Finite volume effects for spatially non-local operators

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Theory Center Cake Seminar
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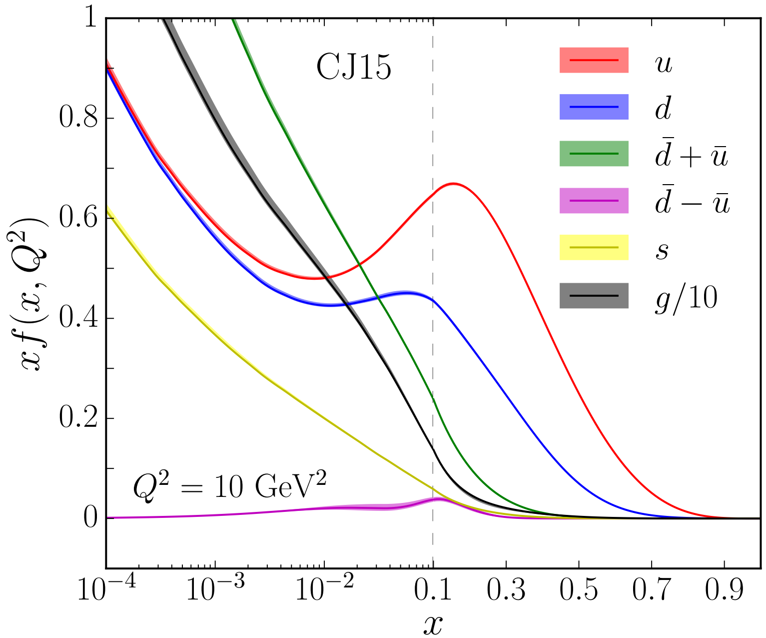
Based on work in progress with:
Raul Briceño, Maxwell Hansen & Chris Monahan (to appear soon)

Parton Distribution Functions (PDFs)

- Describe the internal structure of the nucleons
- Universal
- Non-perturbative objects
- Usually obtained by fitting observables to experimental data



Accardi et al., PRD **93**, 114017 (2016).

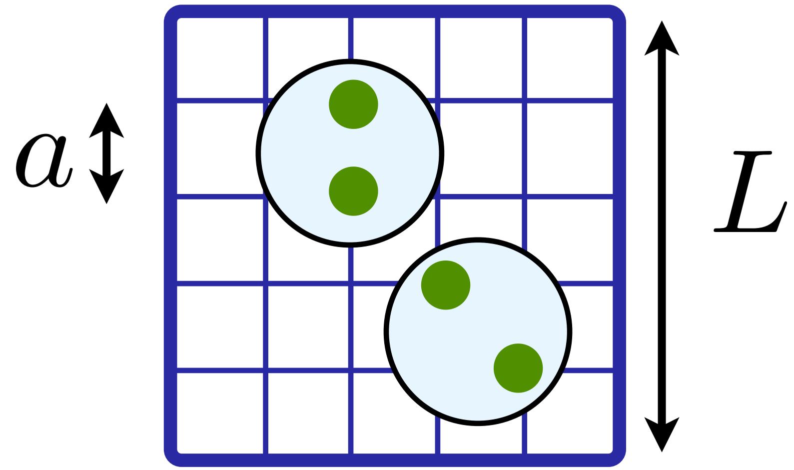
Novel idea: PDFs on the lattice

PDFs from QCD: the only non-perturbative way to study QCD is lattice QCD.

$$t_M \to -it_E$$

Lattice QCD is defined by…

- Discretization
- Euclidean vs Minkowski
- Finite volume
- Quark masses



PDFs on the lattice

- calculations are already on the way… as you saw Monday
- in order to know if we're calculating things correctly, we need to understand systematics
  - Discretization ☑
  - Euclidean vs Minkowski ☑
  - Renormalization ☑
  - Finite volume ? ☆nobody has talked about FV effects…

Scheme to extract PDFs from the lattice

PDFs on the lattice → evaluation of matrix elements of non-local operators

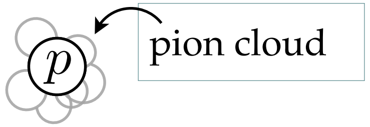
$$\langle N|\mathcal{O}|N\rangle$$

There are different techniques:

- wilson lines: $\langle N|\bar{q}\,W\,q|N\rangle_\infty$
- two current operators: $\langle N|\mathcal{J}(0,\xi)\mathcal{J}(0)|N\rangle_\infty$

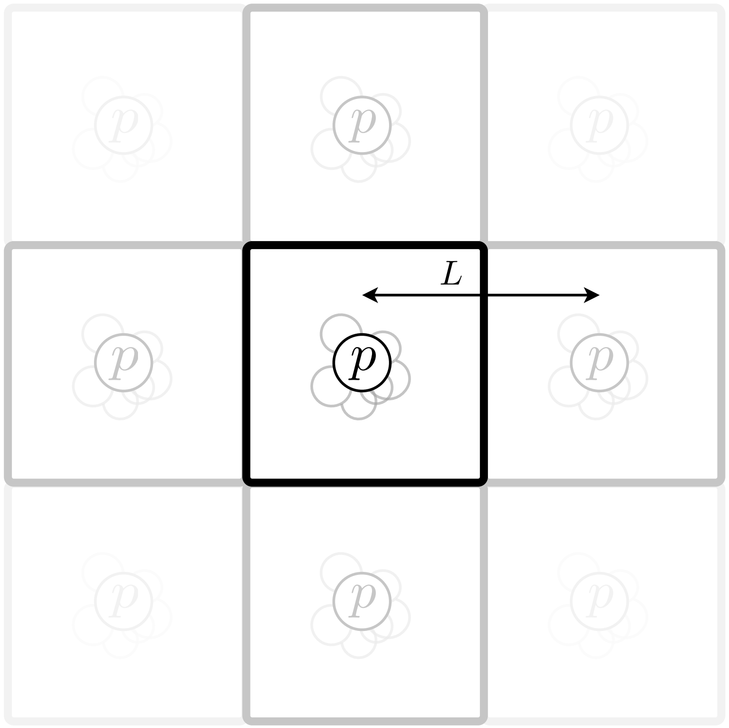
Lattice QCD		Pheno QCD
$\langle N\|\bar{q}\,W\,q\|N\rangle_V$	? →	$\langle N\|\bar{q}\,W\,q\|N\rangle_\infty$
$\langle N\|\mathcal{J}(0,\xi)\mathcal{J}(0)\|N\rangle_V$		$\langle N\|\mathcal{J}(0,\xi)\mathcal{J}(0)\|N\rangle_\infty$

Finite volume: Infrared limit of the theory



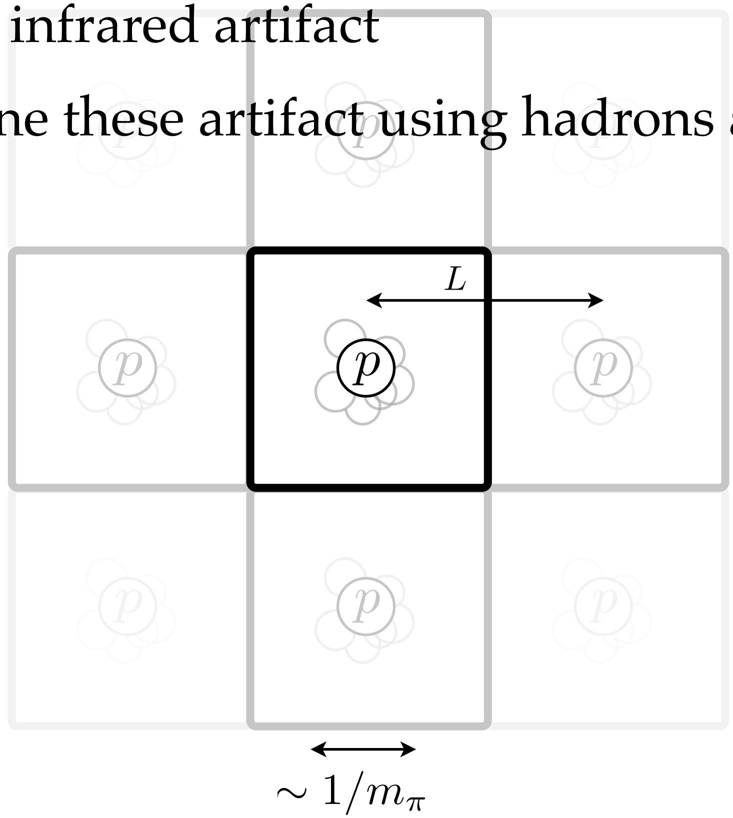
Finite volume: Infrared limit of the theory

- Finite-volume artifacts arise from the interactions with mirror images



Finite volume: Infrared limit of the theory

- Finite-volume artifacts arise from the interactions with mirror images
- Assuming $L >>$ size of the hadrons ~ $1/m_\pi$
  - This is a purely infrared artifact
  - We can determine these artifact using hadrons as the degrees of freedom



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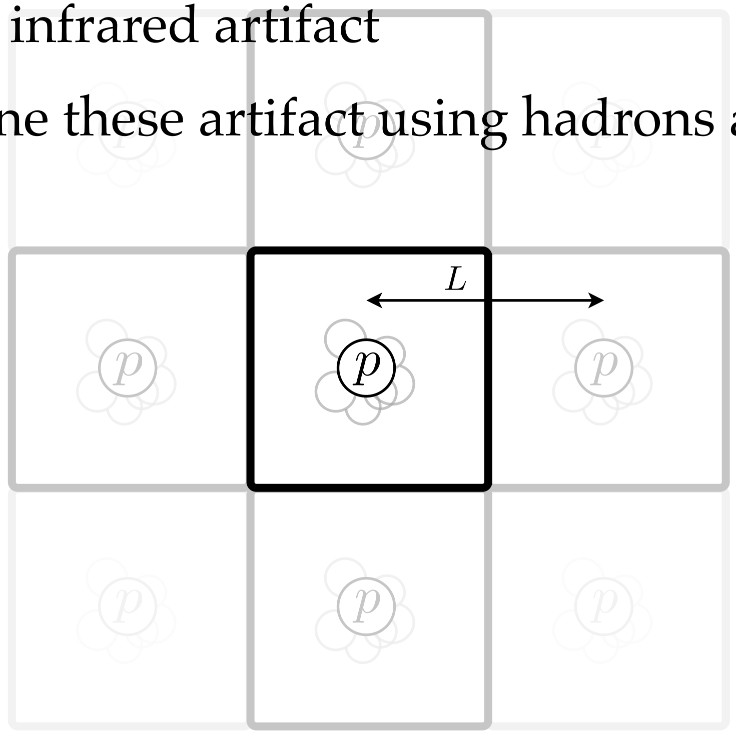


☑ interactions with mirror images: Yukawa

$$m_N(L) - m_N(\infty) \sim \langle N|\hat{V}|N\rangle_L \sim e^{-m_\pi L}$$

Lüscher (1985)

Finite volume effects: Matrix elements

- In general, the masses and matrix elements of stable particles have been observed to have these exponentially supressed corrections.

- But matrix elements of non-local currents suffer of larger FV effects:

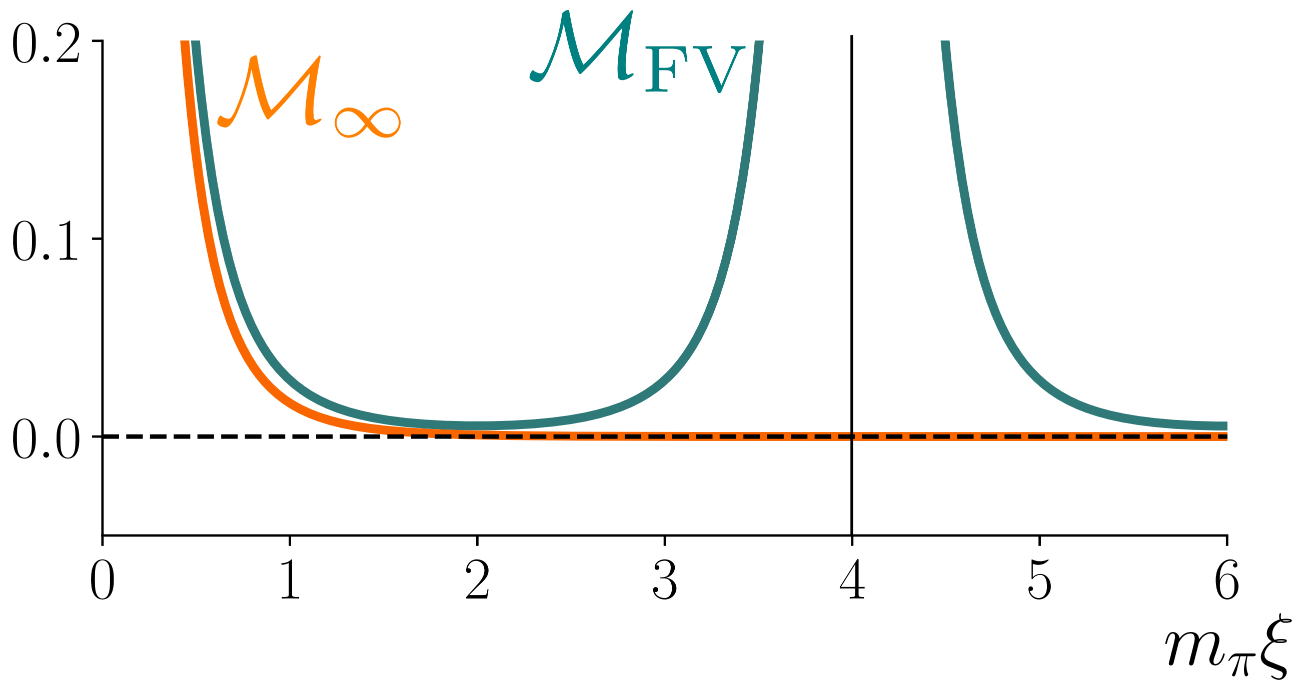
$\langle N|\mathcal{J}(0,\boldsymbol{\xi})\mathcal{J}(0)|N\rangle_\infty$: generally decays as a function of ξ

$\langle N|\mathcal{J}(0,\boldsymbol{\xi})\mathcal{J}(0)|N\rangle_V$: periodic, since

$$\mathcal{J}(t,\mathbf{x}) = \mathcal{J}(t,\mathbf{x}+L\mathbf{e}_i)$$

→ Expect enhanced finite volume effects to keep periodicity!

Finite volume effects: Matrix elements



Expect enhanced finite volume effects to keep periodicity!

Finite volume effects: Matrix elements

Wilson line is not periodic:

$$W[x+\xi\mathbf{e}_i, x] \equiv U_i(x+(\xi-a)\mathbf{e}_i)\, U_i(x+(\xi-2a)\mathbf{e}_i) \times \cdots \times U_i(x+a\mathbf{e}_i)$$

Quark bilinears connected to Wilson Lines:

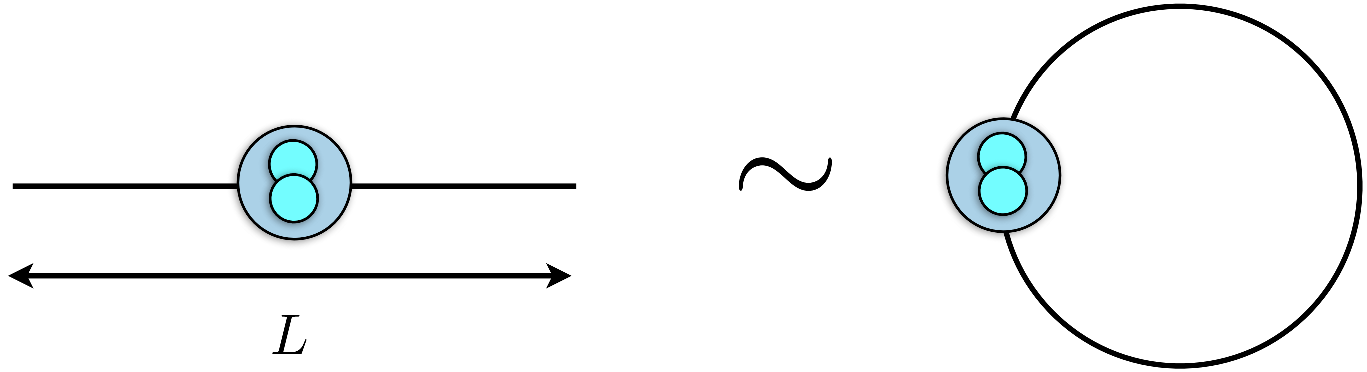
$$\overline{q}\big(x+(\xi+nL)\mathbf{e}_i\big)\, W\big[x+(\xi+nL)\mathbf{e}_i, x\big]\, q\big(x\big) = \overline{q}(x+\xi\mathbf{e}_i)\, W[x+\xi\mathbf{e}_i, x]\,\Big(W[x+L\mathbf{e}_i, x]^n\Big)\, q(x)$$

are no periodic. However,

q(x) and U(x) feel boundary conditions $\xrightarrow{?}$ expect enhanced finite volume effects for large ξ

Physics in a 1D finite box

- Free particle wave function: $\varphi_p(x) = e^{ipx}$



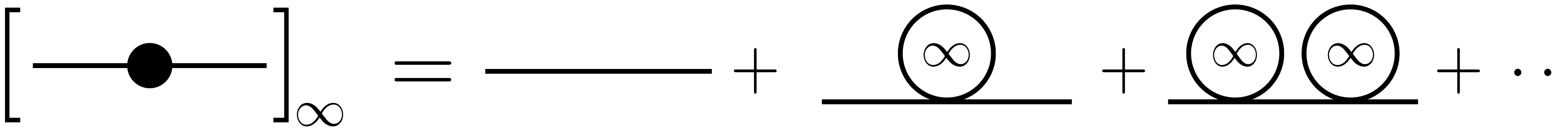
$$\varphi_p(L+x) = e^{ip(x+L)} = \varphi_p(x) = e^{ipx}$$

- Discretized momentum and spectrum: $p = \dfrac{2\pi n}{L}$

- Question: What happens to the masses determined in a finite-volume?

Masses in an infinite volume

mass = pole location of the fully dressed propagator:



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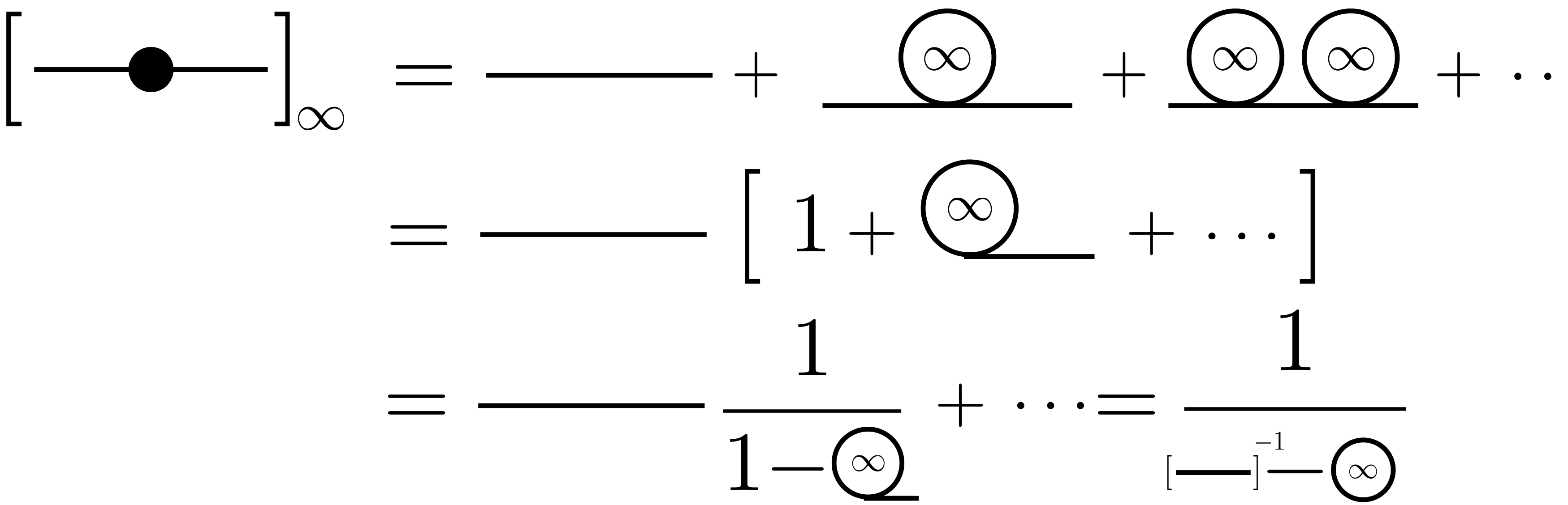

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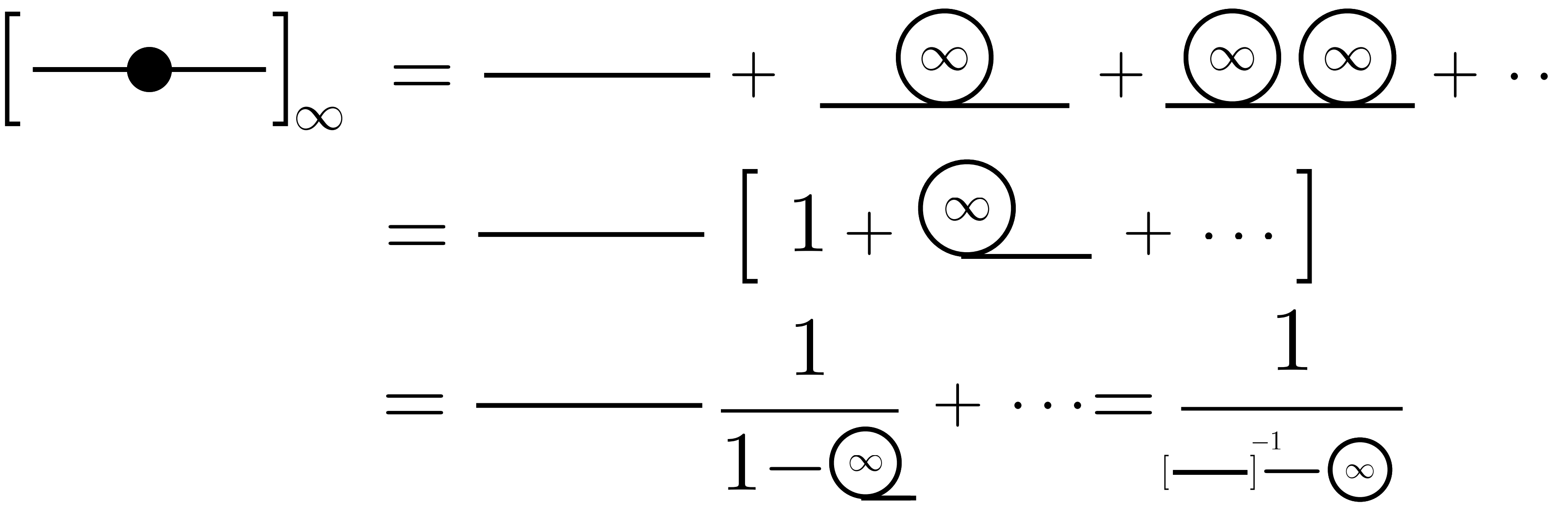
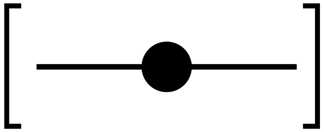


$$= \frac{1}{\frac{p^2 - m_0^2}{i} - iI_\infty} \longrightarrow \frac{i}{p^2 - m_\infty^2}$$

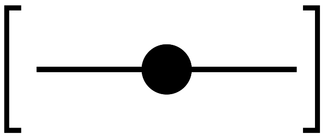
Masses in a finite volume

infinite volume mass = pole of infinite volume propagator:



$$\Big[\quad\Big]_{\infty} = \frac{1}{\frac{p^2-m_0^2}{i} - iI_{\infty}} \longrightarrow \frac{i}{p^2 - m_{\infty}^2}$$

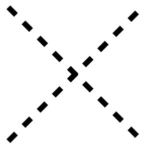
finite volume mass = pole of finite volume propagator:



$$\begin{aligned}\Big[\quad\Big]_{\mathrm{FV}} &= \frac{1}{\frac{p^2-m_0^2}{i} - iI_{\mathrm{FV}}} \\ &= \frac{1}{\frac{p^2-m_0^2}{i} - iI_{\infty} - i(I_{\mathrm{FV}} - I_{\infty})} \\ &= \frac{1}{\frac{p^2-m_{\infty}^2}{i} - i\delta I_{\mathrm{FV}}} \longrightarrow \frac{i}{p^2 - m_{\mathrm{FV}}^2}\end{aligned}$$

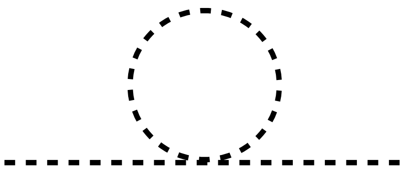
we need to calculate this…

A simple example: mass of a pion

Consider a toy model for mesons $\mathcal{L}_M = \frac{\lambda}{4!}\varphi^4$ 

Bare propagator has no volume dependence:

 $= \Delta_0(p^2) = \dfrac{i}{p^2 - m_0^2 + i\epsilon}$

so we have to have to go to loops… self-energy… 

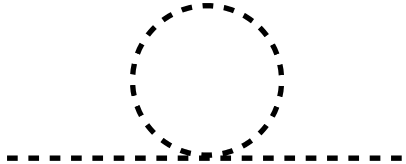
- In a Finite Volume, integrals over momenta become sums:

1D: $\displaystyle \int \frac{dk_i}{2\pi} \to \sum_{k_i} \frac{\Delta k_i}{2\pi} = \sum_{k_i} \frac{2\pi \Delta n}{2\pi L} = \frac{1}{L}\sum_{k_i}$ **3D:** $\displaystyle \int \frac{d^3k}{(2\pi)^3} \to \frac{1}{L^3}\sum_{k_i}$

A simple example: self-energy of a pion

in infinite volume:

$$I_\infty = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m_\pi^2}$$



in finite volume:

Poisson summation

$$I_{\rm FV} = \frac{1}{L^3} \sum_{\mathbf{k}} \int \frac{dk_4}{2\pi} \frac{1}{k^2 + m_\pi^2} = \sum_{\mathbf{n}} \int \frac{d^4k}{(2\pi)^4} \frac{e^{i\mathbf{k}\cdot\mathbf{n}L}}{k^2 + m_\pi^2}$$

finite / infinite volume difference:

$$\delta I_{\rm FV} = I_{\rm FV} - I_\infty = \sum_{\mathbf{n}\neq 0} \int \frac{d^4k}{(2\pi)^4} \frac{e^{i\mathbf{k}\cdot\mathbf{n}L}}{k^2 + m_\pi^2}$$

A simple example: self-energy of a pion

some details…

$$\boxed{l_n = |\mathbf{n}|L}$$

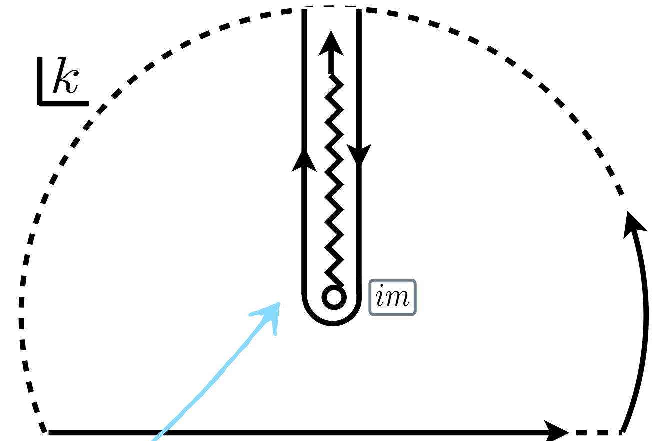
$$\delta I_{\rm FV} = \sum_{\mathbf{n}\neq 0}\int \frac{d^4k}{(2\pi)^4} e^{i\mathbf{k}\cdot L\mathbf{n}} \frac{1}{k^2+m^2}$$

$$= \sum_{\mathbf{n}\neq 0}\int \frac{k^2 dk d\cos\theta}{(2\pi)^2} e^{ik\, l_n \cos\theta} \frac{1}{2\sqrt{k^2+m^2}}$$

$$= \sum_{\mathbf{n}\neq 0}\int_0^\infty \frac{k^2 dk}{(2\pi)^2} \frac{1}{2\sqrt{k^2+m^2}} \frac{1}{ik\, l_n}\left(e^{ik\, l_n} - e^{-ik\, l_n}\right)$$

$$= \sum_{\mathbf{n}\neq 0}\int_{-\infty}^\infty \frac{k dk}{(2\pi)^2} \frac{1}{2\sqrt{k^2+m^2}} \frac{1}{i\, l_n} e^{ik\, l_n}$$

$$= \frac{1}{(2\pi)^2}\sum_{\mathbf{n}\neq 0}\frac{1}{2il_n}\int_{-\infty}^\infty dk \frac{k}{\sqrt{k^2+m^2}} e^{ik\, l_n}$$



$$k_+ = im + ire^{i\epsilon}$$

$$k_- = im + ire^{i(2\pi-\epsilon)}$$

$$\left.\sqrt{k^2+m^2}\right|_{k=k_-} = -\left.\sqrt{k^2+m^2}\right|_{k=k_+}$$

A simple example: self-energy of a pion

almost done…

$$\delta I_{\rm FV} = \frac{1}{(2\pi)^2} \sum_{\mathbf{n}\neq 0} \frac{1}{i l_n} \int_{im}^{i\infty} dk \frac{k}{\sqrt{k^2+m^2}} e^{ik\, l_n}$$

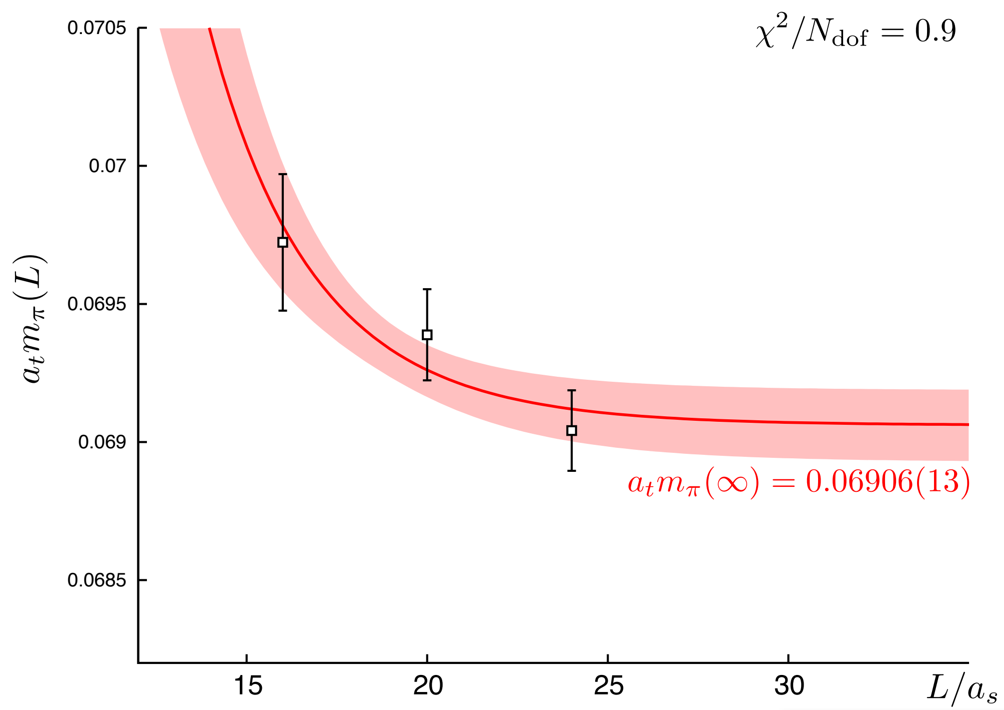
$$= \frac{1}{(2\pi)^2} \sum_{\mathbf{n}\neq 0} \frac{m}{l_n} \int_{1}^{\infty} dq \frac{q}{\sqrt{q^2-1}} e^{-qml_n}$$

final result:

$$\boxed{\delta I_{\rm FV} = \frac{1}{(2\pi)^2} \sum_{\mathbf{n}} \left(\frac{m}{|\mathbf{n}|L}\right) K_1\left(|\mathbf{n}|L\, m\right) \sim e^{-mL}}$$

A simple example: self-energy of a pion

$$m_\pi(L) = m_\pi + c\frac{e^{-m_\pi L}}{(m_\pi L)^{3/2}}$$



Dudek, Edwards & Thomas (2012)

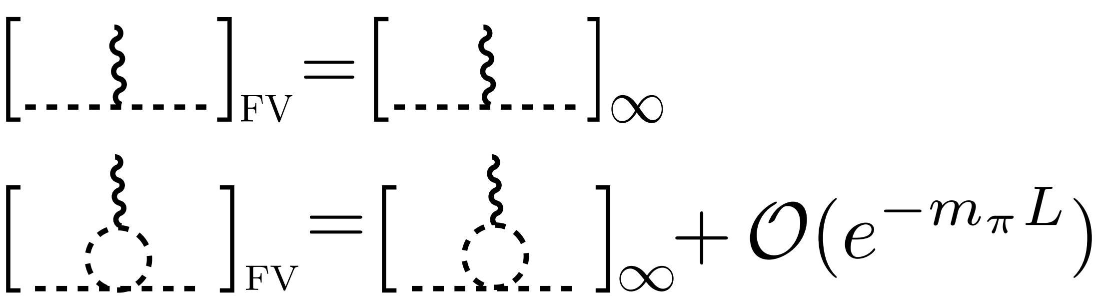
$$m_\pi \sim 390\ MeV,\ a_s \sim 0.12\ \text{fm} \longrightarrow m_\pi L \sim 3.8,\ \ 4.7,\ \ 5.6$$

General observations:

- FV corrections come from *sums/integrals* of momenta.

- For masses and local currents, if the intermediate states cannot go on-shell, the FV corrections are exponentially small

- For example, for the pion form factor :

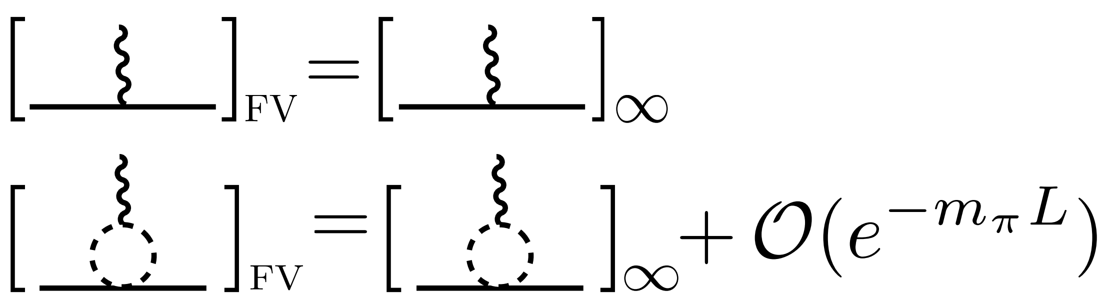


General observations:

- For heavy particles, these observations persists: Nucleon

- Pion cloud in the mass: exponential corrections with the pion mass…not the nucleon mass

- Pion cloud in the form factors…: exponential corrections with **the pion mass**… not the nucleon mass

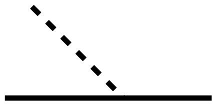


Our toy model

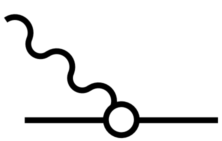
Consider a theory with two scalar particles

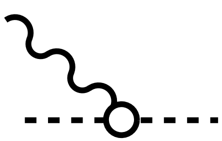
- a light one, φ, analogous to the pion
- a heavy one, χ, analogous to the nucleon
- momentum independent coupling

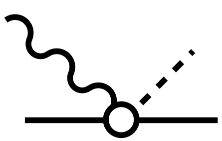
$$m_\varphi \ll m_\chi$$

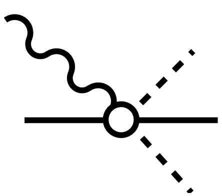
 $= g$

Coupling to an external current :

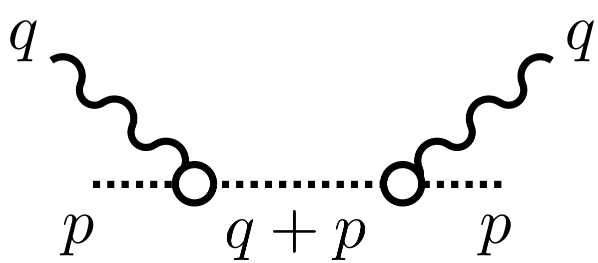
 $= g_\chi$

 $= g_\varphi$

 $= g_{\chi\varphi}$

 $= g_{\chi\varphi\varphi}$

Light external states



$$\mathcal{M}_\infty^{(\mathrm{LO})}(\boldsymbol{\xi},\mathbf{p}) = g_\varphi^2 \int_{q_E} \frac{e^{i\mathbf{q}\cdot\boldsymbol{\xi}}}{(p_E+q_E)^2+m_\varphi^2}$$

Even at LO has an integral → Expect enhanced FV effects

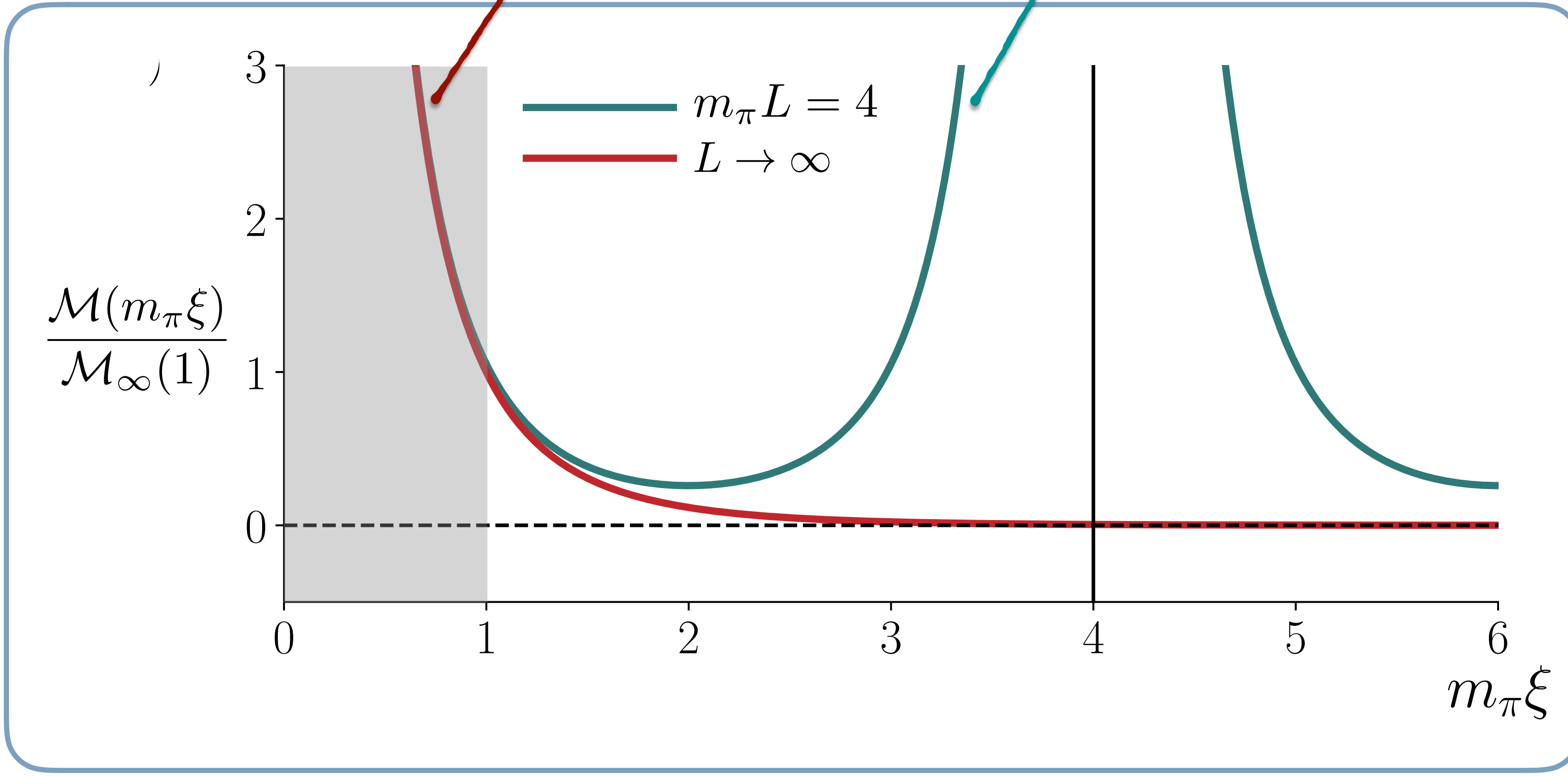
Finite volume correction: $$\delta\mathcal{M}_L^{(\mathrm{LO})}(\boldsymbol{\xi},\mathbf{p}) = g_\varphi^2 \sum_{\mathbf{n}\neq 0} \int_{q_E} \frac{e^{i\mathbf{q}\cdot(\boldsymbol{\xi}+iL\mathbf{n})}}{(p_E+q_E)^2+m_\varphi^2}$$

$$\delta\mathcal{M}_L^{(\mathrm{LO})}(\boldsymbol{\xi},\mathbf{p}) = \frac{m_\varphi g_\varphi^2}{4\pi^2} e^{-i\mathbf{p}\cdot\boldsymbol{\xi}} \sum_{\mathbf{n}\neq 0} \frac{K_1\left(m_\varphi|\boldsymbol{\xi}+L\mathbf{n}|\right)}{|\boldsymbol{\xi}+L\mathbf{n}|} \sim \frac{m_\varphi g_\varphi^2}{4\pi^2} \frac{K_1\left(m_\varphi|L-\xi|\right)}{|L-\xi|}$$

$$\delta\mathcal{M}_L^{(\mathrm{LO})}(\boldsymbol{\xi},\mathbf{p}) \propto \frac{e^{-m_\varphi(L-\xi)}}{(L-\xi)^{3/2}}$$

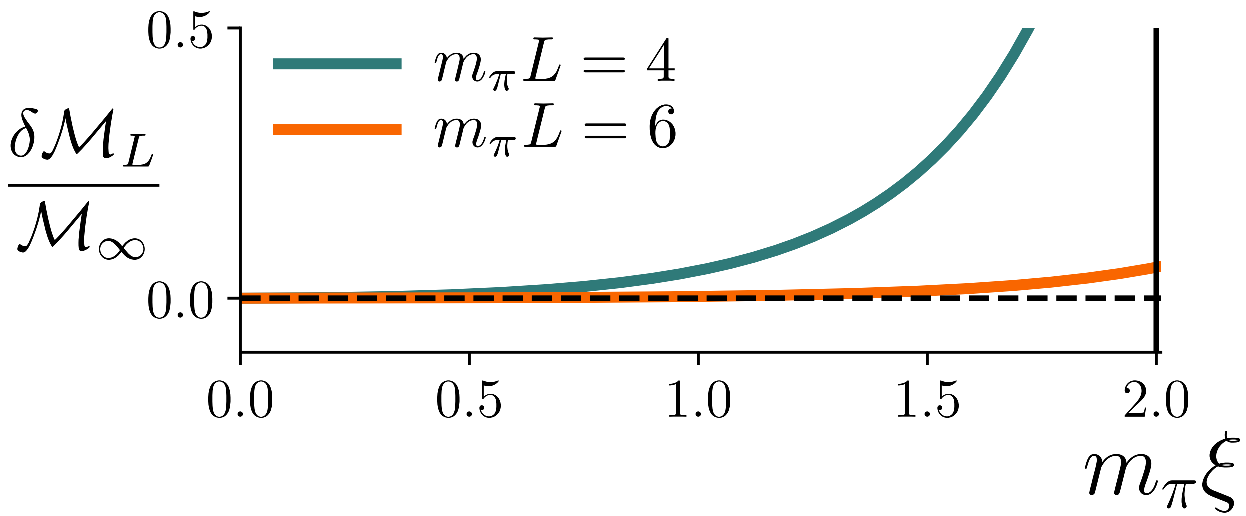
Light external states

$\mathcal{M}_{\infty}^{(\mathrm{LO})}$ $\mathcal{M}_{L}^{(\mathrm{LO})}$

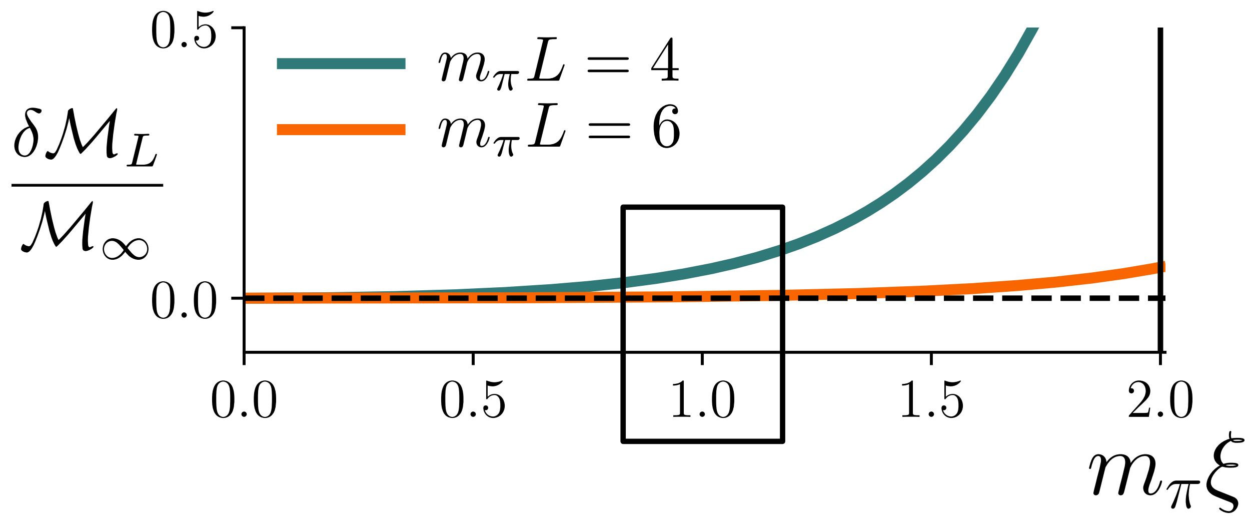


Expected behavior!

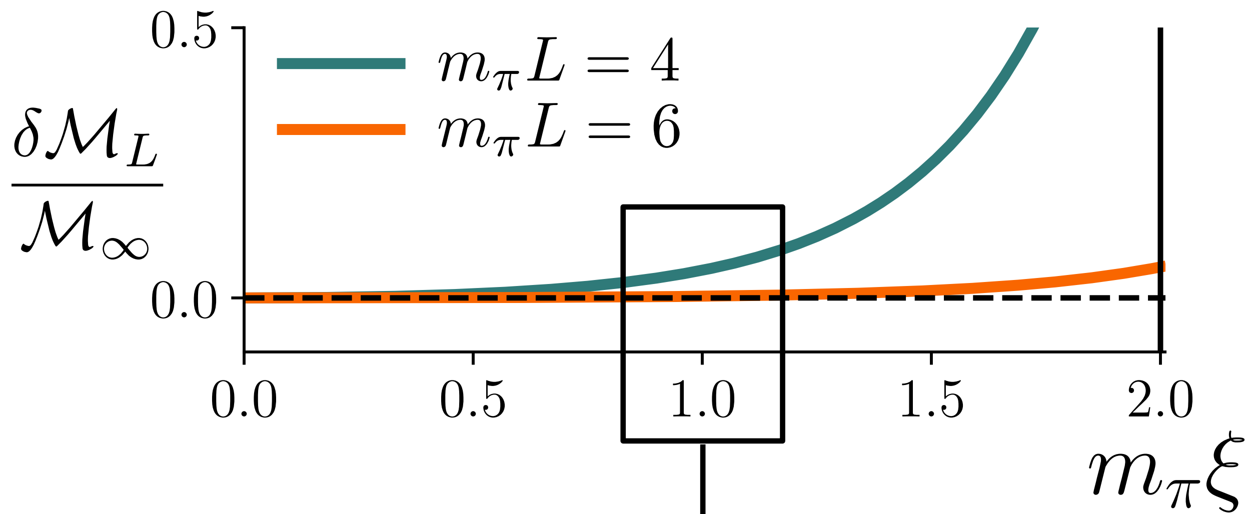
Light external states



Light external states



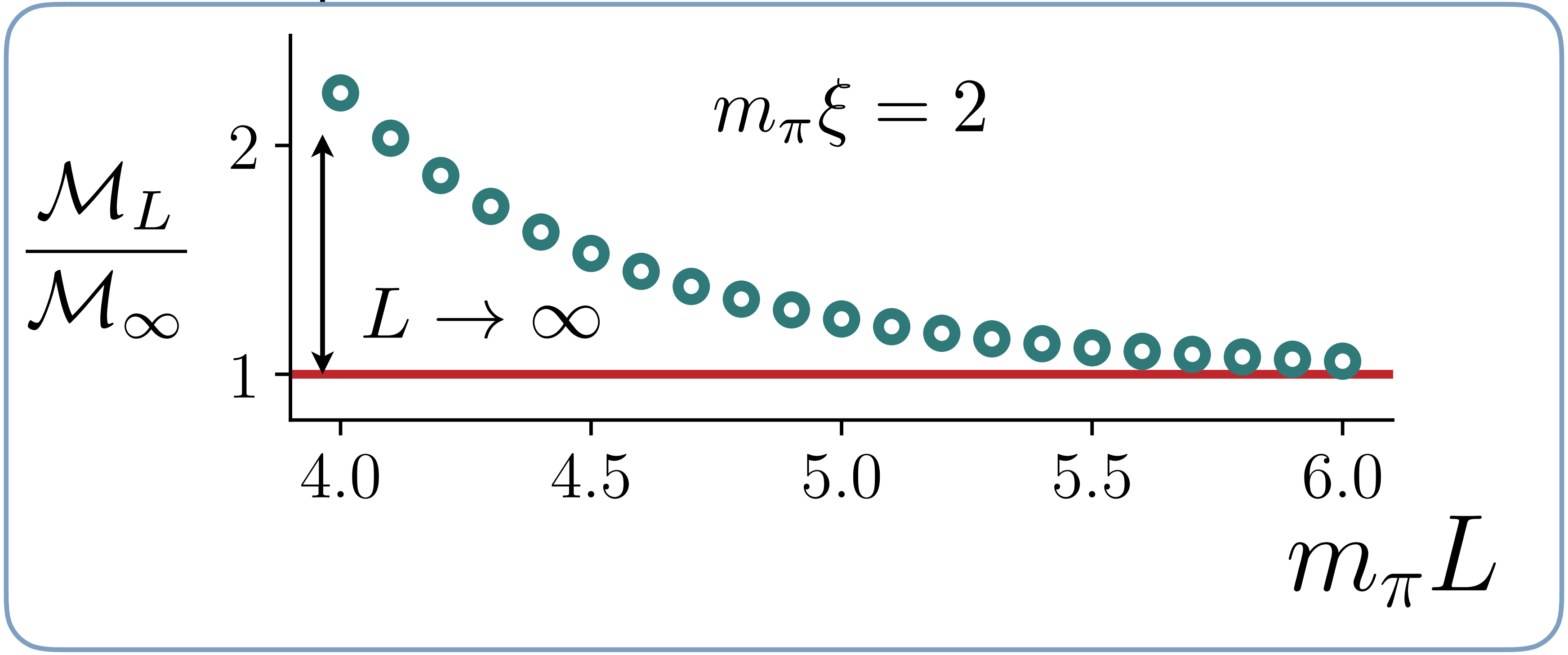
Light external states



$\sim 10\,\%$ when $\dfrac{\xi}{L} = \dfrac{1}{4}$

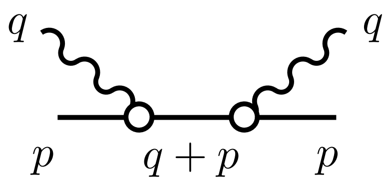
Light external states

100% systematic uncertainty!
inaccurate…despite it being arbitrarily precise!



Heavy external states

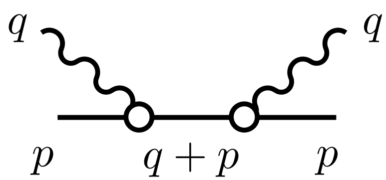
Leading order



$$\delta\mathcal{M}_L^{(\mathrm{LO})}(\boldsymbol{\xi},\mathbf{p}) \propto \frac{e^{-m_\chi(L-\xi)}}{(L-\xi)^{3/2}} \ll e^{-m_\varphi(L-\xi)}$$

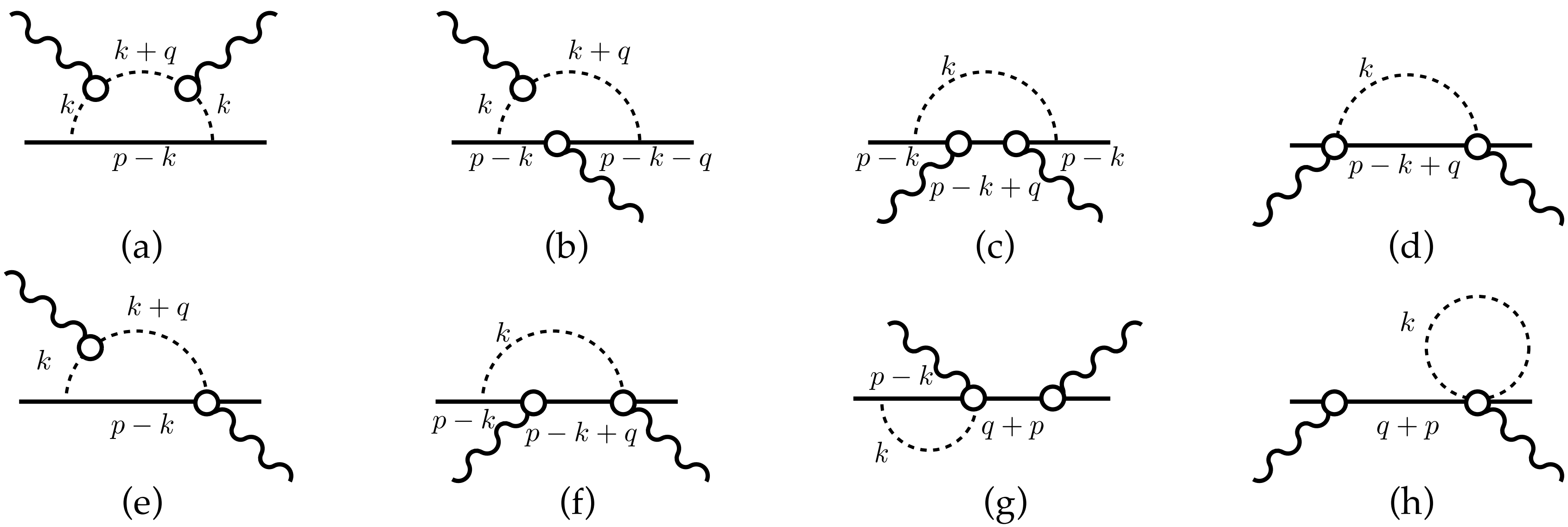
Heavy external states

Leading order



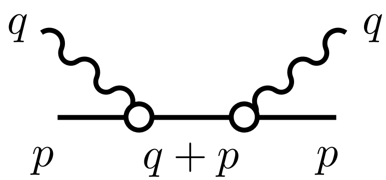
$$\delta\mathcal{M}_L^{(\mathrm{LO})}(\boldsymbol{\xi}, \mathbf{p}) \propto \frac{e^{-m_\chi(L-\xi)}}{(L-\xi)^{3/2}} \ll e^{-m_\varphi(L-\xi)}$$

Next to Leading Order



Heavy external states

Leading order



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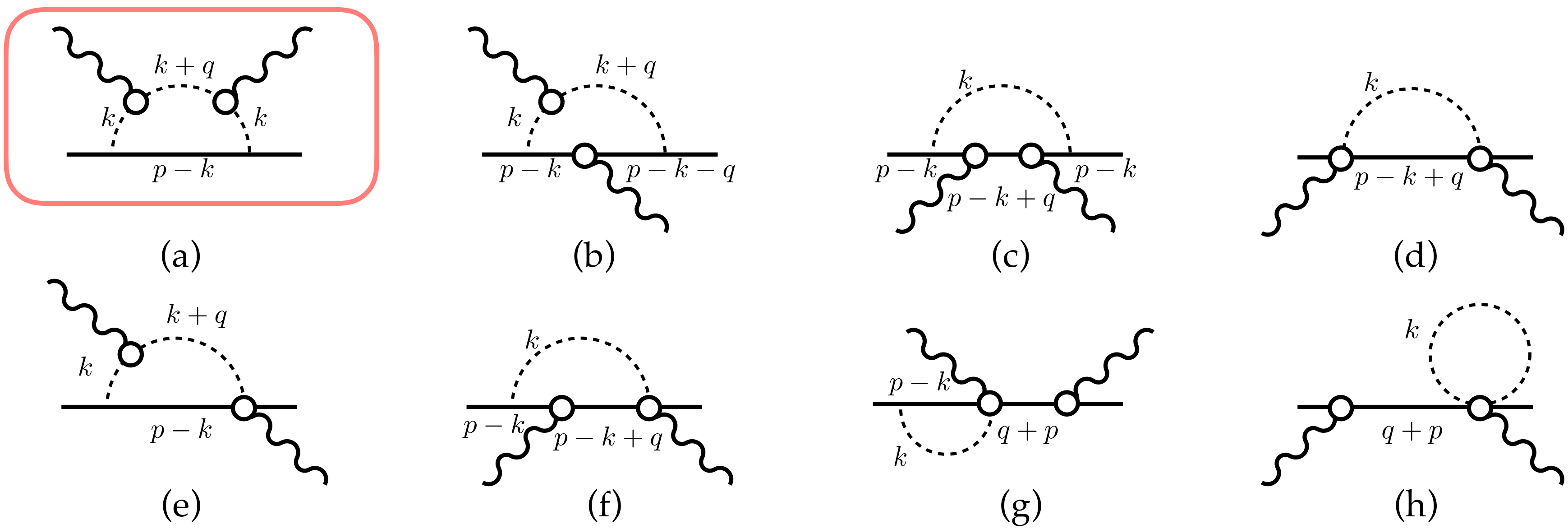
Next to Leading Order



juanvg@jlab.org

In general…

We find that in general the matrix elements…

$$\langle M|\mathcal{J}(0,\boldsymbol{\xi})\mathcal{J}(0)|M\rangle_L - \langle M|\mathcal{J}(0,\boldsymbol{\xi})\mathcal{J}(0)|M\rangle_\infty = P_a(\boldsymbol{\xi},L)e^{-M(L-\xi)} + P_b(\boldsymbol{\xi},L)e^{-m_\pi L} + \cdots\,,$$

Polynomial prefactors $\propto L^m/|L-\xi|^n$

Open questions…

EFT for small separations - sensible?

Wilson lines in EFT…?

Non-zero momenta..p~1/L…momentum and volume dependence will mix…not obvious how to proceed

Flavor changing currents…?

Quark mass dependence will be similar to the FV dependence within χPT…

Summary

- We presented first steps towards understanding finite-volume artifacts that arise in matrix elements of spatially non-local operators.
  - matrix elements of spatially-separated currents, one of the approaches to determine hadron structure from lattice QCD.
- We considered a toy model involving two scalar particles to estimate the size of finite-volume corrections.
  - lightest particle: LO contribution dominant,effects scale like: $P(\xi, L)e^{-m_\pi (L-\xi)}$
  - heaviest particle: NLO contribution dominant,effects scale like: $P(\xi, L)e^{-m_\pi L}$

Thank you!

Backup slides

Asymptotic behaviors

$$\delta\mathcal{M}_L^{(b)}(\boldsymbol{\xi},\mathbf{0}) = g^2 g_\varphi g_\chi \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{n}-\boldsymbol{\xi}|; M(x)\right]\right] \left[\int_0^1 \mathrm{d}y\, \mathcal{I}_2\left[|L\mathbf{m}-\boldsymbol{\xi}|; M(y)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(c)}(\boldsymbol{\xi},\mathbf{0}) = 2g^2 g_\chi^2 \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\chi\right] \left[\int_0^1 \mathrm{d}x\,(1-x)\, \mathcal{I}_3\left[|L\mathbf{m}-\boldsymbol{\xi}|; M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(d)}(\boldsymbol{\xi},\mathbf{0}) = g_{\chi\varphi}^2 \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\chi\right] \mathcal{I}_1\left[|L\mathbf{m}-\boldsymbol{\xi}|; m_\varphi\right] ,$$

$$\delta\mathcal{M}_L^{(e)}(\boldsymbol{\xi},\mathbf{0}) = g g_\varphi g_{\chi\varphi} \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\varphi\right] \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{m}-\boldsymbol{\xi}|; M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(f)}(\boldsymbol{\xi},\mathbf{0}) = g g_\chi g_{\chi\varphi} \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\chi\right] \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{m}-\boldsymbol{\xi}|; M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(g)}(\boldsymbol{\xi},\mathbf{0}) = g g_{\chi\varphi} g_\chi \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\chi\right] \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{m}|; M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(h)}(\boldsymbol{\xi},\mathbf{0}) = \frac{1}{2} g_\chi g_{\chi\varphi\varphi} \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\chi\right] \mathcal{I}_1\left[|L\mathbf{m}|; m_\varphi\right] .$$

Asymptotic behaviors

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi^2}{128\pi^3 m_\varphi}\left[\frac{\xi^{1/2}}{(L-\xi)^{3/2}}H_{x,3/2}(\xi) + \frac{(L-\xi)^{1/2}}{\xi^{3/2}}H_{x,3/2}(L-\xi)\right]e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(b)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi g_\chi}{64\pi^3 m_\varphi}\left[\frac{1}{\xi^{1/2}(L-\xi)^{1/2}}H_{1,1/2}(\xi)H_{1,1/2}(L-\xi)\right]e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(c)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g^2 g_\chi^2}{128\pi^3}\frac{m_\chi^{1/2}}{m_\varphi^{3/2}}\left[\frac{(L-\xi)^{1/2}}{\xi^{3/2}}H_{1-x,3/2}(L-\xi)\right]e^{-\xi(m_\chi-m_\varphi)}e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(d)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g_{\chi\varphi}^2 m_\chi^{1/2} m_\varphi^{1/2}}{32\pi^3}\left[\frac{1}{\xi^{3/2}(L-\xi)^{3/2}}\right]e^{-\xi(m_\chi-m_\varphi)}e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(e)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g g_\varphi g_{\chi\varphi}}{64\pi^3}\left[\frac{1}{\xi^{1/2}(L-\xi)^{3/2}}H_{1,1/2}(\xi) + \frac{1}{\xi^{3/2}(L-\xi)^{1/2}}H_{1,1/2}(L-\xi)\right]e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(f)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g g_\chi g_{\chi\varphi} m_\chi^{1/2}}{64\pi^3 m_\varphi^{1/2}}\left[\frac{1}{\xi^{3/2}(L-\xi)^{1/2}}H_{1,1/2}(L-\xi)\right]e^{-\xi(m_\chi-m_\varphi)}e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(g)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g g_{\chi\varphi} g_\chi m_\chi^{1/2}}{64\pi^3 m_\varphi^{1/2}}\left[\frac{1}{\xi^{3/2}L^{1/2}}H_{1,1/2}(L)\right]e^{-\xi m_\chi}e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(h)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g_\chi g_{\chi\varphi\varphi} m_\varphi^{1/2} m_\chi^{1/2}}{64\pi^3}\left[\frac{1}{\xi^{3/2}L^{3/2}}\right]e^{-m_\chi\xi}e^{-m_\varphi L},$$

Heavy external states: Next to Leading Order