Far-from-Equilibrium Attractors and non-linear dynamical systems approach to relativistic hydrodynamics

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arXiv:1711.01745

Theory Center Seminar
Jefferson Lab, USA

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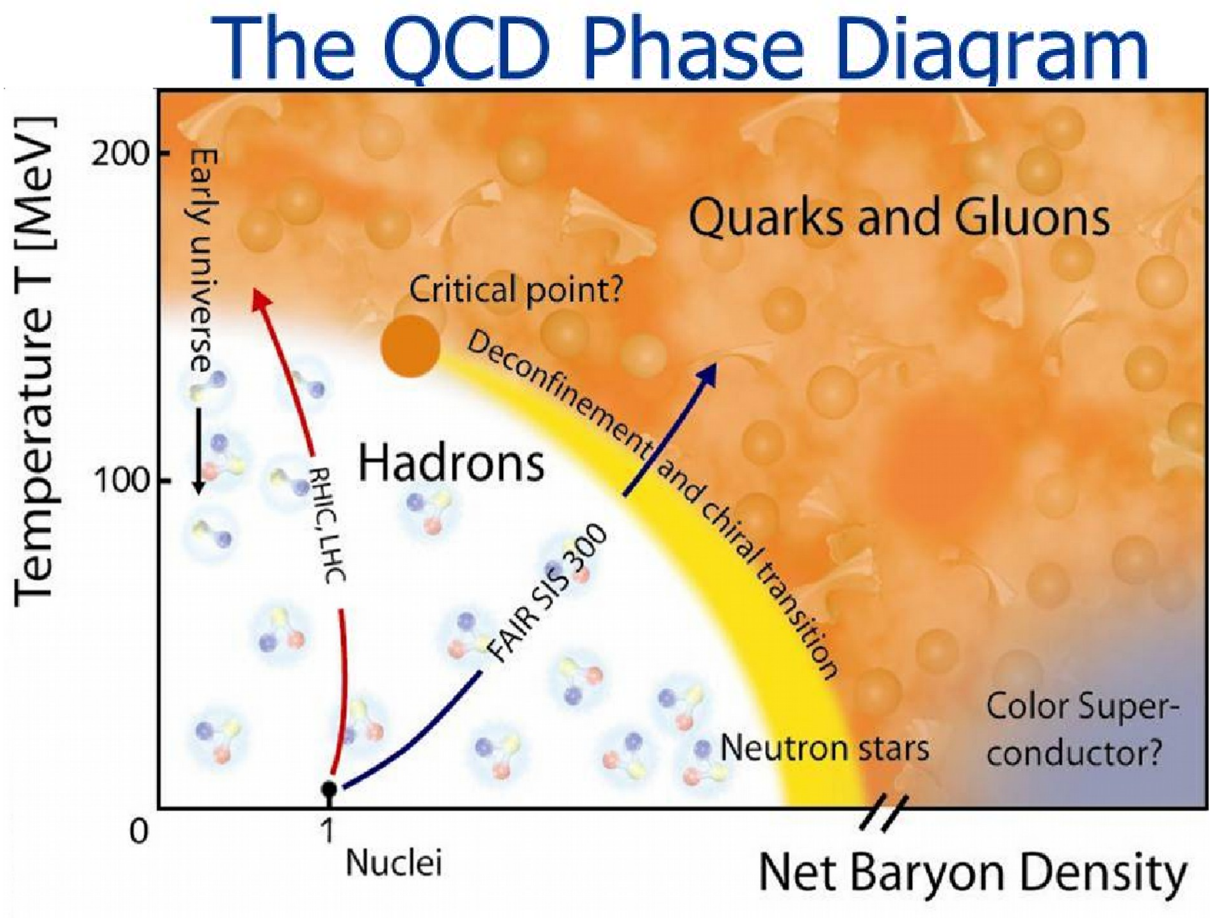
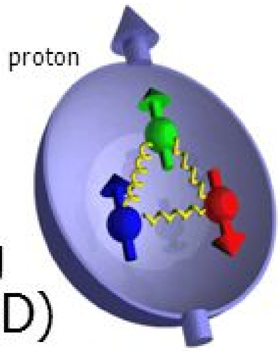
Motivation:
Success of hydrodynamics in high energy nuclear collisions

Determining the QCD phase diagram



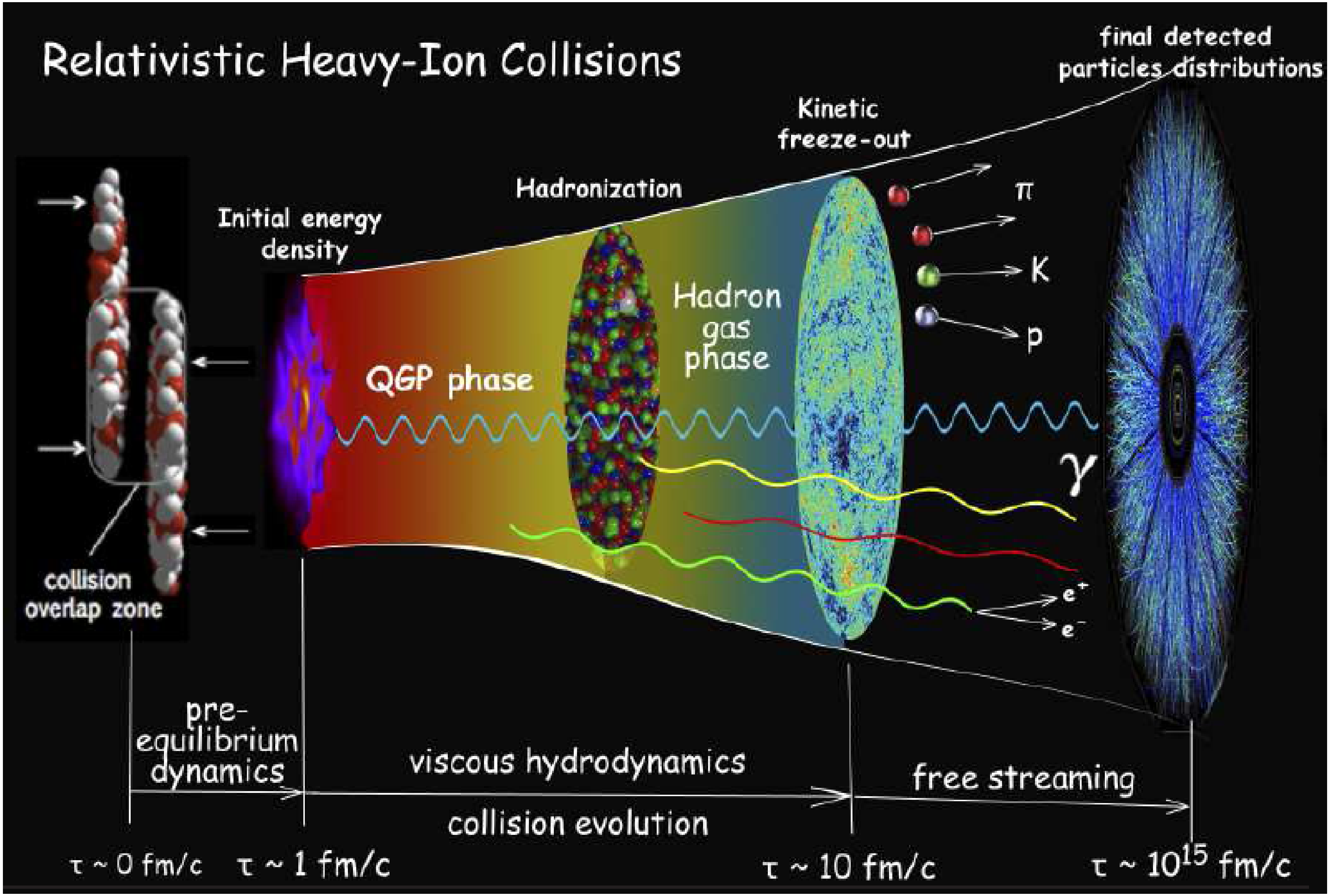


Insights into theory of strong interactions (QCD)

Medium created in heavy-ion (HIC) collisions similar to the one created after Big Bang

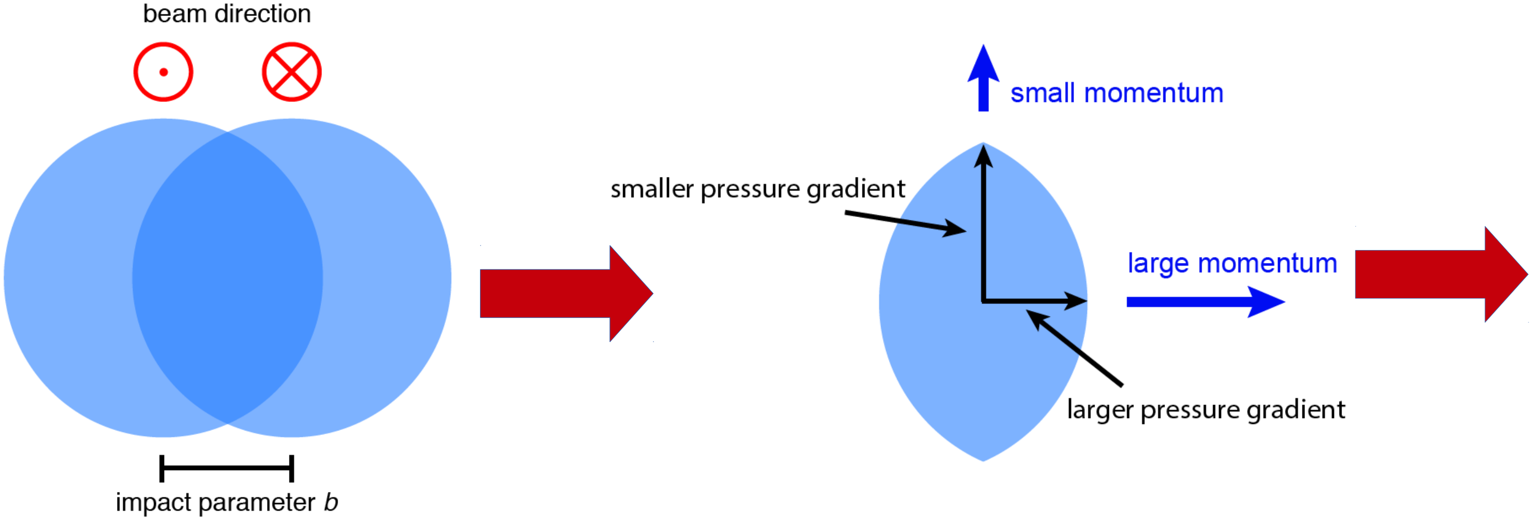
Explore the phase diagram of QCD with HIC

The Little Big Bang



Characterizing observables from the initial seeds

Simple idea: final state observables carry information of the initial state anisotropies!!!



Anisotropies in $dN/d\phi$

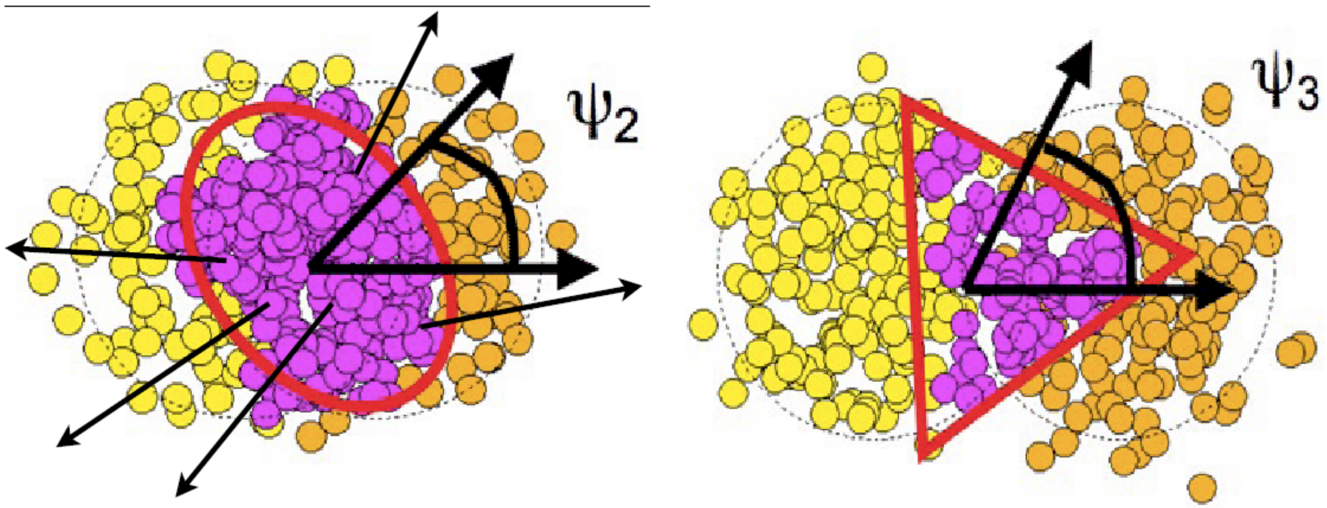
$\Rightarrow$ Anisotropy is quantified through a Fourier analysis of the particle multiplicity

$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left(1 + \sum_n (2v_n \cos(n\phi))\right)$$

$\Rightarrow$ *v_2 characterizes ellipticity*

Characterizing observables from the initial seeds

Nice…. but the description of initial state anisotropies is difficult

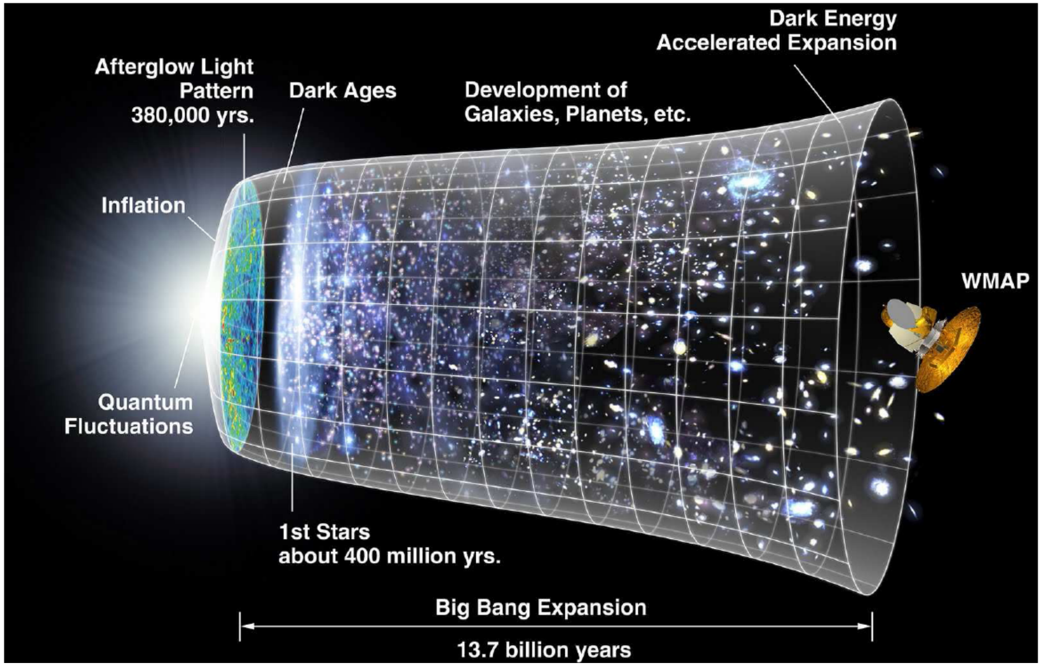


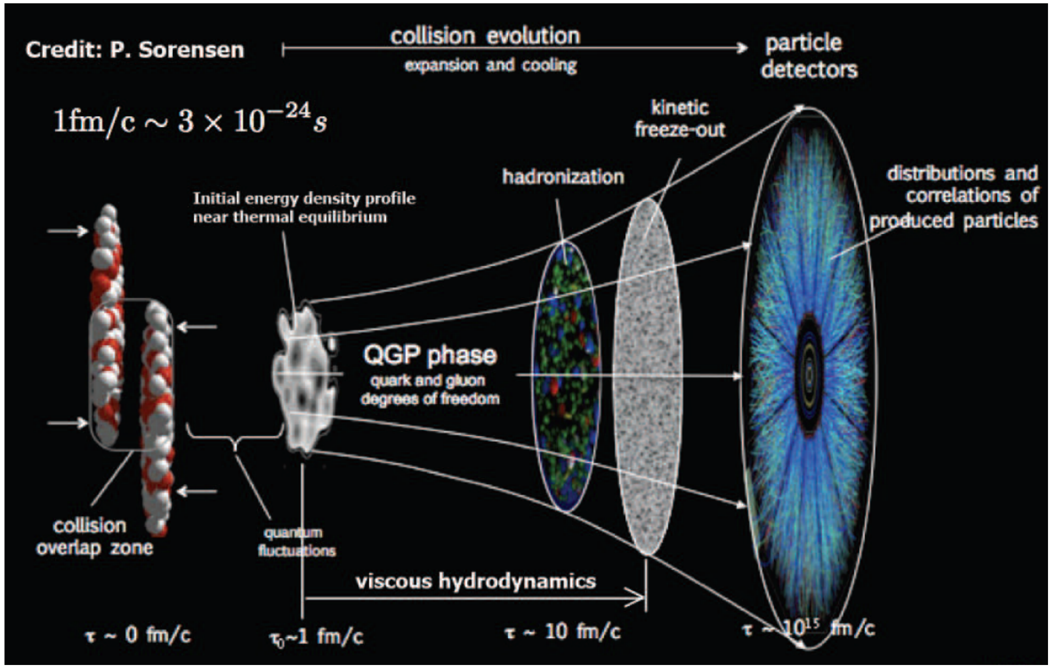
*Alver and Roland, PRC*81 (2010) 054905

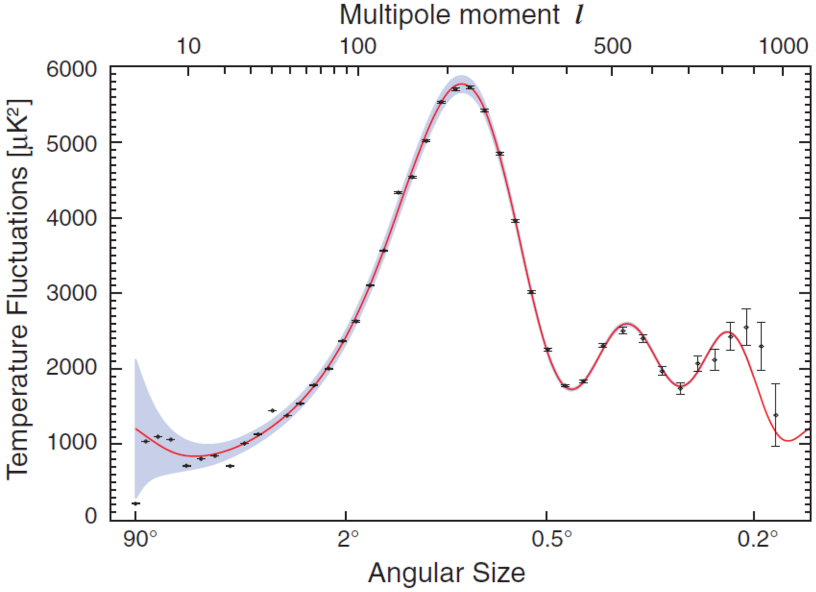
Positions of nucleons fluctuate

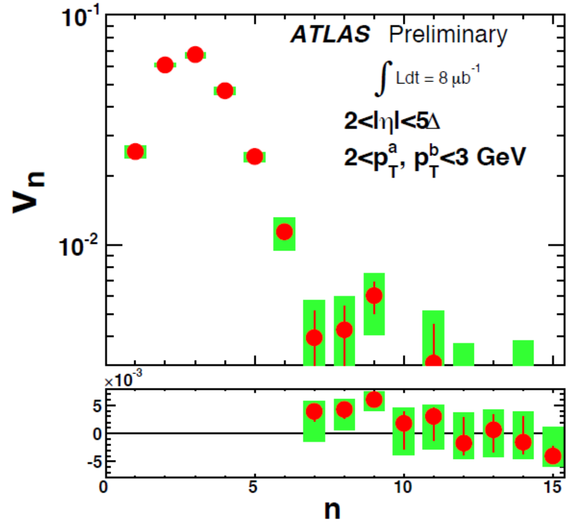
⇛ Each event has its own initial shape

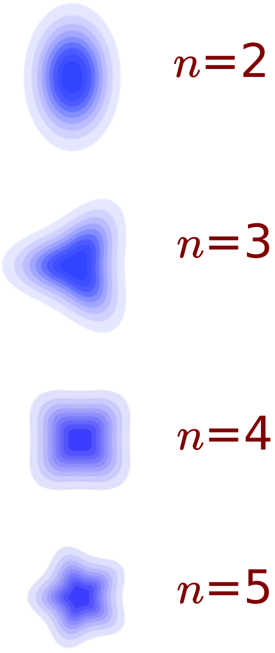
Why do need to study the initial state fluctuations?



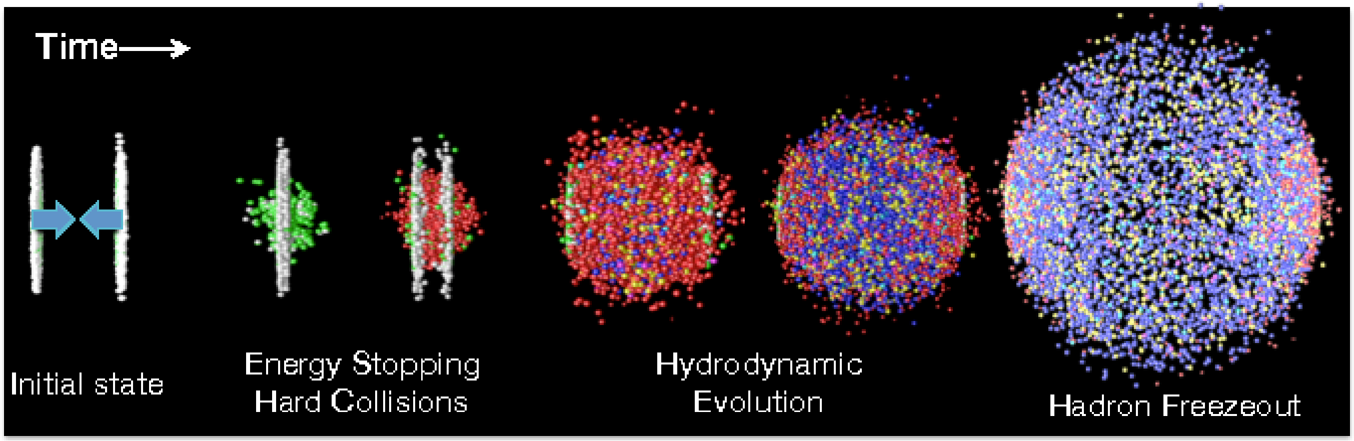








Evolving the fireball



One needs to evolve the bulk of matter deposited in each collision

- **System evolves from an initial state (e.g. initial energy density distribution) according with the energy-momentum conservation law**

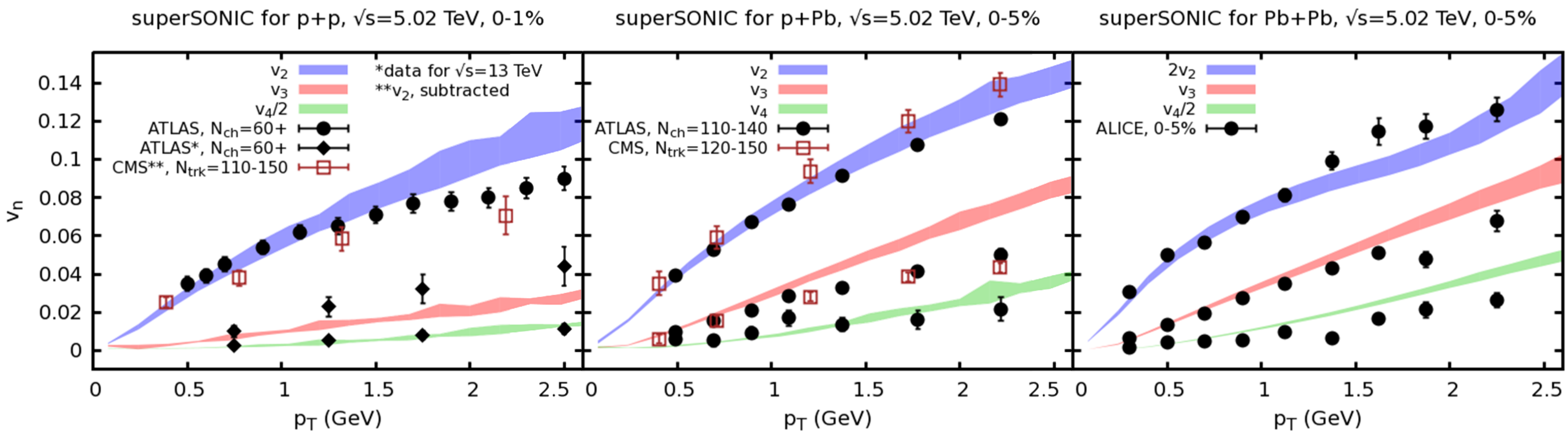
$$\partial_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - P g^{\mu\nu} + \pi^{\mu\nu}$$

Additional equations are needed for $\pi^{\mu\nu}$

"Unreasonable" success of hydrodynamics

***Hydrodynamical models** provide a good description of the low momentum observables in p+p, p+Pb and Pb+Pb collisions*



*Weller and Romatschke, Phys. Lett. B*774 (2017) 351-356

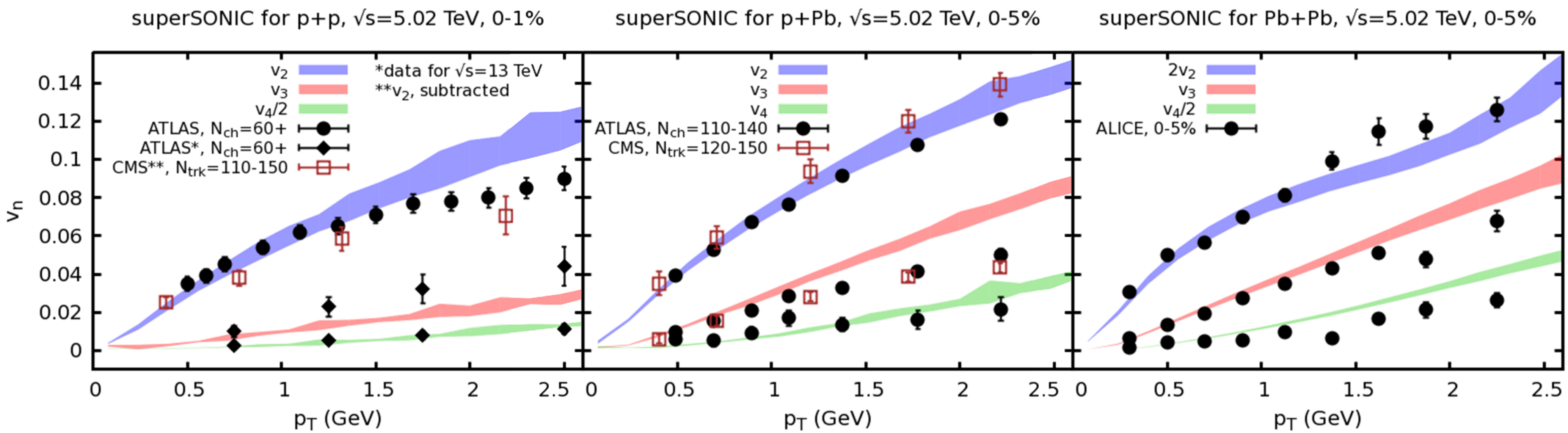
Experimental data is described with a small value of shear viscosity/entropy

$$\frac{\eta}{s} = 0.08 \pm 50\%$$

Quark gluon plasma: the hottest, tiniest and most perfect fluid ever made on Earth

"Unreasonable" success of hydrodynamics

Hydrodynamical models *provide a good description of the low momentum observables in p+p, p+Pb and Pb+Pb collisions*



*Weller and Romatschke, Phys. Lett. B*774 (2017) 351-356

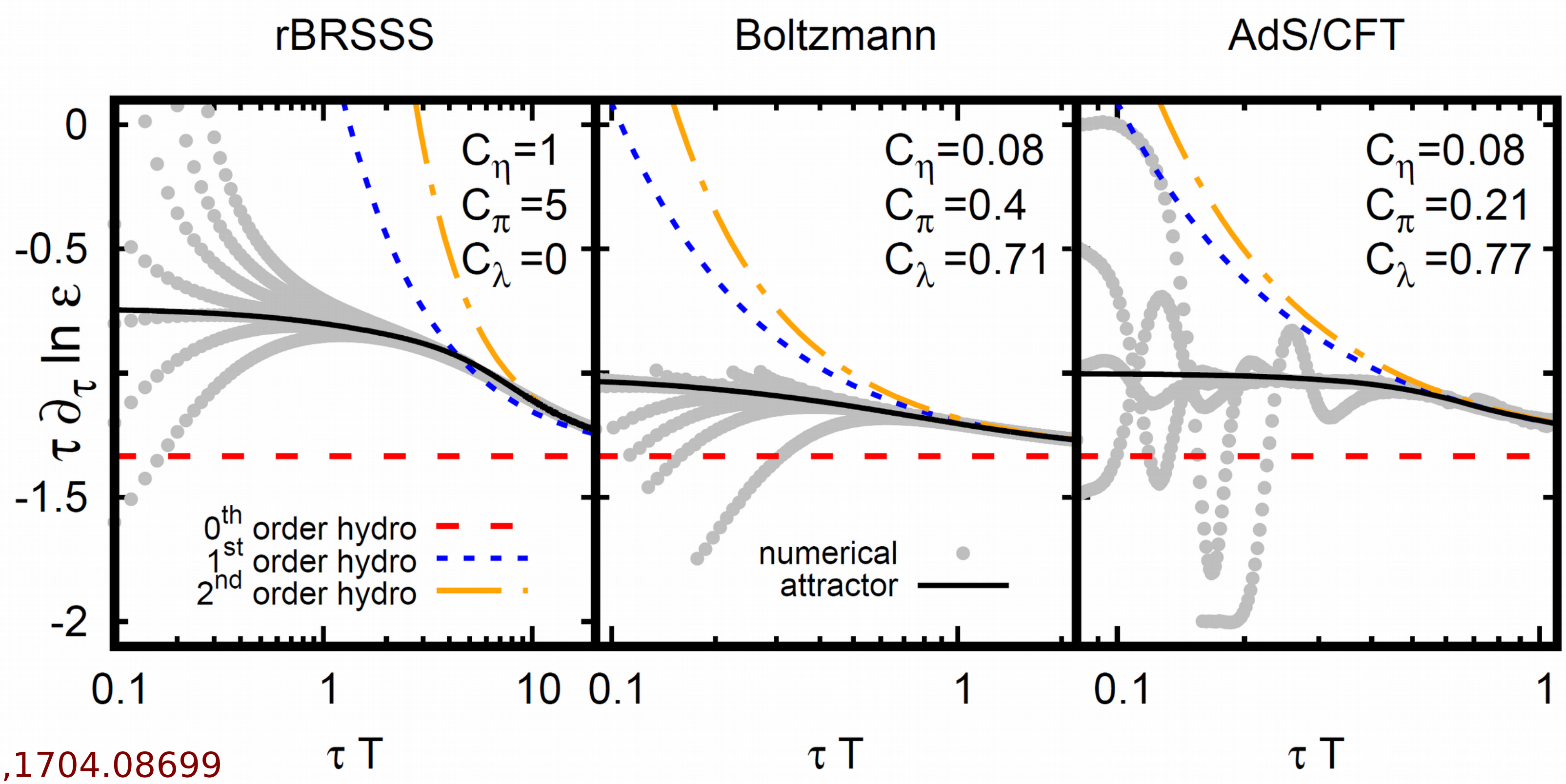
- *how could it be that hydrodynamics works in far-from-equilibrium situations?*

The unreasonable success of hydro might be related with the existence of far-from-equilibrium hydro attractors

In this talk

Attractors in hydrodynamics

Attractors in Hydro: numerical evidence

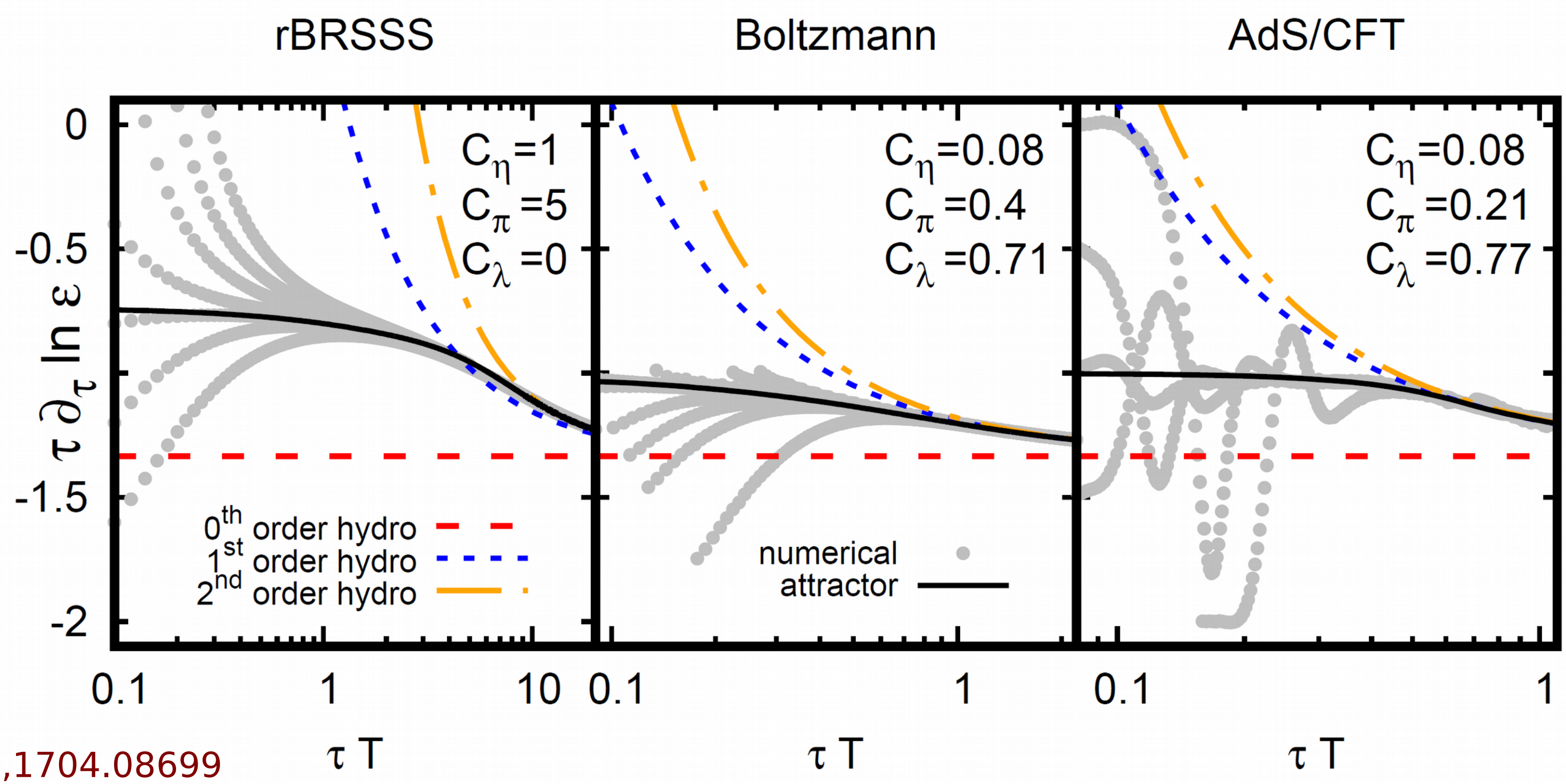


Romatschke,1704.08699

For an effective 0+1 *dim. (Bjorken model) conformal viscous fluid*:

- *arbitrarily far-from-equilibrium initial conditions used to solve hydro equations merge towards a unique line (attractor).*
- *Independent of the coupling regime.*
- *Attractors can be determined from very few terms of the gradient expansion*
- *At the time when hydrodynamical gradient expansion merges to the attractor, the system is far-from-equilibrium, i.e. large pressure anisotropies are present in the system $P_L \neq P_T$*

Attractors in Hydro: numerical evidence



Romatschke,1704.08699

Questions

- *What is the physics of attractors?*
- *How to characterize them?*
- *What happens with more complex expanding fluids (e.g.* 1+1 *dim*, 3+1 *dim.)?*

Attractors in kinetic theory: Gubser flow

arXiv:1711.01745

A bit of kinetic theory

- *Our starting point is the Boltzmann equation in the RTA approximation*

$$p_\mu \partial^\mu f(x^\mu, p_i) = \mathcal{C}[f]$$

$$\mathcal{C}[f] = -\frac{1}{\tau_{rel}} \left(f(x^\mu, p_i) - f_{eq.}(x^\mu, p_i) \right)$$

- *For conformal systems*

$$\tau_{rel} = \frac{c}{T(x^\mu)} \qquad \text{with} \qquad c = 5\frac{\eta}{\mathcal{S}}$$

- *The relevant macroscopic quantities are obtained by considering the hydrodynamical moments*

$$J^\mu = \langle p^\mu \rangle$$

$$T^{\mu\nu} = \langle p^\mu \, p^\nu \rangle$$

$$\langle \mathcal{O}(x^\mu, p_i) \rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{-g}\, p^0} \mathcal{O}(x^\mu, p_i) \, f(x^\mu, p_i)$$

Evolution equations for the moments are obtained from the Boltzmann equation

Deriving hydrodynamics from kinetic theory

Try to solve the Boltzmann equation by expanding around some particular evolving background

$$f(x^\mu, p_i) \approx f_0(x^\mu, p_i) + \delta f \qquad \delta f \lll f_0$$

- f_0 *describes the evolving background.*
- δf *encodes the information of the variations of the macroscopic variables around the background*

$$\delta f \subset \mathcal{O}(Kn, Re^{-1}, Kn^2, Kn \cdot Re^{-1}, Re^{-2}, ...)$$

Two types of dissipative corrections

- *Knudsen number* (Kn): *inhomogeneities of the fluid due to collisions.*

$$Kn \equiv \frac{l_{micro}}{l_{macro}} \sim \frac{\lambda_{m.f.p.}}{L}$$

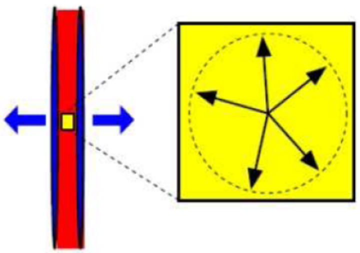
- *Inverse Reynolds number* (Re^{-1}): *space-time inhomogeneities of the macroscopic fluid variables*

$$Rn^{-1} \equiv \frac{|\delta T^{\mu\nu}|}{|T^{\mu\nu}|} \sim \frac{|\Pi^{\mu\nu}|}{P_0}$$

Deriving hydrodynamics from kinetic theory

- *Different expansions of the distribution function do not lead to the same evolution equations of the macroscopic variables*

$$f(x^\mu, p_i) \approx f_0(x^\mu, p_i) + \delta f$$

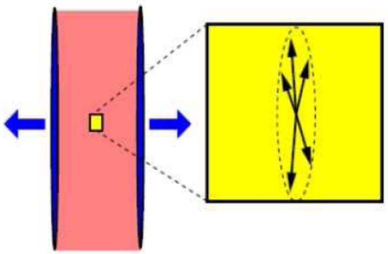


Viscous hydrodynamics

$$f(x,p) = f_{eq} + \delta f(x,p)$$

The equilibrium distribution f_{eq} function is isotropic in momentum space

All the momentum space anisotropies are perturbations around the equilibrium distribution function



Anisotropic hydrodynamics

$$f(x,p) = f_a + \delta \tilde{f}(x,p)$$

The leading order distribution function f_a encodes the largest anisotropies developed at early times

Deviations from the spheroidal form of the leading order term are treated as small perturbations

Gubser flow

- Gubser flow is a boost-invariant longitudinal and azimuthally symmetric transverse flow (Gubser 2010, Gubser & Yarom 2010)

$$SO(3)_q \otimes SO(1,1) \otimes Z_2$$

Special Conformal transformations + rotation along the beam line

Boost invariance

Reflections along the beam line

Gubser flow

- Gubser flow is a boost-invariant longitudinal and azimuthally symmetric transverse flow (Gubser 2010, Gubser & Yarom 2010)

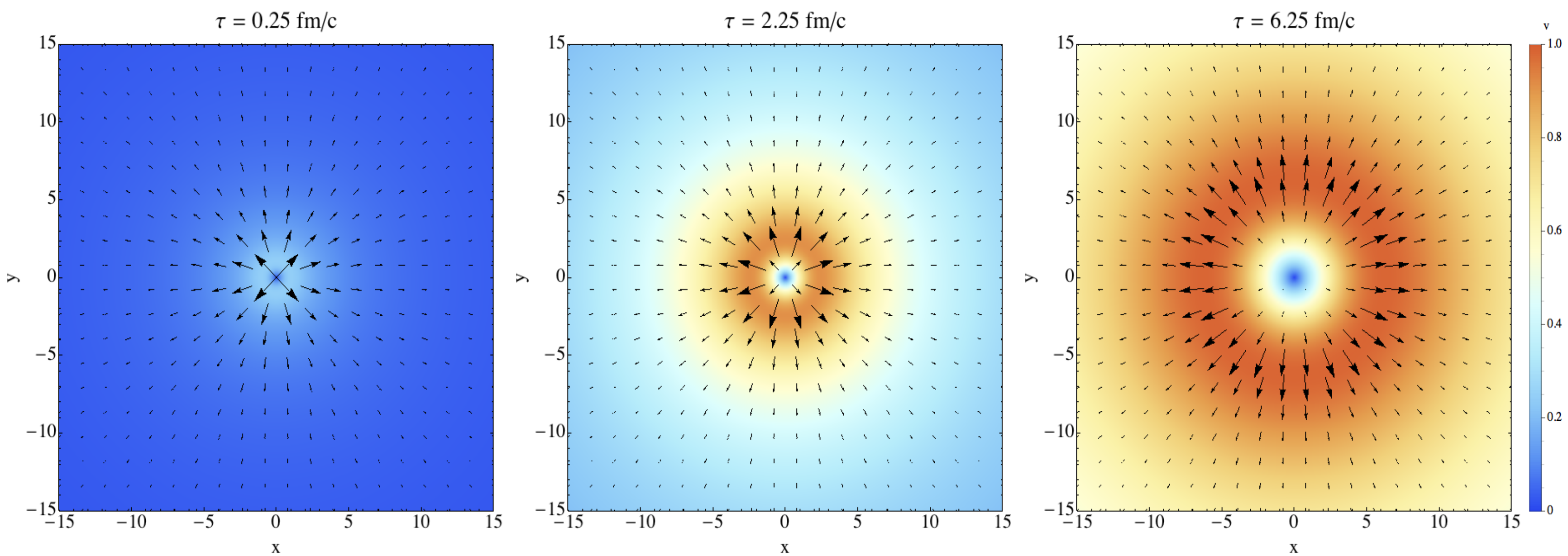
$$SO(3)_q \otimes SO(1,1) \otimes Z_2$$

In polar Milne Coordinates (τ, r, ϕ, η)

$$u^\mu = (\cosh\kappa(\tau,r), \sinh\kappa(\tau,r), 0, 0)$$

$$\kappa(\tau,r) = \tanh^{-1}\left(\frac{2q^2\tau r}{1+(qr)^2+(q\tau)^2}\right)$$

$\boldsymbol{q}$ is a scale parameter

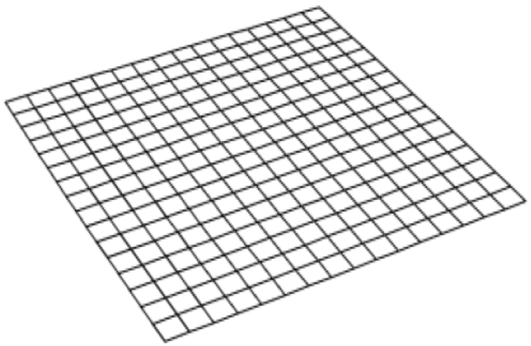


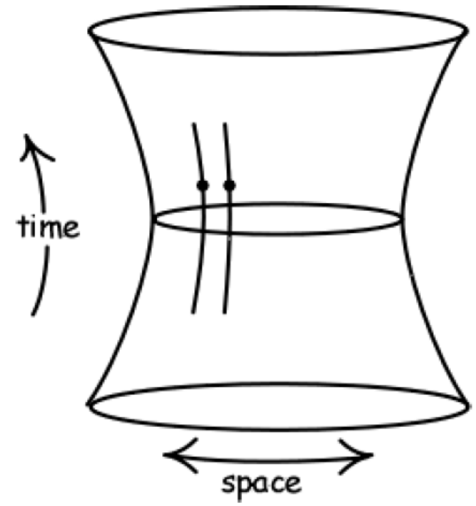
Gubser flow

$$g_{\mu\nu}(x) \rightarrow e^{-2\Omega(x)} g_{\mu\nu}(x)$$

Flat Minkowski space

dS$_3$ x R





$$\sinh\rho = -\frac{1-\tilde{\tau}^2+\tilde{r}^2}{2\tilde{\tau}}, \qquad \tan\theta = \frac{2\tilde{r}}{1+\tilde{\tau}^2-\tilde{r}^2}.$$

Complicated dynamics

3d de Sitter space

$$x^\mu = (\tau, r, \phi, \eta) \quad \Rightarrow \quad \hat{x}^\mu = (\rho, \theta, \phi, \eta)$$

$$ds^2 = -d\tau^2 + dr^2 + r^2\, d\phi^2 + d\eta^2 \quad \Rightarrow \quad d\hat{s}^2 = -d\rho^2 + \cosh^2\rho\,(d\theta^2 + \sin^2\theta d\phi^2) + d\eta^2$$

$$u^\mu = (u^\tau(\tau,r), u^r(\tau,r), 0, 0) \quad \Rightarrow \quad \hat{u}^\mu = (1,0,0,0)$$

$$\epsilon(\tau, r) \quad \Rightarrow \quad \hat{\epsilon}(\rho)$$

Exact Gubser solution

- *In $dS_3 \otimes R$ the dependence of the distribution function is restricted by the symmetries of the Gubser flow*

$$f(\hat{x}^\mu, \hat{p}_i) = f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right)$$

$$\hat{p}_\Omega^2 = \hat{p}_\theta^2 + \frac{\hat{p}_\phi^2}{\sin^2\theta}$$ → *Total momentum in the* (θ,ϕ) *plane*

$\hat{p}_\eta$ → *Momentum along the* η *direction*

- *The RTA Boltzmann equation gets reduced to*

$$\frac{\partial}{\partial\rho} f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right) = -\frac{\hat{T}(\rho)}{c}\left(f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right) - f_{eq}\left(\hat{p}^\rho / \hat{T}(\rho)\right)\right)$$

$$c = 5\frac{\eta}{\mathcal{S}}$$

- *The exact solution to this equation is*

$$f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right) = D(\rho, \rho_0)\, f_0\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right) + \frac{1}{c}\int_{\rho_0}^{\rho} d\rho'\, \hat{T}(\rho')\, D(\rho, \rho')\, f_{eq}\left(\hat{p}^\rho / \hat{T}(\rho)\right)$$

Fluid models for the Gubser flow

E-M conservation law →

$$\frac{\partial_\rho \hat{T}}{\hat{T}} + \frac{2}{3}\tanh\rho = \frac{\bar{\pi}}{3}\tanh\rho$$

$$\hat{\tau}_{\hat{\pi}}\left(\partial_\rho\bar{\pi} + \frac{4}{3}\left(\bar{\pi}\right)^2 \tanh\rho\right) + \bar{\pi} = \frac{4}{3}\frac{\eta}{s\hat{T}}\tanh\rho + \frac{10}{7}\hat{\tau}_{\hat{\pi}}\bar{\pi}\tanh\rho$$

IS theory

Fluid models for the Gubser flow

E-M conservation law

$$\frac{\partial_\rho \hat{T}}{\hat{T}} + \frac{2}{3}\tanh\rho = \frac{\bar{\pi}}{3}\tanh\rho$$

DNMR theory

$$\hat{\tau}_{\hat{\pi}}\left(\partial_\rho \bar{\pi} + \frac{4}{3}(\bar{\pi})^2 \tanh\rho\right) + \bar{\pi} = \frac{4}{3}\frac{\eta}{s\hat{T}}\tanh\rho + \frac{10}{7}\hat{\tau}_{\hat{\pi}}\bar{\pi}\tanh\rho$$

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IS theory

Anisotropic hydrodynamics

$$\partial_\rho\bar{\pi} + \frac{\bar{\pi}}{\hat{\tau}_r} = \frac{4}{3}\tanh\rho\left(\frac{5}{16} + \bar{\pi} - \bar{\pi}^2 - \frac{9}{16}\mathcal{F}(\bar{\pi})\right)$$

Non-linear dynamical system analysis of the IS theory

arXiv:1711.01745

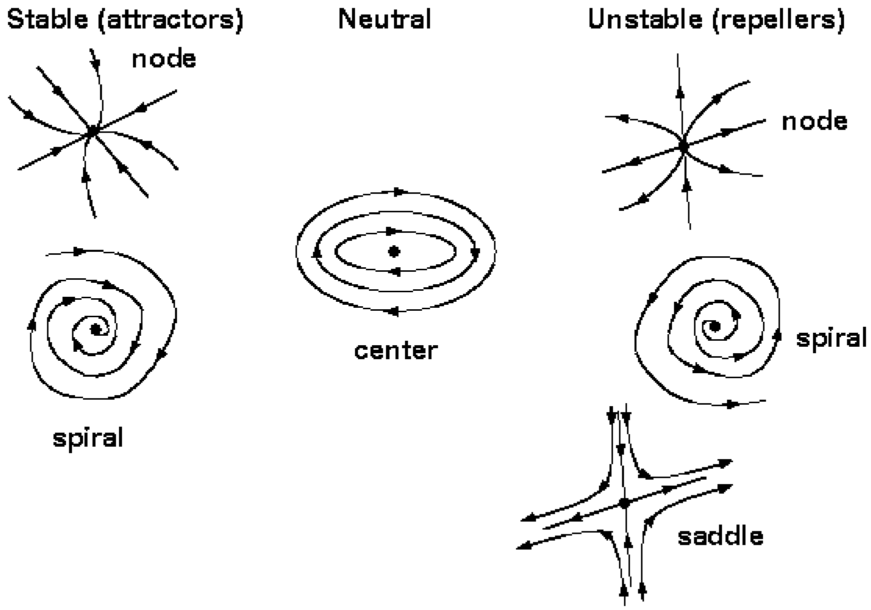
IS theory as a 2d non-autonomous system

IS evolution eqs. can be re-written as $\tau = \tanh\rho$

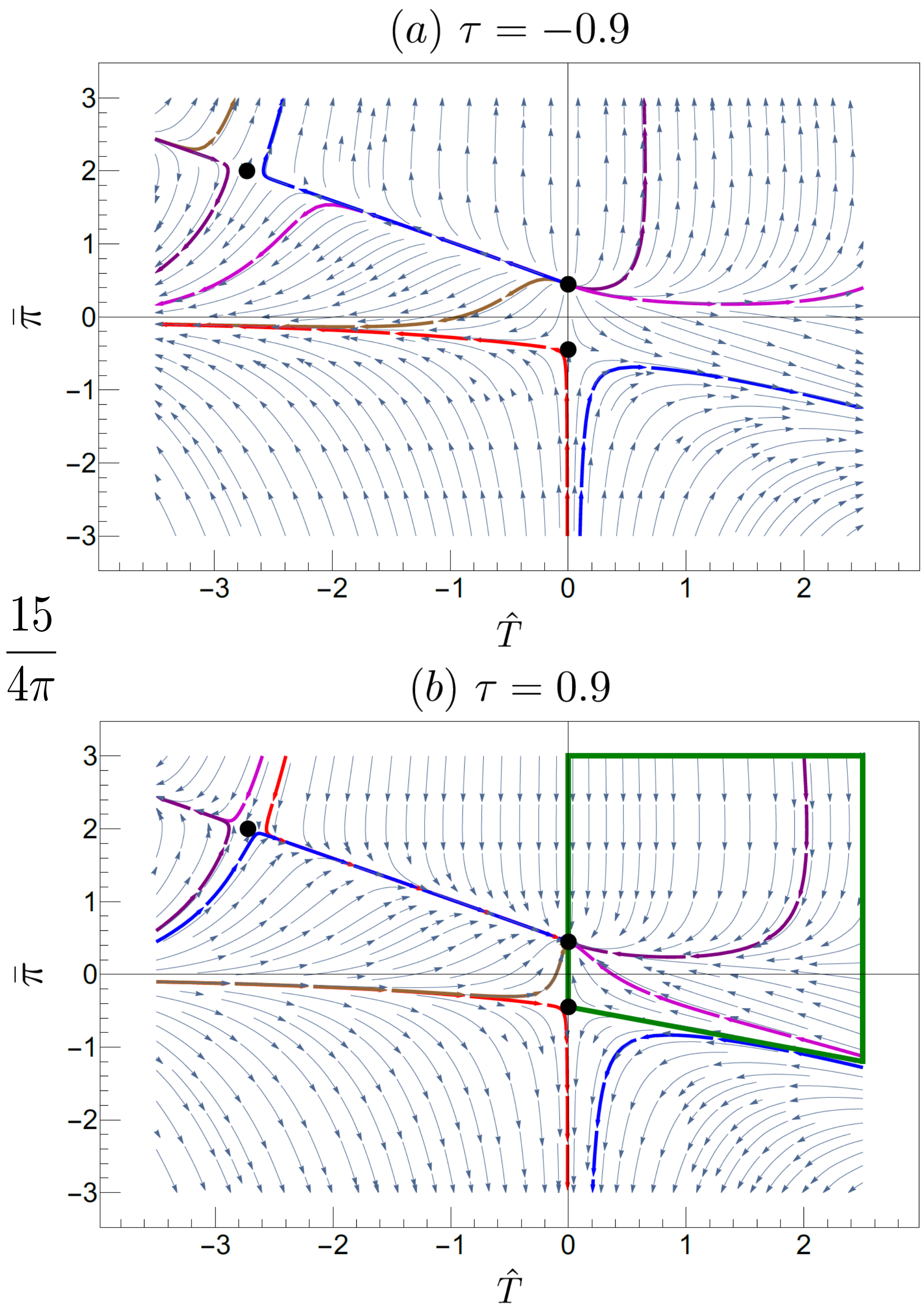
$$\frac{d\hat{T}}{d\tau} = \frac{\tau\hat{T}}{3(1-\tau^2)}\left(\bar{\pi}(\tau) - 2\right),$$

$$\frac{d\bar{\pi}}{d\tau} = -\frac{1}{1-\tau^2}\left(\frac{4}{3}\,\bar{\pi}^2(\tau)\,\tau + \frac{1}{c}\,\bar{\pi}(\tau)\,\hat{T}(\tau) - \frac{4}{15}\,\tau\right).$$

Before continuing, let's remember some basic of flow lines in the phase space of the dynamical variables



IS theory as a 2d non-autonomous system



$$c = \frac{15}{4\pi}$$

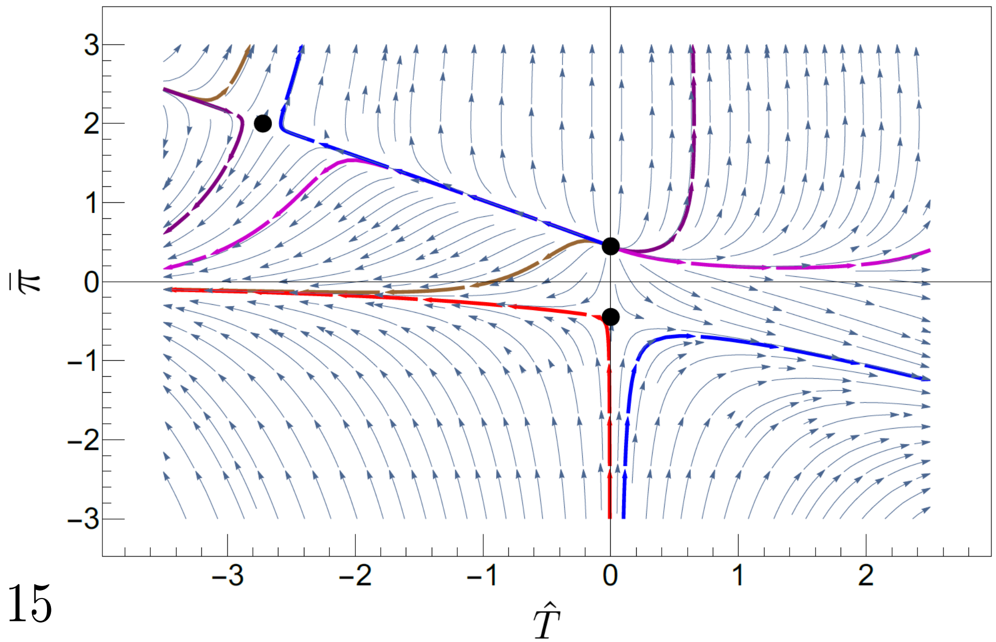
Fixed points are determined from the null-line conditions:

$$\frac{d\hat{T}}{d\tau} = 0$$

$$\frac{d\bar{\pi}}{d\tau} = 0$$

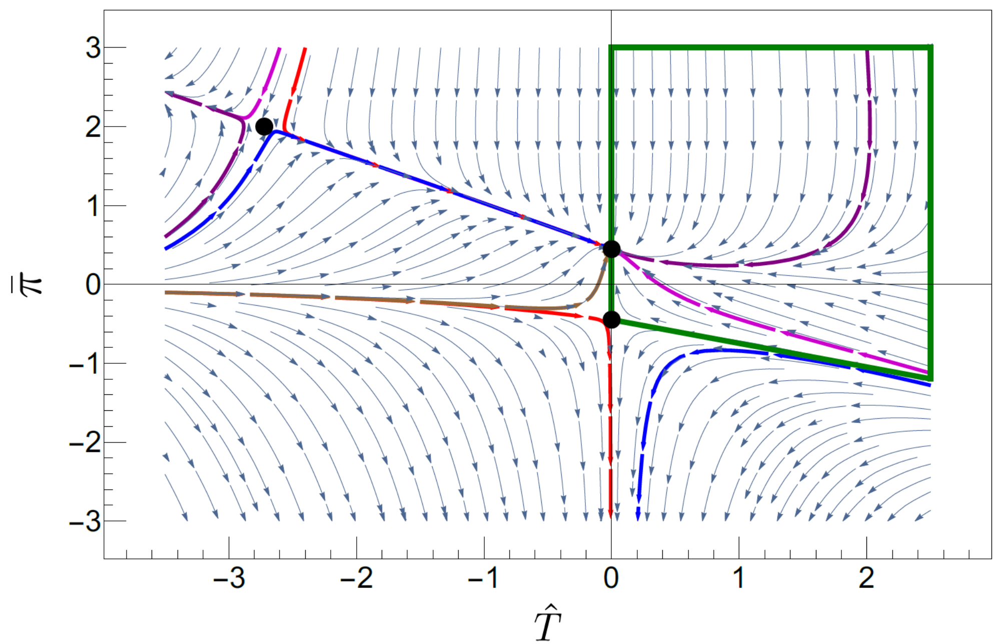
IS theory as a 2d non-autonomous system

$(a)\ \tau = -0.9$



$$c = \frac{15}{4\pi}$$

$(b)\ \tau = 0.9$



$$\text{Fixed points}: \quad \bar{\pi}_c^{\pm} = \pm\frac{1}{\sqrt{5}}, \quad \hat{T}_c = 0,$$

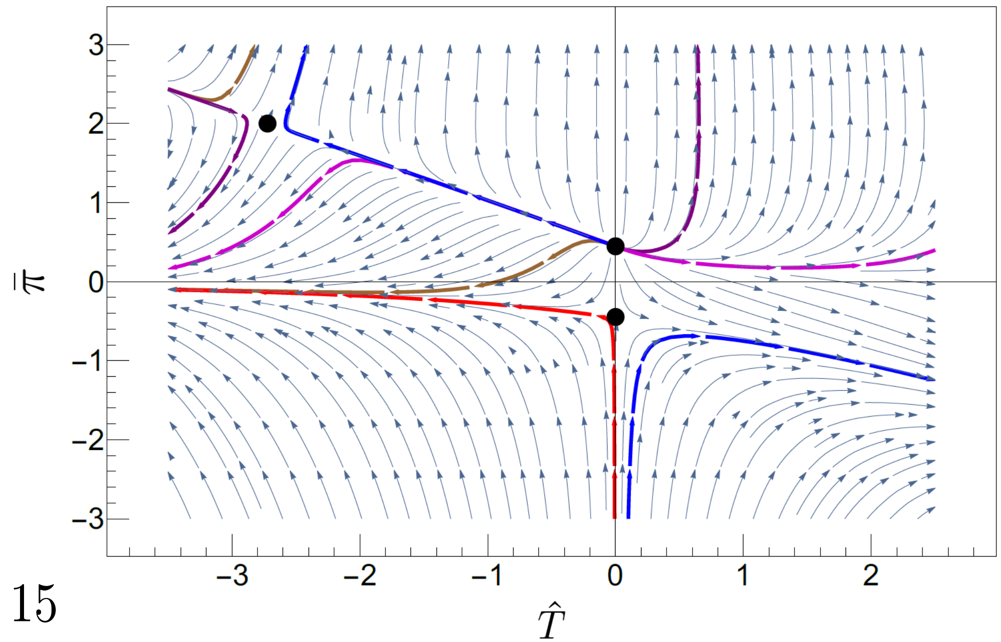
$$\bar{\pi}_c = 2, \quad \hat{T}_c = -\frac{38c}{15}.$$

Early times:

- *Three unstable fixed points*: 2 *saddle fixed point and one source*

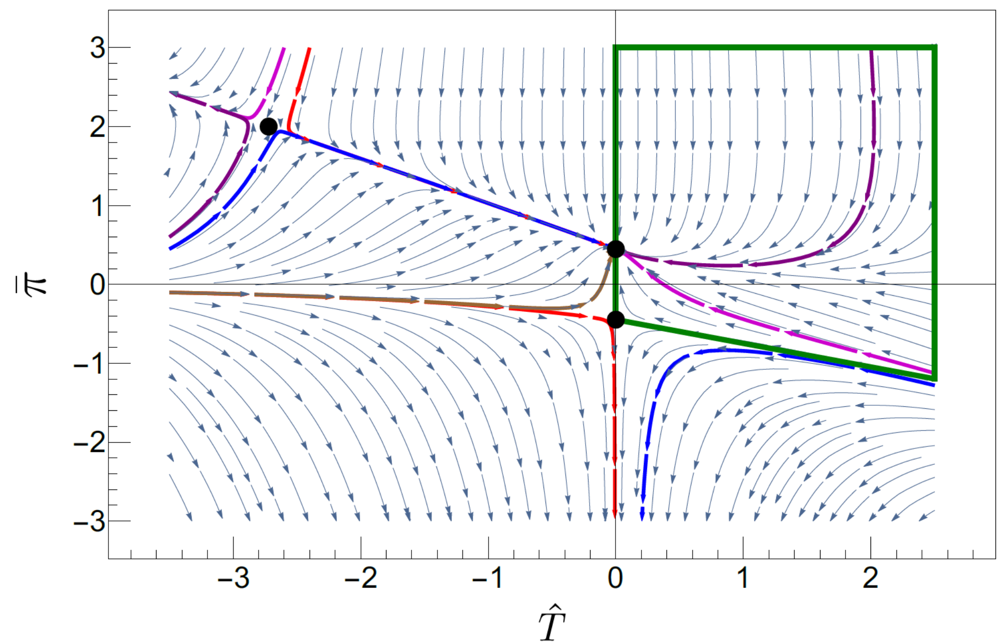
IS theory as a 2d non-autonomous system

(a) $\tau = -0.9$



$$c = \frac{15}{4\pi}$$

(b) $\tau = 0.9$



$$\text{Fixed points}: \quad \bar{\pi}_c^{\pm} = \pm\frac{1}{\sqrt{5}}, \quad \hat{T}_c = 0,$$

$$\bar{\pi}_c = 2, \quad \hat{T}_c = -\frac{38c}{15}.$$

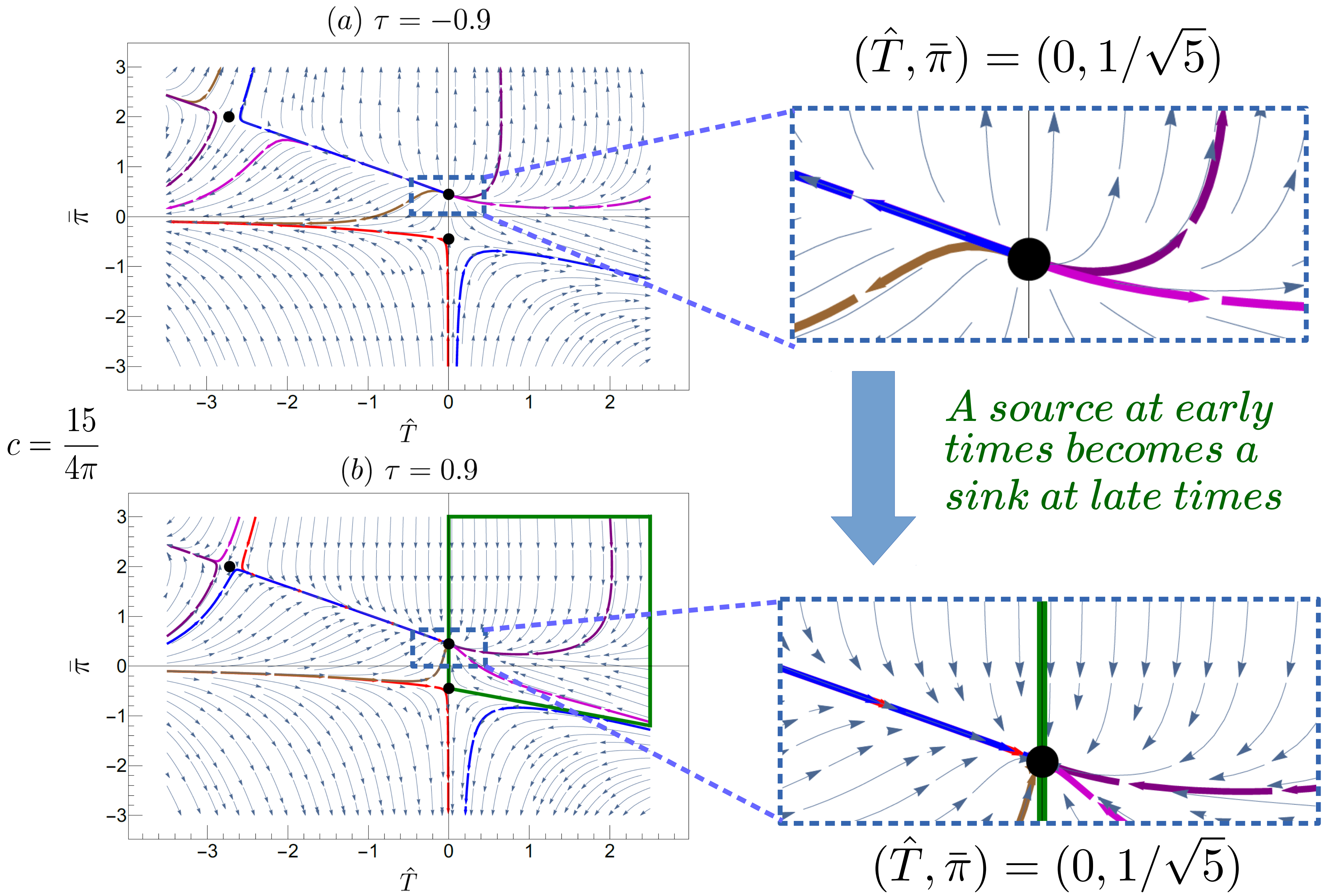
***Late times*:**

- *Two unstable fixed points (saddle) and one stable fixed point (sink)*
- *Stable point correspond to*

$$(\hat{T}, \bar{\pi}) = (0, 1/\sqrt{5})$$

$\Rightarrow$ *system never reaches thermal equilibrium.* *Steady non-equilibrium state*!!!

IS theory as a 2d non-autonomous system



Subtle issue of IS theory for Gubser flow

For the Gubser flow IS can be combined into one equation

$$\frac{\partial_\rho \hat{T}}{\hat{T}} + \frac{2}{3}\tanh\rho = \frac{\bar{\pi}}{3}\tanh\rho$$

$$\hat{\tau}_{\hat{\pi}}\left(\partial_\rho\bar{\pi} + \frac{4}{3}(\bar{\pi})^2\tanh\rho\right) + \bar{\pi} = \frac{4}{3}\frac{\eta}{s\,\hat{T}}\tanh\rho$$

$$\mathcal{A}(w) = \frac{1}{\tanh\rho}\frac{\partial_\rho\hat{T}}{\hat{T}} = \frac{d\log(\hat{T})}{d\log(\cosh\rho)}$$

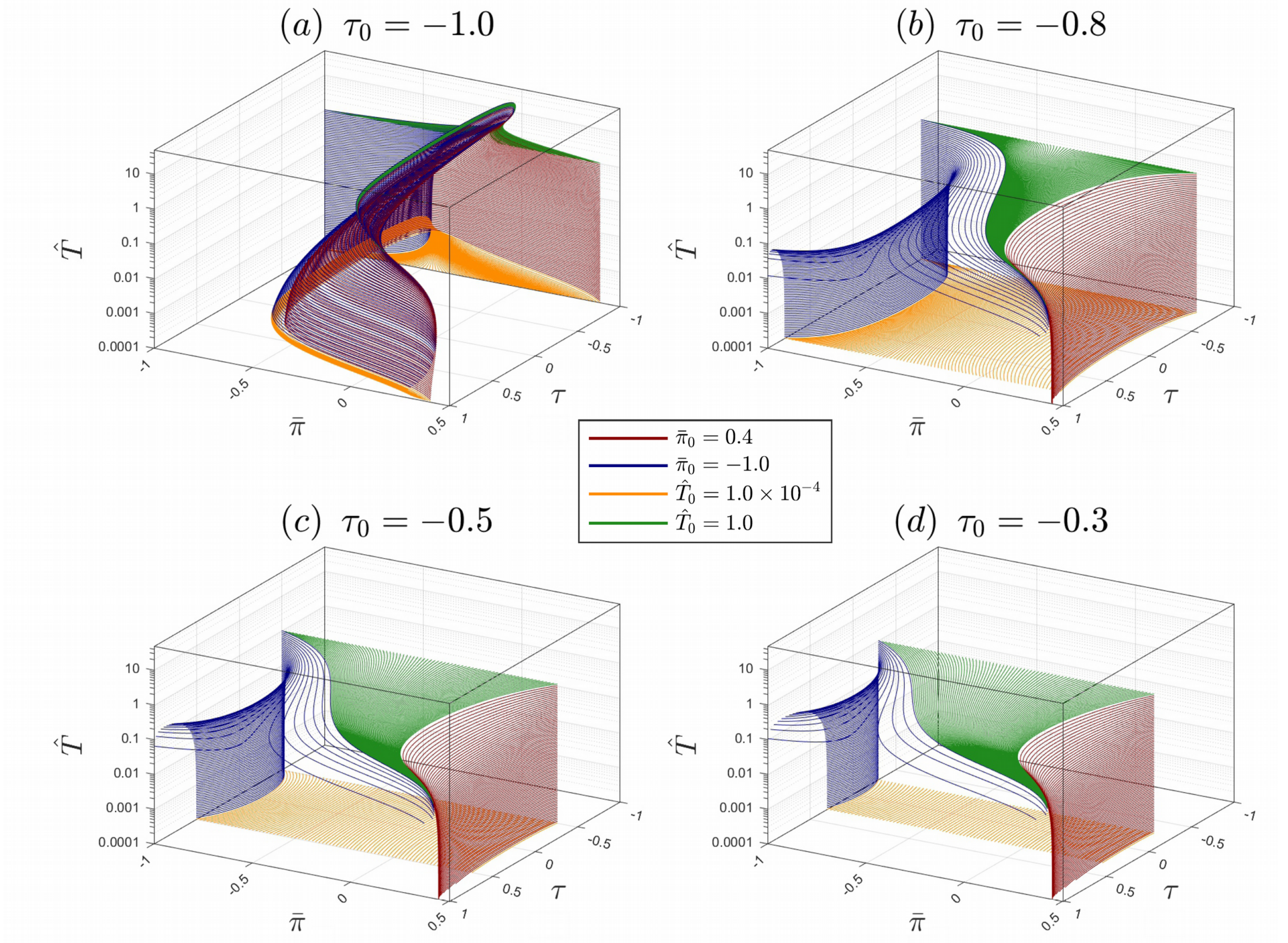
$$3w\left(\coth^2\rho - 1 - \mathcal{A}(w)\right)\frac{d\mathcal{A}(w)}{dw} + \frac{4}{3}(3\mathcal{A}(w)+2)^2 + \frac{3\mathcal{A}(w)+2}{cw} - \frac{4}{15} = 0$$

The solution of this ODE depends on ρ

- *$dS_3 \otimes R$ is a curved space whose expansion rate does not vanish asymptotically (non-equilibrium steady state)*
- *This did not happen for the 0+1 dim. system (Bjorken)*

Rethinking IS Eqs. as a 3d DOE system

$$\frac{d\hat{T}}{d\rho} = \frac{1}{3}\hat{T}(\bar{\pi} - 2)\tau, \qquad \frac{d\bar{\pi}}{d\rho} = \frac{4}{3}\left(\frac{1}{5} - \bar{\pi}^2\right)\tau - \frac{1}{c}\pi\hat{T}, \qquad \frac{d\tau}{d\rho} = 1 - \tau^2.$$



Rethinking IS Eqs. as a 3d DOE system

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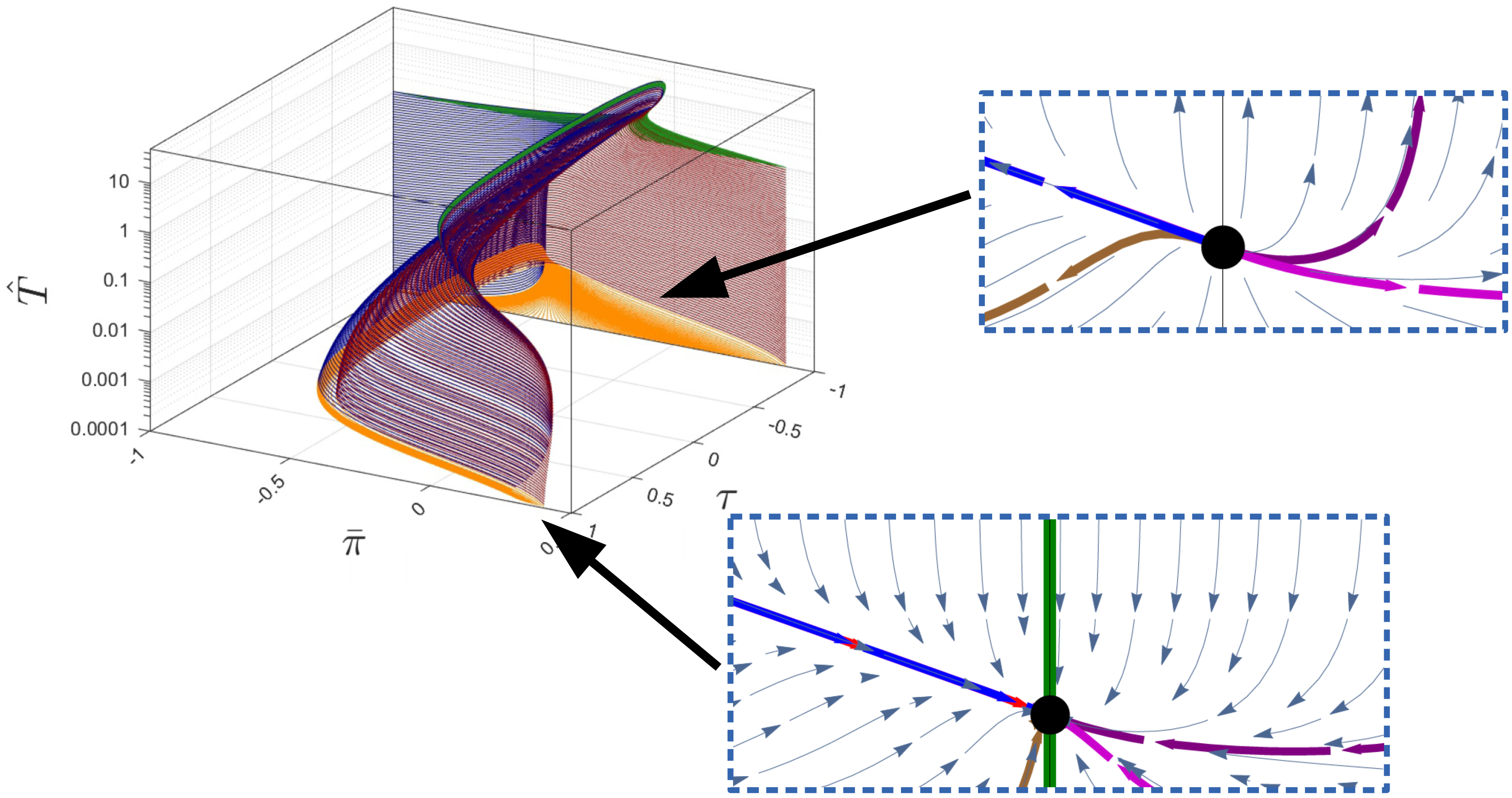
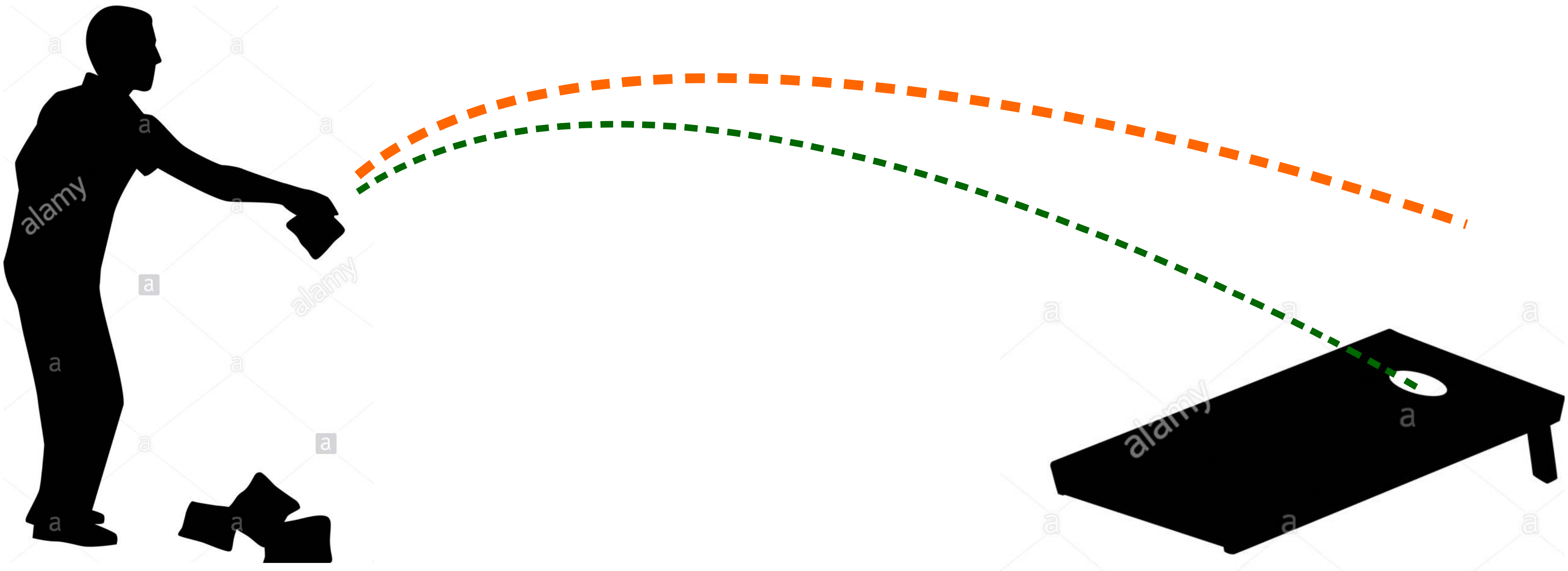
Basin of attraction for the Gubser flow is* 3 *dim.



Parenthesis:
Attractors in real life

Cornhole as a dynamical system

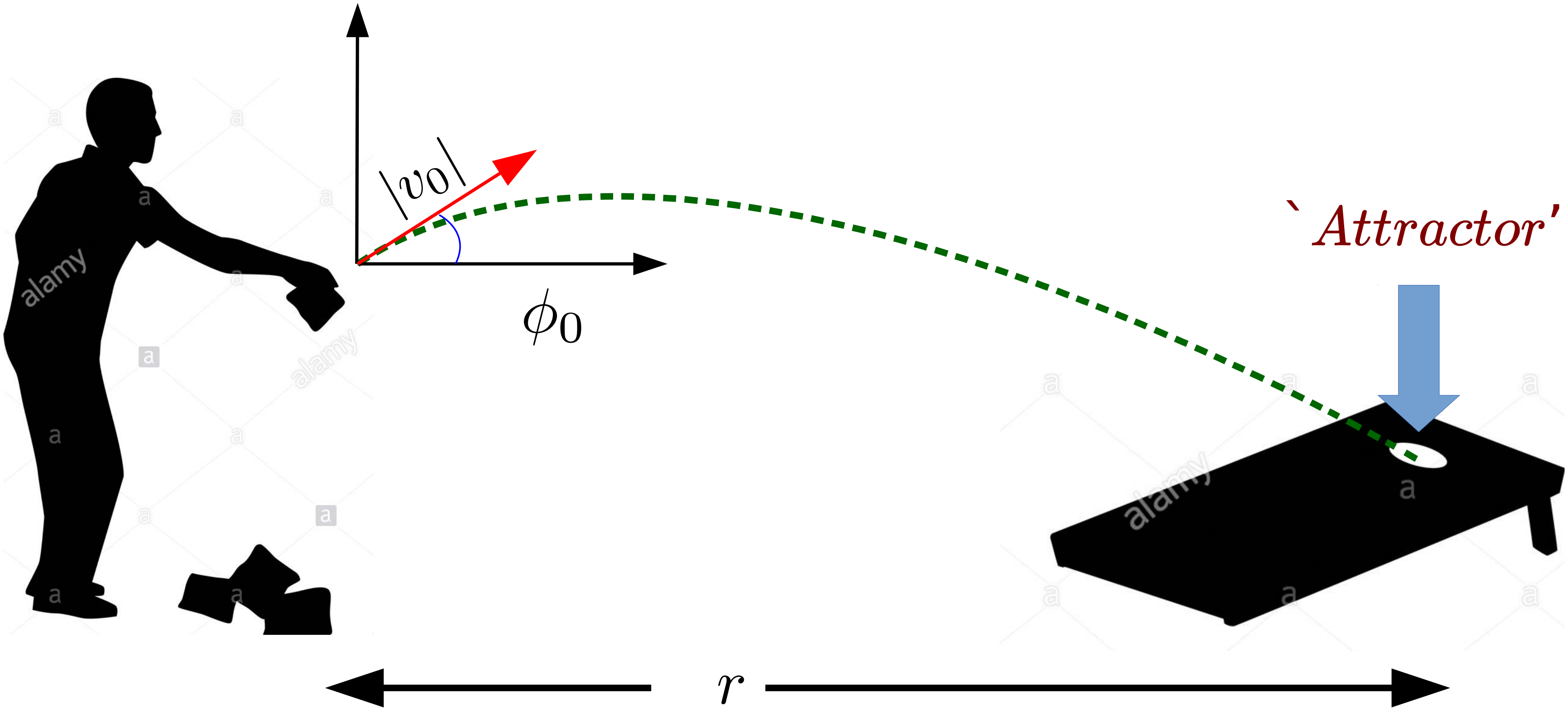


The goal: *score points by throwing a bean bag into the hole of a board*

- *However, not all the trajectories go into the hole....*

There are `privileged trajectories' (stable) and `non-privileged one' (unstable)

Cornhole as a dynamical system



- *The hole in the board is the* ***attractor*** *of the 'privileged trajectories'*
- *The `privileged trajectories' form a set (****basis of attraction****)*
- *Each `privileged trajectory' is characterized by the initial angle* $\boldsymbol{\phi_0}$*, initial velocity* $\boldsymbol{v_0}$ *and distance* $\boldsymbol{r}$ *between player and the hole of the board (****dimension of the basis of attraction****)*

Non-linear dynamical system analysis of the IS theory

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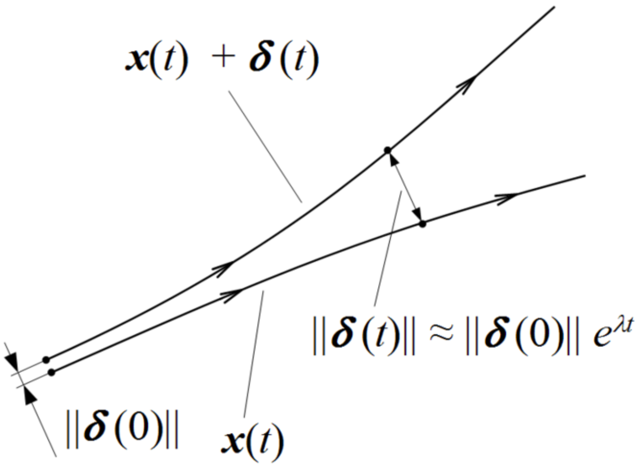
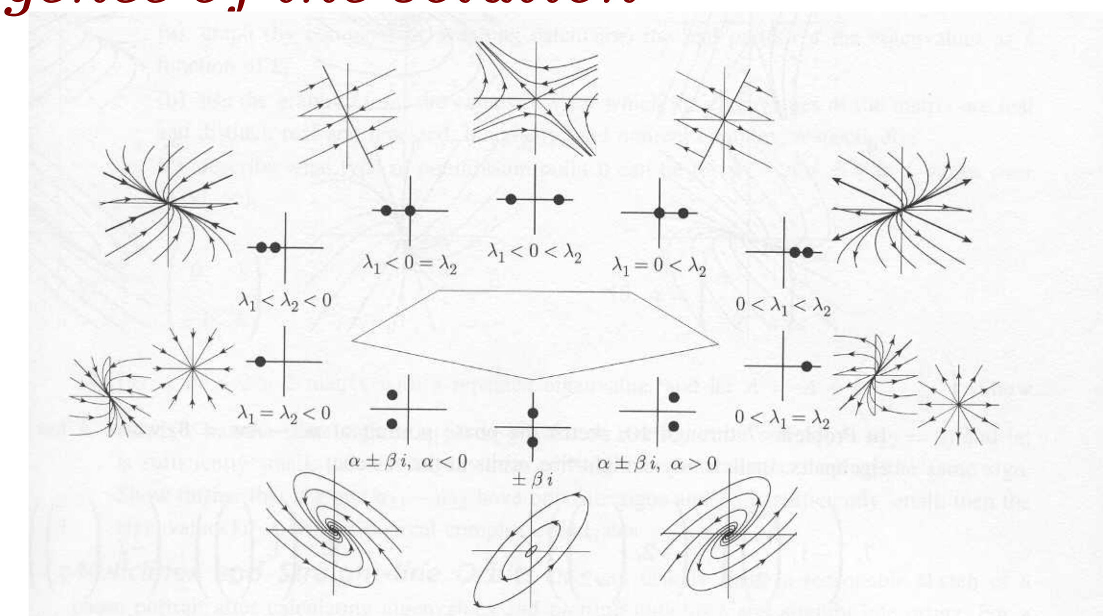
Lyapunov exponents of IS theory



Lyapunov exponent *measures the distance between two trajectories in the phase space*
Stability of the DOE's *depend on the value of the Lyapunov exponent*

$$\frac{dx}{dt} = Ax$$

Eigenvalues of matrix A determine the stability and convergence of the solution



Lyapunov exponents of IS theory

We can linearize our 3d system around the fixed points of the IS theory for the Gubser flow

$$\begin{pmatrix} \partial_\rho \hat{T} \\ \partial_\rho \bar{\pi} \\ \partial_\rho \tau \end{pmatrix} = \begin{pmatrix} \frac{1(\bar{\pi}-2)}{3}\tau & \frac{\hat{T}\tau}{3} & \frac{\hat{T}(\bar{\pi}-2)}{3} \\ -\frac{\bar{\pi}}{c} & -\frac{\hat{T}}{c} - \frac{8\bar{\pi}\tau}{3} & \frac{4}{15} - \frac{4\bar{\pi}^2}{3} \\ 0 & 0 & -2\tau \end{pmatrix}_{(\hat{T}_c,\bar{\pi}_c,\tau_c)} \begin{pmatrix} \hat{T} - \hat{T}_c \\ \bar{\pi} - \bar{\pi}_c \\ \tau - \tau_c \end{pmatrix}$$

The eigenvalues of this matrix at $\tau \longrightarrow 1$

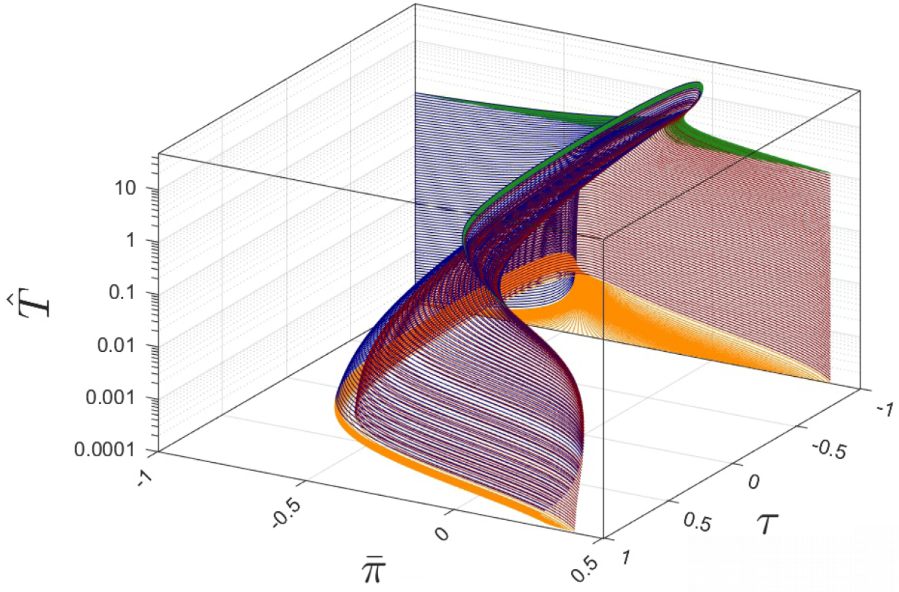
$$A : \left\{ -2, -\frac{8}{3\sqrt{5}}, -\frac{2}{3} + \frac{1}{3\sqrt{5}} \right\}$$ → *Stable (sink)*

$$B : \left\{ -2, : \frac{8}{3\sqrt{5}}, -\frac{2}{3} - \frac{1}{3\sqrt{5}} \right\}$$

$$C : \left\{ -2, \frac{7}{5} - \frac{\sqrt{821}}{15}, -\frac{7}{5} + \frac{\sqrt{821}}{15} \right\}.$$

(B, C) → *Unstable (saddle)*

Lyapunov exponents of IS theory



Lyapunov exponents of the attractor are read off from the eigenvalues of the matrix

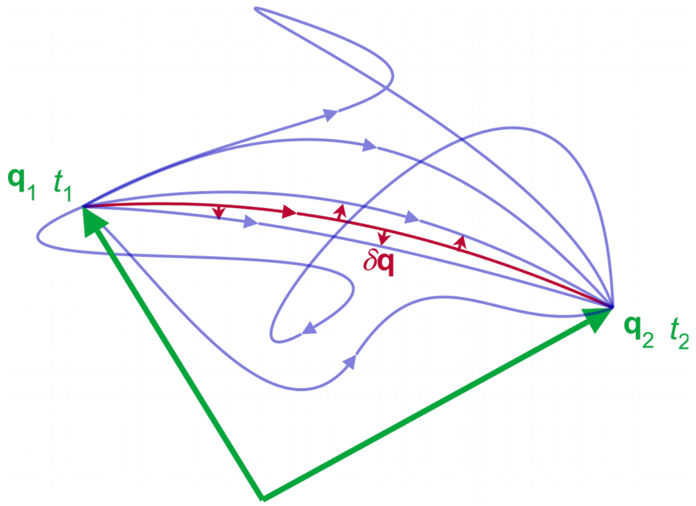
$$A : \left\{-2, -\frac{8}{3\sqrt{5}}, -\frac{2}{3} + \frac{1}{3\sqrt{5}}\right\} \longrightarrow \textit{Stable (sink)}$$

Attractor: $\mathcal{A} \sim \hat{T}_0\, e^{\lambda_{\hat{T}}\rho}\mathbf{u_1} + (\frac{1}{\sqrt{5}} - \bar{\pi}_0\, e^{\lambda_{\bar{\pi}}\rho})\mathbf{u_2} + \mathbf{u_3},$

$$\lambda_{\hat{T}} = -\frac{2}{3} + \frac{1}{3\sqrt{5}}, \quad \lambda_{\bar{\pi}} = -\frac{8}{3\sqrt{5}}, \quad \lambda_{\tau} = -2.$$

Why is the basin of attraction so interesting?

$$Z = \int_M D\phi \, e^{-S[\phi]}$$



- *M defines the space of fields or paths over which the integral is evaluated*
- *Saddle points (classical path) are determined from the action principle*

$$\frac{\delta S[\phi]}{\delta \phi} = 0$$

$\Rightarrow$ *M is a stable manifold of integration shaped by the solutions to the saddle point approximation*

Why is the basin of attraction so interesting?

Using this analogy the partition function for hydrodynamics

$$Z_{\text{eff}}(c) = \int_M D\hat{T} D\bar{\pi} Dt \; e^{-\int d\rho \left(\left(\frac{d\mathbf{x}}{d\rho} \right)^2 - \mathcal{V}(\mathbf{x}, c) \right)}.$$

$\mathcal{V}$ is the Lyapunov function which due to stability has to satisfy

$$\frac{d\mathcal{V}}{d\rho} \leq 0,$$

Thus M is the manifold whose paths are determined by the basin of attraction of the hydrodynamical equations!!!!

For the Gubser flow and IS theory local Lyapunov function was obtained

see arXiv:1711.01745

Universal asymptotic attractors of different fluid models

arXiv:1711.01745

Determining attractors I

- *IS, DNMR and anisotropic hydro equations can be recombined into a unique equation*

$$3w\left(\coth^2\rho - 1 - \mathcal{A}(w)\right)\frac{d\mathcal{A}(w)}{dw} + H(\mathcal{A}(w), w) = 0 \qquad (1)$$

Remember, we evaluate the asymptotic attractor $\coth^2\rho \longrightarrow 1$

- *The function H depends on the hydro model*

$$H_{\rm IS} = \tfrac{4}{3}\left(3\mathcal{A}(w)+2\right)^2 + \tfrac{3\mathcal{A}(w)+2}{cw} - \tfrac{4}{15}\,,$$

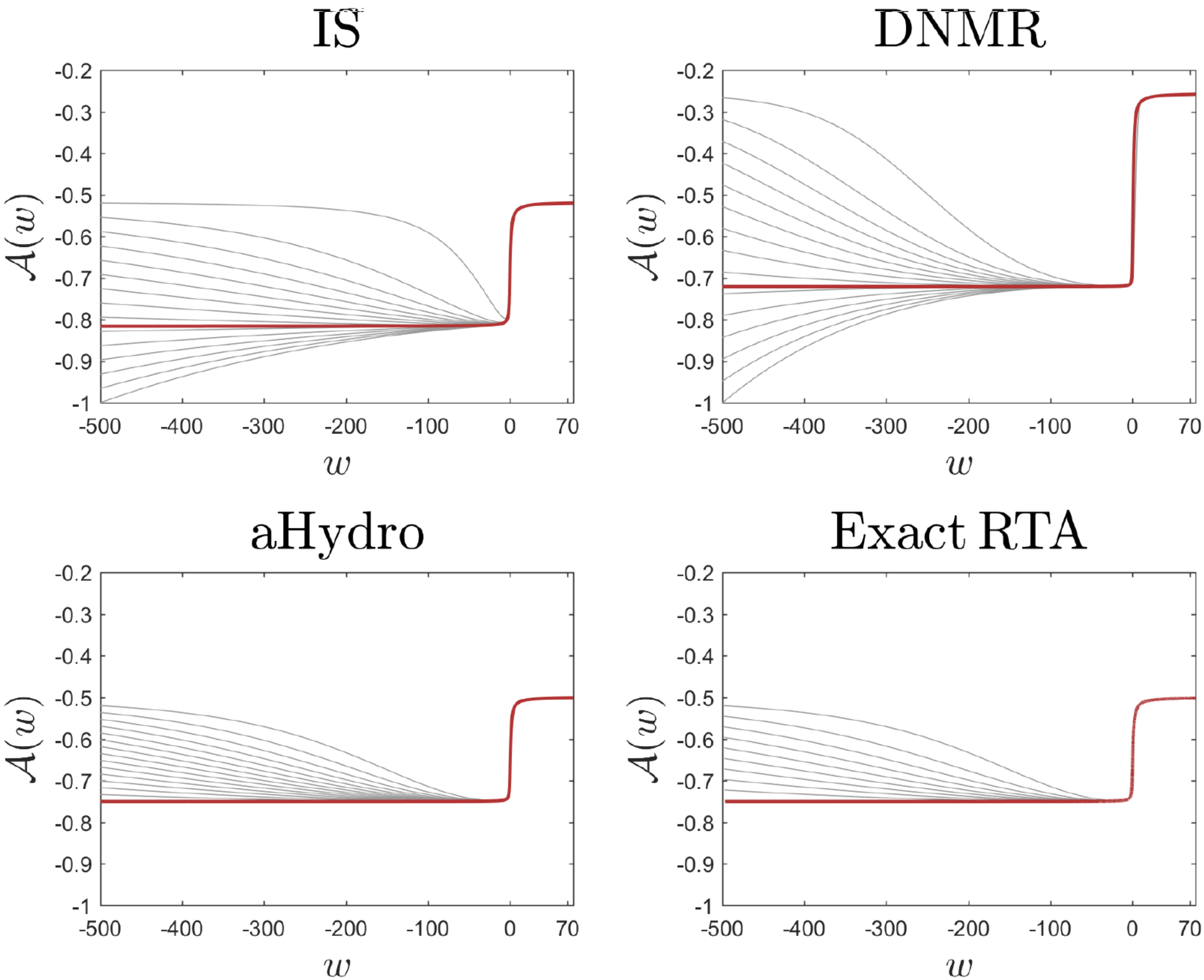
$$H_{\rm DNMR} = \tfrac{4}{3}\left(3\mathcal{A}(w)+2\right)^2 + \left(3\mathcal{A}(w)+2\right)\left[\tfrac{1}{c\,w} - \tfrac{10}{7}\right] - \tfrac{4}{15}\,,$$

$$H_{\rm aHydro} = \tfrac{4}{3}\left(3\mathcal{A}(w)+2\right)^2 + \left(3\mathcal{A}(w)+2\right)\left[\tfrac{1}{c\,w} - \tfrac{4}{3}\right] - \\ - \tfrac{5}{12} + \tfrac{3}{4}\mathcal{F}\left(3\mathcal{A}(w)+2\right)\,.$$

Attractors are found by a two-step process:

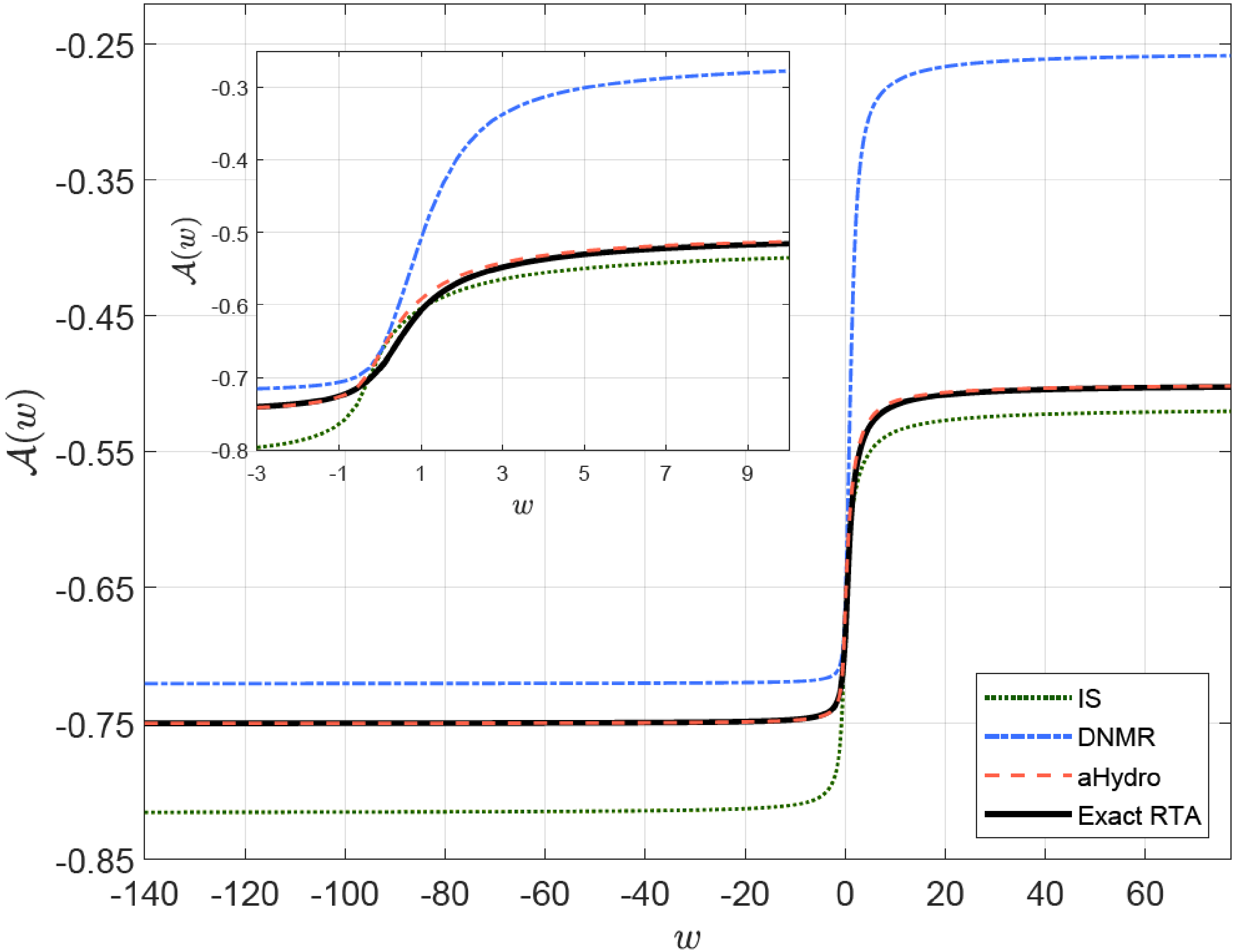
- *Finding null-lines with slow-roll down approx.* $d\mathcal{A}/dw=0$
- *The initial condition for solving* (1) *is obtained from the stable solution of the null-line* $\mathcal{A}_i=\mathcal{A}_+(w\longrightarrow-\infty)$

Universal attractors for Gubser flow



$$c = \frac{15}{4\pi}$$

Comparing attractors



$$c = \frac{15}{4\pi}$$

Anisotropic hydrodynamics matches almost exactly the exact attractor

Conclusions

- *We study the non-equilibrium attractors of different fluid dynamical models undergoing Gubser flow.*
- *The stability properties of the IS theory were studied by considering well-known methods of non-linear dynamical systems: fixed points, flow lines around those, Lyapunov exponents, Lyapunov function and dimension of the basin of attraction (3 dim for Gubser flow)*
- *Our work opens the possibility to study hydrodynamics as an EFT by using the relation between the path integral and the Lyapunov function*
- *Anisotropic hydrodynamics is able to describe the asymptotic exact attractor to high numerical accuracy*

  $\Rightarrow$ *Anisotropic hydrodynamics resums effectively the inverse Reynolds and Knudsen number to all orders*

Backup slides

Divergence of gradient expansion I

Gradient expansion diverges (*zero convergence radius*)

Consider conformal viscous fluid in a 0+1 *dim.*(*Bjorken model*)

$$\tau\dot{\epsilon} = -\frac{4}{3}\epsilon + \phi\,,$$

$$\tau_\Pi\dot{\phi} = \frac{4\eta}{3\tau} - \frac{\lambda_1\phi^2}{2\eta^2} - \frac{4\tau_\Pi\phi}{3\tau} - \phi$$

Last equations can be used to derive a single EOM

$$C_{\tau\Pi} w f f' + 4C_{\tau\Pi} f^2 + \left(w - \frac{16C_{\tau\Pi}}{3}\right) f - \frac{4C_\eta}{9} + \frac{16C_{\tau\Pi}}{9} - \frac{2w}{3} = 0$$

$$w = \tau T, \qquad f = \tau\frac{\dot{w}}{w}$$

The function $f = f(w)$

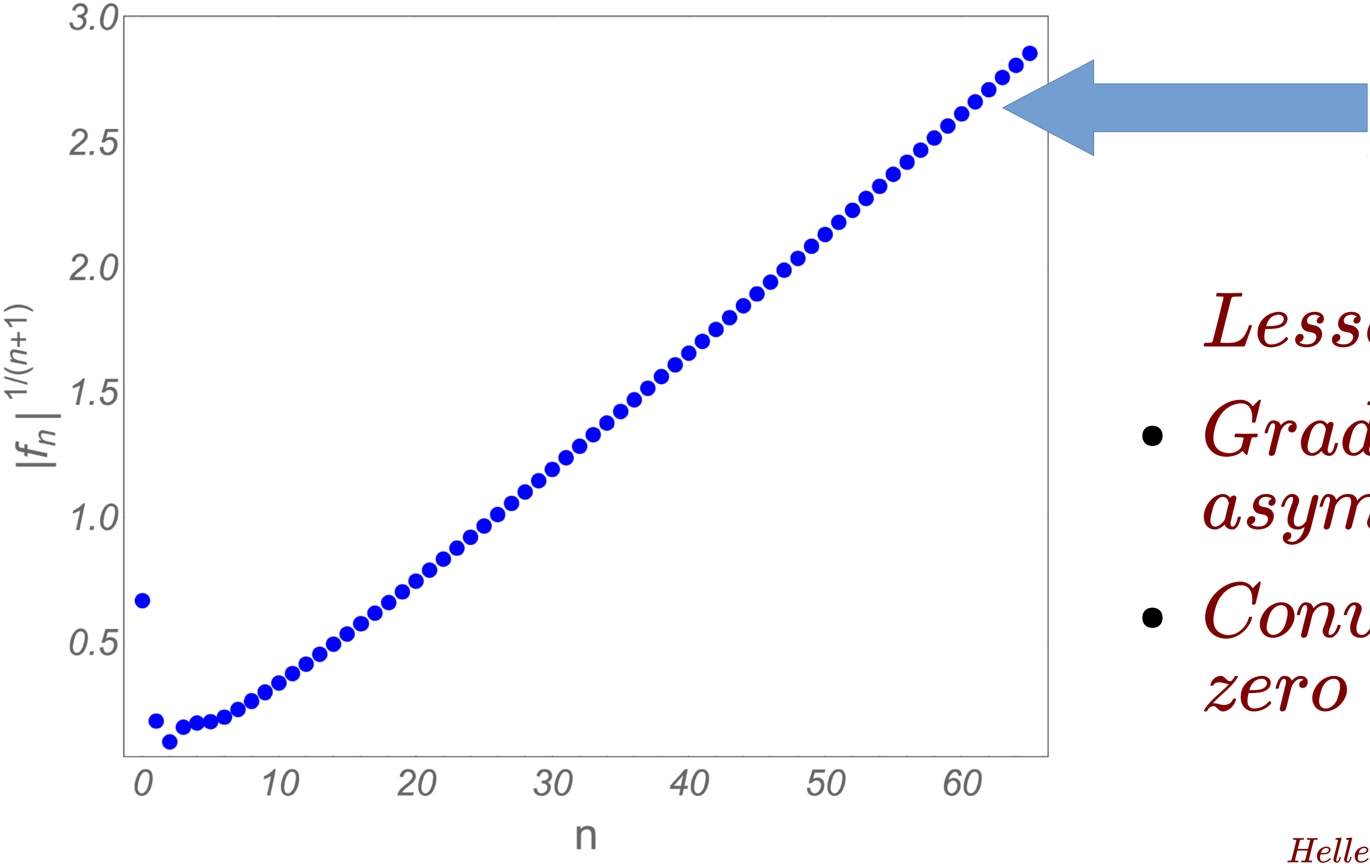
Heller and Spalinski, *PRL* 115 (2015) 072501

Divergence of gradient expansion I

$$C_{\tau\Pi} w f f' + 4C_{\tau\Pi} f^2 + \left(w - \frac{16C_{\tau\Pi}}{3}\right) f - \frac{4C_\eta}{9} + \frac{16C_{\tau\Pi}}{9} - \frac{2w}{3} = 0$$

Consider a series ansatz solution of the form

$$f(w) = \sum^{\infty} f_n w^{-n}$$



The coefficients f_n diverge as $n!$

Lessons to bear in mind

- *Gradient series is an asymptotic series*
- *Convergence radius is zero*

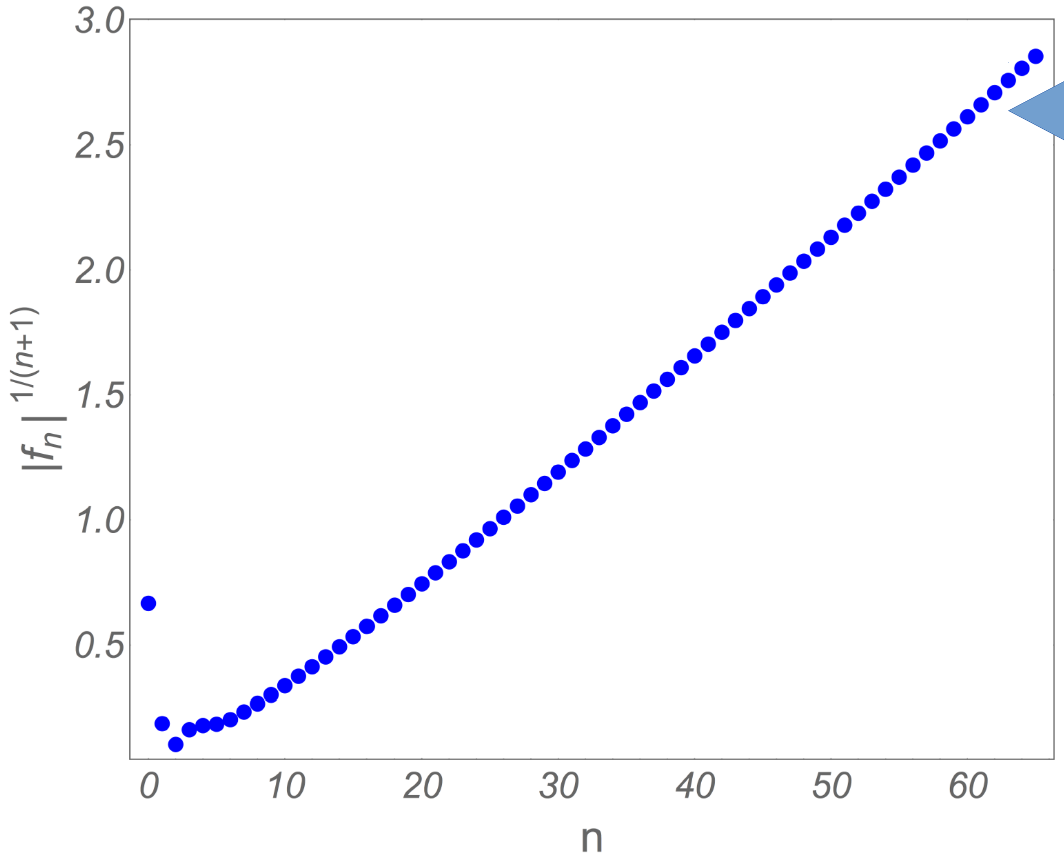
Heller and Spalinski, PRL 115 (2015) 072501

Divergence of gradient expansion II

$$C_{\tau\Pi} w f f' + 4C_{\tau\Pi} f^2 + \left(w - \frac{16C_{\tau\Pi}}{3}\right) f - \frac{4C_\eta}{9} + \frac{16C_{\tau\Pi}}{9} - \frac{2w}{3} = 0$$

Consider a series ansatz solution of the form

$$f(w) = \sum^{\infty} f_n w^{-n}$$



The coefficients f_n diverge as $n!$

Lessons to bear in mind

- *Gradient series is an asymptotic series*
- *Convergence radius is zero*

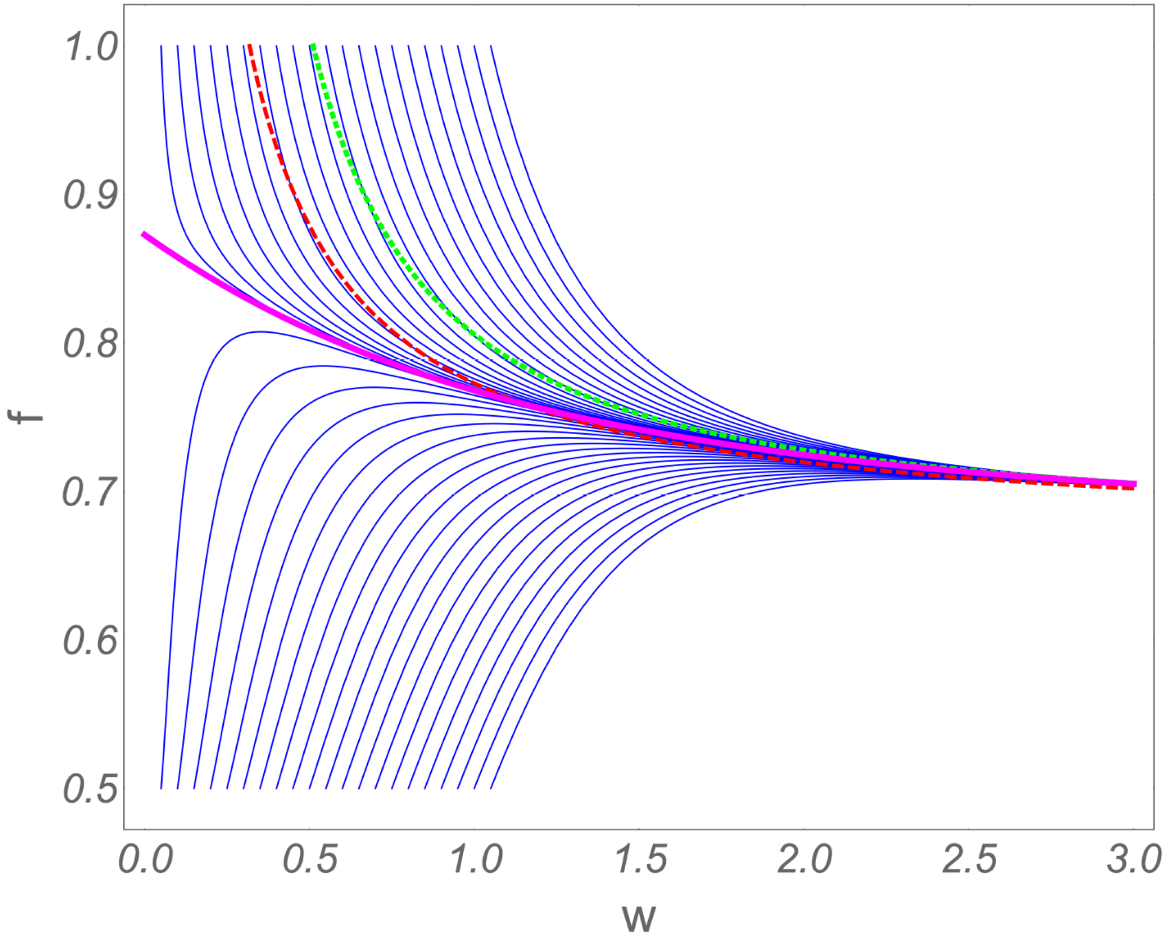
Heller and Spalinski, PRL 115 (2015) 072501

Attractor in 0+1 Bjorken model

$$C_{\tau\Pi} w f f' + 4C_{\tau\Pi} f^2 + \left(w - \frac{16C_{\tau\Pi}}{3}\right) f - \frac{4C_\eta}{9} + \frac{16C_{\tau\Pi}}{9} - \frac{2w}{3} = 0$$

Numerical solutions for arbitrary initial conditions

- *Gradient series is an asymptotic series*
- *Convergence radius is zero*



Heller and Spalinski, PRL 115 (2015) 072501

Hydrodynamics as an effective theory

- *Hydro is derived as a gradient expansion of the macroscopic dynamical degrees of freedom (DOF)*

$$\mathrm{T}^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu} \qquad \delta T^{\mu\nu} << T_0^{\mu\nu}$$

- *If the system is close to equilibrium, the (DOF) are the energy density, particle density, pressure, fluid velocity, etc*

$$\mathrm{T}^{\mu\nu} = \underbrace{(\epsilon + p)u^\mu u^\nu + pg^{\mu\nu}}_{\text{Ideal fluid}} + \underbrace{\Pi^{\mu\nu}}$$

→ Shear viscous stress + bulk viscous pressure

$$\Pi^{\mu\nu} \subset \mathcal{O}(\partial\epsilon, \partial u, ..., \partial^2\epsilon, \partial^2 u....)$$

- *For a conformal fluid up to second order in gradients (BRSSS, 2008)*

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$$

→ **First order (Navier-Stokes)**

$$+\eta\tau_\Pi\left[{}^{\langle}D\sigma^{\mu\nu\rangle} + \frac{1}{d-1}\sigma^{\mu\nu}(\nabla\cdot u)\right] + \kappa\left[R^{\langle\mu\nu\rangle} - (d-2)u_\alpha R^{\alpha\langle\mu\nu\rangle\beta}u_\beta\right]$$
$$+\lambda_1\sigma^{\langle\mu}{}_\lambda\sigma^{\nu\rangle\lambda} + \lambda_2\sigma^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda} + \lambda_3\Omega^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda}\,.$$

Second order ←

Matching prescriptions

***"Standard" Viscous Hydrodynamics*:**
Expansion around the equilibrium (ξ=0)

- ***Denicol-Niemi-Molnar-Rischke (DNMR) approach*:** *Nonlinear expansion of δf in terms of method of moments together with a systematic power counting in Knudsen and inverse Reynolds number*.

***Viscous Anisotropic Hydrodynamics*:**
Expansion around an anisotropic state ($\xi\neq$0).

- **$\mathcal{P}_L$ *matching*** (*MNR, See Niemi's talk*): *matches ξ evolution to that of the longitudinal pressure. For the Gubser flow it means that the energy-momentum tensor receives no residual dissipative corrections. The evolution equations can be written in terms of macroscopic variables just as in standard viscous hydrodynamics*.