Dispersion relation for hadronic light-by-light scattering

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in collaboration with G. Colangelo, M. Hoferichter, and M. Procura

JHEP **04** (2017) 161, [arXiv:1702.07347 [hep-ph]]
Phys. Rev. Lett. **118** (2017) 232001, [arXiv:1701.06554 [hep-ph]]
JHEP **09** (2015) 074, [arXiv:1506.01386 [hep-ph]]
JHEP **09** (2014) 091, [arXiv:1402.7081 [hep-ph]]

and with G. Colangelo, M. Hoferichter, B. Kubis, and M. Procura

Phys. Lett. **B738** (2014) 6, [arXiv:1408.2517 [hep-ph]]

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Jefferson Lab Theory Seminar, Newport News

Magnetic moment

- relation of spin and magnetic moment of a lepton:

$$\vec{\mu}_\ell = g_\ell \frac{e}{2m_\ell}\vec{s}$$

  g_ℓ: Landé factor, gyromagnetic ratio

- Dirac's prediction: $g_e = 2$
- anomalous magnetic moment: $a_\ell = (g_\ell - 2)/2$
- helped to establish QED and QFT as the framework for elementary particle physics
- today: probing not only QED but entire SM

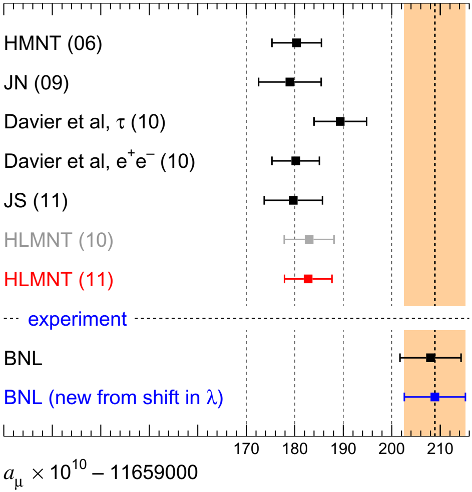
Electron vs. muon magnetic moments

- influence of heavier virtual particles of mass M scales as

$$\frac{\Delta a_\ell}{a_\ell} \propto \frac{m_\ell^2}{M^2}$$

- a_e used to determine α_{QED}
- $(m_\mu/m_e)^2 \approx 4 \times 10^4 \Rightarrow$ muon is much more sensitive to new physics, but also to EW and hadronic contributions
- a_τ experimentally not yet known precisely enough

$(g-2)_\mu$: comparison of theory and experiment

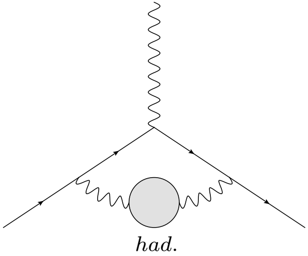


→ Hagiwara et al. 2012

$(g-2)_\mu$: theory vs. experiment

- discrepancy between SM and experiment $\sim 3\sigma$
- hint to new physics?
- new experiments (FNAL, J-PARC) aim at reducing the experimental error by a factor of 4
- theory error completely dominated by hadronic effects

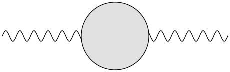
Hadronic vacuum polarisation: $\mathcal{O}(\alpha^2)$



- problem: QCD is non-perturbative at low energies
- first principle calculations (lattice QCD) may become competitive in the future
- current evaluations based on dispersion relations and data

Hadronic vacuum polarisation: $\mathcal{O}(\alpha^2)$

Photon hadronic vacuum polarisation function:

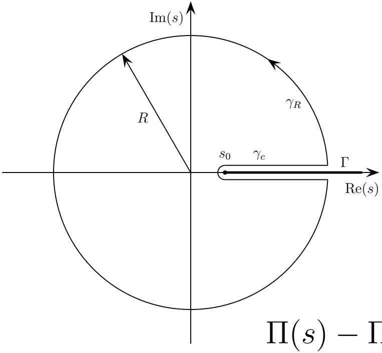
 $= -i(q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2)$

Unitarity of the S-matrix implies the optical theorem:

$$\mathrm{Im}\Pi(s) = \frac{s}{e(s)^2}\sigma_{\mathrm{tot}}(e^+e^- \to \gamma^* \to \mathrm{hadrons})$$

Dispersion relation

Causality implies analyticity:



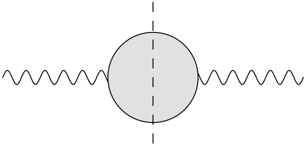
Cauchy integral formula:

$$\Pi(s) = \frac{1}{2\pi i} \oint_\gamma \frac{\Pi(s')}{s'-s} ds'$$

Deform integration path:

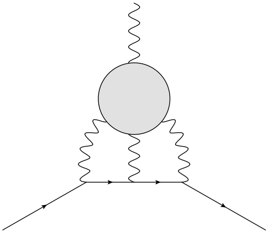
$$\Pi(s) - \Pi(0) = \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{Im}\Pi(s')}{(s'-s-i\epsilon)s'} ds'$$

Hadronic vacuum polarisation: $\mathcal{O}(\alpha^2)$



- basic principles: unitarity and analyticity
- direct relation to experiment: total hadronic cross section $\sigma_{\text{tot}}(e^+e^- \to \gamma^* \to \text{hadrons})$
- at present: dominant theoretical uncertainty
- can be systematically improved: dedicated e^+e^- program (BaBar, Belle, BESIII, CMD3, KLOE2, SND)

Hadronic light-by-light (HLbL) scattering



- up to now only model calculations
- uncertainty estimate based rather on consensus than on a systematic method
- with recent progress on vacuum polarisation, HLbL starts to dominate theory error

SM contributions to $(g-2)_\mu$

	$10^{11} \times a_\mu$	$10^{11} \times \Delta a_\mu$	
BNL E821	$116\,592\,089$	63	→ PDG 2016
QED total	$116\,584\,718.95$	0.08	→ Kinoshita et al. 2012
EW	153.6	1.0	→ Gnendiger et al. 2013
LO HVP	$6\,949$	43	→ Hagiwara et al. 2011
NLO HVP	-98	1	→ Hagiwara et al. 2011
NNLO HVP	12.4	0.1	→ Kurz et al. 2014
LO HLbL	116	40	→ Jegerlehner, Nyffeler 2009
NLO HLbL	3	2	→ Colangelo et al. 2014
Hadronic total	6982	59	
Theory total	$116\,591\,855$	59	

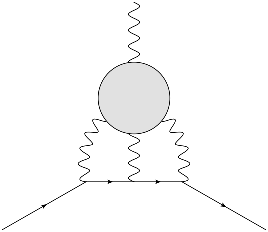
Model calculations of HLbL

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops + other subleading in N_c	–	–	–	0±10	–	–	–
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	–	15±10	22± 5
scalars	−6.8±2.0	–	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	110±40	105±26	116±39

→ Jegerlehner, Nyffeler (2009)

- pseudoscalar pole contribution most important
- pion-loop second most important
- differences between models, large uncertainties

How to improve HLbL calculation?



- make use of fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- relate HLbL to experimentally accessible quantities

The HLbL tensor: definitions

- hadronic four-point function:

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)$$
$$= -i \int dx dy dz e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \langle 0 | T j_{\text{em}}^{\mu}(x) j_{\text{em}}^{\nu}(y) j_{\text{em}}^{\lambda}(z) j_{\text{em}}^{\sigma}(0) | 0 \rangle$$

- EM current:

$$j_{\text{em}}^{\mu} = \sum_{i=u,d,s} Q_i \bar{q}_i \gamma^{\mu} q_i$$

The HLbL tensor: definitions

- helicity amplitudes for the process
  $\gamma^*(q_1,\lambda_1)\gamma^*(q_2,\lambda_2) \to \gamma^*(-q_3,\lambda_3)\gamma(q_4,\lambda_4)$:

$$H_{\lambda_1\lambda_2\lambda_3\lambda_4} = \epsilon_\mu^{\lambda_1}\epsilon_\nu^{\lambda_2}\epsilon_\lambda^{\lambda_3 *}\epsilon_\sigma^{\lambda_4 *}\Pi^{\mu\nu\lambda\sigma}$$

- Mandelstam variables:
  $s=(q_1+q_2)^2$, $t=(q_1+q_3)^2$, $u=(q_2+q_3)^2$
- for $(g-2)_\mu$, the external photon is on shell:
  $q_4^2=0$, where $q_4=q_1+q_2+q_3$

The HLbL tensor

- a priori 138 'naive' Lorentz structures:

$$\begin{aligned}\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 \\ + \sum_{i,k,l,m} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} \\ + \sum_{i,j} g^{\lambda\sigma} q_i^\mu q_j^\nu \Pi^5_{ij} + \dots\end{aligned}$$

- in 4 space-time dimensions: 2 linear relations among the 138 Lorentz structures $\rightarrow$ Eichmann et al. (2014)
- six dynamical variables, e.g. two Mandelstam variables s, t and the photon virtualities q_1^2, q_2^2, q_3^2, q_4^2

HLbL tensor: gauge invariance

- Ward identities

$$\{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\}\Pi_{\mu\nu\lambda\sigma} = 0$$

  imply 95 linear relations between scalar functions Π_i
- off-shell basis: 138 − 95 − 2 = 41 structures
- corresponding to 41 helicity amplitudes
- relations between Π_i imply kinematic zeros

HLbL tensor: Lorentz decomposition

Solution for the Lorentz decomposition, following a recipe by Bardeen, Tung (1968) and Tarrach (1975):

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- Lorentz structures manifestly gauge invariant
- crossing symmetry manifest: only 7 distinct structures, 47 follow from crossing
- scalar functions Π_i free of kinematic singularities $\Rightarrow$ ideal quantities for a dispersive treatment

Master formula: contribution to $(g-2)_\mu$

- from gauge invariance:

$$\Pi_{\mu\nu\lambda\rho} = -q_4^\sigma \frac{\partial}{\partial q_4^\rho} \Pi_{\mu\nu\lambda\sigma}$$

- for $(g-2)_\mu$: afterwards take $q_4 \to 0$
- no kinematic singularities in scalar functions: perform these steps with the derived Lorentz decomposition
- only 12 linear combinations of the scalar functions Π_i contribute to $(g-2)_\mu$

Master formula: contribution to $(g-2)_\mu$

$$a_\mu^{\mathrm{HLbL}} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_\mu^2][(p - q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known integration kernel functions
- five loop integrals can be performed with Gegenbauer polynomial techniques
- Wick rotation possible even in the presence of anomalous thresholds

Master formula: contribution to $(g-2)_\mu$

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \times \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

- T_i: known integration kernels
- $\bar{\Pi}_i$: linear combinations of the scalar functions Π_i
- Euclidean momenta: $Q_i^2 = -q_i^2$
- $Q_3^2 = Q_1^2 + Q_2^2 + 2Q_1Q_2\tau$

Analytic properties of scalar functions

- right- and left-hand cuts in each Mandelstam variable
- double-spectral regions (box topologies)
- anomalous thresholds for large photon virtualities

Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \dots$$

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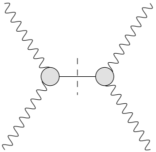
one-pion intermediate state:



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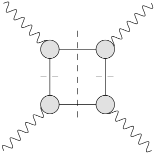
two-pion intermediate state in both channels:



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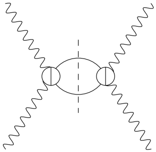
two-pion intermediate state in first channel:



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future work: higher intermediate states

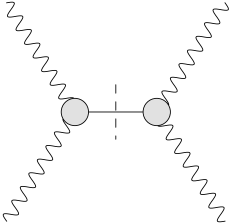
Mandelstam representation

- we limit ourselves to intermediate states of at most two pions
- writing down a double-spectral (Mandelstam) representation allows us to split up the HLbL tensor:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \ldots$$

- the limit $q_4 \to 0$ for $(g-2)_\mu$ is taken in the end

Pion pole



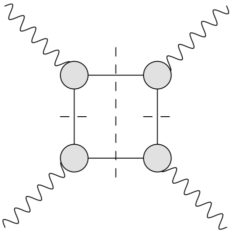
$$\Pi_1^{\pi^0\text{-pole}} = \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)}{s - M_\pi^2}$$

$\Pi_{2,3}^{\pi^0\text{-pole}}$ via crossing symmetry

- input: doubly-virtual and singly-virtual pion transition form factors $\mathcal{F}_{\gamma^*\gamma^*\pi^0}$ and $\mathcal{F}_{\gamma^*\gamma\pi^0}$
- pion is on shell
- dispersive analysis of transition form factor:

  → Hoferichter et al., EPJC **74** (2014) 3180

Box contributions



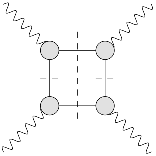
- simultaneous two-pion cuts in two channels
- Mandelstam representation explicitly constructed

$$\Pi_i^{\pi\text{-box}} = \frac{1}{\pi^2}\int ds'dt' \frac{\rho_i^{st}(s',t')}{(s'-s)(t'-t)} + (t \leftrightarrow u) + (s \leftrightarrow u)$$

- q^2-dependence: pion vector form factors $F_\pi^V(q_i^2)$ for each off-shell photon factor out

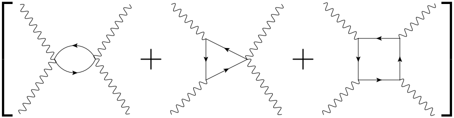
Box contributions

- sQED loop projected on BTT basis fulfils the same Mandelstam representation
- only difference are factors of F_π^V
- $\Rightarrow$ box topologies are identical to FsQED:



$$\equiv F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)$$

$\times$



- model-independent definition of pion loop

Box contributions

Very simple expressions for box contributions in terms of Feynman parameter integrals

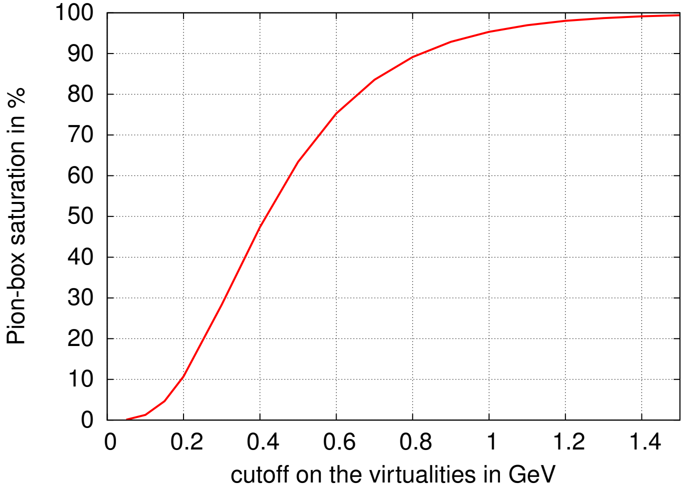
$$\Pi_i^{\pi\text{-box}}(q_1^2, q_2^2, q_3^2) = F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2) \times \frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy\, I_i(x, y),$$

with e.g.

$$I_7(x, y) = -\frac{4}{3} \frac{(1-2x)^2(1-2y)^2 y(1-y)}{\Delta_{123}^3},$$

$$\Delta_{ijk} = M_\pi^2 - xyq_i^2 - x(1-x-y)q_j^2 - y(1-x-y)q_k^2.$$

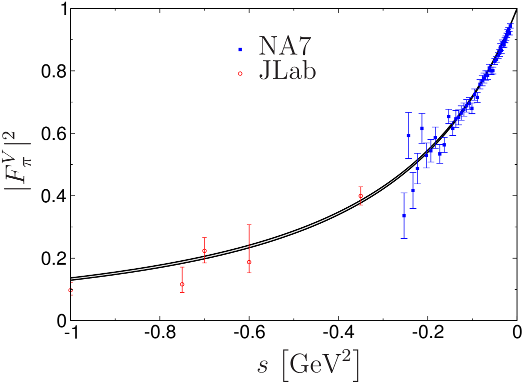
Pion-box saturation with photon virtualities



Box contributions

F_π^V: fit of dispersive representation to time- and space-like data

Result: $a_\mu^{\pi\text{-box}} = -15.9(2) \times 10^{-11}$



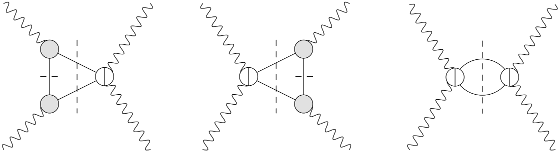
Helicity formalism and sum rules

- construction of singly-on-shell basis: unphysical helicity amplitudes drop out, 27 elements remain
- uniform asymptotic behaviour of the full tensor together with BTT tensor decomposition leads to 15 HLbL sum rules:

$$0 = \int ds' \mathrm{Im}\check{\Pi}_i(s') \Big|_{t=q_2^2, q_4^2=0}$$

- can be expressed in terms of helicity amplitudes

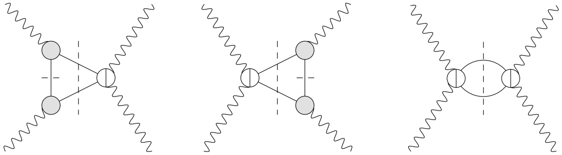
Rescattering contribution



- neglect left-hand cut due to multi-particle intermediate states in crossed channel
- two-pion cut in only one channel:

$$\Pi_i^{\pi\pi} = \frac{1}{2}\Bigg(\frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dt' \, \frac{\mathrm{Im}\Pi_i^{\pi\pi}(s,t',u')}{t'-t} + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} du' \, \frac{\mathrm{Im}\Pi_i^{\pi\pi}(s,t',u')}{u'-u}$$
$$+ \text{fixed-}t$$
$$+ \text{fixed-}u\Bigg)$$

Rescattering contribution



- unitarity gives imaginary parts in terms of helicity amplitudes for $\gamma^*\gamma^{(*)} \to \pi\pi$
- basis change to helicity amplitudes calculated
- expansion into partial waves
- framework valid for arbitrary partial waves
- resummation of PW expansion reproduces full result: checked for pion box

Convergence of partial-wave expansion

Relative deviation from full result: $1 - \frac{a_{\mu, J_{\text{max}}}^{\pi\text{-box, PW}}}{a_\mu^{\pi\text{-box}}}$

J_{max}	fixed-s	fixed-t	fixed-u	average
0	100.0%	-6.2%	-6.2%	29.2%
2	26.1%	-2.3%	7.3%	10.4%
4	10.8%	-1.5%	3.6%	4.3%
6	5.7%	-0.7%	2.1%	2.4%
8	3.5%	-0.4%	1.3%	1.5%
10	2.3%	-0.2%	0.9%	1.0%
12	1.7%	-0.1%	0.7%	0.7%
14	1.3%	-0.1%	0.5%	0.6%
16	1.0%	-0.0%	0.4%	0.4%

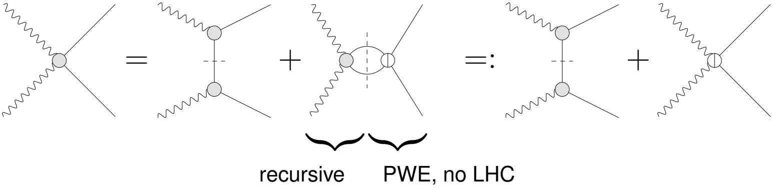
The subprocess

Helicity amplitudes for $\gamma^*\gamma^* \to \pi\pi$: dispersive solution of the S-wave unitarity relation with Omnès methods

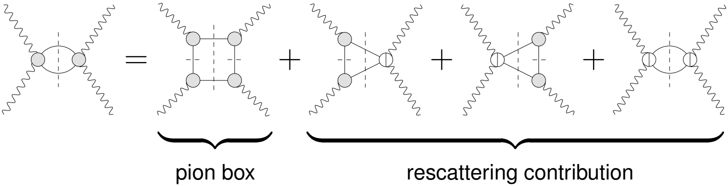
- pion-pole approximation to left-hand cut
  $\Rightarrow q^2$-dependence again given by F_π^V
- phase shifts based on modified inverse-amplitude method
- low-energy properties accurately reproduced, including $f_0(500)$ parameters
- fully consistent with $\pi^\pm$ polarisabilities
- result for S-waves: $a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$

Topologies in the rescattering contribution

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ provides the following:



Two-pion contributions to HLbL:



Results for two-pion contributions

Pion-box contribution:

$$a_\mu^{\pi\text{-box}} = -15.9(2) \times 10^{-11}$$

S-wave rescattering contribution:

$$a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$$

Summary

- our dispersive approach to HLbL scattering is based on fundamental principles:
  - gauge invariance, crossing symmetry
  - unitarity, analyticity
- we take into account the lowest intermediate states: π^0-pole and $\pi\pi$-cuts
- relation to experimentally accessible (or again with data dispersively reconstructed) quantities
- precise numerical evaluation of two-pion contributions
- a step towards a model-independent calculation of a_μ

Outlook

- higher pseudoscalar poles can be included directly
- two-particle intermediate states:
  - include kaons in a coupled-channel system
  - numerics for D-waves
  - generalisation to heavier left-hand cuts
- higher intermediate states in direct channel
  - framework needs to be extended
  - e.g. $3\pi \Rightarrow$ axials
- match the total to OPE/pQCD constraints

Backup

HLbL tensor: BTT Lorentz decomposition

Problem: find a decomposition

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_i T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

with the following properties:

- Lorentz structures $T_i^{\mu\nu\lambda\sigma}$ manifestly gauge invariant:

$$\{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\} T^i_{\mu\nu\lambda\sigma} = 0$$

- scalar functions Π_i free of kinematic singularities and zeros

HLbL tensor: BTT Lorentz decomposition

Recipe by Bardeen, Tung (1968) and Tarrach (1975):

- construct gauge projectors:

$$I_{12}^{\mu\nu} = g^{\mu\nu} - \frac{q_2^\mu q_1^\nu}{q_1 \cdot q_2}, \quad I_{34}^{\lambda\sigma} = g^{\lambda\sigma} - \frac{q_4^\lambda q_3^\sigma}{q_3 \cdot q_4}$$

- gauge invariant themselves, e.g.

$$q_1^\mu I^{12}_{\mu\nu} = 0$$

- leave HLbL tensor invariant, e.g.

$$I_{12}^{\mu\mu'} \Pi_{\mu'\nu\lambda\sigma} = \Pi^\mu{}_{\nu\lambda\sigma}$$

HLbL tensor: BTT Lorentz decomposition

Following Bardeen, Tung (1968):

- apply gauge projectors to the 138 initial structures: 95 immediately project to 0
- remove $1/q_1 \cdot q_2$ and $1/q_3 \cdot q_4$ poles by taking appropriate linear combinations
- BT basis: degenerate in the limits $q_1 \cdot q_2 \to 0$, $q_3 \cdot q_4 \to 0$

HLbL tensor: BTT Lorentz decomposition

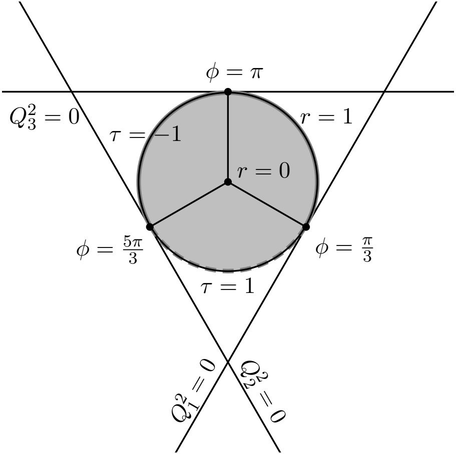
According to Tarrach (1975):

- degeneracies in the limits $q_1 \cdot q_2 \to 0$, $q_3 \cdot q_4 \to 0$:

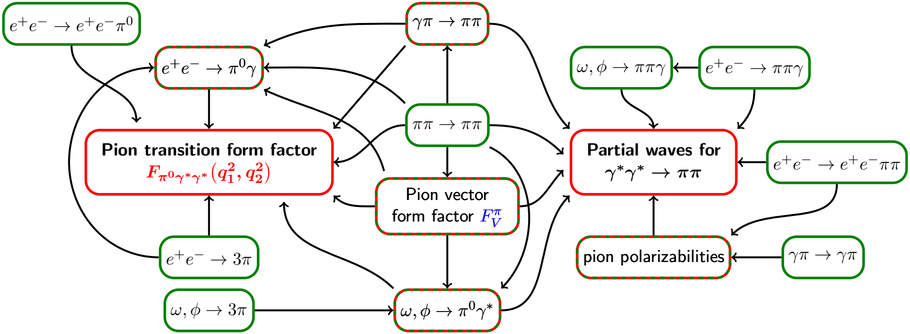
$$\sum_k c_k^i T_k^{\mu\nu\lambda\sigma} = q_1 \cdot q_2 X_i^{\mu\nu\lambda\sigma} + q_3 \cdot q_4 Y_i^{\mu\nu\lambda\sigma}$$

- extend basis by additional structures $X_i^{\mu\nu\lambda\sigma}$, $Y_i^{\mu\nu\lambda\sigma}$ taking care of remaining kinematic singularities
- equivalent: implementing crossing symmetry

$(g-2)_\mu$ integration region in polar coordinates



A roadmap for HLbL



$\to$ flowchart by M. Hoferichter