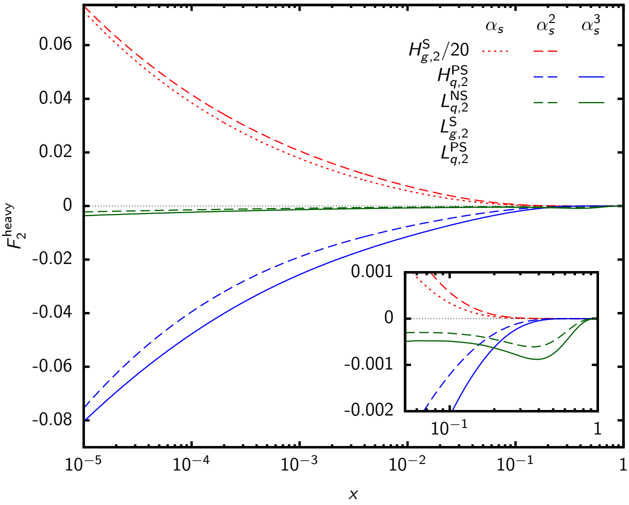
$H_{g,2}^{S}$ not yet known at $\mathcal{O}(\alpha_s^3)$
$\rightarrow$ in progress

$H_{q,2}^{PS}$ [Ablinger et al. '14c]



$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs
Renormalisation: α_s in $\overline{\text{MS}}$ scheme, m_c in on-shell scheme

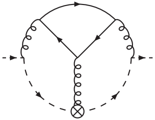
Contributions to the structure function F_2



$H_{g,2}^{S}$ not yet known at $\mathcal{O}(\alpha_s^3)$
$\rightarrow$ in progress

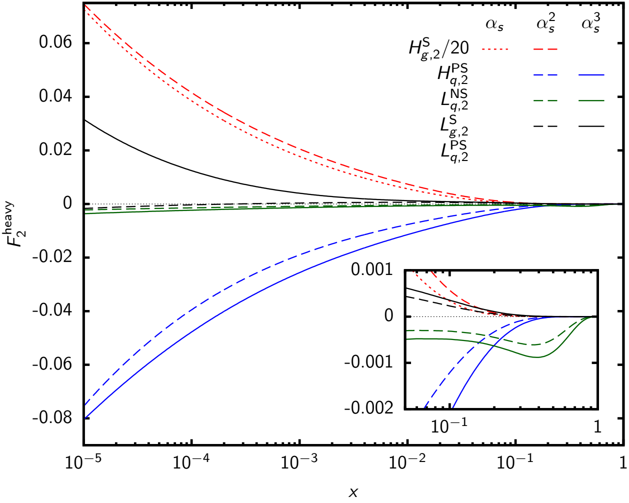
$H_{q,2}^{PS}$ [Ablinger et al. '14c]

$L_{q,2}^{NS}$ [Ablinger et al. '14b]



$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs
Renormalisation: α_s in $\overline{\text{MS}}$ scheme, m_c in on-shell scheme

Contributions to the structure function F_2

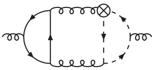


$H_{g,2}^{S}$ not yet known at $\mathcal{O}(\alpha_s^3)$
$\rightarrow$ in progress

$H_{q,2}^{PS}$ [Ablinger et al. '14c]

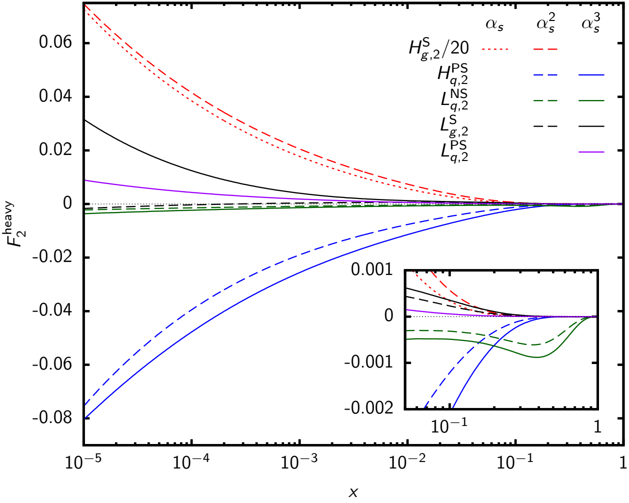
$L_{q,2}^{NS}$ [Ablinger et al. '14b]

$L_{g,2}^{S}$ [Ablinger et al. '10] [Behring et al. '14]



$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs
Renormalisation: α_s in $\overline{\text{MS}}$ scheme, m_c in on-shell scheme

Contributions to the structure function F_2



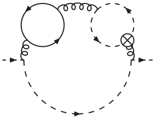
$H_{g,2}^{S}$ not yet known at $\mathcal{O}(\alpha_s^3)$ → in progress

$H_{q,2}^{PS}$ [Ablinger et al. '14c]

$L_{q,2}^{NS}$ [Ablinger et al. '14b]

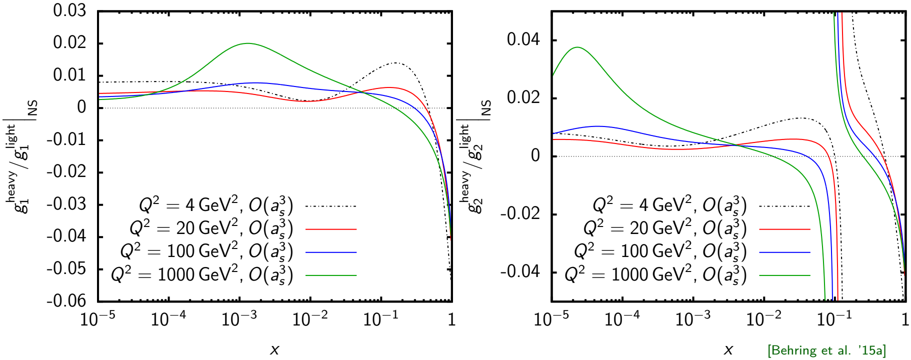
$L_{g,2}^{S}$ [Ablinger et al. '10] [Behring et al. '14]

$L_{q,2}^{PS}$ [Ablinger et al. '10] [Behring et al. '14]



$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs
Renormalisation: α_s in $\overline{\text{MS}}$ scheme, m_c in on-shell scheme

Non-singlet part of polarised structure functions g_1 & g_2

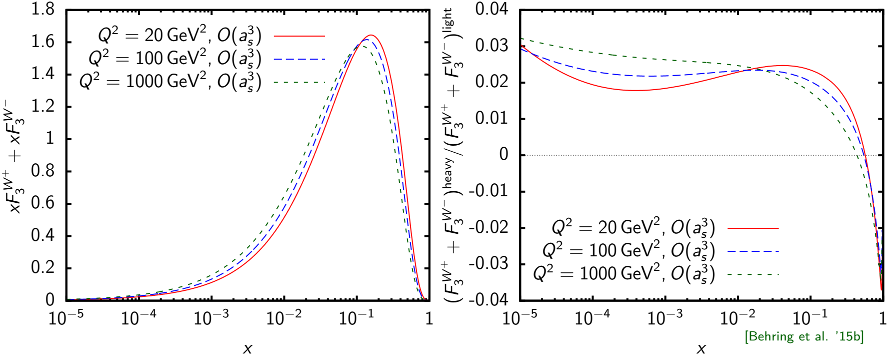


[Behring et al. '15a]

- Odd moments of $A_{qq,Q}^{\text{NS}}$ calculated as well [Ablinger et al. '14b]
- They enter the non-singlet contribution to g_1
- Twist-2 part of g_2 determined via Wandzura-Wilczek relation:

$$g_2(x, Q^2) = -g_1(x, Q^2) + \int_x^1 \frac{dy}{y} g_1(y, Q^2)$$

Charged current function xF_3



- Odd moments of $A_{qq,Q}^{\text{NS}}$ enter also $xF_3^{W^+} + xF_3^{W^-}$
- Two non-singlet Wilson coefficients:
  - $L_{q,3}^{\text{NS}}$: W couples to light quarks ($u \to d, \ldots$)
  - $H_{q,3}^{\text{NS}}$: W couples to heavy quark ($s \to c, \ldots$)

Sum rules

Polarised Bjorken sum rule [Bjorken '70]

$$\int_0^1 \mathrm{d}x \left[g_1^{ep}(x, Q^2) - g_1^{en}(x, Q^2)\right] = \frac{1}{6}\left|\frac{g_A}{g_V}\right| C_{\mathrm{pBj}}(\hat{a}_s)$$

Gross-Llewellyn-Smith sum rule [Gross, Llewellyn-Smith '69]

$$\int_0^1 \mathrm{d}x \left[F_3^{\bar{\nu}p}(x, Q^2) + F_3^{\nu p}(x, Q^2)\right] = 6 C_{\mathrm{GLS}}(\hat{a}_s)$$

- QCD corrections: Coefficients C_{pBj} and C_{GLS} follow from moment $N = 1$ of the Wilson coefficients
- Massless corrections known to $\mathcal{O}(\alpha_s^4)$
- $A_{qq,Q}^{NS}(N = 1) = 0$ due to fermion number conservation

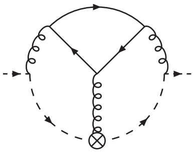
$\Rightarrow$ Corrections from heavy quarks at $Q^2 \gg m^2$ reduce to shift $N_F \to N_F + 1$ in massless coefficient [Behring et al. '15a '15b]

Transversity

- Tensor operator ($\rightarrow$ transversity h_1):

$$O_{q,r}^{\mathrm{TR,NS},\mu\mu_1\ldots\mu_N} = \frac{1}{2} i^{N-1} S\left[\bar{\psi}\sigma^{\mu\mu_1} D^{\mu_2}\ldots D^{\mu_N}\frac{\lambda_r}{2}\psi\right] - \text{trace terms}$$

- We calculated its massive operator matrix element $A_{qq,Q}^{\mathrm{TR,NS}}$
[Ablinger et al. '14b]



- Results for transversity:
  - N_F-dependent parts of the 3-loop anomalous dimension $\gamma_{qq}^{\mathrm{TR,NS}}$
  - 3-loop massive operator matrix element $A_{qq,Q}^{\mathrm{TR,NS},(3)}$
  - Once the corresponding massless Wilson coefficients are known, also the asymptotic heavy flavour Wilson coefficients for transversity can be constructed using our results

Variable flavour number scheme (VFNS)

NNLO matching condition for non-singlet case:
[Buza et al. '96] [Bierenbaum, Blümlein, Klein, '09a, '09b]

$$\begin{aligned} f_k(N_F+1,\mu^2) + \bar{f}_k(N_F+1,\mu^2) \\ &= A_{qq,Q}^{\mathrm{NS}} \otimes [f_k(N_F,\mu^2) + \bar{f}_k(N_F,\mu^2)] \\ &+ \frac{1}{N_F}\left(A_{qq,Q}^{\mathrm{PS}} \otimes \Sigma(N_F,\mu^2) + A_{qg,Q} \otimes G(N_F,\mu^2)\right) \end{aligned}$$

- Ingredients at NNLO are now complete for the above relation [Ablinger et al. '14b]
- $A_{qq,Q}^{\mathrm{PS}}$ and $A_{qg,Q}$ start at NNLO
  $\to \Sigma$ and G contribute only from NNLO on

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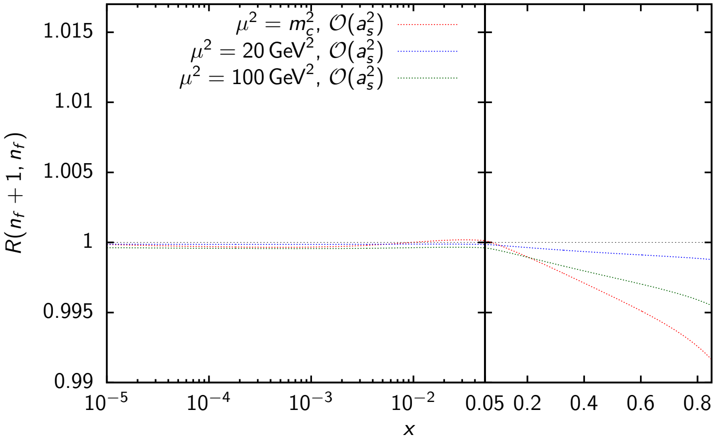
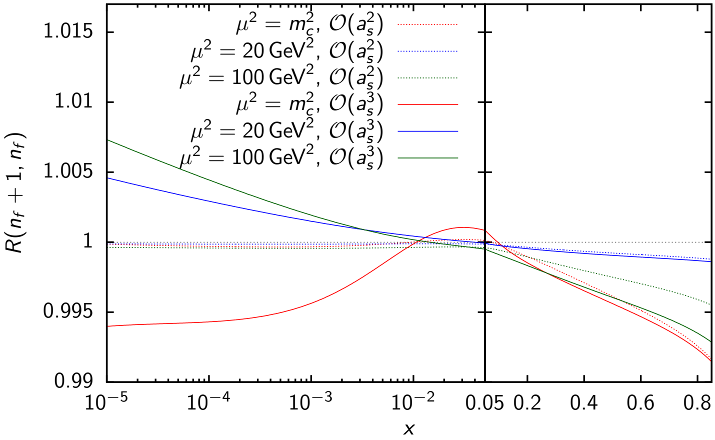


[Ablinger et al. '14b]

$$R(N_F+1, N_F) = \frac{u(N_F+1,\mu^2) + \bar{u}(N_F+1,\mu^2)}{u(N_F,\mu^2) + \bar{u}(N_F,\mu^2)}, \qquad \text{here } N_F = 3$$

Variable Flavour Number Scheme (VFNS)



[Ablinger et al. '14b]

$$R(N_F+1, N_F) = \frac{u(N_F+1,\mu^2) + \bar{u}(N_F+1,\mu^2)}{u(N_F,\mu^2) + \bar{u}(N_F,\mu^2)}, \qquad \text{here } N_F = 3$$

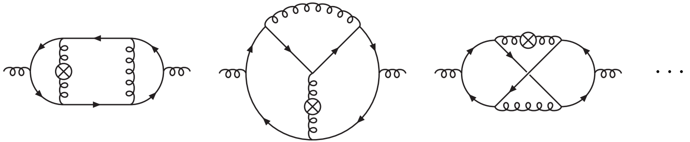
Conclusion

- Heavy quarks yield important contributions to DIS
  $\Rightarrow$ essential for precision measurements
  of α_s ($\sim 1\%$) and m_c ($\sim 3\%$) [Alekhin et al. '12]
- Analytical calculation of the 3-loop corrections requires modern computer-algebraic methods and tools
- Completed massive operator matrix elements:
  $A_{qq,Q}^{\mathrm{PS}}$, $A_{qg,Q}$, $A_{qq,Q}^{\mathrm{NS}}$, $A_{qq,Q}^{\mathrm{TR}}$, A_{Qq}^{PS}, $A_{gq,Q}$ and $A_{gg,Q}$
- Calculation of ladder- and V-diagrams for A_{Qg}
  lead to improvements of tools and methods
  $\Rightarrow$ Allowed calculation of all $A_{gg,Q}$ and many A_{Qg} diagrams
- Applications:
  - Calculation of anomalous dimensions
  - Heavy flavour Wilson coefficients for F_2, g_1 and xF_3
  - Sum rules for g_1 and xF_3
  - Matching relations in the Variable Flavour Number Scheme
- The new results are an important step towards an $\mathcal{O}(\alpha_s^3)$ description of the heavy quarks in DIS.

Backup

- Operator matrix element $A_{gg,Q} \rightarrow 29$
- Non-planarity of diagram 12 $\rightarrow$ 31
- Feynman rules $\rightarrow$ 32

Gluonic operator matrix element $A_{gg,Q}$



- Important building block for the VFNS
  $\rightarrow$ enters the matching relation of the gluon PDF
  [Buza et al. '96] [Bierenbaum, Blümlein, Klein, '09a, '09b]

$$G(N_F+1,\mu^2) = A_{gq,Q} \otimes \Sigma(N_F,\mu^2) + A_{gg,Q} \otimes G(N_F,\mu^2)$$

- 642 diagrams $\rightarrow$ 67212 scalar integrals $\rightarrow$ 139 master integrals
- 2 crossed-box diagrams
- MI partly overlap with earlier calculations ($\sim$ 25%)
- Remaining MI calculated mainly via differential/difference equations

$\Rightarrow$ Diagrams are all done

$\Rightarrow$ Unrenormalised OME is known for all even N; vanishes for odd N

Constant term of the gluonic OME $A_{gg,Q}$

$$
\begin{aligned}
a_{gg,Q}^{(3)} = \frac{1+(-1)^N}{2}\Bigg\{ & C_F^2 T_F \Bigg[\frac{16(N^2+N+2)}{N^2(N+1)^2}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} - \frac{4P_{69}S_1^2}{3(N-1)N^4(N+1)^4(N+2)} \\
& + \tilde{\gamma}_{gq}^{(0)}\left(\frac{128\,(S_{-4}-S_{-3}S_1+S_{-3,1}+2S_{-2,2})}{3N(N+1)(N+2)} + \frac{4(5N^2+5N-22)S_1^2 S_2}{3N(N+1)(N+2)} + \cdots\right) + \cdots\Bigg] \\
& + C_A C_F T_F\Bigg[\frac{16P_{42}}{3(N-1)N^2(N+1)^2(N+2)}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} + \frac{32P_2 S_{-2,2}}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{64P_{14}S_{-2,1,1}}{3(N-1)N^2(N+1)^2(N+2)} - \frac{16P_{23}S_{-4}}{3(N-1)N^2(N+1)^2(N+2)} + \frac{4P_{63}S_4}{3(N-2)(N-1)N^2(N+1)^2(N+2)} + \cdots\Bigg] \\
& + C_A^2 T_F\Bigg[-\frac{4P_{46}}{3(N-1)N^2(N+1)^2(N+2)}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} + \frac{256P_5 S_{-2,2}}{9(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{32P_{30}S_{-2,1,1}+16P_{35}S_{-3,1}+16P_{44}S_{-4}}{9(N-1)N^2(N+1)^2(N+2)} + \frac{16P_{52}S_{-2}^2}{27(N-1)N^2(N+1)^2(N+2)} + \frac{8P_{36}S_2^2}{9(N-1)N^2(N+1)^2} + \cdots\Bigg] \\
& + C_F T_F^2\Bigg[-\frac{16P_{48}\binom{2N}{N}4^{-N}\left(\sum_{i=1}^{N}\frac{4^i S_1(i-1)}{\binom{2i}{i}i^2}-7\zeta_3\right)}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} - \frac{32P_{86}S_1}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \\
& + \frac{16P_{45}S_1^2}{27(N-1)N^3(N+1)^3(N+2)} - \frac{16P_{45}S_2}{9(N-1)N^3(N+1)^3(N+2)} + \cdots\Bigg] + \cdots\Bigg\}
\end{aligned}
$$

Constant term of the gluonic OME $A_{gg,Q}$

$$a_{gg,Q}^{(3)} = \frac{1+(-1)^N}{2}\Bigg\{C_F^2 T_F\Bigg[\frac{16(N^2+N+2)}{N^2(N+1)^2}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} - \frac{4P_{69}S_1^2}{3(N-1)N^4(N+1)^4(N+2)}$$

$$+\tilde{\gamma}_{gq}^{(0)}\left(\frac{128\,(S_{-4}-S_{-3}S_1+S_{-3,1}+2S_{-2,2})}{3N(N+1)\ldots} + \frac{4(5N^2+5N-22)S_1^2S_2}{\ldots}+\cdots\right)+\cdots\Bigg]$$

$$+C_AC_FT_F\Bigg[\frac{1\ldots}{3(N-1)N^2(\ldots} \ldots \frac{32P_2S_{-2,2}}{\ldots N-1)N^2(N+1)^2(N+2)}$$

$$-\frac{64P_{14}S_{-2,1,1}}{3(N-1)N^2(N+1)^2(\ldots} \ldots \frac{4P_{63}S_4}{\ldots(N-1)N^2(N+1)^2(N+2)}+\cdots\Bigg]$$

$$\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2}$$

$$\ldots\cdots\Bigg] \quad \ldots N-1)$$

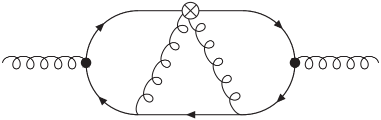
$$+\frac{16P_{45}S_1^2}{27(N-1)N^3(N+1)^3(N+2)} - \frac{16P_{45}S_2}{9(N-1)N^3(N+1)^3(N+2)}+\cdots\Bigg]+\cdots\Bigg\}$$

Binomial sums

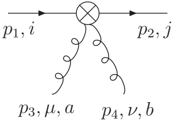
- Two objects involving binomial weights appear
- One of them already occurred in the T_F^2 colour factor [Ablinger et al. '14d]

Nicht-Planarität von Diagramm 12

- The diagram is planar



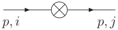
- The Feynman rule for the operator reads



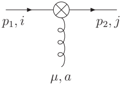
$$g^2 \Delta^\mu \Delta^\nu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2}$$
$$\left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1}\right],$$
$$N \geq 3$$

- $(p_1 + p_4)^{(l-j-1)}$ part mixes momenta from two different loops
- Shifting the momenta allows to put these mixed momenta into the operator or into one of the propagators, but one cannot get rid of this structure
- Effective non-planarity arises

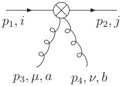
Feynman rules for operator insertions



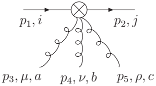
$$\delta^{ij}\not{\Delta}\gamma_{\pm}(\Delta\cdot p)^{N-1}\ , \quad N\geq 1$$



$$g t^a_{ji}\Delta^{\mu}\not{\Delta}\gamma_{\pm}\sum_{j=0}^{N-2}(\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-j-2}\ , \quad N\geq 2$$



$$g^2\Delta^{\mu}\Delta^{\nu}\not{\Delta}\gamma_{\pm}\sum_{j=0}^{N-3}\sum_{l=j+1}^{N-2}(\Delta p_2)^j(\Delta p_1)^{N-l-2}$$
$$\left[(t^a t^b)_{ji}(\Delta p_1+\Delta p_4)^{l-j-1}+(t^b t^a)_{ji}(\Delta p_1+\Delta p_3)^{l-j-1}\right]\ ,$$
$$N\geq 3$$

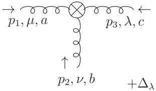


$$g^3\Delta^{\mu}\Delta^{\nu}\Delta^{\rho}\not{\Delta}\gamma_{\pm}\sum_{j=0}^{N-4}\sum_{l=j+1}^{N-3}\sum_{m=l+1}^{N-2}(\Delta.p_2)^j(\Delta.p_1)^{N-m-2}$$
$$\Big[\ \ (t^a t^b t^c)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}$$
$$+(t^a t^c t^b)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}$$
$$+(t^b t^a t^c)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}$$
$$+(t^b t^c t^a)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}$$
$$+(t^c t^a t^b)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}$$
$$+(t^c t^b t^a)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}\Big]\ ,$$
$$N\geq 4$$

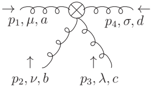
$\gamma_+ = 1\ ,\quad \gamma_- = \gamma_5\ .$ For transversity, one has to replace: $\not{\Delta}\gamma_{\pm}\to\ \sigma^{\mu\nu}\Delta_{\nu}$.

Feynman rules for operator insertions

$$\frac{1+(-1)^N}{2}\delta^{ab}(\Delta\cdot p)^{N-2}$$
$$\Big[g_{\mu\nu}(\Delta\cdot p)^2-(\Delta_\mu p_\nu+\Delta_\nu p_\mu)\Delta\cdot p+p^2\Delta_\mu\Delta_\nu\Big]\,,\quad N\geq 2$$



$$-ig\frac{1+(-1)^N}{2}f^{abc}\Bigg(\Big[(\Delta_\nu g_{\lambda\mu}-\Delta_\lambda g_{\mu\nu})\Delta\cdot p_1+\Delta_\mu(p_{1,\nu}\Delta_\lambda-p_{1,\lambda}\Delta_\nu)\Big](\Delta\cdot p_1)^{N-2}$$
$$+\Delta_\lambda\Big[\Delta\cdot p_1 p_{2,\mu}\Delta_\nu+\Delta\cdot p_2 p_{1,\nu}\Delta_\mu-\Delta\cdot p_1\Delta\cdot p_2 g_{\mu\nu}-p_1\cdot p_2\Delta_\mu\Delta_\nu\Big]$$
$$\times\sum_{j=0}^{N-3}(-\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-3-j}$$
$$+\left\{\begin{smallmatrix}p_1\to p_2\to p_3\to p_1\\ \mu\to\nu\to\lambda\to\mu\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}p_1\to p_3\to p_2\to p_1\\ \mu\to\lambda\to\nu\to\mu\end{smallmatrix}\right\}\Bigg)\,,\quad N\geq 2$$



$$g^2\frac{1+(-1)^N}{2}\Bigg(f^{abe}f^{cde}O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)$$
$$+f^{ace}f^{bde}O_{\mu\lambda\nu\sigma}(p_1,p_3,p_2,p_4)+f^{ade}f^{bce}O_{\mu\sigma\nu\lambda}(p_1,p_4,p_2,p_3)\Bigg)\,,$$
$$O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)=\Delta_\nu\Delta_\lambda\Bigg\{-g_{\mu\sigma}(\Delta\cdot p_3+\Delta\cdot p_4)^{N-2}$$
$$+[p_{4,\mu}\Delta_\sigma-\Delta\cdot p_4 g_{\mu\sigma}]\sum_{i=0}^{N-3}(\Delta\cdot p_3+\Delta\cdot p_4)^i(\Delta\cdot p_4)^{N-3-i}$$
$$-[p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1 g_{\mu\sigma}]\sum_{i=0}^{N-3}(-\Delta\cdot p_1)^i(\Delta\cdot p_3+\Delta\cdot p_4)^{N-3-i}$$
$$+[\Delta\cdot p_1\Delta\cdot p_4 g_{\mu\sigma}+p_1\cdot p_4\Delta_\mu\Delta_\sigma-\Delta\cdot p_4 p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1 p_{4,\mu}\Delta_\sigma]$$
$$\times\sum_{i=0}^{N-4}\sum_{j=0}^{i}(-\Delta\cdot p_1)^{N-4-i}(\Delta\cdot p_3+\Delta\cdot p_4)^{i-j}(\Delta\cdot p_4)^j\Bigg\}$$
$$-\left\{\begin{smallmatrix}p_1\leftrightarrow p_2\\ \mu\leftrightarrow\nu\end{smallmatrix}\right\}-\left\{\begin{smallmatrix}p_3\leftrightarrow p_4\\ \lambda\leftrightarrow\sigma\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}p_1\leftrightarrow p_2,\ p_3\leftrightarrow p_4\\ \mu\leftrightarrow\nu,\ \lambda\leftrightarrow\sigma\end{smallmatrix}\right\}\,,\quad N\geq 2$$