Precision light and heavy meson physics from Lattice QCD

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QCD is a key part of the Standard Model but quark confinement is a complication/interesting feature.

Connecting observed hadron properties to those of quarks requires full nonperturbative treatment of Quantum Chromodynamics.

Some properties of hadrons can be very accurately measured and are calculable in lattice QCD - can test SM and determine parameters very accurately (1%).
Lattice QCD: fields defined on 4-d discrete space-(Euclidean) time.
Lagrangian parameters: $\alpha_s$, $m_q a$

1) Generate sets of gluon fields for Monte Carlo integrn of Path Integral (inc effect of u, d, s, (c) sea quarks)

2) Calculate valence quark propagators and combine for “hadron correlators”. Fit for hadron masses and amplitudes

- Determine $a$ to convert results in lattice units to physical units. Fix $m_q$ from hadron mass

*numerically extremely challenging*

- cost increases as $a \to 0, m_{u/d} \to \text{phys}$ and with statistics, volume.
Darwin@Cambridge, part of UK’s £15m HPC facility for theoretical particle physics and astronomy

State-of-the-art commodity cluster: 9600 Intel Sandybridge cores, infiniband interconnect, fast switch and 2 Pbytes storage

Allows us to calculate quark propagators rapidly and store them for flexible re-use.

www.dirac.ac.uk
Quark formalisms

Many ways to discretise Dirac Lagrangian onto lattice. All should give same answers.

Issues are: Discretisation errors at power $a^n$
Numerical speed of matrix inversion
Chiral symmetry
Quark doubling

We use Highly Improved Staggered Quarks (HISQ) for u, d, s and c. Also (with extrapolation) for b.
Disc. errors $\alpha_s^2 a^2, a^4$. Numerically fast. Chiral symm.
Some complications from doublers (‘tastes’).

For b quarks, also use NRQCD. Fast for b quarks at moderate a, disc. errors now $\alpha_s^2 a^2$. Current normln perturbative, through $\alpha_s$
Example parameters for ‘2nd generation’ calculations now being done with staggered quarks.

\[ m_u = m_d = m_l \]

mass of u,d quarks

real world

\[ m_{\pi^0} = 135 \text{ MeV} \]

Volume:

\[ m_\pi L > 3 \]

“2nd generation” lattices inc. c quarks in sea

HISQ = Highly improved staggered quarks - very accurate discretisation

E. Follana, et al, HPQCD, hep-lat/0610092.

\[ m_u,d \approx m_s/10 \]

\[ m_u,d \approx m_s/27 \]

other formalisms: clover (Fermilab), domain-wall, twisted-mass …
Hadron correlation functions (‘2point functions’) give masses and decay constants.

\[
\langle 0| H^\dagger(T) H(0)|0 \rangle = \sum_n A_n e^{-m_n T} \rightarrow A_0 e^{-m_0 T}
\]

masses of all hadrons with quantum numbers of H

\[
A_n = \frac{\langle 0| H| n \rangle^2}{2m_n} = \frac{f_n^2 m_n}{2}
\]

decay constant parameterises amplitude to annihilate - a property of the meson calculable in QCD. Can often relate to experimental decay rate.

1% accurate experimental info. for f and m for many mesons!

Need accurate determination from lattice QCD to match
Example (state-of-the-art) calculation

R. Dowdall et al, HPQCD, 1303.1670.

Extract meson mass and amplitude=decay constant from correlator for multiple lattice spacings and $m_{u/d}$. Very high statistics (16,000 samples of the correlator).

Convert decay constant to GeV units using $w_0$ to fix relative lattice spacing. Very small discretisation errors. PCAC reln means no renormln factors needed.
The gold-plated meson spectrum

Uncties at few MeV level: Future: inc QED and $m_u \neq m_d$
Meson decay constants
Parameterises hadronic information needed for annihilation rate to W or photon:

\[ \Gamma \propto f^2 \]

Experiment: weak decays
Lattice QCD: predictions

Meson decay constants parameterise hadronic information needed for annihilation rate to W or photon:

\[ f_{\pi}^2, f_{\eta}^2, f_{\eta'}^2, f_{\phi}^2, f_{\psi}^2, f_{\psi'}^2, f_{c}^2, f_{b}^2 \]

\[ 1503.05762, 1408.5768, 1302.2644, 1303.1670 \]

Decay constants of vector mesons now being pinned down

2% accurate
\[ B(s) \to \mu^+\mu^- \]

0.2% accurate
\[ V_{us} \]

0.5% accuracy from lattice QCD now: FNAL/MILC 1407.3772
BES will improve expt. \( V_{cd} \) \( V_{cs} \)
Vector meson decay constants

HPQCD 1312.5264, 1503.05762

We calculate:

\[ \frac{f_{D_s^*}}{f_{D_s}} = 1.10(2) \]

\[ \Gamma(D_s^* \rightarrow D_s \gamma) = 0.066(26) \text{ keV} \]

\[ \text{Br}(D_s^* \rightarrow \ell \nu) = 3.4(1.4) \times 10^{-5} \]

We also find:

\[ \frac{f_{B_s^*}}{f_{B_s}} = 0.953(23) \]

\[ < 1 \]

\[ \frac{f_{B_s^*} \sqrt{M_{B_s^*}}}{f_{B_s} \sqrt{M_{B_s}}} \]

This allows us to fix the total width of the decay. We can test HQET perturbation matching uncertainty dominates.

The size of the hyperfine coefficient \( f \) is less than the naive estimate of 12%. This is due to the hyperfine term contributing to the perturbative matching calculation for the NRQCD currents and we do not have the \( C_3 \) parameter automatically included in the perturbative matching calculation.

The key sources of systematic error that need to be allowed for, by inclusion in our fit function, are the variations in the mass splittings between vector and pseudoscalar mesons and some difference in the \( \delta_x \) mass splittings. The same values of \( \delta_x \) are used for all values of \( a \), with \( a = 0.08 \) and \( a = 0.10 \). The \( \delta_a \) term is not included in the fit. We write the factor as \( \alpha \) as a function of the mass splitting, allowing for this by simply taking a fractional increase of 0.06 for \( \delta_x \) and 0.05 for \( \delta_a \).

The missing term into the fit is obtained by incorporating a factor to take account of potentially the largest source of uncertainty here. This represents the uncertainty of additional current corrections that only appear first order in the \( a \) expansion.

Finally, we include the current corrections, which are small, but they have opposite sign. This may be possible to see this?

We also find:

\[ \frac{f_{B_s^*}}{f_{B_s}} = 0.953(23) \]

\[ < 1 \]

\[ \frac{f_{B_s^*} \sqrt{M_{B_s^*}}}{f_{B_s} \sqrt{M_{B_s}}} \]

can test HQET.

The total width may be possible to see this?
Further probes of hadron structure: form factors as functions of $q^2$

Needs ‘3-point’ functions e.g. for pion to pion transition via vector current $J$

Need to calculate correlators for multiple $T$ values and $0 < t < T$ and fit as a function of $t$, $T$ simultaneously with 2pt.

$$C_{3pt} = \sum_{i,j} b_i J_{ij} b_j e^{-E_i t} e^{-E_j (T-t)}$$

$$\langle \pi | V_\mu | \pi \rangle / (2Z \sqrt{E_i E_j})$$

Normaln of $J$ must be fixed, e.g. here $f_+(0) = 1$ from charge cons.
\( \pi \) electromagnetic form factor at small \( q^2 \)

\( \pi - e \) scattering probes \( \pi \) electric charge distn

\[ \langle \pi(p_1) | V_\mu | \pi(p_2) \rangle = f_+(q^2)(p_1 + p_2)_\mu \]

Working at physical u/d quark masses on HISQ 2+1+1 configs, lattice QCD raw results on top of experiment

J. Koponen et al, HPQCD, 1511.07382
At small $q^2$ can fit pole:

\[ f_+(q^2) = \frac{1}{1 - q^2/M^2} \]

\[ M^2 = 6/\left\langle r^2 \right\rangle \approx M^2_\rho \]

\[ \left\langle r^2 \right\rangle_{V, \text{NA7}} = 0.431(10)\text{fm}^2 \]
Meson form factors at high (space-like) $q^2$

To ‘understand’ form factor want to map it experimentally and theoretically up to high $q^2$ where perturbative QCD becomes valid. But where is this point?

EXPT: JLAB E12-06-101

Perturbative QCD: High $q^2$ photon must be accompanied by high momentum gluon exchange

$$Q^2 = -q^2 \rightarrow \infty$$

$$f_+(q^2) = F_\pi(Q^2) = \frac{8\pi f_\pi^2 \alpha_s(Q/2)}{Q^2} \left| 1 + \sum_{n=2}^{\infty} a_{\pi n}(Q/2) \right|^2$$

Lepage, Brodsky, PRD22(1980)2157

coeffs of expansion of distn amplitude in Gegenbauer polyn., run to 0 with inc. $Q^2$
Enter lattice QCD …

Need a formalism with small discretisation errors that is numerically fast. *HISQ*

Instead of $\pi$ use $\eta_s$ pseudoscalar made of $s$ quarks for added speed.

$\eta_s$ prevented from mixing with light states on lattice so not physical, but properties can be mapped out. $M_{\eta_s} = 688.5(2.2)\text{ MeV}$

$f_{\eta_s} = 181.14(55)\text{ MeV}$

R. Dowdall et al, HPQCD, 1303.1670.
Work at two values of lattice spacing (0.12fm and 0.09fm) and two values of light sea quark mass (m_s/5 and m_s/10)

Use ‘twisted boundary conditions’ to insert momentum and test discretisation errors as a function of (pa)

For 3pt functions use Breit frame \( \vec{p}_i = -\vec{p}_f \)

Maximises \( Q^2 \) for a given (pa)

J. Koponen et al, HPQCD, 1701.04250.
Results - Can reach $Q^2$ of 6 GeV$^2$ for $(pa)<0.6$
Disc. and sea quark mass effects very small

\[ F = \frac{1}{1 + \frac{Q^2}{M^2}} \]

physical point limit of ‘z-expansion’ fit

pert. QCD inc.
\[ \phi_\pi(2\text{GeV}) = [x(1 - x)]^{0.52} \]
Braun et al, 1503.03656

asymp. pert. QCD
J. Koponen et al, HPQCD, 1701.04250.
z-expansion

\[ z(t, t_{\text{cut}}) = \frac{\sqrt{t_{\text{cut}} - t} - \sqrt{t_{\text{cut}}} - \sqrt{t_{\text{cut}}}}{\sqrt{t_{\text{cut}} - t} + \sqrt{t_{\text{cut}}}} \]

Maps \( t \) region into \(-1 < z < 1\)

Fit:

\[ (1 + Q^2/M_\phi^2) F = 1 + \sum_i z^i A_i \left[ 1 + B_i (a\Lambda)^i + C_i (a\Lambda)^4 + D_i \frac{\delta m_{\text{sea}}}{10} \right] \]

Further tests of pert. QCD underway: scaling with decay constant as change mass and ff of helicity-nonconserving scalar current.
Quark masses and strong coupling constant

Multiple lattice QCD methods now that agree to high accuracy

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\alpha_{\text{MS}}(M_Z, n_f = 5)
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Muon anomalous magnetic moment

\[ \vec{\mu} = g \frac{e}{2m} \vec{S} \]

\[ a_\mu = \frac{g - 2}{2} \]

Measure using polarised muons circulating in E and B fields. At a momentum where \( \beta \times E \) terms cancel, difference between precession and cyclotron frequencies:

\[ \omega_a = -\frac{e}{m} a_\mu B \]

BNL result:

\[ a_\mu^{exp} = 11659208.9(6.3) \times 10^{-10} \]

E989 (FNAL) will reduce exptl uncty to 1.6, starting 2017.
Standard Model theory expectations

Contributions from QED, EW and QCD interactions.

QED dominates.

QCD contribs start at $\alpha_{QCD}^2$.

**QED**

$$\frac{\alpha_{QED}}{2\pi} = 0.00116$$

**QED**

$\alpha_{QED}^2 = 11658471.885(4) \times 10^{-10}$

**EW**

$\alpha_{\mu}^{EW} = 15.4(2) \times 10^{-10}$

**QCD**

$\alpha_{\mu}^{QCD} = 11659208.9(6.3) \times 10^{-10}$

LO Hadronic vacuum polarisation (HVP) dominates uncertainty in SM result.

Blum et al, 1301.2607
Hadronic (and other) contributions = EXPT - QED - EW

\[ a_{\mu}^{\text{expt}} - a_{\mu}^{\text{QED}} - a_{\mu}^{\text{EW}} = 721.7(6.3) \times 10^{-10} \]

\[ = a_{\mu}^{\text{HV P}} + a_{\mu}^{\text{HOHV P}} + a_{\mu}^{\text{HLBL}} + a_{\mu}^{\text{new physics}} \]

Focus on lowest order hadronic vacuum polarisation, so assume:

\[ a_{\mu}^{\text{HLbL}} = 10.5(2.6) \times 10^{-10} \]

\[ a_{\mu}^{\text{HOHV P}} = -8.85(9) \times 10^{-10} \] \text{NLO+NNLO}

\[ a_{\mu}^{\text{HV P, no new physics}} = 719.8(6.8) \times 10^{-10} \]

\text{Kurz et al, 1403.6400}
Lattice calculation of HVP,LO

Analytically continue to Euclidean $q^2$.

$$a_{µ}^{HV \mu i} = \frac{α}{π} \int_{0}^{∞} dq^2 f(q^2) (4\pi α e_i^2) \hat{Π}_i(q^2)$$

connected contribution for flavour $i$

$f(q^2)$ divergent function with scale set by $m_µ$

$$\hat{Π}(q^2) = Π(q^2) - Π(0)$$

HPQCD method: time-moments of vector $JJ$ correlators give expansion around $q^2=0$

$$G_n ≡ \sum_{t,\vec{x}} t^n Z_V^2 \langle J^j(\vec{x}, t) J^j(0) \rangle$$

$$\hat{Π}(q^2) = \sum_{k=1}^{∞} q^k Π_k$$

Chakraborty et al, HPQCD, 1403.1778

replace with [2,2] Padé
Test on STRANGE contribution

HISQ valence quarks on MILC 2+1+1 HISQ configs. Local $J_v$ - nonpert. $Z_v$.
multiple $a$ (fixed by $w_0$), $m_l$ (inc. phys.), volumes.
Tune $s$ from $\eta_s$ up to (5.8 fm)$^3$

<table>
<thead>
<tr>
<th>Uncertainty in lattice spacing $(w_0, r_1)$:</th>
<th>$a^s_\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncertainty in $Z_V$:</td>
<td>1.0%</td>
</tr>
<tr>
<td>Monte Carlo statistics:</td>
<td>0.4%</td>
</tr>
<tr>
<td>$a^2 \rightarrow 0$ extrapolation:</td>
<td>0.1%</td>
</tr>
<tr>
<td>QED corrections:</td>
<td>0.1%</td>
</tr>
<tr>
<td>Quark mass tuning:</td>
<td>0.0%</td>
</tr>
<tr>
<td>Finite lattice volume:</td>
<td>&lt; 0.1%</td>
</tr>
<tr>
<td>Padé approximants:</td>
<td>&lt; 0.1%</td>
</tr>
<tr>
<td>Total:</td>
<td>1.1%</td>
</tr>
</tbody>
</table>

$\alpha^HVP,s = 53.41(59) \times 10^{-10}$
Also
$\alpha^HVP,c = 14.4(4) \times 10^{-10}$
$\alpha^HVP,b = 0.27(4) \times 10^{-10}$
**UP/DOWN contribution**

Much noisier and sensitive to u/d mass. Use

\[
G(t) = \begin{cases} 
  G_{\text{data}}(t) & \text{for } t \leq t^* \\
  G_{\text{fit}}(t) & \text{for } t > t^*
\end{cases}
\]

\(t^* = 1.5\text{fm} = 6/m_\rho\)

so 70% of result from \(G_{\text{data}}\)

Must correct for finite vol.

effects in \(\pi \pi\) contribn using scalar QED (7%)

Rescale \(\Pi_j\) by

\[
(m_\rho^{\text{latt}}/m_\rho^{\text{exp}})^{2j}
\]

to reduce u/d mass effects.

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**Chakraborty et al, HPQCD, 1601.03071**
Combining numbers for a total

3.5σ discrepancy with no new physics

$\alpha_{\mu}^{\text{HVP,LO}} \times 10^{-10}$

598(11) $u/d$

53.4(6) $s$

14.4(4) $c$

0.27(4) $b$

Total 666(6)(12)

add syst from
disc. diags
(1.5%) in quad

Chakraborty et al, HPQCD, 1601.03071

no new physics
Conclusion

- Lattice QCD calculations now on ‘2nd generation’ gluon configs with charm in the sea and $m_{u,d}$ at physical value (so no extrapoln).

- Gold-plated hadron masses and decay constants provide stringent tests of QCD/SM.

- NEW calculation of light meson electromagnetic form factor up to $Q^2 = 6 \text{ GeV}^2$ shows behaviour inconsistent with asymptotic pert. QCD. Further calc. underway …

- Accurate quark mass determinations being tested with multiple methods

- sub-1% uncertainties on lattice QCD calculations for HVP contribution to $a_\mu$ are within sight.
Spares
Mass ratio direct from lattice QCD using same formalism for both quarks (HISQ). Not possible to connect heavy and light masses without lattice.

\[
m_c/m_s
\]

Good consistency between different lattice actions

\[
\frac{m_c}{m_s} = 11.652(65)
\]

HPQCD, 1408.4169
B meson decay constants: HPQCD results from NRQCD b and physical u/d quarks

Use 2+1+1 configs with u/d down to physical values + improved NRQCD

meson mass difference correct to 2%

B_s to B decay constant ratio accurate to 0.6% - since Z factors cancel.
Separate decay constants to 2%
B, B_s decay constant world averages

PDG av. branching fraction + unitarity Vub

HPQCD NRQCD 1302.2644
HPQCD HISQ 1110.4510
FNAL/MILC 1112.3051
RBC/UKQCD 1404.4670
ETMC 1308.1851
ALPHA 1404.3590

Different approaches to b quark being used and to normalising weak current

NOTE: 
\( f_{B_s} < f_{D_s} \) now quite clear
Constraining new physics with lattice QCD

* results at physical u/d quark masses*

\[ f_K / f_\pi \]

Annihilation of \( K/\pi \) to W allows CKM element determination given decay constants from lattice QCD expt for \( \frac{\Gamma(K^+ \rightarrow \ell\nu)}{\Gamma(\pi^+ \rightarrow \ell\nu)} \)

\[ |V_{us}| \frac{f_{K^+}}{|V_{ud}| f_{\pi^+}} = 0.27598(35) Br(K^+) (25)_{EM} \]

from lattice gives CKM

R.Dowdall et al, HPQCD: 1303.1670
Charm contribution to

$$R = \sigma(e^+ e^- \rightarrow \text{hadrons})/\sigma_{pt}$$

Subtract u,d,s with pert. th. to isolate charm contribution.

Calculate inverse-s-moment:

$$\mathcal{M}_n \equiv \int \frac{ds}{s^{n+1}} R_c(s)$$

Lattice calcln:

$$\mathcal{M}_n = \sum t^n G(t)$$

G. Donald et al, HPQCD,1208.2855
Error budget for HVP,LO calculation

TABLE III: Error budget for the connected contributions to the muon anomaly $a_\mu$ from vacuum polarization of $u/d$ quarks.

<table>
<thead>
<tr>
<th>Error Source</th>
<th>$a_\mu^{\text{HVP,LO}}(u/d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED corrections</td>
<td>1.0%</td>
</tr>
<tr>
<td>Isospin breaking corrections</td>
<td>1.0%</td>
</tr>
<tr>
<td>Staggered pions, finite volume</td>
<td>0.7%</td>
</tr>
<tr>
<td>Noise reduction ($t^*$)</td>
<td>0.5%</td>
</tr>
<tr>
<td>Valence $m_\ell$ extrapolation</td>
<td>0.4%</td>
</tr>
<tr>
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<td>0.4%</td>
</tr>
<tr>
<td>Correlator fits</td>
<td>0.2%</td>
</tr>
<tr>
<td>Tuning sea-quark masses</td>
<td>0.2%</td>
</tr>
<tr>
<td>Lattice spacing uncertainty</td>
<td>$&lt; 0.05%$</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>1.9%</strong></td>
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