Nucleon Form Factors at Low and High Momenta.

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Cake Seminar

Proton EM form factors

- Nucleon Pauli and Dirac Form Factors described in terms of matrix element of vector current

$$\langle N \mid V_\mu \mid N\rangle(\vec{q}) = \bar{u}(\vec{p}_f)\left[F_q(q^2)\gamma_\mu + \sigma_{\mu\nu} q_\nu \frac{F_2(q^2)}{2m_N}\right] u(\vec{p}_i)$$

- Alternatively, Sach's form factors determined in experiment

$$\begin{aligned} G_E(Q^2) &= F_1(Q^2) - \frac{Q^2}{4M^2} F_2(Q^2) \\ G_M(Q^2) &= F_1(Q^2) + F_2(Q^2) \end{aligned}$$

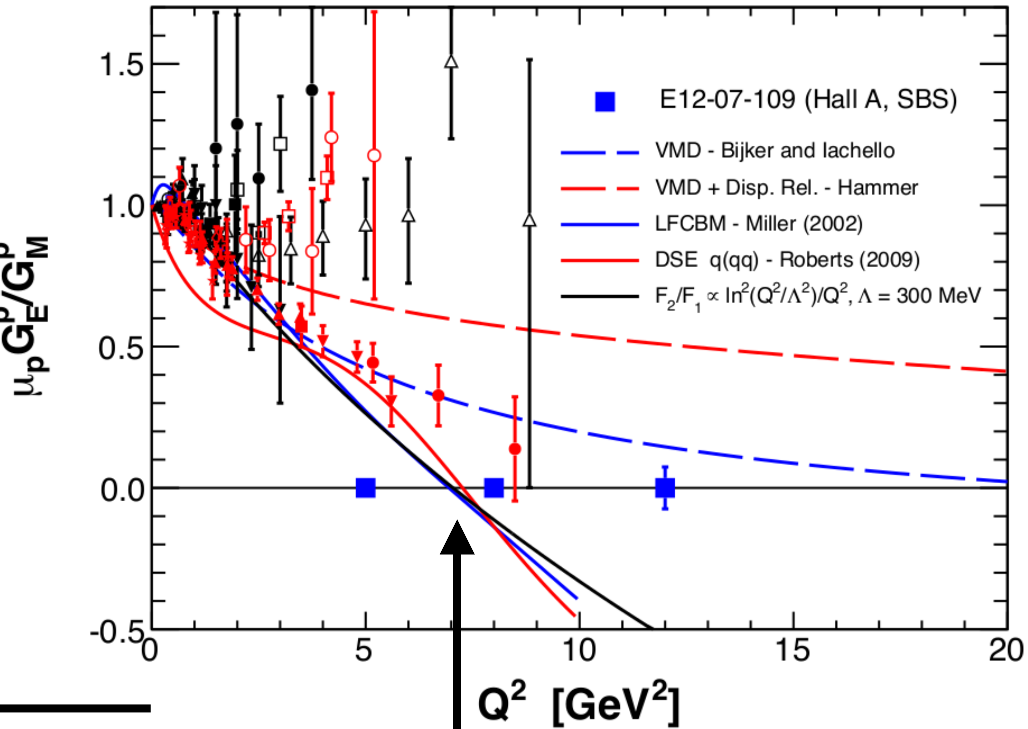
Charge radius is slope at $Q^2 = 0$

$$\left.\frac{\partial G_E(Q^2)}{\partial Q^2}\right|_{Q^2=0} = -\frac{1}{6}\langle r^2\rangle = \left.\frac{\partial F_1(Q^2)}{\partial Q^2}\right|_{Q^2=0} - \frac{F_2(0)}{4M^2}$$

EM Form factors - Expt

PRAD: E12-11-106

Approved expt E12-07-109



Nucleon Charge Radius

Quark and Gluon DOF

Cross zero

Lattice QCD

Observables in lattice QCD are then expressed in terms of the path integral as

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \prod_{n,\mu} dU_\mu(n) \prod_n d\psi(n) \prod_n d\bar{\psi}(n) \mathcal{O}(U, \psi, \bar{\psi}) e^{-\left(S_G[U] + S_F[U,\psi,\bar{\psi}]\right)}$$

Integrate out the Grassmann variables:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \prod_{n,\mu} dU_\mu(n) \mathcal{O}(U, G[U]) \det M[U] e^{-S_G[U]}$$

Importance Sampling

where $G(U,x,y)^{ij}_{\alpha\beta} \equiv \langle \psi^i_\alpha(x) \bar{\psi}^j_\beta(y) \rangle = M^{-1}(U)$

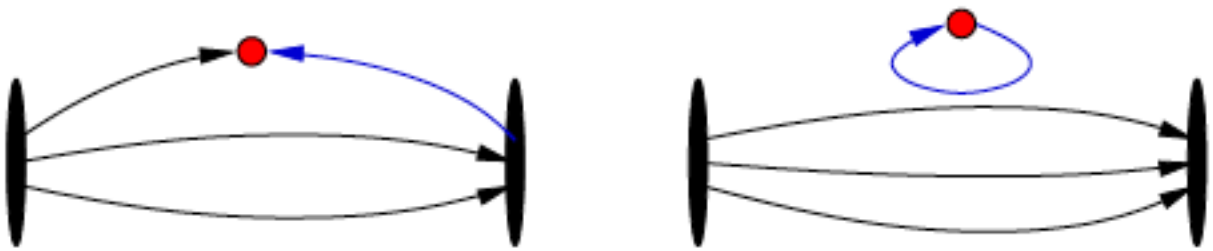
- Generate an ensemble of gauge configurations

$$P[U] \propto \det M[U] e^{-S_G[U]}$$

- Calculate observable

$$\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{n=1}^{N} \mathcal{O}(U^n, G[U^n])$$

Hadron Structure



$$C_{3\text{pt}}(t_{sep}, t; \vec{p}, \vec{q}) = \sum_{\vec{x},\vec{y}} \langle 0 \mid N(\vec{x}, t_{\text{sep}}) V_\mu(\vec{y}, t) \bar{N}(\vec{0}, 0) \mid 0 \rangle e^{-i\vec{p}\cdot\vec{x}} e^{-i\vec{q}\cdot\vec{y}}$$

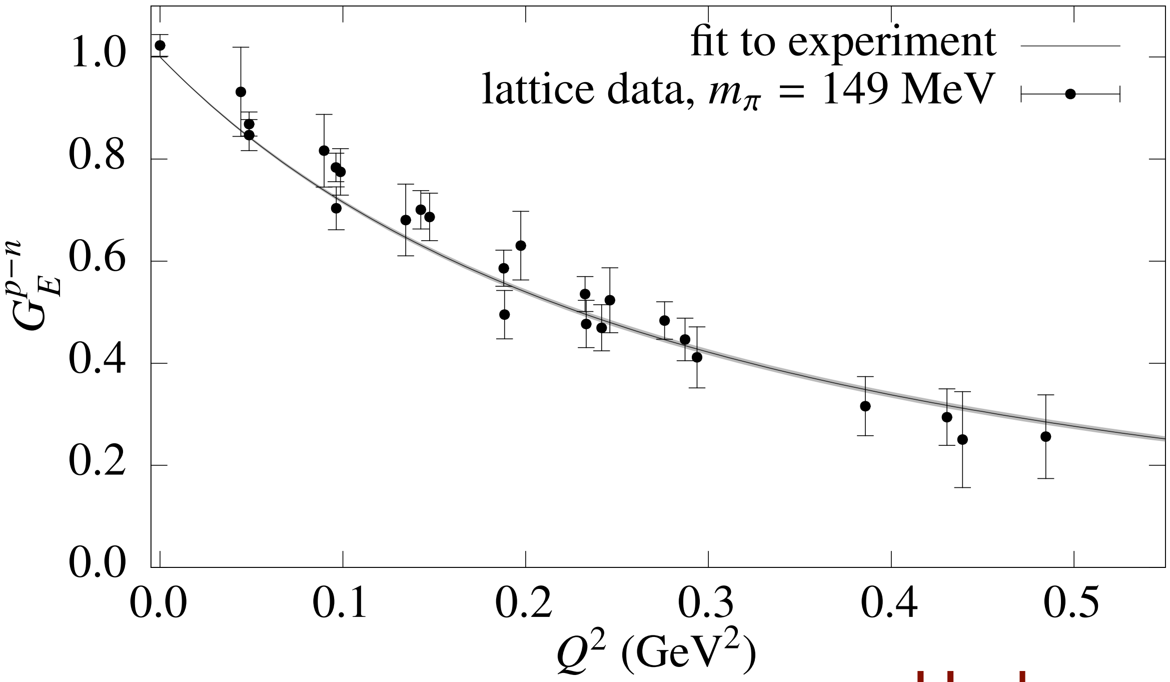
Resolution of unity – insert states

$$\longrightarrow \langle 0 \mid N \mid N, \vec{p}+\vec{q}\rangle \langle N, \vec{p}+\vec{q} \mid V_\mu \mid N\vec{p}\rangle \langle N, \vec{p} \mid \bar{N} \mid 0\rangle e^{-E(\vec{p}+\vec{q})(t_{\text{sep}}-t)} e^{-E(\vec{p})t}$$

1D Structure: EM Form Factors

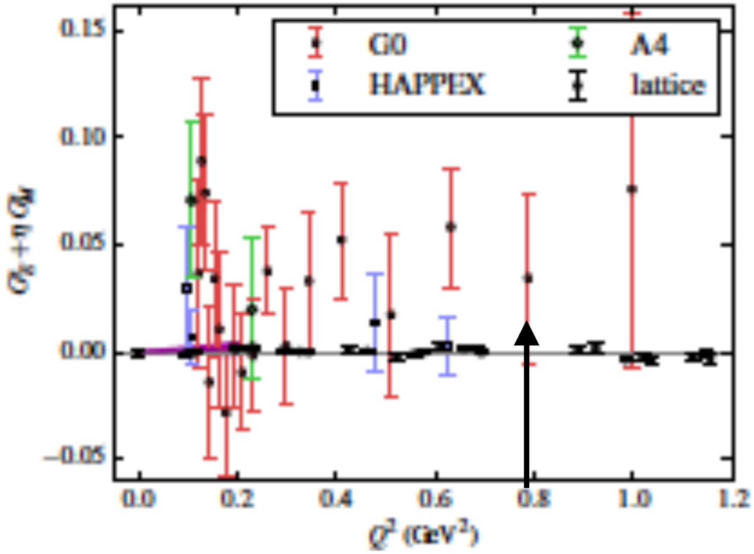
Wilson-clover lattices from BMW

Green et al (LHPC), Phys. Rev. D 90, 074507 (2014)



Hadron structure at nearly-physical quark masses

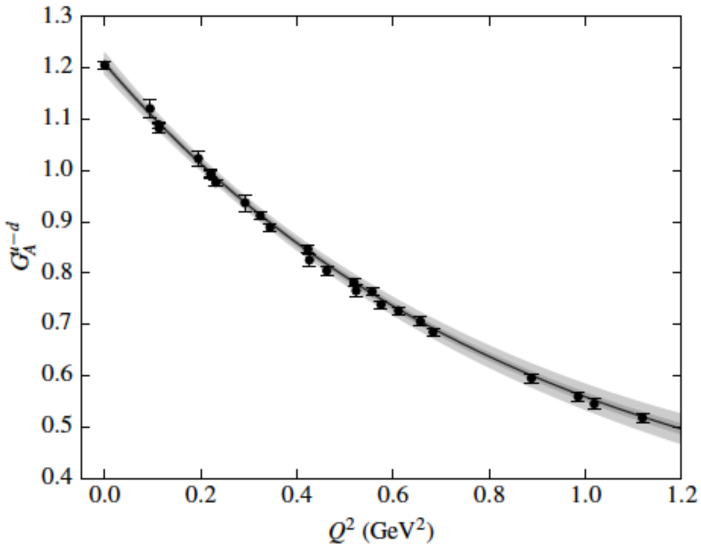
Sea Quark Contributions

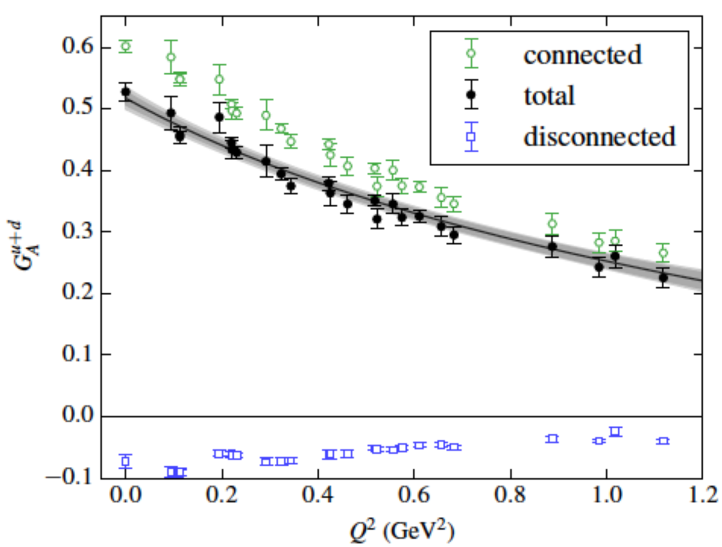


J. Green, K. Orginos *et al.*, Phys. Rev. D 92, 031501 (2015); Phys. Rev. D 95, 114502 (2017

Using *Hierarchical Probing* - A. Stathopoulos, J. Laeuchli, K. Orginos (2013)

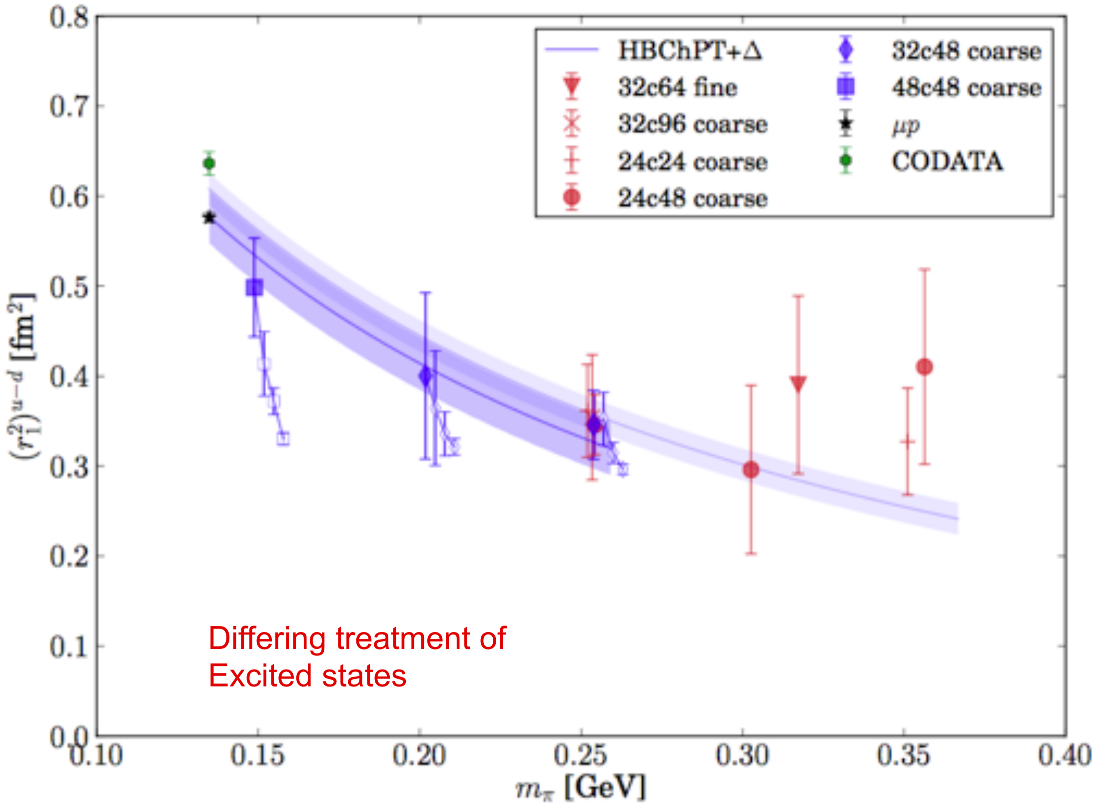
Combination *measured* in expt



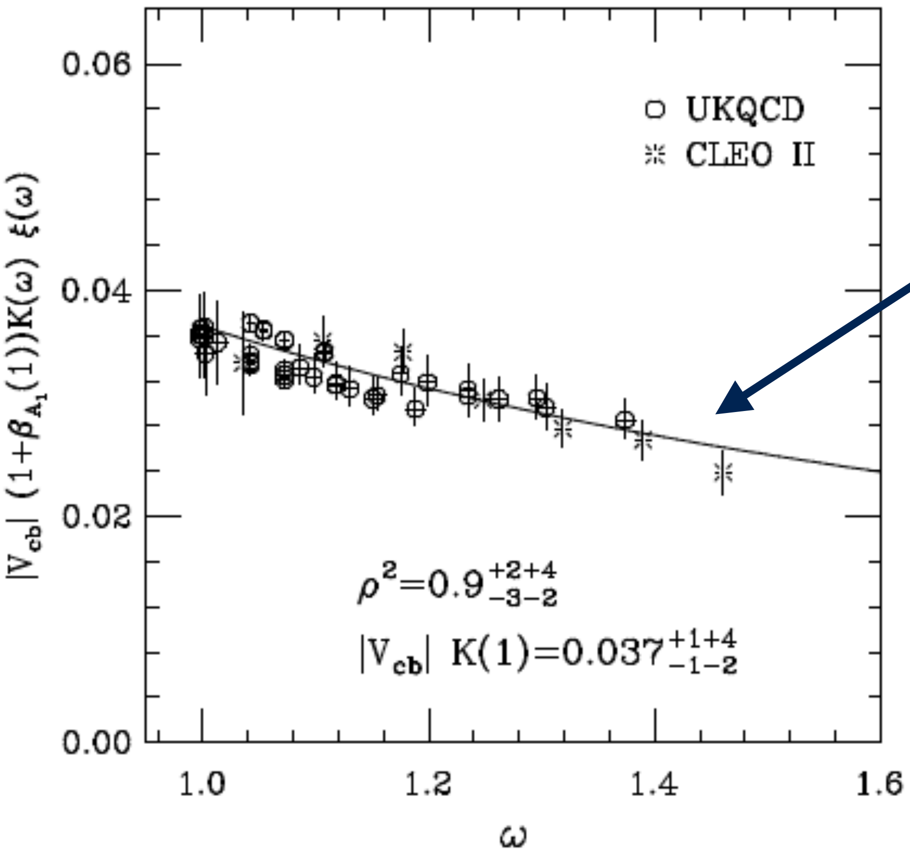


EM Form factors

Green et al, arXiv:1404.40



Isgur-Wise Function and CKM matrix



Extract V_{cb} if know intercept at zero recoil

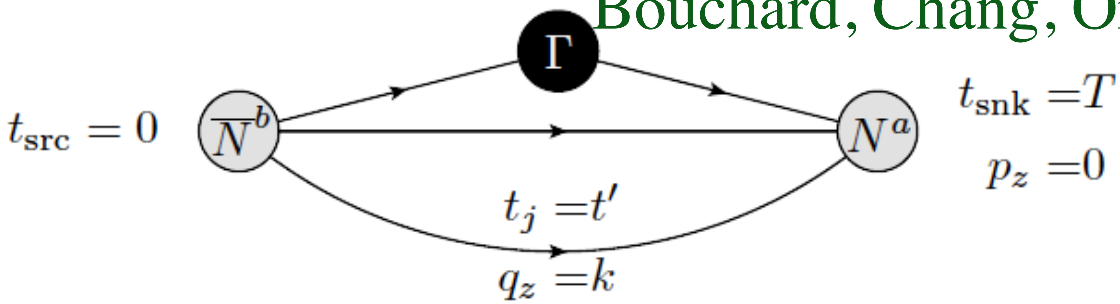
Lattice

Calculate slope at zero recoil..

UKQCD, L. Lellouch et al., Nucl. Phys. B444, 401 (1995), hep-lat/9410013

Moment Methods

Bouchard, Chang, Orginos, Richards



- Introduce three-momentum projected three-point function

$$C^{3\rm pt}(t,t') = \sum_{\vec{x},\vec{x}'} \left\langle N^a_{t,\vec{x}} \Gamma_{t',\vec{x}'} \overline{N}^b_{0,\vec{0}} \right\rangle e^{-ikx'_z}$$

- Now take derivative w.r.t. k^2

$$C'_{3\rm pt}(t,t') = \sum_{\vec{x},\vec{x}'} \frac{-x'_z}{2k} \sin\left(kx'_z\right) \left\langle N^a_{t,\vec{x}} \Gamma_{t',\vec{x}'} \overline{N}^b_{0,\vec{0}} \right\rangle$$

whence

$$\lim_{k^2\to 0} C'_{3\rm pt}(t,t') = \sum_{\vec{x},\vec{x}'} \frac{-x'^2_z}{2} \left\langle N^a_{t,\vec{x}} \Gamma_{t',\vec{x}'} \overline{N}^b_{0,\vec{0}} \right\rangle .$$

Odd moments vanish by symmetry

Moment Methods - II

- Analogous expressions for two-point functions:

$$C_{2\mathrm{pt}}(t) = \sum_{\vec{x}} \left\langle N^b_{t,\vec{x}} \overline{N}^b_{0,\vec{0}} \right\rangle e^{-ikx_z}$$

$\Longrightarrow$

$$C'_{2\mathrm{pt}}(t) = \sum_{\vec{x}} \frac{-x_z}{2k} \sin\left(kx_z\right) \left\langle N^b_{t,\vec{x}} \overline{N}^b_{0,\vec{0}} \right\rangle$$

$\Longrightarrow$

$$\lim_{k^2 \to 0} C'_{2\mathrm{pt}}(t) = \sum_{\vec{x}} \frac{-x_z^2}{2} \left\langle N^b_{t,\vec{x}} \overline{N}^b_{0,\vec{0}} \right\rangle .$$

Lowest coordinate-space moment ⇔ slope at zero momentum

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Lowest coordinate-space moment ⇔ slope at zero momentum

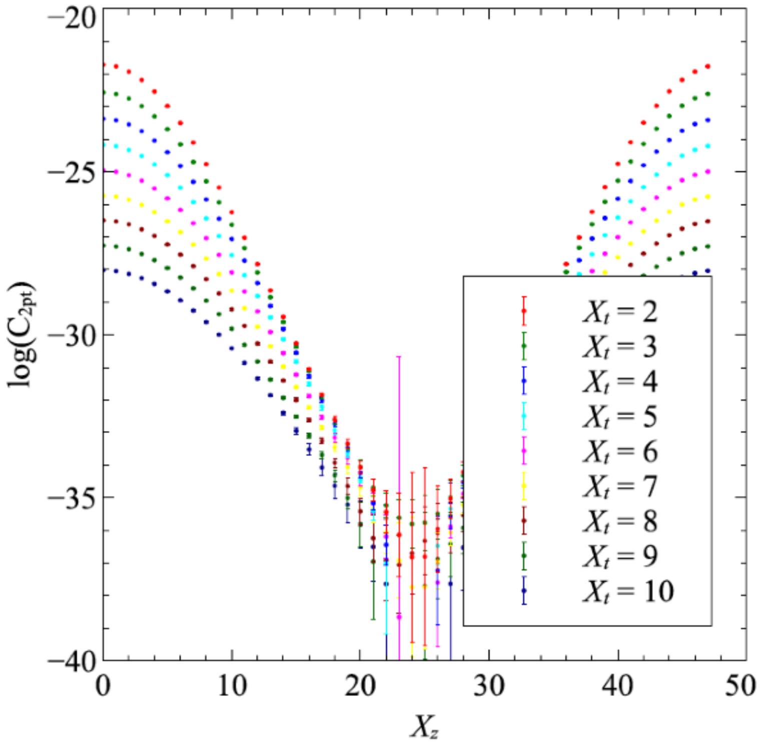
Lattice Details

- Two degenerate light-quark flavors, and strange quark set to its physical value

$$
\begin{aligned}
a &\simeq 0.12 \text{ fm} \\
m_\pi &\simeq 400 \text{ MeV} \\
\text{Lattice Size} &: 24^3 \times 64
\end{aligned}
$$

- To gain control over finite-volume effects, replicate in z direction: $24 \times 24 \times 48 \times 64$

Two-point correlator

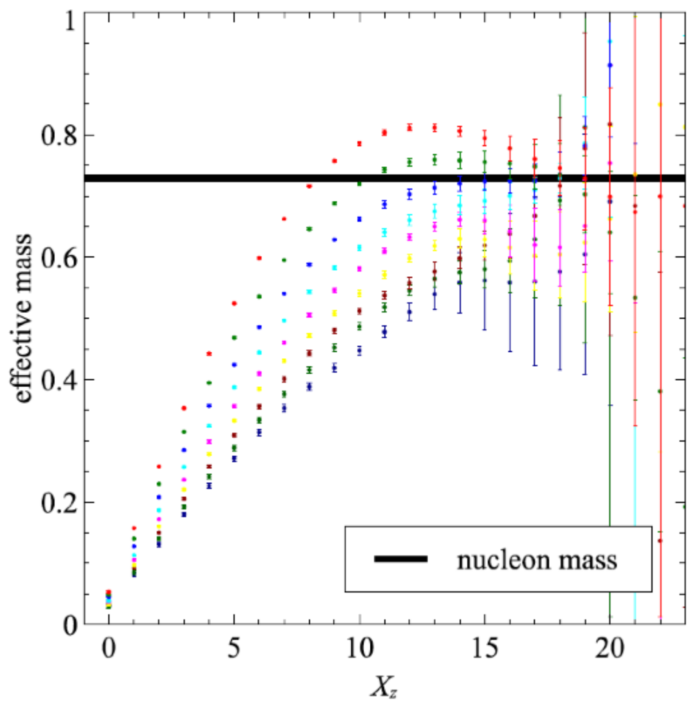


$$\ln\left[C_{\rm 2pt}(t,x_z)\right]$$

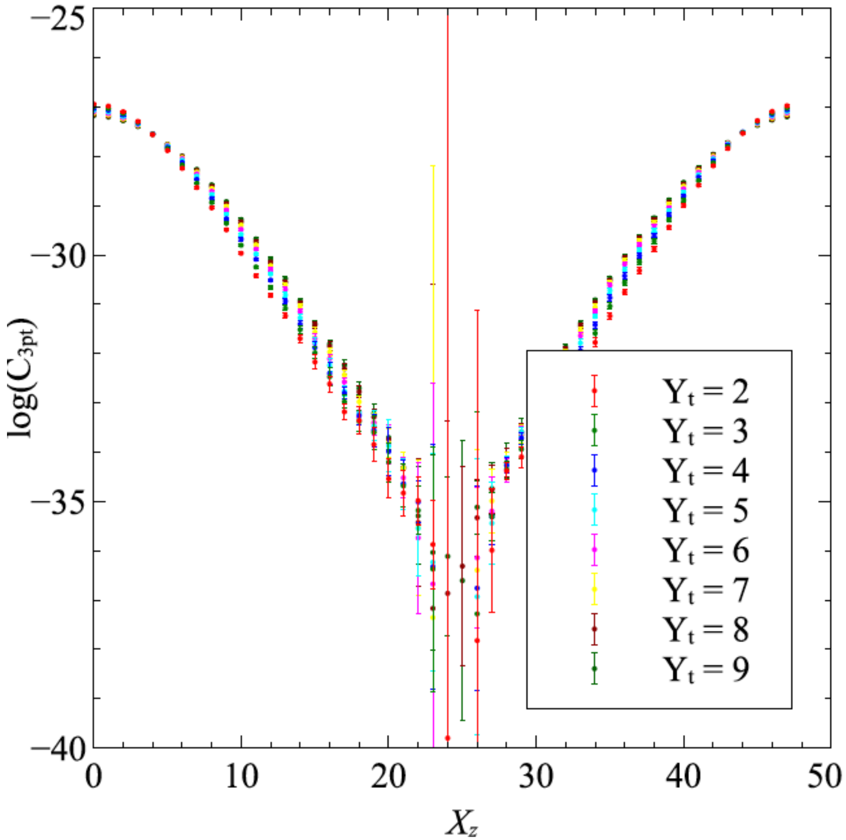
Any polynomial moment in x_z converges

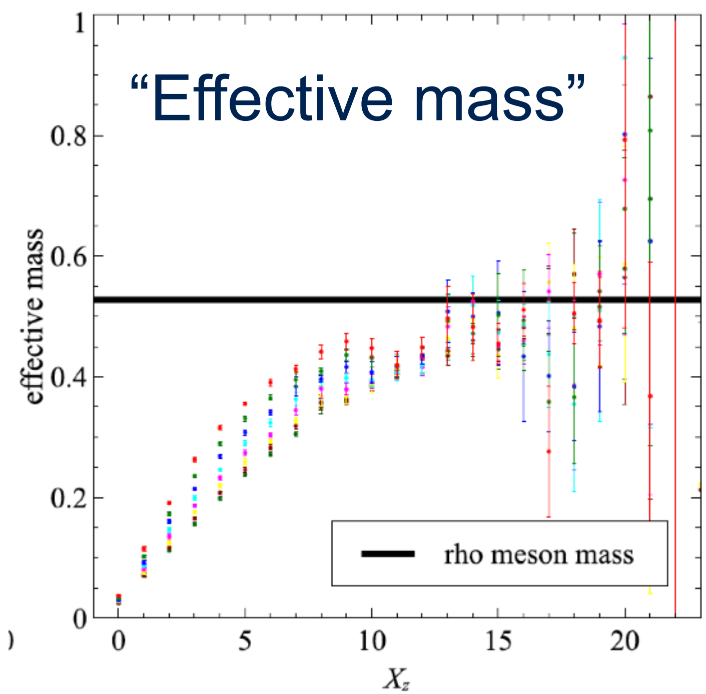
"Effective mass"

$$\ln C_{\rm 2pt}(t,x_z)/C_{\rm 2pt}(t,x_z+1)$$



Three-point correlator





- Spatial moments push the peak of the correlator away from origin
- Larger finite volume corrections compared to regular correlators

Fitting the data…

$$C^{3\mathrm{pt}}(t,t') = \sum_{n,m} \frac{Z_n^{\dagger a}(0)\Gamma_{nm}(k^2)Z_m^b(k^2)}{4M_n(0)E_m(k^2)} e^{-M_n(0)(t-t')} e^{-E_m(k^2)t'}$$

$$C_{2\mathrm{pt}}(t) = \sum_m \frac{Z_m^{b\dagger}(k^2)Z_m^b(k^2)}{2E_m(k^2)} e^{-E_m(k^2)t}$$

where

$$Z_n^{\dagger a}(0) \equiv \langle \Omega | N^a | n, p_i = (0,0,0) \rangle$$

$$Z_m^b(k^2) \equiv \langle m, p_i = (0,0,k) | \overline{N}^b | \Omega \rangle$$

$$\boxed{\Gamma_{nm}(k^2) \equiv \langle n, p_i = (0,0,0) | \Gamma | m, p_i = (0,0,k) \rangle}$$

Allow for multi-state contributions in the fit

Fitting - II

- Now look at the functional form of derivatives:

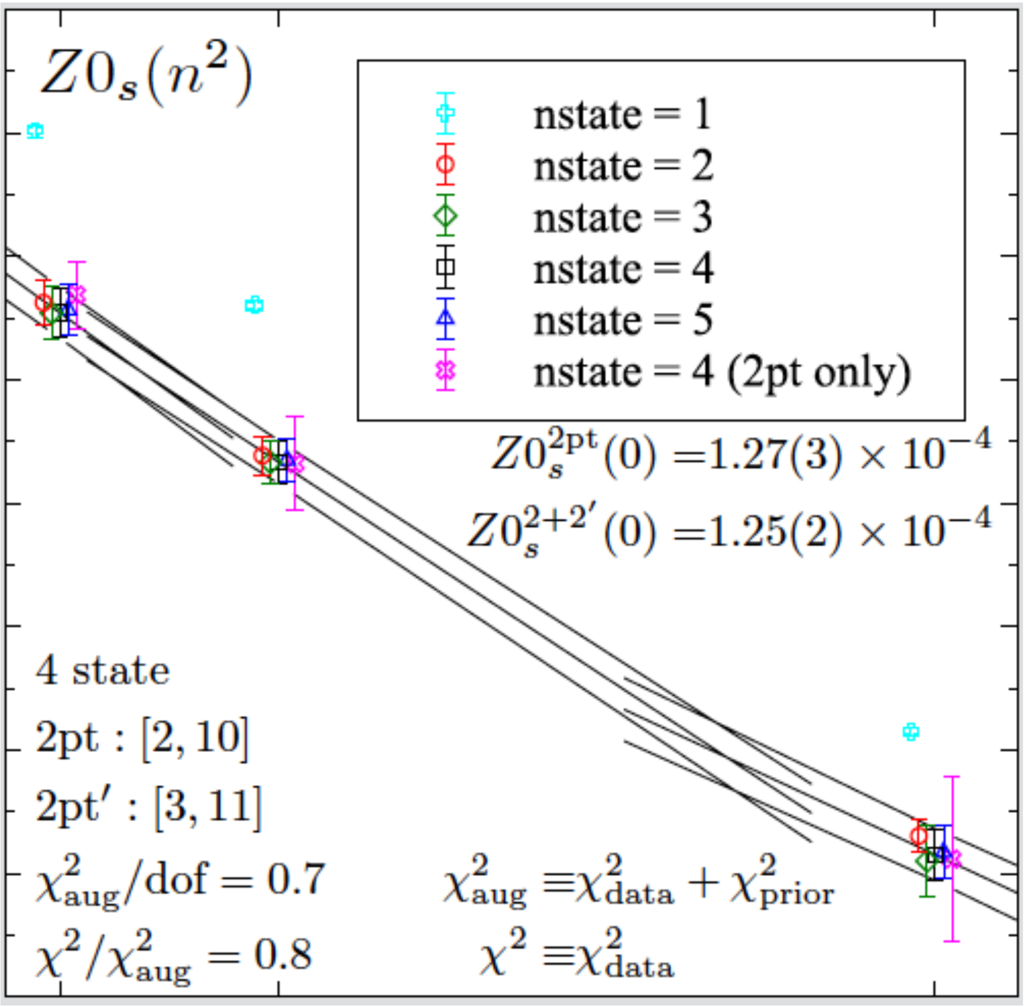
$$C'_{2\mathrm{pt}}(t) = \sum_m C_m^{2\mathrm{pt}}(t)\left(\frac{2Z_m^{b\prime}(k^2)}{Z_m^b(k^2)} - \frac{1}{2[E_m(k^2)]^2} - \frac{t}{2E_m(k^2)}\right)$$

$$C'_{3\mathrm{pt}}(t,t') = \sum_{n,m} C_{nm}^{3\mathrm{pt}}(t,t')\left\{\frac{\Gamma'_{nm}(k^2)}{\Gamma_{nm}(k^2)} + \frac{Z_m^{b\prime}(k^2)}{Z_m^b(k^2)} - \frac{1}{2[E_m(k^2)]^2} - \frac{t'}{2E_m(k^2)}\right\}$$

spatially extended sources

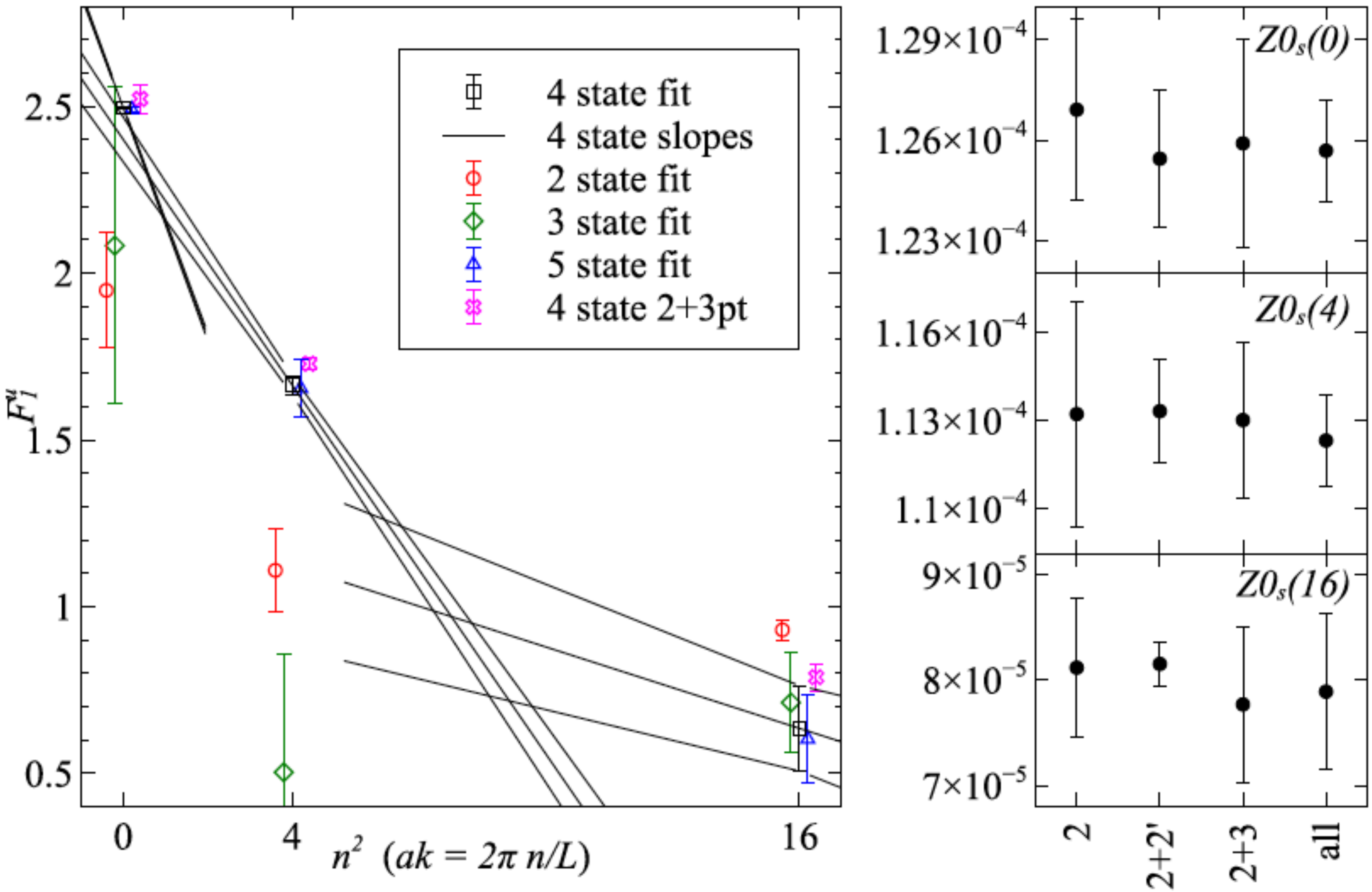
Second distance scale

Fitting - III



In practice we use multi-exponential, Bayesian fits

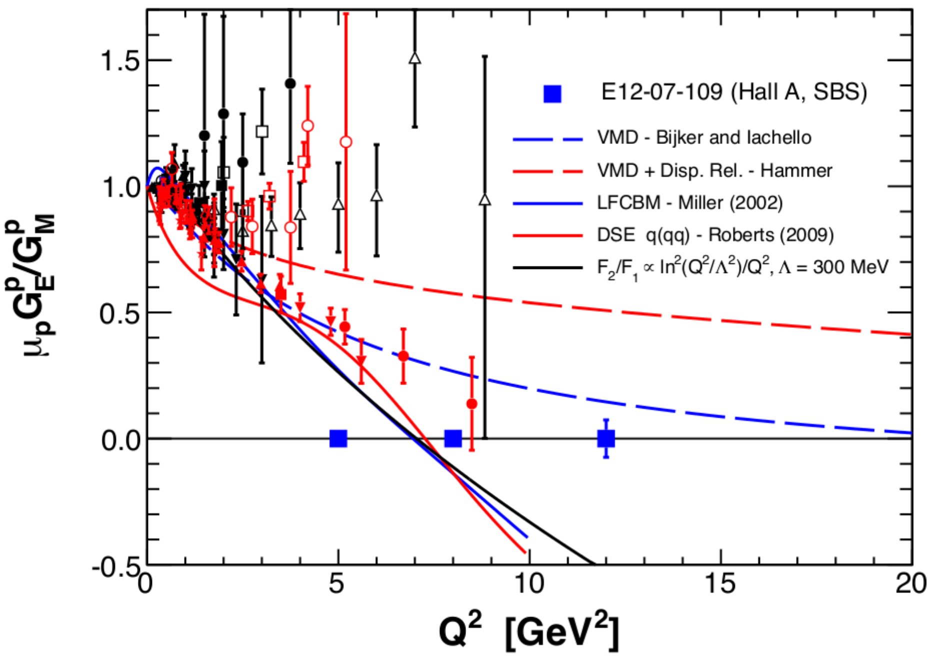
F_1 Form Factor



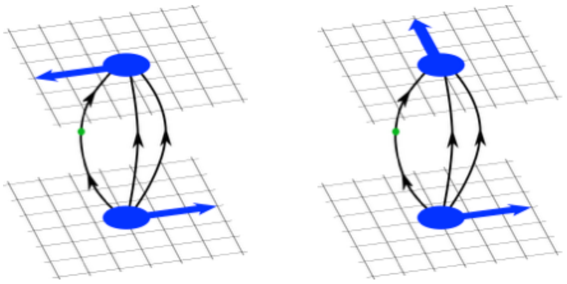
Nucleon Form Factors at High Momenta

Work in progress…

EM Form factors at Low and High Momenta



$Q^2 \lesssim 8.2$ GeV^2 $\quad Q^2 \lesssim 4.1$ GeV^2



Boosted interpolating operators

Lattice Challenges

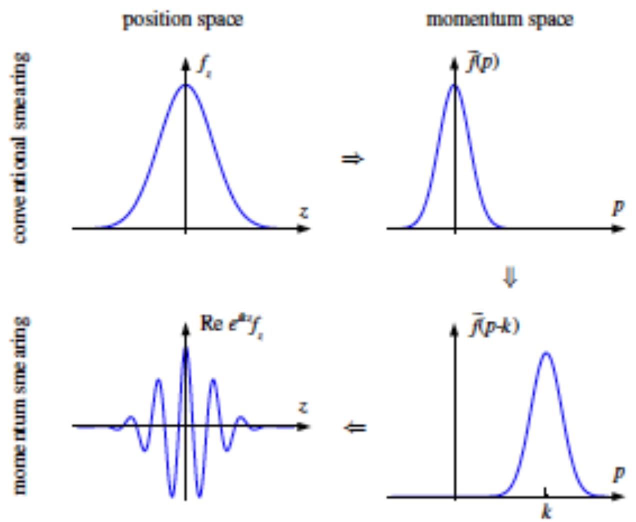
Discretisation Uncertainties $\qquad \mathcal{O}(q^2a^2, |\, p_i a\,|^2)$

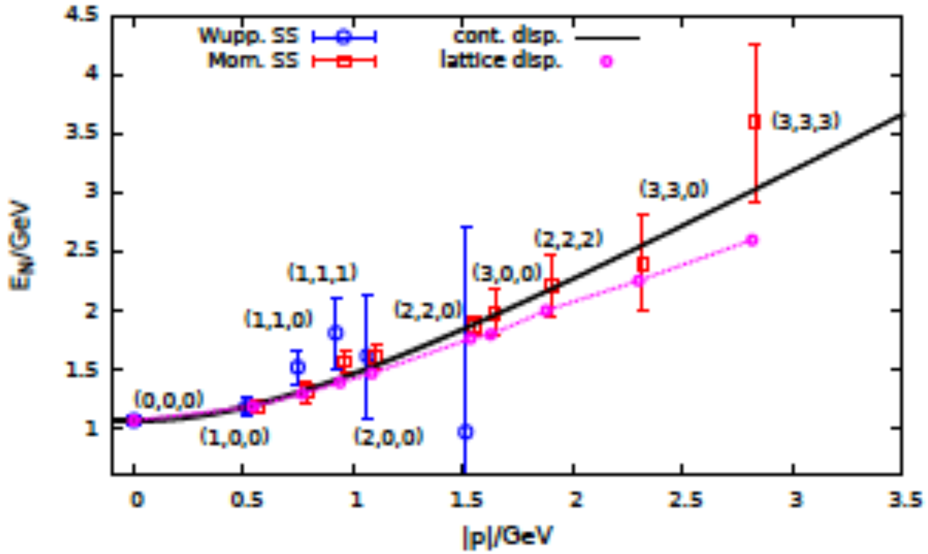
$$C_{2\mathrm{pt}}(t,\vec{p})/C_{\sigma^2}(t) \longrightarrow e^{-((E_N(\vec{p})-3m_\pi/2)t)}$$

Boosted Sources

Replace quark field by spatially extended (smeared) quark field

$$\psi \longrightarrow (1 - \sigma^2 \nabla^2 / 4N)^N \psi$$





Bali et al., Phys. Rev. D 93, 094515 (2016)

Variational Method

Subleading terms → *Excited* states

Construct matrix of correlators: *different smearing radii*

$$C_{ij}(t) = \sum_{\vec{x}} \langle N_i(\vec{x}, t) \bar{N}_j(0) \rangle = \sum_n A_n^i A_n^{j\dagger} e^{-E_n t}$$

Delineate contributions using *variational method*: solve

$$C(t) v^{(N)}(t, t_0) = \lambda_N(t, t_0) C(t_0) v^{(N)}(t, t_0).$$

$$\lambda_N(t, t_0) \rightarrow e^{-E_N(t-t_0)}(1 + \mathcal{O}(e^{-\Delta E(t-t_0)}))$$

Eigenvectors, with metric $C(t_0)$, are orthonormal and project onto the respective states

$$v^{(N')\dagger} C(t_0) v^{(N)} = \delta_{N,N'}$$

Baryon Operators

$$\langle 0 \mid O^{JM} \mid J', M' \rangle = Z^J \delta_{J,J'} \delta_{M,M'}$$

Starting point $\quad B = \left(\mathcal{F}_{\Sigma_\mathsf{F}} \otimes \mathcal{S}_{\Sigma_\mathsf{S}} \otimes \mathcal{D}_{\Sigma_\mathsf{D}}\right)\{\psi_1\psi_2\psi_3\}$

Introduce circular basis:

$$\overleftrightarrow{D}_{m=-1} = \frac{i}{\sqrt{2}}\left(\overleftrightarrow{D}_x - i\overleftrightarrow{D}_y\right)$$
$$\overleftrightarrow{D}_{m=0} = i\overleftrightarrow{D}_z$$
$$\overleftrightarrow{D}_{m=+1} = -\frac{i}{\sqrt{2}}\left(\overleftrightarrow{D}_x + i\overleftrightarrow{D}_y\right).$$

Straighforward to project to definite spin: *J = 1/2, 3/2, 5/2*

$$\big|\,[J,M]\big\rangle = \sum_{m_1,m_2} \big|\,[J_1,m_1]\big\rangle \otimes \big|\,[J_2,m_2]\big\rangle \big\langle J_1 m_1; J_2 m_2 \big| JM \big\rangle$$

R.G.Edwards et al., arXiv:1104.5152

Distillation for Baryons?

Measure matrix of correlation functions:

$$C_{ij}(t) \equiv \sum_{\vec{x},\vec{y}} \langle N_i(\vec{x},t)\bar{N}_j(\vec{y},0)\rangle$$

M. Peardon *et al.*, PRD80,054506 (2009)

Perambulators

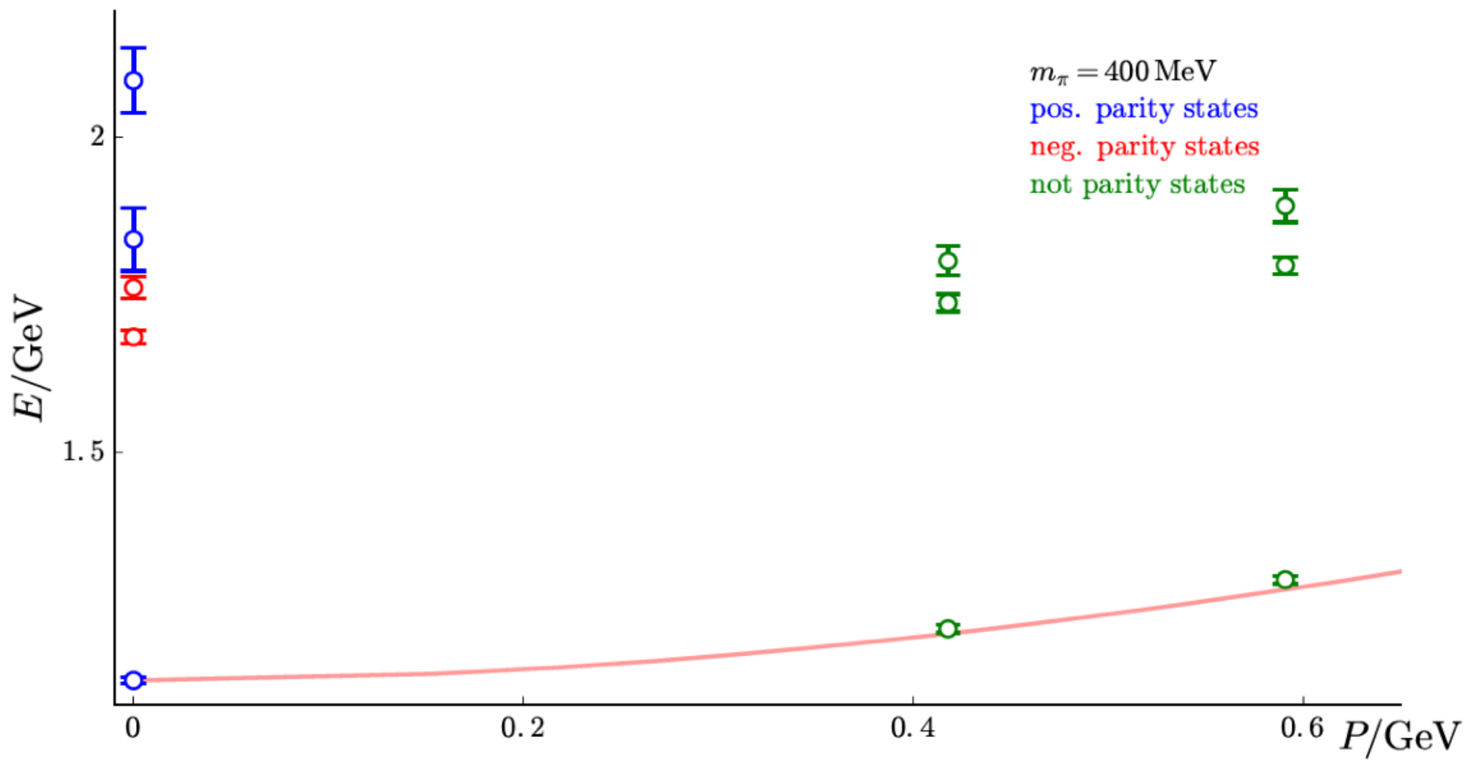
$$\tau^{ij}_{\alpha\beta}(t,0) = \xi^{*i}(t)M^{-1}(t,0)_{\alpha\beta}\xi^j$$

$$C_{ij}(t) = \phi^{i,(pqr)}_{\alpha\beta\gamma)}(t)\phi^{j,(\bar{p}\bar{q}\bar{r})}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}(0) \times \left[\tau^{p\bar{p}}_{\alpha\bar{\alpha}}(t,0)\tau^{q\bar{q}}_{\beta\bar{\beta}}(t,0)\tau^{r\bar{r}}_{\gamma\bar{\gamma}}(t,0) + \ldots\right]$$

- Meson correlation functions N^3
- Baryon correlation functions N^4

Severely constrains baryon lattice sizes

Nucleon Dispersion Relation



Isotropic Clover Production

ID	m_l	β	a (fm)	M_π (MeV)	L	T	$M_\pi L$	Split at	N_{traj}	On Titan
C12	-0.2800	6.1	0.118	430	48	96	12.4		20000	
C13	-0.2850	6.1	0.114	300	32	96	5.6			1762 - 2104
C13a	-0.2850	6.1	0.114	300	32	96	5.6			1100 - 1870
C13b	-0.2850	6.1	0.114	300	32	96	5.6			1000 - 2618
C13-W	-0.2850	6.1	0.114	300	32	96	5.6			2108 - 3164
C13a-W	-0.2850	6.1	0.114	300	32	96	5.6			1872 - 3564
C13b-W	-0.2850	6.1	0.114	300	32	96	5.6			2620 - 3980
D4	-0.2350	6.3	0.085	400	32	64	5.5		5164	
D5	-0.2390	6.3	0.081	310	32	64	4.0		6020	1000 - 6020
D6	-0.2416	6.3	0.080	210	48	96	3.7		2312 (a)	1000 - 2312
D6a	-0.2416	6.3	0.080	210	48	96	3.7	1000	866 (a)	254 - 866
D6b	-0.2416	6.3	0.080	210	48	96	3.7	1200	956 (a)	284 - 956
D7	-0.2416	6.3	0.080	210	64	128	4.9		1514 (a)	1112 - 1514
D7b	-0.2416	6.3	0.080	210	64	128	4.9	700	640 (a)	330 - 640
D7c	-0.2416	6.3	0.080	210	64	128	4.9	750	732 (a)	288 - 592
D7d	-0.2416	6.3	0.080	210	64	128	4.9	800	762 (a)	328 - 736
D8	-0.2424	6.3	0.080	140	72	196	4.1		370 (b)	

Add third lattice spacing: β = 6.5, a ~0.06

SUMMARY

- Controlling systematic uncertainties key at both low momenta and high momenta
- Momentum methods for direct calculation of form factors
- Can we get to high momenta? Exploring “distillation” for pion (see Bipasha)…