Dispersion relation for hadronic light-by-light scattering

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Outline

1. Introduction
2. Hadronic contributions to \((g - 2)_\mu\)
3. Lorentz structure of the HLbL tensor
4. Master formula for \((g - 2)_\mu\)
5. Dispersive representation
6. Conclusion and outlook
Overview

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6 Conclusion and outlook
Magnetic moment

- relation of spin and magnetic moment of a lepton:

\[ \vec{\mu}_\ell = g_\ell \frac{e}{2m_\ell} \vec{s} \]

\( g_\ell \): Landé factor, gyromagnetic ratio

- Dirac’s prediction: \( g_e = 2 \)
- anomalous magnetic moment: \( a_\ell = (g_\ell - 2)/2 \)
- helped to establish QED and QFT as the framework for elementary particle physics
- today: probing not only QED but entire SM
Electron vs. muon magnetic moments

- influence of heavier virtual particles of mass $M$
scales as

$$\frac{\Delta a_\ell}{a_\ell} \propto \frac{m_\ell^2}{M^2}$$

- $a_e$ used to determine $\alpha_{\text{QED}}$

- $(m_\mu/m_e)^2 \approx 4 \times 10^4 \Rightarrow$ muon is much more sensitive to new physics, but also to EW and hadronic contributions

- $a_\tau$ experimentally not yet known precisely enough
$\left( g - 2 \right)_\mu$: comparison of theory and experiment

![Diagram showing comparison of $g - 2$ results for various experiments and models.](image)

- HMNT (06)
- JN (09)
- Davier et al, $\tau$ (10)
- Davier et al, $e^+e^-$ (10)
- JS (11)
- HLMNT (10)
- HLMNT (11)

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**Experimental results**

- BNL
- BNL (new from shift in $\lambda$)

$a_\mu \times 10^{10} = 11659000$

→ Hagiwara et al. 2012
Introduction

\((g - 2)_\mu\): theory vs. experiment

- discrepancy between SM and experiment \(\sim 3\sigma\)
- hint to new physics?
- new experiments (FNAL, J-PARC) aim at reducing the experimental error by a factor of 4
- theory error completely dominated by hadronic effects
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Hadronic contributions to $(g - 2)_\mu$

Hadronic vacuum polarisation: $\mathcal{O}(\alpha^2)$

- problem: QCD is non-perturbative at low energies
- first principle calculations (lattice QCD) may become competitive in the future
- current evaluations based on dispersion relations and data
Hadronic contributions to $(g - 2)_\mu$

**Hadronic vacuum polarisation: $\mathcal{O}(\alpha^2)$**

Photon hadronic vacuum polarisation function:

$$\Pi(q^2) = -i(q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2)$$

Unitarity of the $S$-matrix implies the optical theorem:

$$\text{Im}\Pi(s) = \frac{s}{e(s)^2}\sigma_{\text{tot}}(e^+ e^- \rightarrow \gamma^* \rightarrow \text{hadrons})$$
Hadronic contributions to $(g - 2)_μ$

Dispersion relation

Causality implies analyticity:

Cauchy integral formula:

$$\Pi(s) = \frac{1}{2\pi i} \oint_\gamma \frac{\Pi(s')}{s' - s} ds'$$

Deform integration path:

$$\Pi(s) - \Pi(0) = \frac{s}{\pi} \int_{\text{Re}}^\infty \frac{\text{Im}\Pi(s')}{(s' - s - i\epsilon) s'} ds'$$
Hadronic contributions to \((g - 2)_\mu\)

Hadronic vacuum polarisation: \(\mathcal{O}(\alpha^2)\)

- basic principles: unitarity and analyticity
- direct relation to experiment: total hadronic cross section \(\sigma_{\text{tot}}(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})\)
- at present: dominant theoretical uncertainty
- can be systematically improved: dedicated \(e^+e^-\) program (BaBar, Belle, BESIII, CMD3, KLOE2, SND)
Hadronic contributions to \( (g - 2)_\mu \)

Hadronic light-by-light (HLbL) scattering

- up to now only model calculations
- uncertainty estimate based rather on consensus than on a systematic method
- with recent progress on vacuum polarisation, HLbL starts to dominate theory error
## Hadronic contributions to \((g - 2)_\mu\)

### SM contributions to \((g - 2)_\mu\)

<table>
<thead>
<tr>
<th></th>
<th>(10^{11} \times a_\mu)</th>
<th>(10^{11} \times \Delta a_\mu)</th>
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<td>BNL E821</td>
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<tr>
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<td>EW</td>
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<td>LO HVP</td>
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<td>1</td>
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<td>LO HLbL</td>
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<td>40</td>
</tr>
<tr>
<td>NLO HLbL</td>
<td>3</td>
<td>2</td>
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<tr>
<td>Hadronic total</td>
<td>6982</td>
<td>59</td>
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<tr>
<td>Theory total</td>
<td>116 591 855</td>
<td>59</td>
</tr>
</tbody>
</table>

→ PDG 2016

→ Kinoshita et al. 2012

→ Gnendiger et al. 2013

→ Hagiwara et al. 2011

→ Hagiwara et al. 2011

→ Kurz et al. 2014

→ Jegerlehner, Nyffeler 2009

→ Colangelo et al. 2014
Hadronic contributions to $(g - 2)_\mu$

## Model calculations of HLbL

<table>
<thead>
<tr>
<th>Contribution</th>
<th>BPP</th>
<th>HKS</th>
<th>KN</th>
<th>MV</th>
<th>BP</th>
<th>PdRV</th>
<th>N/JN</th>
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<tr>
<td>$\pi^0, \eta, \eta'$</td>
<td>85±13</td>
<td>82.7±6.4</td>
<td>83±12</td>
<td>114±10</td>
<td>–</td>
<td>114±13</td>
<td>99±16</td>
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<td>$\pi, K$ loops</td>
<td>–19±13</td>
<td>–4.5±8.1</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–19±19</td>
<td>–19±13</td>
</tr>
<tr>
<td>$\pi, K$ loops + other subleading in $N_c$</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0±10</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>axial vectors</td>
<td>2.5±1.0</td>
<td>1.7±1.7</td>
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<td>22±5</td>
<td>–</td>
<td>15±10</td>
<td>22±5</td>
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<td>scalars</td>
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<td>–</td>
<td>–</td>
<td>–</td>
<td>–7±7</td>
<td>–7±2</td>
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<tr>
<td>quark loops</td>
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<td>9.7±11.1</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>2.3</td>
<td>21±3</td>
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<tr>
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<td>83±32</td>
<td>89.6±15.4</td>
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<td>136±25</td>
<td>110±40</td>
<td>105±26</td>
<td>116±39</td>
</tr>
</tbody>
</table>

→ Jegerlehner, Nyffeler (2009)

- pseudoscalar pole contribution most important
- pion-loop second most important
- differences between models, large uncertainties
How to improve HLbL calculation?

- make use of fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- relate HLbL to experimentally accessible quantities
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The HLbL tensor: definitions

- **hadronic four-point function:**

\[
\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int dx dy dz e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \langle 0 | T j_{em}^\mu(x) j_{em}^\nu(y) j_{em}^\lambda(z) j_{em}^\sigma(0) | 0 \rangle
\]

- **EM current:**

\[
j^\mu_{em} = \sum_{i=u,d,s} Q_i \bar{q}_i \gamma^\mu q_i
\]
The HLbL tensor: definitions

- helicity amplitudes for the process

\[ \gamma^*(q_1, \lambda_1) \gamma^*(q_2, \lambda_2) \rightarrow \gamma^*(-q_3, \lambda_3) \gamma(q_4, \lambda_4) : \]

\[ H_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} = \epsilon^{\lambda_1}_{\mu} \epsilon^{\lambda_2}_{\nu} \epsilon^{\lambda_3}_{\lambda} \epsilon^{\lambda_4*}_{\sigma} \Pi^{\mu \nu \lambda \sigma} \]

- Mandelstam variables:

\[ s = (q_1 + q_2)^2, \quad t = (q_1 + q_3)^2, \quad u = (q_2 + q_3)^2 \]

- for \((g - 2)_\mu\), the external photon is on shell:

\[ q_4^2 = 0, \text{ where } q_4 = q_1 + q_2 + q_3 \]
Lorentz structure of the HLbL tensor

The HLbL tensor

- a priori 138 ‘naive’ Lorentz structures:

\[ \Pi_{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 \]

\[ + \sum_{i,k,l,m} q_i^{\mu} q_j^{\nu} q_k^{\lambda} q_l^{\sigma} \Pi^4_{ijkl} \]

\[ + \sum_{i,j} g^{\lambda\sigma} q_i^{\mu} q_j^{\nu} \Pi^5_{ij} + \ldots \]

- in 4 space-time dimensions: 2 linear relations among the 138 Lorentz structures \( \rightarrow \) Eichmann et al. (2014)

- six dynamical variables, e.g. two Mandelstam variables \( s, t \) and the photon virtualities \( q_1^2, q_2^2, q_3^2, q_4^2 \)
HLbL tensor: gauge invariance

- Ward identities

\[ \{ q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma \} \Pi_{\mu\nu\lambda\sigma} = 0 \]

imply 95 linear relations between scalar functions \( \Pi_i \)

- off-shell basis: \( 138 - 95 - 2 = 41 \) structures
- corresponding to 41 helicity amplitudes
- relations between \( \Pi_i \) imply kinematic zeros
Lorentz structure of the HLbL tensor

HLbL tensor: Lorentz decomposition

Solution for the Lorentz decomposition, following a recipe by Bardeen, Tung (1968) and Tarrach (1975):

\[ \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2) \]

- Lorentz structures manifestly gauge invariant
- crossing symmetry manifest: only 7 distinct structures, 47 follow from crossing
- scalar functions \( \Pi_i \) free of kinematic singularities
  ⇒ ideal quantities for a dispersive treatment
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Master formula: contribution to \((g - 2)_\mu\)

- from gauge invariance:
\[
\Pi_{\mu\nu\lambda\rho} = -q_4^\sigma \frac{\partial}{\partial q_4^\rho} \Pi_{\mu\nu\lambda\sigma}
\]

- for \((g - 2)_\mu\): afterwards take \(q_4 \to 0\)

- no kinematic singularities in scalar functions: perform these steps with the derived Lorentz decomposition

- only 12 linear combinations of the scalar functions \(\Pi_i\) contribute to \((g - 2)_\mu\)
Master formula: contribution to \((g - 2)_{\mu}\)

\[
a_{\mu}^{\text{HLbL}} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2) \\
\frac{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_{\mu}^2][(p - q_2)^2 - m_{\mu}^2]}{(q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_{\mu}^2][(p - q_2)^2 - m_{\mu}^2]}
\]

- \(\hat{T}_i\): known integration kernel functions
- five loop integrals can be performed with Gegenbauer polynomial techniques
- Wick rotation possible even in the presence of anomalous thresholds
Master formula: contribution to \((g - 2)_\mu\)

\[
a_{\mu}^{\text{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1 - \tau^2} Q_1^3 Q_2^3 \\
\times \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)
\]

- \(T_i\): known integration kernels
- \(\bar{\Pi}_i\): linear combinations of the scalar functions \(\Pi_i\)
- Euclidean momenta: \(Q_i^2 = -q_i^2\)
- \(Q_3^2 = Q_1^2 + Q_2^2 + 2Q_1Q_2\tau\)
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Analytic properties of scalar functions

- right- and left-hand cuts in each Mandelstam variable
- double-spectral regions (box topologies)
- anomalous thresholds for large photon virtualities
Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

\[
\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \ldots
\]
Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\pi} + \ldots$$

one-pion intermediate state:
Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

\[
\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0}\text{-pole} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\pi} + \ldots
\]

two-pion intermediate state in both channels:
Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \Pi_{\mu\nu\lambda\sigma}^{\pi\pi} + \ldots \]

two-pion intermediate state in first channel:
Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

\[ \Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \cdots \]

future work: higher intermediate states
Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

\[
\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{box}} + \Pi_{\mu\nu\lambda\sigma}^{\pi\pi} + \ldots
\]

- the limit \( q_4 \to 0 \) for \( (g - 2)_\mu \) is taken in the end
\[ \Pi_{1}^{\pi^0}\text{-pole} = \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)}{s - M_{\pi}^2} \]

\[ \Pi_{2,3}^{\pi^0}\text{-pole} \text{ via crossing symmetry} \]

- input: doubly-virtual and singly-virtual pion transition form factors \( \mathcal{F}_{\gamma^*\gamma^*\pi^0} \) and \( \mathcal{F}_{\gamma^*\gamma\pi^0} \)
- pion is on shell
- dispersive analysis of transition form factor:
  \[ \rightarrow \text{Hoferichter et al., EPJC 74 (2014) 3180} \]
Box contributions

- simultaneous two-pion cuts in two channels
- Mandelstam representation explicitly constructed

\[ \Pi_i^{\pi\text{-box}} = \frac{1}{\pi^2} \int ds' dt' \frac{\rho_{st}^{i}(s', t')}{(s' - s)(t' - t)} + (t \leftrightarrow u) + (s \leftrightarrow u) \]

- \( q^2 \)-dependence: pion vector form factors \( F^V_\pi(q^2_i) \) for each off-shell photon factor out
Dispersive representation

Box contributions

- sQED loop projected on BTT basis fulfills the same Mandelstam representation
- only difference are factors of $F_\pi^V$
- $\Rightarrow$ box topologies are identical to FsQED:

\[
\equiv F_\pi^V (q_1^2) F_\pi^V (q_2^2) F_\pi^V (q_3^2)
\]

- model-independent definition of pion loop
Box contributions

Very simple expressions for box contributions in terms of Feynman parameter integrals

\[
\Pi_{\pi}^{\text{box}}(q_1^2, q_2^2, q_3^2) = F^V_\pi (q_1^2) F^V_\pi (q_2^2) F^V_\pi (q_3^2) \\
\times \frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy I_i(x, y),
\]

with e.g.

\[
I_7(x, y) = -\frac{4}{3} \frac{(1 - 2x)^2(1 - 2y)^2y(1 - y)}{\Delta^3_{123}},
\]

\[
\Delta_{ijk} = M^2_\pi - xyq_i^2 - x(1 - x - y)q_j^2 - y(1 - x - y)q_k^2.
\]
Pion-box saturation with photon virtualities

Pion-box saturation in %

cutoff on the virtualities in GeV
Box contributions

$F_{\pi}^{V}$: fit of dispersive representation to time- and space-like data

Result: $a_{\mu}^{\pi-\text{box}} = -15.9(2) \times 10^{-11}$
Helicity formalism and sum rules

- construction of singly-on-shell basis: unphysical helicity amplitudes drop out, 27 elements remain
- uniform asymptotic behaviour of the full tensor together with BTT tensor decomposition leads to 15 HLbL sum rules:

\[ 0 = \int ds' \text{Im} \tilde{\Pi}_i(s') \bigg|_{t=q_2^2, q_4^2=0} \]

- can be expressed in terms of helicity amplitudes
Rescattering contribution

- neglect left-hand cut due to multi-particle intermediate states in crossed channel
- two-pion cut in only one channel:

\[
\Pi_{i\pi\pi} = \frac{1}{2} \left( \frac{1}{\pi} \int_{4M_{\pi}^2}^{\infty} dt' \frac{\text{Im}\Pi_{i\pi\pi}(s, t', u')}{t' - t} + \frac{1}{\pi} \int_{4M_{\pi}^2}^{\infty} du' \frac{\text{Im}\Pi_{i\pi\pi}(s, t', u')}{u' - u} \right) + \text{fixed-}t + \text{fixed-}u
\]
Rescattering contribution

- unitarity gives imaginary parts in terms of helicity amplitudes for $\gamma^*\gamma^{(*)} \to \pi\pi$
- basis change to helicity amplitudes calculated
- expansion into partial waves
- framework valid for arbitrary partial waves
- resummation of PW expansion reproduces full result: checked for pion box
Convergence of partial-wave expansion

Relative deviation from full result: \( 1 - \frac{a_{\pi-\text{box}, J_{\max}}}{a_{\mu-\text{box}}} \)

<table>
<thead>
<tr>
<th>( J_{\max} )</th>
<th>fixed-( s )</th>
<th>fixed-( t )</th>
<th>fixed-( u )</th>
<th>average</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>100.0%</td>
<td>-6.2%</td>
<td>-6.2%</td>
<td>29.2%</td>
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<td>2</td>
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<td>4</td>
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<td>16</td>
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<td>-0.0%</td>
<td>0.4%</td>
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</tr>
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</table>
The subprocess

Helicity amplitudes for $\gamma^* \gamma^* \rightarrow \pi \pi$: dispersive solution of the $S$-wave unitarity relation with Omnès methods

- pion-pole approximation to left-hand cut
  \[ q^2 \]-dependence again given by $F_V^\pi$

- phase shifts based on modified inverse-amplitude method

- low-energy properties accurately reproduced, including $f_0(500)$ parameters

- fully consistent with $\pi^\pm$ polarisabilities

- result for $S$-waves: $a_{\mu,J=0}^{\pi \pi,\pi^\text{-pole LHC}} = -8(1) \times 10^{-11}$
Topologies in the rescattering contribution

Omnès solution for $\gamma^* \gamma^* \rightarrow \pi\pi$ provides the following:

\[
\begin{align*}
\text{recursive} \quad \text{and} \quad \text{PWE, no LHC}
\end{align*}
\]

Two-pion contributions to HLbL:

\[
\begin{align*}
\text{pion box} \quad \text{and} \quad \text{rescattering contribution}
\end{align*}
\]
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Results for two-pion contributions

Pion-box contribution:

\[ a_{\mu}^{\pi-\text{box}} = -15.9(2) \times 10^{-11} \]

\textit{S}-wave rescattering contribution:

\[ a_{\mu, J=0}^{\pi\pi, \pi-\text{pole LHC}} = -8(1) \times 10^{-11} \]
Conclusion and outlook

Summary

- our dispersive approach to HLbL scattering is based on fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- we take into account the lowest intermediate states: $\pi^0$-pole and $\pi\pi$-cuts
- relation to experimentally accessible (or again with data dispersively reconstructed) quantities
- precise numerical evaluation of two-pion contributions
- a step towards a model-independent calculation of $a_\mu$
Conclusion and outlook

Outlook

- higher pseudoscalar poles can be included directly
- two-particle intermediate states:
  - include kaons in a coupled-channel system
  - numerics for $D$-waves
  - generalisation to heavier left-hand cuts
- higher intermediate states in direct channel
  - framework needs to be extended
  - e.g. $3\pi \Rightarrow$ axials
- match the total to OPE/pQCD constraints
Backup
HLbL tensor: BTT Lorentz decomposition

Problem: find a decomposition

\[ \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_i T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2) \]

with the following properties:

- Lorentz structures \( T_i^{\mu\nu\lambda\sigma} \) manifestly gauge invariant:
  \[ \{ q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma \} T_i^{\mu\nu\lambda\sigma} = 0 \]

- scalar functions \( \Pi_i \) free of kinematic singularities and zeros
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Recipe by Bardeen, Tung (1968) and Tarrach (1975):

- construct gauge projectors:

\[ I_{12}^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}, \quad I_{34}^{\lambda\sigma} = g^{\lambda\sigma} - \frac{q_4^\lambda q_3^\sigma}{q_3 \cdot q_4} \]

- gauge invariant themselves, e.g.

\[ q_1^\mu I_{12}^{\mu\nu} = 0 \]

- leave HLbL tensor invariant, e.g.

\[ I_{12}^{\mu\mu'} \Pi_{\mu'\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma} \]
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Following Bardeen, Tung (1968):

• apply gauge projectors to the 138 initial structures: 95 immediately project to 0
• remove $1/q_1 \cdot q_2$ and $1/q_3 \cdot q_4$ poles by taking appropriate linear combinations
• BT basis: degenerate in the limits

$q_1 \cdot q_2 \rightarrow 0$, $q_3 \cdot q_4 \rightarrow 0$
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According to Tarrach (1975):

- degeneracies in the limits $q_1 \cdot q_2 \to 0$, $q_3 \cdot q_4 \to 0$:

$$\sum_k c^i_k T^\mu_\nu_\lambda_\sigma_k = q_1 \cdot q_2 X^\mu_\nu_\lambda_\sigma_i + q_3 \cdot q_4 Y^\mu_\nu_\lambda_\sigma_i$$

- extend basis by additional structures $X^\mu_\nu_\lambda_\sigma_i$, $Y^\mu_\nu_\lambda_\sigma_i$ taking care of remaining kinematic singularities

- equivalent: implementing crossing symmetry
$(g - 2)_\mu$ integration region in polar coordinates
A roadmap for HLbL

\[ e^+ e^- \rightarrow e^+ e^- \pi^0 \]
\[ e^+ e^- \rightarrow \pi^0 \gamma \]

Pion transition form factor
\[ F_{\pi^0 \gamma^* \gamma^*} (q_1^2, q_2^2) \]

\[ e^+ e^- \rightarrow 3\pi \]
\[ \omega, \phi \rightarrow 3\pi \]

\[ \gamma \pi \rightarrow \pi \pi \]
\[ \pi \pi \rightarrow \pi \pi \]

Pion vector form factor \( F_V^{\pi} \)
\[ \omega, \phi \rightarrow \pi^0 \gamma^* \]

Partial waves for \( \gamma^* \gamma^* \rightarrow \pi \pi \)
\[ e^+ e^- \rightarrow \pi \pi \gamma \]
\[ e^+ e^- \rightarrow e^+ e^- \pi \pi \]

\[ \omega, \phi \rightarrow \pi \pi \gamma \]

Pion polarizabilities
\[ \gamma \pi \rightarrow \gamma \pi \]

→ flowchart by M. Hoferichter