Bound states and QCD

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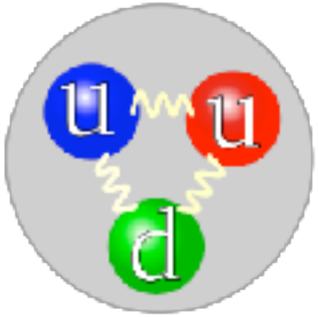
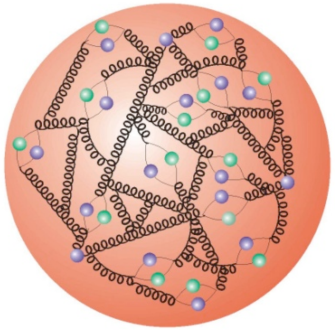


Hadrons are bound states of QCD



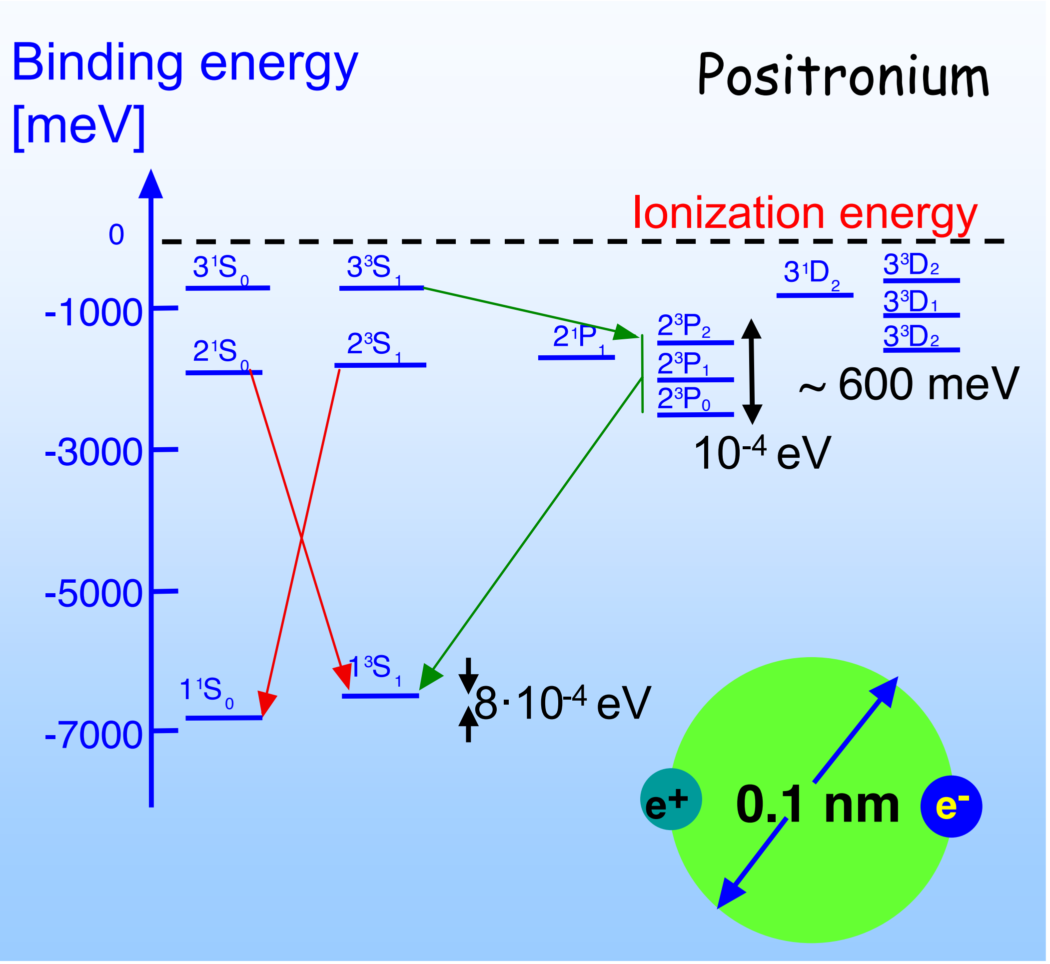
Methods:

- PQCD in hard collisions: Factorization
- Lattice QCD: Hadron spectrum, form factors, …
- Effective theories: χPT, HQET, …
- Models: "Inspired" by QCD and data   Quark model, Duality, …

PH 1605.01532, 1711.10851 and to appear

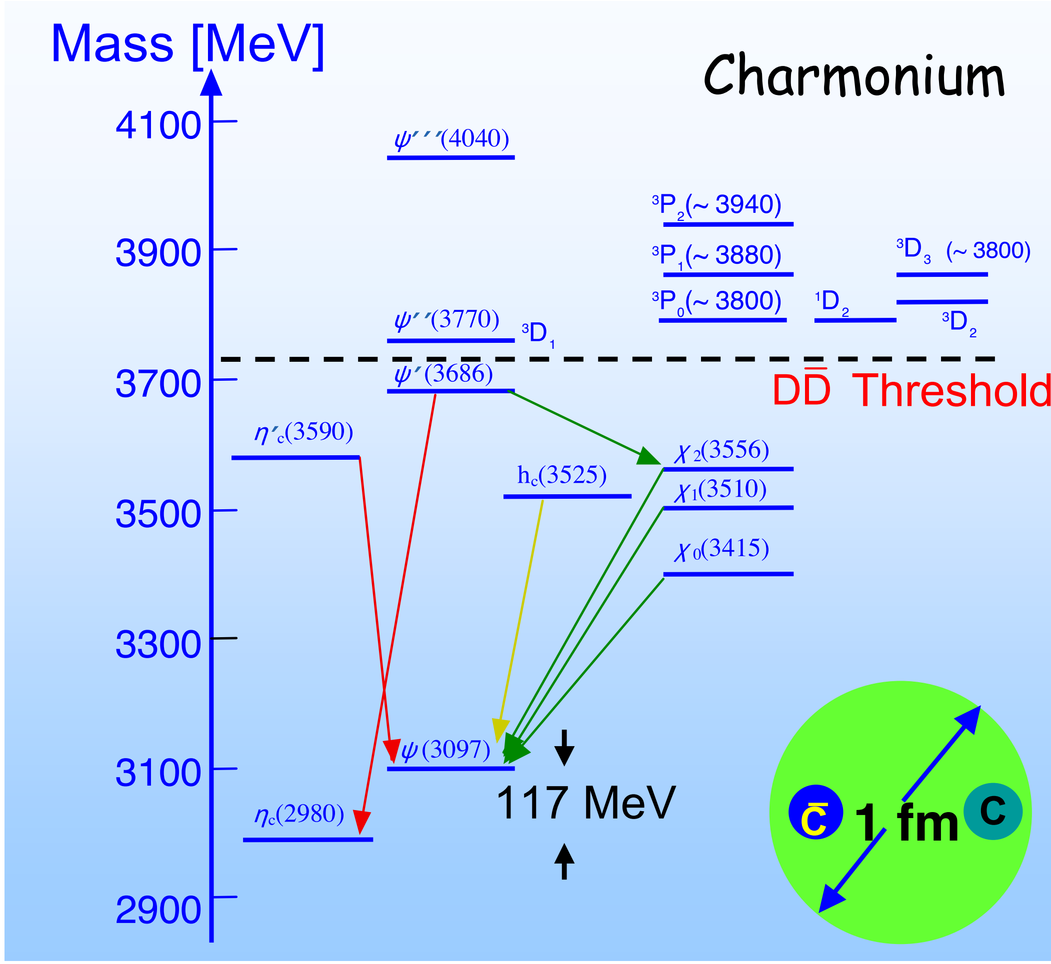
"The J/ψ is the Hydrogen atom of QCD"

QED



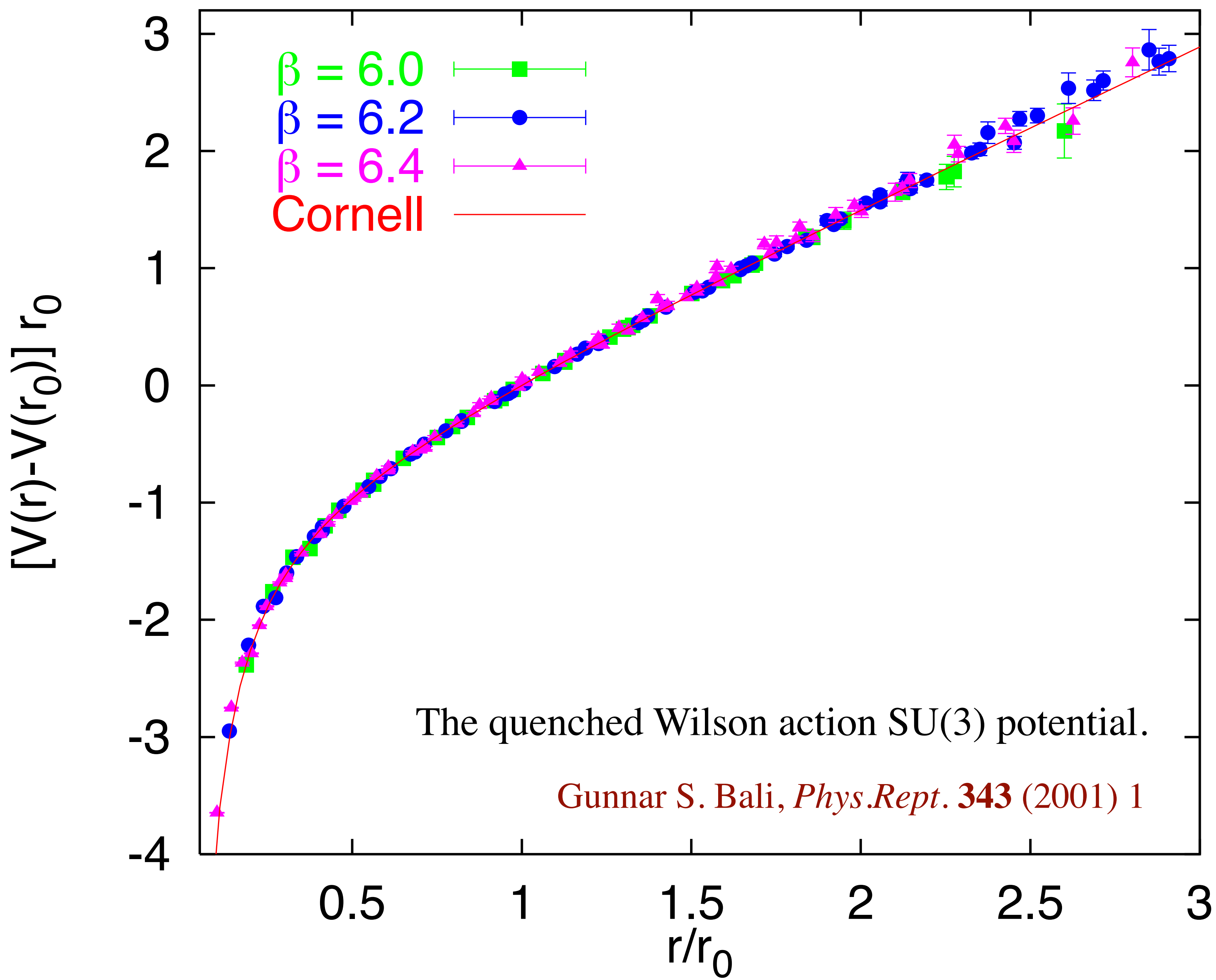
$$V(r) = -\frac{\alpha}{r}$$

QCD



$$V(r) = c\,r - \frac{4}{3}\frac{\alpha_s}{r}$$

Linear Cornell potential agrees with Lattice QCD



QED works for atoms

Example: Hyperfine splitting in Positronium

G. S. Adkins,
Hyperfine Interact. **233** (2015) 59

$$\Delta\nu_{QED} = m_e\alpha^4\left\{\frac{7}{12}-\frac{\alpha}{\pi}\left(\frac{8}{9}+\frac{\ln 2}{2}\right) + \frac{\alpha^2}{\pi^2}\left[-\frac{5}{24}\pi^2\ln\alpha+\frac{1367}{648}-\frac{5197}{3456}\pi^2+\left(\frac{221}{144}\pi^2+\frac{1}{2}\right)\ln 2-\frac{53}{32}\zeta(3)\right] - \frac{7\alpha^3}{8\pi}\ln^2\alpha+\frac{\alpha^3}{\pi}\ln\alpha\left(\frac{17}{3}\ln 2-\frac{217}{90}\right)+\mathcal{O}\left(\alpha^3\right)\right\} = 203.39169(41)\ \text{GHz}$$

$\Delta\nu_{EXP} = 203.394 \pm .002$ GHz

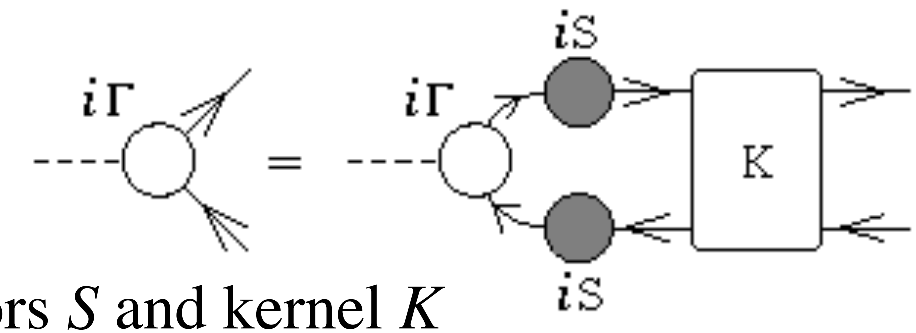
- Binding energy is perturbative in α and $\log(\alpha)$ (measurable)
- Wave function $\psi(r) \propto \exp(-m\alpha r)$ is of $\mathcal{O}(\alpha^\infty)$ (gauge dependent)

There are many ways to (re)organize an expansion that starts with $\mathcal{O}(\alpha^\infty)$

NRQED chooses to start from Schrödinger equation with $V(r) = -\alpha/r$

Some developments in bound state QED

1951: Salpeter & Bethe



Perturbatively expand propagators S and kernel K

Explicit Lorentz covariance ensured

1975: Caswell & Lepage: Not unique: ∞ # of equivalent equations, $S \leftrightarrow K$

We may start from Schrödinger atoms

1986: Caswell & Lepage NRQED: Effective NR field theory

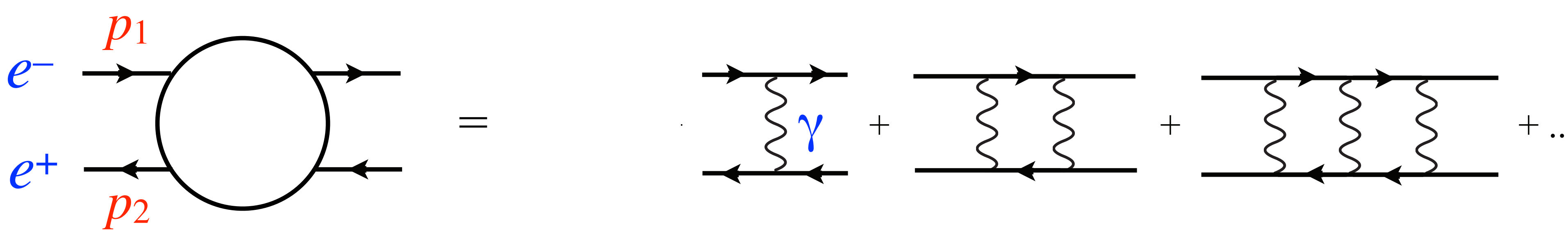
Relativistic electrons are rare in atomic wave functions

Expand QED action in powers of ∇/m_e

QED origins of the Schrödinger equation

In atomic (rest frame) kinematics: $|\boldsymbol{p}| \sim \alpha\, m_e$ $\quad$ $p^0 - m_e \sim \alpha^2\, m_e$

"Ladder diagrams" are distinguished by being of leading order in α :



$$= \frac{|\varphi_{e^+e^-}|^2}{p^0 - E + i\varepsilon} + \ldots$$

Their geometric sum gives the Schrödinger equation

$$E = 2m_e - \tfrac{1}{4} m_e \alpha^2 + \mathcal{O}\left(\alpha^4\right)$$

Divergent series?

Unique result?

Feynman diagrams: The Interaction Picture

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \qquad\qquad \mathcal{H}_0 \left|i\right\rangle_{in} = E_i \left|i\right\rangle_{in}$$

$$S_{fi} = {}_{out}\langle f, t \to \infty| \left\{ \mathrm{T} \exp\left[-i \int_{-\infty}^{\infty} dt \, \mathcal{H}_I(t) \right] \right\} |i, t \to -\infty\rangle_{in}$$

Formally exact expression, provided the *in*- and *out*-states at $t = \pm\infty$ have a non-vanishing overlap with the the physical i, f states.

Bound states have no overlap with free *in*- and *out*-states at $t = \pm\infty$

Expanding around free states is inadequate for soft processes, which are influenced by classical fields (Maxwell's equations).

Expanding around a stationary action

A stationary action implies a classical gauge field:

$$\frac{\delta \mathcal{S}[A^\mu]}{\delta A^\mu} = 0 \qquad \int [dA^\mu]\, \exp\big(iS[A^\mu]/\hbar\big) \quad \Longrightarrow \quad \hbar \to 0$$

Positronium is bound by its classical potential $V(r) = -\alpha/r$

We should expand around *in* and *out* states with their classical gauge field

The $\hbar \to 0$ limit selects an optimal expansion for bound states.

The "Potential Picture"

$$\mathcal{H} = \mathcal{H}_V + \mathcal{H}_I \qquad\qquad \mathcal{H}_V = \mathcal{H}_0 + \mathcal{H}_I(A_{cl})$$

$$S_{fi} = {}_V\langle f, t \to \infty| \left\{ \mathrm{T}\exp\left[-i \int_{-\infty}^{\infty} dt\, \mathcal{H}_I(t) \right] \right\} |i, t \to -\infty\rangle_V$$

$$\mathcal{H}_V\,|i\rangle_V = E_i\,|i\rangle_V$$

Particles will propagate in the classical field, as appropriate for bound states.

Here: Stay at $(\mathcal{H}_I)^0$ (Born) level. Consider bound asymptotic states.

Postpone a derivation and studies of higher order contributions in the PP.

Illustration: Positronium

$$|M\rangle_V = \int \frac{d\boldsymbol{k}}{(2\pi)^3}\,\phi(\boldsymbol{k})\,b^\dagger_{\boldsymbol{k},\lambda_1}\,d^\dagger_{-\boldsymbol{k},\lambda_2}\,|0\rangle$$

$\boldsymbol{\phi(k)}$ is the Schrödinger wave function

$$= \int d\boldsymbol{x}_1\,d\boldsymbol{x}_2\,\overline{\psi}_\alpha(0,\boldsymbol{x}_1)\,\Phi_{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\psi_\beta(0,\boldsymbol{x}_2)\,|0\rangle$$

where Φ is given by the Schrödinger wave function as

$$\Phi_{\alpha\beta}(\boldsymbol{x}) = {}_\alpha\big[\gamma^0 u(-i\,\boldsymbol{\nabla},\lambda_1)\big]\big[\bar{v}(i\,\boldsymbol{\nabla},\lambda_2)\gamma^0\big]_\beta\,\phi(\boldsymbol{x})$$

Check: $\mathcal{H}_V\,|M\rangle_V = M\,|M\rangle_V$

where $\mathcal{H}_V$ includes the classical photon field.

The classical field for Positronium

$$\frac{\delta \mathcal{S}_{QED}}{\delta \hat{A}^0(t,\boldsymbol{x})} = 0 \quad \Rightarrow \quad -\boldsymbol{\nabla}^2 \hat{A}^0(t,\boldsymbol{x}) = e\psi^\dagger(t,\boldsymbol{x})\psi(t,\boldsymbol{x})$$

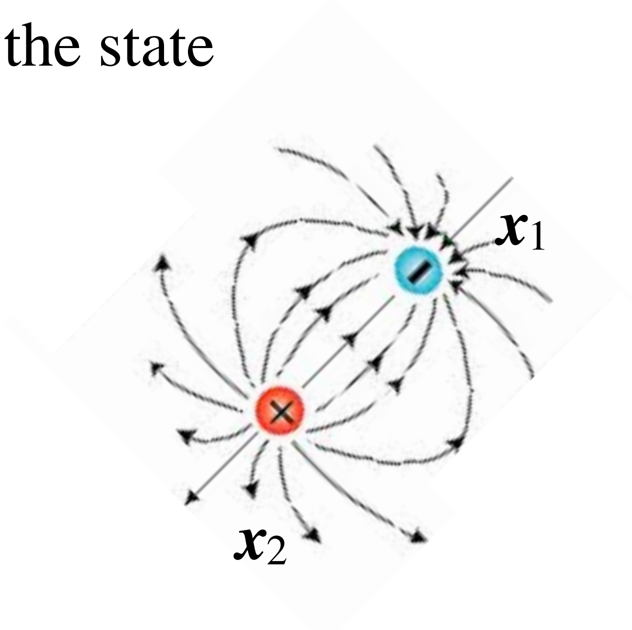
$$\hat{A}^0(t,\boldsymbol{x}) = \int d^3\boldsymbol{y}\, \frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\psi^\dagger\psi(t,\boldsymbol{y})$$

The classical field is the expectation value of $\hat{A}^0$ in the state

$$|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle = \bar{\psi}(t,\boldsymbol{x}_1)\psi(t,\boldsymbol{x}_2)\,|0\rangle$$

$$\frac{\langle \boldsymbol{x}_1,\boldsymbol{x}_2|\, e\hat{A}^0(\boldsymbol{x})\,|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle}{\langle \boldsymbol{x}_1,\boldsymbol{x}_2|\,\boldsymbol{x}_1,\boldsymbol{x}_2\rangle} = \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_2|}$$

$$\equiv eA^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2)$$



Note:
- A^0 is determined instantaneously for all $\boldsymbol{x}$
- It depends on $\boldsymbol{x}_1, \boldsymbol{x}_2$
- $eA^0(\boldsymbol{x}_1) = -eA^0(\boldsymbol{x}_2) = -\dfrac{\alpha}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}$ is the classical $-\alpha/r$ potential

The Schrödinger equation

The classical field determines $\mathscr{H}_V$, operating on $|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle$

$$\mathcal{H}_V(t;\boldsymbol{x}_1,\boldsymbol{x}_2) = \int d\boldsymbol{x}\,\psi^\dagger(t,\boldsymbol{x})\big[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha} + m\gamma^0 + \tfrac{1}{2}eA^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2)\big]\psi(t,\boldsymbol{x})$$

$$|M\rangle_V = \int d\boldsymbol{x}_1\,d\boldsymbol{x}_2\,\bar\psi(\boldsymbol{x}_1)\,\Phi(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\psi(\boldsymbol{x}_2)\,|0\rangle$$

$\mathcal{H}_V\,|M\rangle_V = M\,|M\rangle_V$ gives the bound state equation for $\Phi(\boldsymbol{x}_1 - \boldsymbol{x}_2)$:

$$\big[i\gamma^0\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0\big]\Phi(\boldsymbol{x}) + \Phi(\boldsymbol{x})\big[i\gamma^0\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}} - m\gamma^0\big] = \big[M - V(|\boldsymbol{x}|)\big]\Phi(\boldsymbol{x})$$

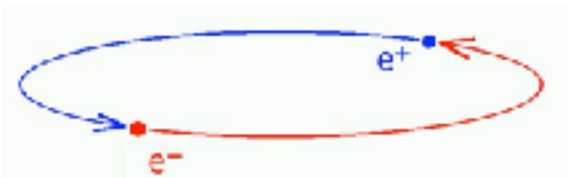
with $V(|\boldsymbol{x}|) = -\dfrac{\alpha}{|\boldsymbol{x}|}$

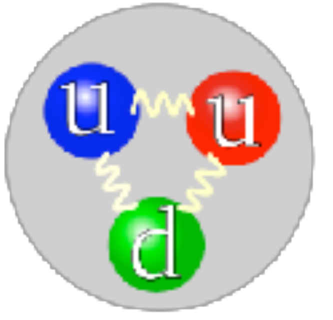
This BSE reduces to the Schrödinger equation for non-relativistic kinematics.

The $\hbar \to 0$ limit is required for its derivation.

Classical field in QCD

Global gauge invariance allows classical gauge field for neutral atoms, but not for color singlet hadrons in QCD





$$A^0 = \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_2|}$$

$$A_a^0(\boldsymbol{x}) = 0$$

Positronium
QED

Proton
QCD

However, a classical gluon field is allowed for quarks of fixed colors C:

$$A_a^0(\boldsymbol{x}; C) \neq 0 \qquad \sum_C A_a^0(\boldsymbol{x}; C) = 0$$

Three consequences of $\hbar \to 0$ in QCD

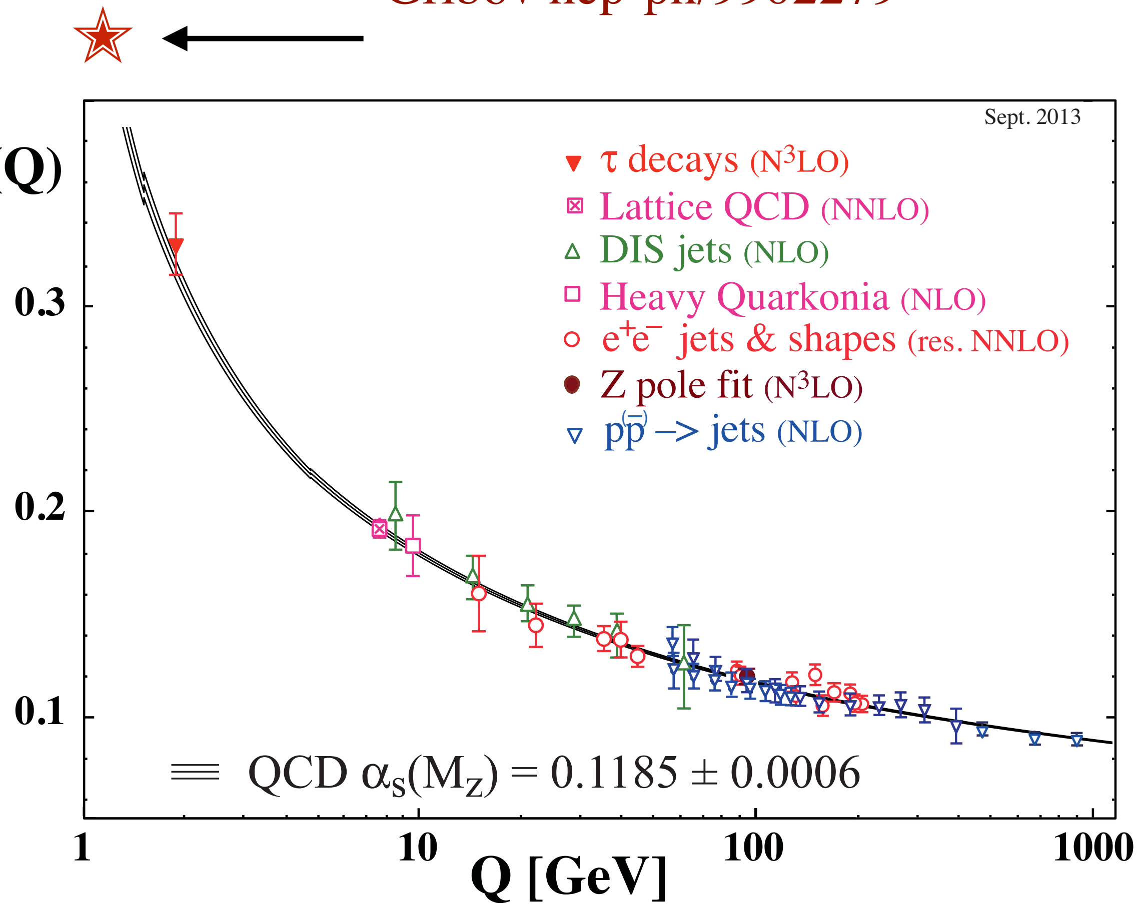
1. The suppression of loops,
   stops the running of α_s

   Estimates for the frozen coupling indicate

   $$\alpha_s(0)/\pi \approx 0.14$$

$\Rightarrow$ PQCD corrections to $\mathcal{O}(\hbar^0)$ can be relevant.

$\alpha_s^{crit} \approx 0.43$ ← Gribov hep-ph/9902279



2. In the absence of loops, the QCD scale Λ_{QCD} cannot arise from renormalization.

3. Poincaré invariance, unitarity etc. should hold at each power of $\hbar$

The QCD scale Λ_{QCD}

At $O(\hbar^0)$ (no loops) the QCD scale can arise only via a boundary condition

$$\frac{\delta}{\delta A_a^0} S_{QCD} = 0 \qquad \Rightarrow \qquad \partial_i F_a^{i0} = -g f_{abc} A_b^i F_c^{i0} + g \psi_A^\dagger T_a^{AB} \psi_B$$

A homogeneous, $\mathcal{O}\left(\alpha_s{}^0\right)$ solution with $\hat{A}_a^i = 0$ and hence $\boldsymbol{\nabla}^2 \hat{A}_a^0 = 0$

$$\hat{A}_a^0(\boldsymbol{x}) = \kappa \sum_{B,C} \int d\boldsymbol{y} \, (\boldsymbol{x} \cdot \boldsymbol{y}) \, \psi_B^\dagger(\boldsymbol{y}) \, T_a^{BC} \psi_C(\boldsymbol{y})$$

- Linear in $\boldsymbol{x}$ for translation invariance: $\hat{A}_a^0(\boldsymbol{x}_1) - \hat{A}_a^0(\boldsymbol{x}_2) \neq f(\boldsymbol{x}_1 + \boldsymbol{x}_2)$
- $\boldsymbol{x} \cdot \boldsymbol{y}$ for rotational invariance
- $\boldsymbol{x}$-independent field energy density $\sum_a |\boldsymbol{\nabla} \hat{A}_a^0(\boldsymbol{x})|^2$ must be universal

$\Rightarrow$ determines $\varkappa$ up to a scale Λ [GeV]

Classical color field for mesons

$$|M\rangle = \sum_{A,B}\int d\boldsymbol{x}_1\,d\boldsymbol{x}_2\,\bar\psi^A(\boldsymbol{x}_1)\,\Phi^{AB}(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\psi^B(\boldsymbol{x}_2)\,|0\rangle \qquad \Phi^{AB}(\boldsymbol{x}) = \frac{1}{\sqrt{N_C}}\,\delta^{AB}\Phi(\boldsymbol{x})$$

$$\hat{A}^0_a(\boldsymbol{x}) = \kappa\sum_{B,C}\int d\boldsymbol{y}\,(\boldsymbol{x}\cdot\boldsymbol{y})\,\psi^\dagger_B(\boldsymbol{y})\,T_a^{BC}\psi_C(\boldsymbol{y})$$

$$\frac{\langle \boldsymbol{x}_1^A,\boldsymbol{x}_2^A|\,\hat{A}^0_a(\boldsymbol{x})\,|\boldsymbol{x}_1^A,\boldsymbol{x}_2^A\rangle}{\langle \boldsymbol{x}_1^A,\boldsymbol{x}_2^A|\,\boldsymbol{x}_1^A,\boldsymbol{x}_2^A\rangle} = \kappa(\boldsymbol{x}_1,\boldsymbol{x}_2)\,\boldsymbol{x}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)\,T_a^{AA}$$

for each quark color A

$$\Rightarrow \quad A^0_a(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2,A) = \left[\boldsymbol{x}-\tfrac{1}{2}(\boldsymbol{x}_1+\boldsymbol{x}_2)\right]\cdot\frac{\boldsymbol{x}_1-\boldsymbol{x}_2}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}\,T_a^{AA}\,6\Lambda^2 \qquad \mathcal{O}\left(\alpha_s^0\right)$$

$$\sum_a\left[\boldsymbol{\nabla}_x A^0_a(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2,A)\right]^2 = 12\Lambda^4$$

Universal field energy

$$\sum_A A^0_a(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2,A) \propto \mathrm{Tr}\,T^{AA} = 0$$

Another hadron feels no field at any $\boldsymbol{x}$

$$V(\boldsymbol{x}_1-\boldsymbol{x}_2) = \tfrac{1}{2}g\sum_a T_a^{AA}\left[A^0_a(\boldsymbol{x}_1;\boldsymbol{x}_1,\boldsymbol{x}_2,A) - A^0_a(\boldsymbol{x}_2;\boldsymbol{x}_1,\boldsymbol{x}_2,A)\right] = g\Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2|$$

Linear potential, independent of quark color component A

Classical color field for baryons

$$|M\rangle = \sum_{A,B,C}\int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, d\boldsymbol{x}_3\, \psi_A^\dagger(\boldsymbol{x}_1)\psi_B^\dagger(\boldsymbol{x}_2)\psi_C^\dagger(\boldsymbol{x}_3)\,\Phi^{ABC}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)\,|0\rangle \qquad \Phi^{ABC} = \epsilon^{ABC}\Phi$$

Expectation value of $\hat{A}_a^0(\boldsymbol{x}) = \kappa\sum_{B,C}\int d\boldsymbol{y}\,(\boldsymbol{x}\cdot\boldsymbol{y})\,\psi_B^\dagger(\boldsymbol{y})\,T_a^{BC}\psi_C(\boldsymbol{y})$

in $\psi_A^\dagger(\boldsymbol{x}_1)\psi_B^\dagger(\boldsymbol{x}_2)\psi_C^\dagger(\boldsymbol{x}_3)\,|0\rangle$ $(A \neq B \neq C)$ determines the classical field:

$$A_a^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,ABC) = \left[\boldsymbol{x} - \tfrac{1}{3}(\boldsymbol{x}_1+\boldsymbol{x}_2+\boldsymbol{x}_3)\right]\cdot\left(T_a^{AA}\boldsymbol{x}_1 + T_a^{BB}\boldsymbol{x}_2 + T_a^{CC}\boldsymbol{x}_3\right)\frac{6\Lambda^2}{d(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)}$$

where $d(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = \frac{1}{\sqrt{2}}\sqrt{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2 + (\boldsymbol{x}_2-\boldsymbol{x}_3)^2 + (\boldsymbol{x}_3-\boldsymbol{x}_1)^2}$

$$\sum_a \left|\boldsymbol{\nabla}_x A_a^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,ABC)\right|^2 = 12\Lambda^4$$ Universal field energy

$$\sum_{A,B,C}\epsilon^{ABC}A_a^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,ABC) = 0$$ No classical field for singlet state

$$V(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = g\Lambda^2 d(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)$$

Bound state equation for mesons (rest frame)

$\mathcal{H}_V\,|M\rangle_V = M\,|M\rangle_V$ Bound state condition implies, with $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$

$$i\boldsymbol{\nabla}\cdot\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\} + m\,[\gamma^0,\Phi(\boldsymbol{x})] = [M - V(\boldsymbol{x})]\,\Phi(\boldsymbol{x})$$

$$V(\boldsymbol{x}) = g\Lambda^2|\boldsymbol{x}| \equiv V'|\boldsymbol{x}|$$

Expanding the 4×4 wave function in a basis of 16 Dirac structures $\Gamma_i(\boldsymbol{x})$

$$\Phi(\boldsymbol{x}) = \sum_i \Gamma_i(\boldsymbol{x})F_i(r)Y_{j\lambda}(\hat{\boldsymbol{x}})$$

we may use rotational, parity and charge conjugation invariance to determine which $\Gamma_i(\boldsymbol{x})$ may occur for a state of given j^{PC}:

0^{-+} trajectory $[s=0,\ \ell=j]:$	$-\eta_P = \eta_C = (-1)^j$	$\gamma_5,\ \gamma^0\gamma_5,\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L}$
0^{--} trajectory $[s=1,\ \ell=j]:$	$\eta_P = \eta_C = -(-1)^j$	$\gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \boldsymbol{\alpha}\cdot\boldsymbol{L},\ \gamma^0\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$
0^{++} trajectory $[s=1,\ \ell=j\pm1]:$	$\eta_P = \eta_C = +(-1)^j$	$1,\ \boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$
0^{+-} trajectory [exotic] :	$\eta_P = -\eta_C = (-1)^j$	$\gamma^0,\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$

$\Rightarrow$ There are no solutions for quantum numbers that would be exotic in the quark model (despite the relativistic dynamics)

Example: 0^{-+} trajectory wf's

$$\Phi_{-+}(\boldsymbol{x}) = \Big[\frac{2}{M-V}(i\boldsymbol{\alpha}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0) + 1\Big]\gamma_5\, F_1(r)Y_{j\lambda}(\hat{\boldsymbol{x}})$$

$$\eta_P = (-1)^{j+1}$$
$$\eta_C = (-1)^j$$

Radial equation: $$F_1'' + \Big(\frac{2}{r} + \frac{V'}{M-V}\Big)F_1' + \Big[\tfrac{1}{4}(M-V)^2 - m^2 - \frac{j(j+1)}{r^2}\Big]F_1 = 0$$

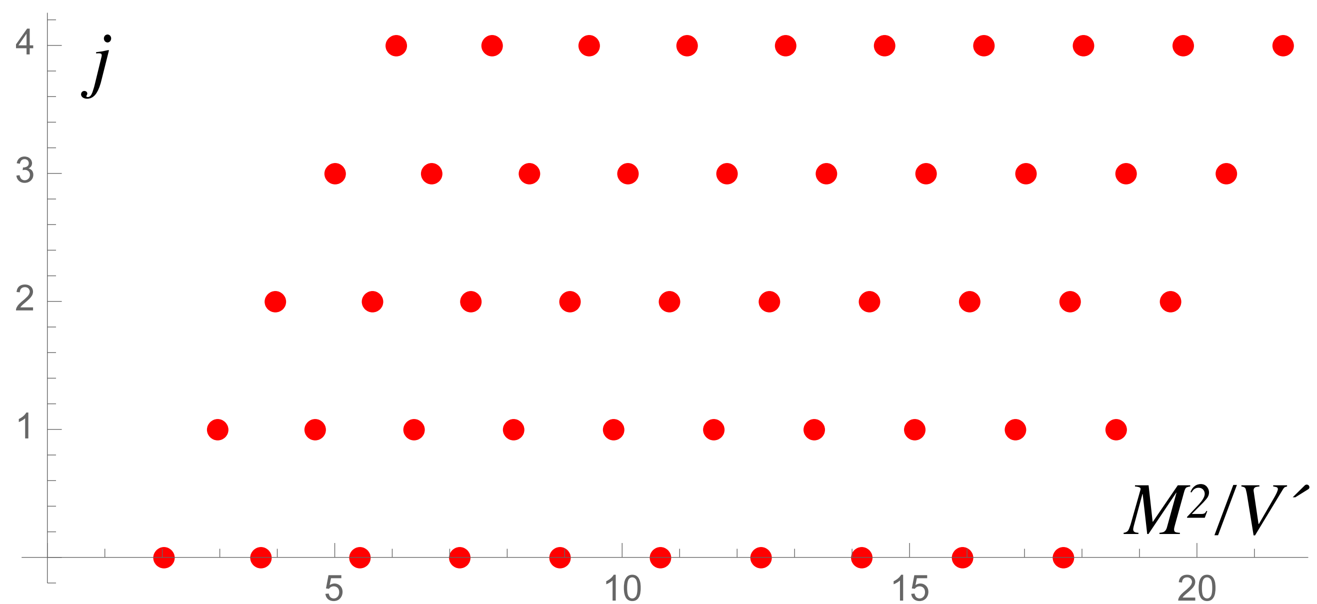
Local normalizability at $r = 0$ and at $V(r) = M$ determines the discrete M

Mass spectrum:

Linear Regge trajectories with daughters

Spectrum similar to dual models

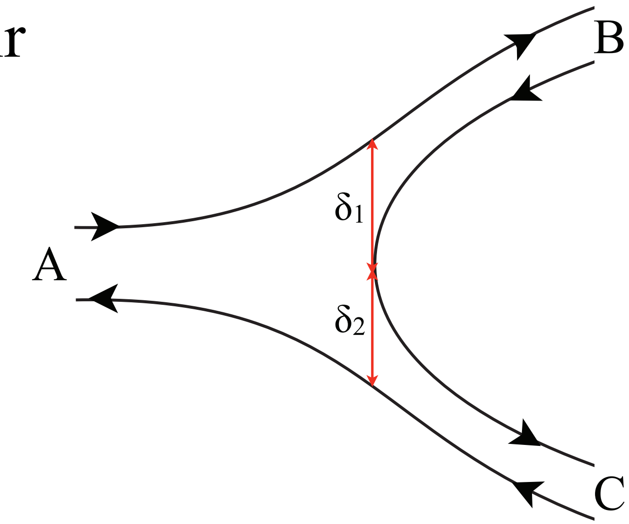
$m = 0$



String breaking: Pair production

The bound state equation was obtained neglecting pair production (string breaking).

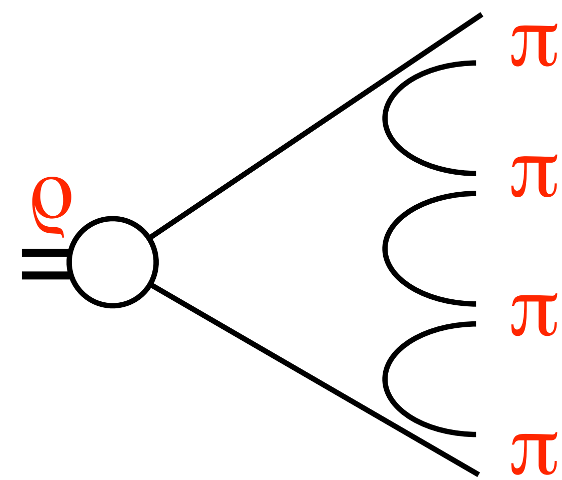
There is an $\mathcal{O}\left(1/\sqrt{N_C}\right)$ coupling between the states:



$$\langle B,C|A\rangle = -\frac{(2\pi)^3}{\sqrt{N_C}}\delta^3(\boldsymbol{P}_A-\boldsymbol{P}_B-\boldsymbol{P}_C)\int d\boldsymbol{\delta}_1 d\boldsymbol{\delta}_2\, e^{i\boldsymbol{\delta}_1\cdot\boldsymbol{P}_C/2-i\boldsymbol{\delta}_2\cdot\boldsymbol{P}_B/2}\mathrm{Tr}\left[\gamma^0\Phi_B^\dagger(\boldsymbol{\delta}_1)\Phi_A(\boldsymbol{\delta}_1+\boldsymbol{\delta}_2)\Phi_C^\dagger(\boldsymbol{\delta}_2)\right]$$

When squared, this gives a $1/N_C$ hadron loop unitarity correction.

As in the Dirac eq. with a linear potential, the pairs show up indirectly, via a constant norm of the wave function as $|\boldsymbol{x}| \to \infty$



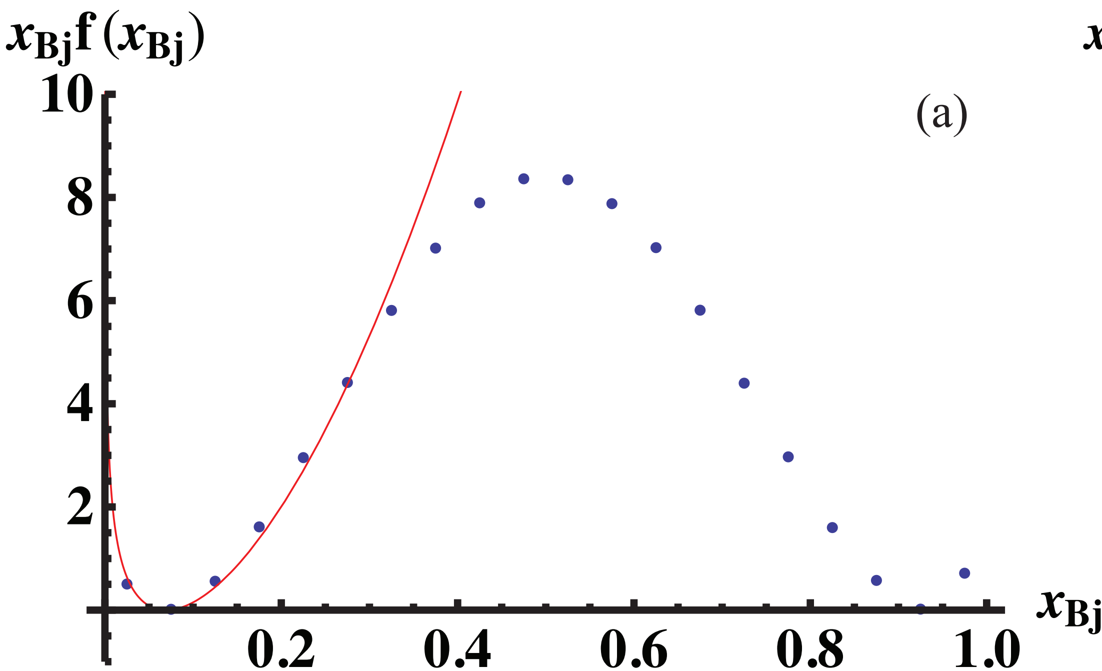
This seems related to duality.

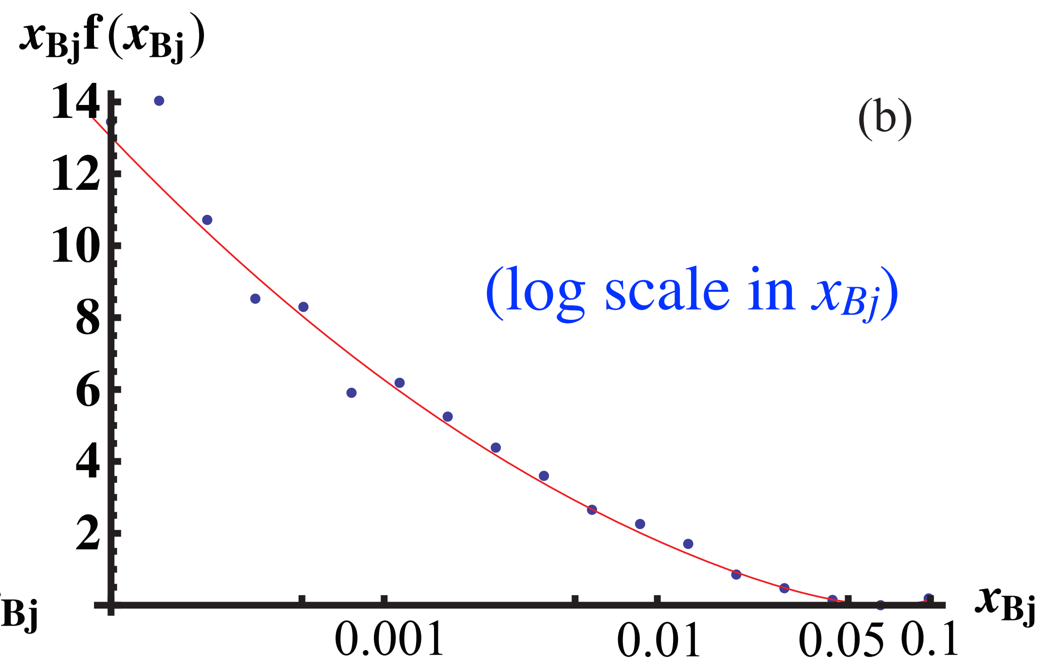
Parton distributions have a sea component

In D=1+1 dimensions the sea component is prominent at low *m*/*e* :

$m/e = 0.1$

D. D. Dietrich, PH, M. Järvinen
arXiv 1212.4747





The red curve is an analytic approximation, valid in the $x_{Bj} \to 0$ limit.

Note: Enhancement at low x is due to bd (sea), not to $b^\dagger d^\dagger$ (valence) component.
String breaking is not included.

Bound states in motion

A q$\bar{q}$ bound state with CM momentum $\boldsymbol{P}$ may be expressed as

$$|M,P\rangle_V \equiv \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, \bar{\psi}(t=0,\boldsymbol{x}_1)\, e^{i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2}\Phi^{(P)}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi(t=0,\boldsymbol{x}_2)\,|0\rangle$$

Note: States are defined at equal time in all frames.

The potential Hamiltonian is

$$\mathcal{H}_V = \int d\boldsymbol{x}\, \psi^\dagger(t,\boldsymbol{x})\big[-i\boldsymbol{\alpha}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0 + \tfrac{1}{2}\gamma^0 g \not{A}_{(P)}\big]\psi(t,\boldsymbol{x})$$

What is the classical field $A^{\mu}_{(P)}$?

The answer depends on the frame of the observer.

1. The classical field is independent of P

The component $\bar{\psi}(\boldsymbol{x}_1)\psi(\boldsymbol{x}_2)\,|0\rangle$ specifies positions, not momenta.

It is independent of $\boldsymbol{P}$ and so is the instantaneous A^0 field.

The bound state equation has a $\boldsymbol{P}$-independent potential $V(\boldsymbol{x}) = V'|\boldsymbol{x}|$

$$i\boldsymbol{\nabla}\cdot\left\{\boldsymbol{\alpha},\Phi_1^{(P)}(\boldsymbol{x})\right\} - \tfrac{1}{2}\boldsymbol{P}\cdot\left[\boldsymbol{\alpha},\Phi_1^{(P)}(\boldsymbol{x})\right] + m\left[\gamma^0,\Phi_1^{(P)}(\boldsymbol{x})\right] = \left[E - V(\boldsymbol{x})\right]\Phi_1^{(P)}(\boldsymbol{x})$$

$\Phi_1^{(P)}(\boldsymbol{x})$ determines the states with momentum $\boldsymbol{P}$ in the original frame.

$\boldsymbol{P}$ breaks rotational symmetry: angular & radial dependence does not separate.

The solution for $\Phi_1^{(P)}(\boldsymbol{x})$ in D = 1+1 dimensions is not simply Lorentz contracting.

It provides a boundary condition at $\boldsymbol{x}_\perp = 0$ on $\Phi_1^{(P)}(\boldsymbol{x})$ in D = 3+1 dimensions.

2. Rest frame dynamics as seen by a moving observer

Define boost ξ taking $\boldsymbol{P} = (0, 0, P)$ along the z-axis: $P = M\sinh(\xi)$

A moving observer sees a boosted rest frame A^0 field ($\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$):

$$A^0_{(P)}(\boldsymbol{x}) = \cosh\xi\, A^0(\boldsymbol{x}_R) \qquad\qquad A^3_{(P)}(\boldsymbol{x}) = \sinh\xi\, A^0(\boldsymbol{x}_R)$$

where the rest frame (Lorentz dilated) separation is $\boldsymbol{x}_R = (x, y, z\cosh\xi)$

The P-dependence of the $\Phi_2^{(P)}(\boldsymbol{x})$ solution is found analytically from the BSE:

$$\Phi_2^{(P)}(\boldsymbol{x}) = e^{-\xi\gamma^0\gamma^3/2}\,\Phi^{(0)}(\boldsymbol{x}_R)\,e^{\xi\gamma^0\gamma^3/2}$$

The wave function classically Lorentz contracts: $\boldsymbol{x}_R \to \boldsymbol{x}$.

Extra twist: The magnetic field $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ causes quark spins to precess in time

States with P = M = 0

We required the wave function to be normalizable at $r = 0$ and $V'r = M$

For $M = 0$ the two points coincide. Regular, massless solutions are found.

The massless 0^{++} meson "σ" may mix with the perturbative vacuum. This spontaneously breaks chiral invariance.

$$|\sigma\rangle = \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, \bar\psi(\boldsymbol{x}_1)\, \Phi_\sigma(\boldsymbol{x}_1 - \boldsymbol{x}_2)\, \psi(\boldsymbol{x}_2)\, |0\rangle \equiv \hat\sigma\, |0\rangle$$

For $m = 0$ and $V' = 1$: $$\Phi_\sigma(\boldsymbol{x}) = N_\sigma \Big[J_0(\tfrac{1}{4}r^2) + \boldsymbol{\alpha}\cdot\boldsymbol{x}\frac{i}{r}\, J_1(\tfrac{1}{4}r^2)\Big]$$

where J_0 and J_1 are Bessel functions.

$\hat P^\mu\, |\sigma\rangle = 0$ State has *vanishing four-momentum* in any frame

It may form a non-trivial condensate.

A chiral condensate (m = 0)

Since | σ ⟩ has vacuum quantum numbers and zero momentum it can mix with the perturbative vacuum without violating Poincaré invariance

Ansatz: $|\chi\rangle = \exp(\hat{\sigma})\,|0\rangle$ implies $\langle\chi|\bar{\psi}\psi\,|\chi\rangle = 4N_\sigma$

An infinitesimal chiral rotation of the condensate generates a pion

$$U_\chi(\beta) = \exp\left[i\beta\int d\boldsymbol{x}\,\psi^\dagger(\boldsymbol{x})\gamma_5\psi(\boldsymbol{x})\right] \qquad U_\chi(\beta)\,|\chi\rangle = (1-2i\beta\,\hat{\pi}\,|\chi\rangle$$

where $\hat{\pi}$ is the massless 0^{-+} state with wave function $\Phi_\pi = \gamma_5\Phi_\sigma$

Small quark mass: m > 0

The massless ($M_\sigma = 0$) sigma 0^{++} state has wave function

$$\Phi_\sigma(\boldsymbol{x}) = f_1(r) + i\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\, f_2(r) + i\,\boldsymbol{\gamma}\cdot\boldsymbol{x}\, g_2(r)$$

Radial functions are Laguerre fn's

An $M_\pi > 0$ pion 0^{-+} state has rest frame wave function

$$\Phi_\pi(\boldsymbol{x}) = \big[F_1(r) + i\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\, F_2(r) + \gamma^0\, F_4(r)\big]\gamma_5$$

$$F_4(0) = \frac{2m}{M} F_1(0)$$

$$F_1'' + \Big(\frac{2}{r} + \frac{1}{M-r}\Big)F_1' + \big[\tfrac{1}{4}(M-r)^2 - m^2\big]F_1 = 0$$

$$\langle\chi|j_5^\mu(x)\hat{\pi}\,|\chi\rangle = iP^\mu f_\pi\, e^{-iP\cdot x} \quad\Rightarrow\quad F_4(0) = \tfrac{1}{4} i M_\pi f_\pi$$

$$\langle\chi|\bar{\psi}(x)\gamma_5\psi(x)\,\hat{\pi}\,|\chi\rangle = -i\,\frac{M^2}{2m}\, f_\pi\, e^{-iP\cdot x} \quad\Rightarrow\quad F_1(0) = i\,\frac{M^2}{8m}\, f_\pi$$

Also the *P*-dependence is correct.

A smooth $m \to 0$ requires $M_\pi^2 \propto m$, which is allowed at lowest order in m.

Bound states built on $|\chi\rangle$

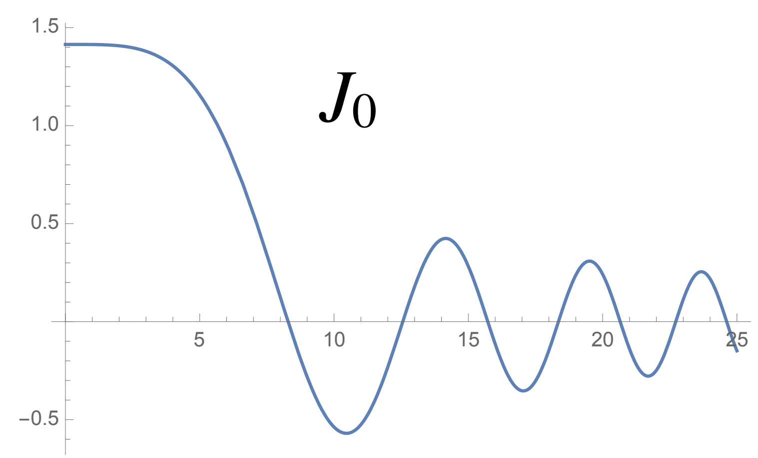
$$|M\rangle_\chi = \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, \bar\psi(\boldsymbol{x}_1)\, \Phi(\boldsymbol{x}_1-\boldsymbol{x}_2)\, \psi(\boldsymbol{x}_2)\, |\chi\rangle$$

The fields in $|\chi\rangle$ will break chiral invariance (no parity doublets).

For low momentum transfers Φ_σ may be approximated to be pointlike

$$\Phi_\sigma(\boldsymbol{x}) \to \Phi_{\sigma 0}(\boldsymbol{x}) = \delta^3(\boldsymbol{x})\phi_0$$

$$|\chi\rangle \to |\chi_0\rangle = \exp\left[\phi_0 \int d\boldsymbol{x}\, \bar\psi(\boldsymbol{x})\psi(\boldsymbol{x})\right] |0\rangle$$



The contractions of $\bar\psi(\boldsymbol{x}_1)\psi(\boldsymbol{x}_2)$ with $\bar\psi\psi$ in $|\chi\rangle$ have the effect of a mass term in $\mathscr{H}_V$

⇒ Momentum dependent mass term as in the DSE approach?

Some topical issues

- Expanding the perturbative S-matrix around fields with stationary action

$$\mathcal{H} = \mathcal{H}_V + \mathcal{H}_I \qquad\qquad \mathcal{H}_V = \mathcal{H}_0 + \mathcal{H}_I(A_{cl})$$

- Equal-time bound states in motion
  - $\boldsymbol{P}$-dependence of wave function (classical, $\boldsymbol{P}$-independent field)
  - Precession of state (as seen by a moving observer)

- Meson spectrum with chiral symmetry breaking and $m_u, m_d \neq 0$

- Baryon spectrum

- Duality and Parton distributions

- Hadron decays and scattering amplitudes (string breaking)

Time to teach bound states in QFT?

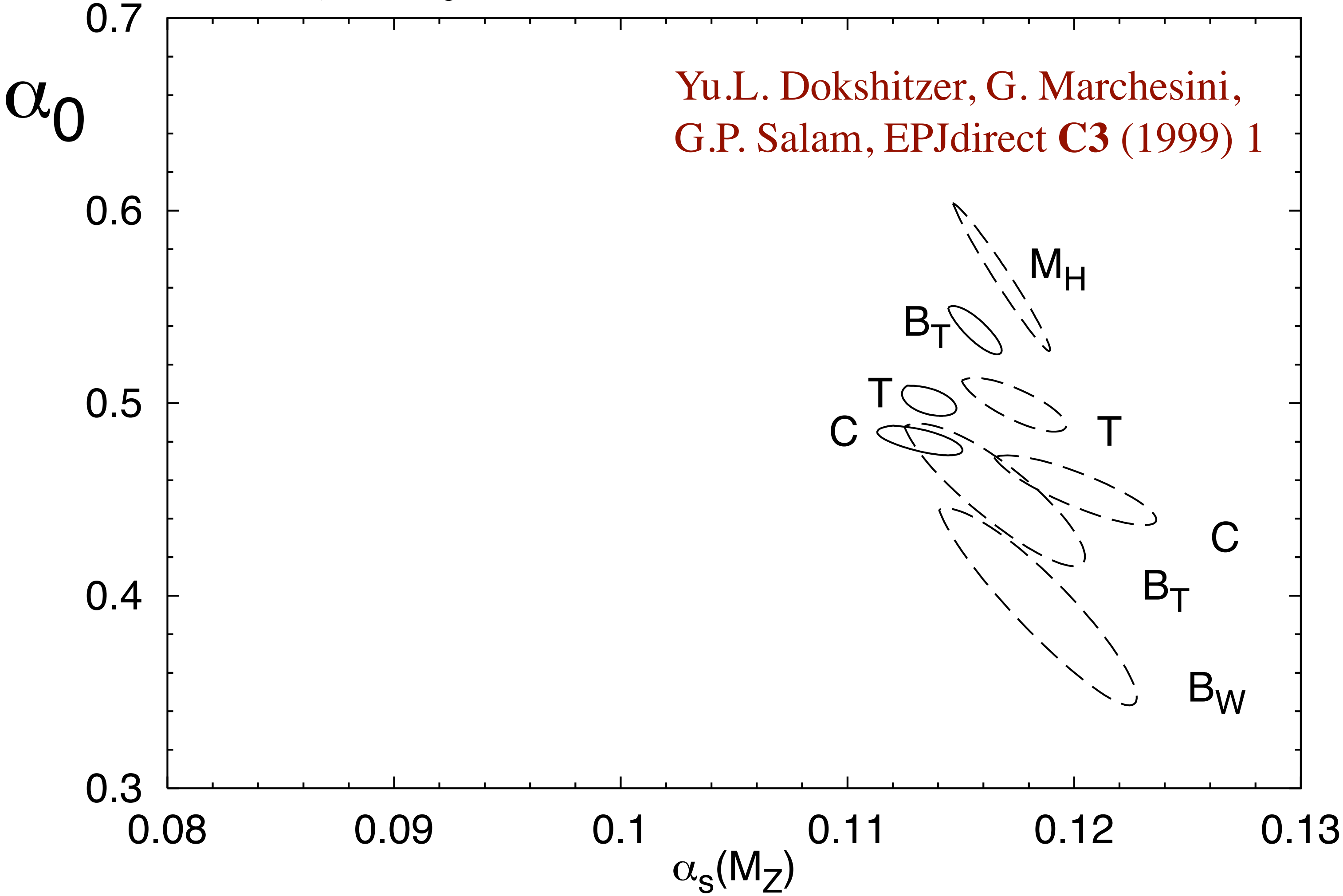
To understand hadrons we need the methods of bound state QFT.

- Wave functions: Equal-time, Light Front, Bethe-Salpeter, …
- Deriving the Schrödinger equation from QED
- The states described by the Dirac wave function
- Poincaré invariance for bound states

Back-up slides

α_s in the infrared from event shapes

$$\frac{1}{\mu_I}\int_0^{\mu_I} \mathrm{d}Q\,\alpha_{\rm eff}(Q^2) = \alpha_0(\mu_I)$$



$$\alpha_s(M_Z) = 0.1153 \pm 0.0017(\mathrm{exp}) \pm 0.0023(\mathrm{th})$$

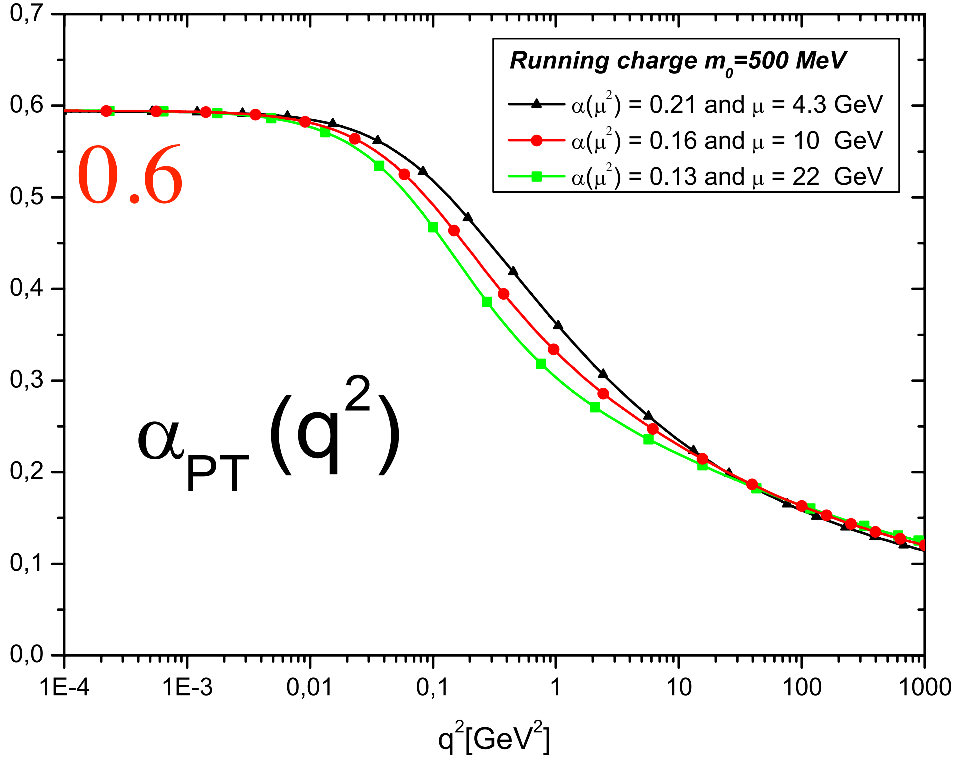
$$\alpha_0 = 0.5132 \pm 0.0115(\mathrm{exp}) \pm 0.0381(\mathrm{th})$$

T. Gehrmann, M. Jaquier, G. Luisoni,
Eur. Phys. J. C **67** (2010) 57

$$\mu_I = 2\ \mathrm{GeV}$$

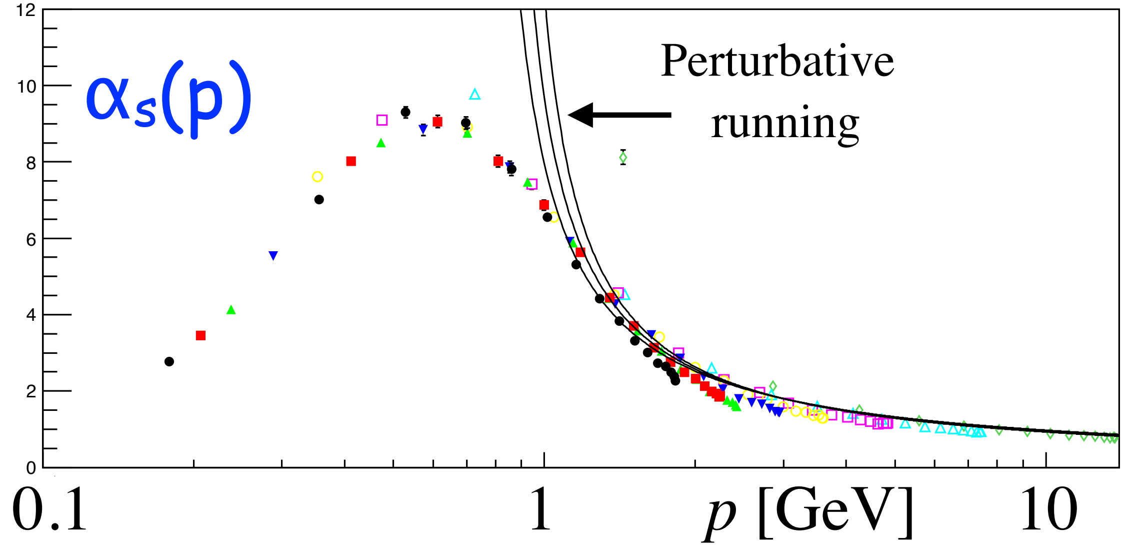
α_s "freezes" in the infrared

Pinch Technique



J. M. Cornwall;
A. C. Aguilar, D. Binosi, J. Papavassiliou,
J. Rodriguez-Quintero, PRD 80 (2009) 085018

Lattice QCD

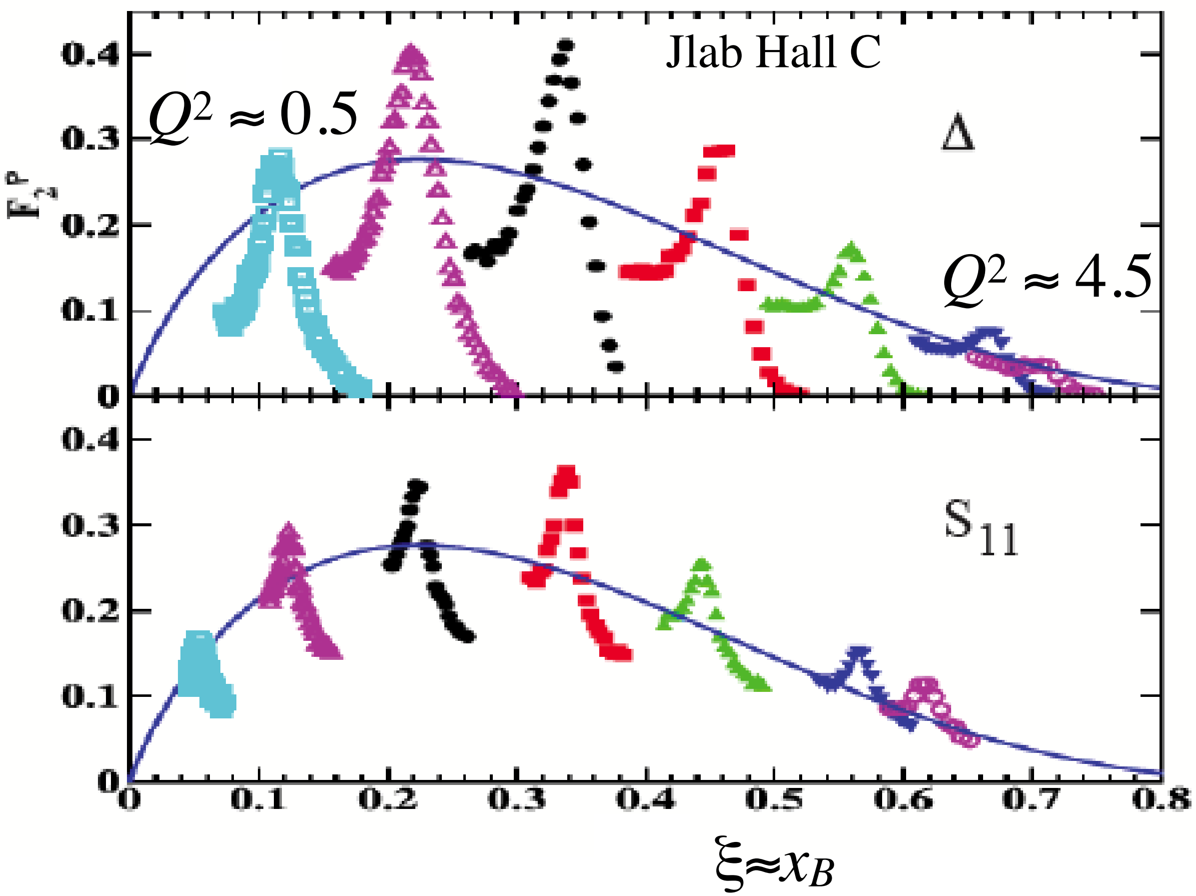


A. Maas, PRD 91 (2015) 034502
[arXiv:1402.5050v2]

The coupling is scheme-dependent for small as well as large Q^2.

Bloom-Gilman Duality

W. Melnitchouk et al, Phys. Rep. 406 (2005) 127

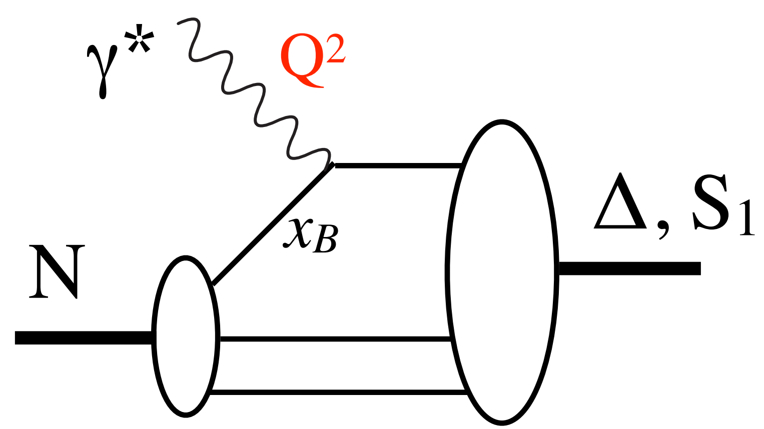


Resonance contributions

$ep \to eN^*$

build DIS scaling in

$ep \to eX$



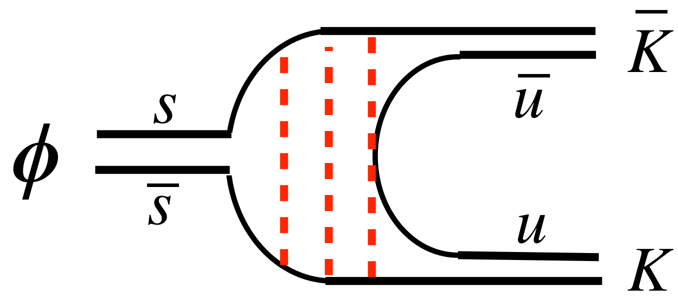
$$m_{N^*}^2 = m_N^2 + Q^2\left(\frac{1}{x_B} - 1\right)$$

Scattering dynamics is built into hadron wave functions.

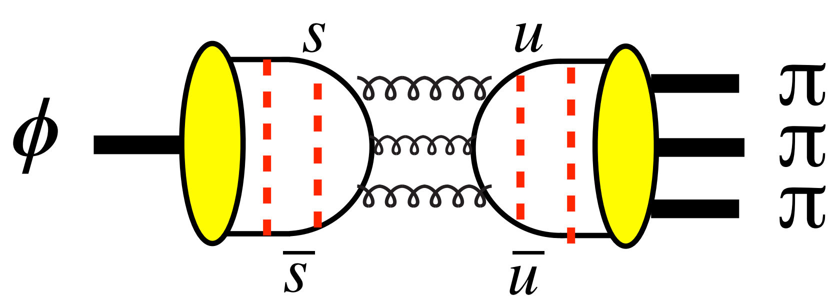
We must understand relativistic bound states in motion.

Rules of Thumb - e.g., OZI

Connected diagrams: Unsuppressed, string breaking from confining potential

		ΔE	Br
$\phi(1020) \to K\bar{K}$		26 MeV	83.1 %

Disconnected, perturbative diagrams are suppressed

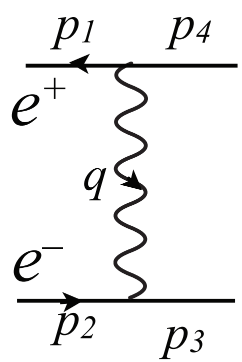
$\phi(1020) \not\to \pi\pi\pi$		610 MeV	15.3 %

This suggests that perturbative corrections are small even in the soft regime.

Ladder diagrams (rest frame)

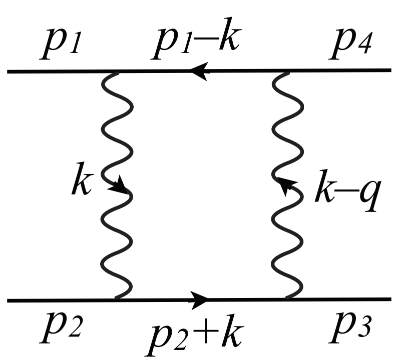
The Bohr momentum scale is $|\boldsymbol{p}| \sim \alpha m$, kinetic energy $|\boldsymbol{p}|^2/2m \sim \alpha^2 m \sim E_B$

With momenta $\propto \alpha$, the propagators bring inverse powers of α :



$$\sim \frac{e^2}{q^2} \sim \frac{\alpha}{\boldsymbol{q}^2} \sim \frac{1}{\alpha}$$

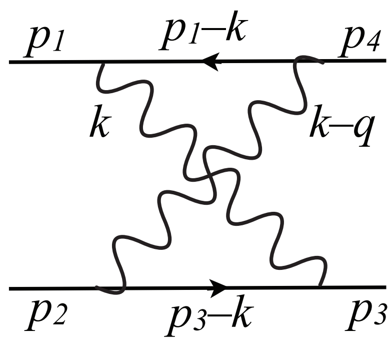
Note: $q^0 \sim \alpha^2 \ll |\boldsymbol{q}| \sim \alpha$

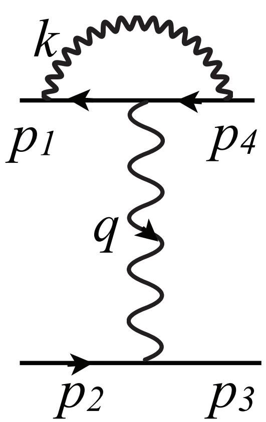


$$\sim \int dk^0\, d^3\boldsymbol{k}\, \frac{e^4}{(\boldsymbol{k}^2)^2\,(\Delta E_e)^2} \sim \alpha^2\,\alpha^3\,\frac{\alpha^2}{(\alpha^2)^2\,(\alpha^2)^2} \sim \frac{1}{\alpha}$$

All “ladder diagrams” are of order $1/\alpha$ $\Rightarrow$ Sum can diverge!

Non-ladders are suppressed by α





These diagrams have the same number of propagators and vertices as the 2-photon ladder. A similar counting would again give ~ 1/α .

However, the O(1/α) term vanishes:

$$\propto \int \frac{dk^0}{2\pi} \frac{1}{(k^0 - a + i\varepsilon)(k^0 - b + i\varepsilon)} = 0$$

In the straight ladders the integration contour is pinched:

$$\propto \int \frac{dk^0}{2\pi} \frac{1}{(k^0 - a + i\varepsilon)(k^0 - b - i\varepsilon)} \neq 0$$

⇒ Only straight ladders are of the leading order, 1/α .

Gribov's View of Confinement (1991-95)

According to Gribov, confinement sets in when a strong Coulomb interaction between fermions causes a rearrangement of the vacuum:

$$\alpha^{crit}(\mathrm{QED}) = \pi\left(1-\sqrt{\frac{2}{3}}\right) \simeq 0.58 \qquad \gg \frac{1}{137}$$

$$\alpha_s^{crit}(\mathrm{QCD}) = \frac{\pi}{C_F}\left(1-\sqrt{\frac{2}{3}}\right) \simeq 0.43 \quad \gtrsim \alpha_s(m_\tau^2) \simeq 0.33$$

$\alpha_s^{crit}/\pi = 0.14$ may allow PQCD down to Q^2=0.

See Yu. Dokshitzer, sect. 2.4 of hep-ph/0306287

Baryons

For baryons the homogeneous classical solution gives:

$$V_{\mathcal{B}}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = \frac{g\Lambda^2}{\sqrt{2}}\sqrt{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2+(\boldsymbol{x}_2-\boldsymbol{x}_3)^2+(\boldsymbol{x}_3-\boldsymbol{x}_1)^2}$$

It agrees with the meson potential when two quarks coincide:

$$V_{\mathcal{B}}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_2) = V_{\mathcal{M}}(\boldsymbol{x}_1-\boldsymbol{x}_2)$$

Translation invariance requires color singlet meson and baryon states.

The "external" color field vanishes also for the qqq states.

For SU(3) this type of solution only exists for $q\bar{q}$ and qqq states.

AUGUST 1, 1932 *PHYSICAL REVIEW* *VOLUME 41*

The Dirac Electron in Simple Fields*

By MILTON S. PLESSET
Sloane Physics Laboratory, Yale University

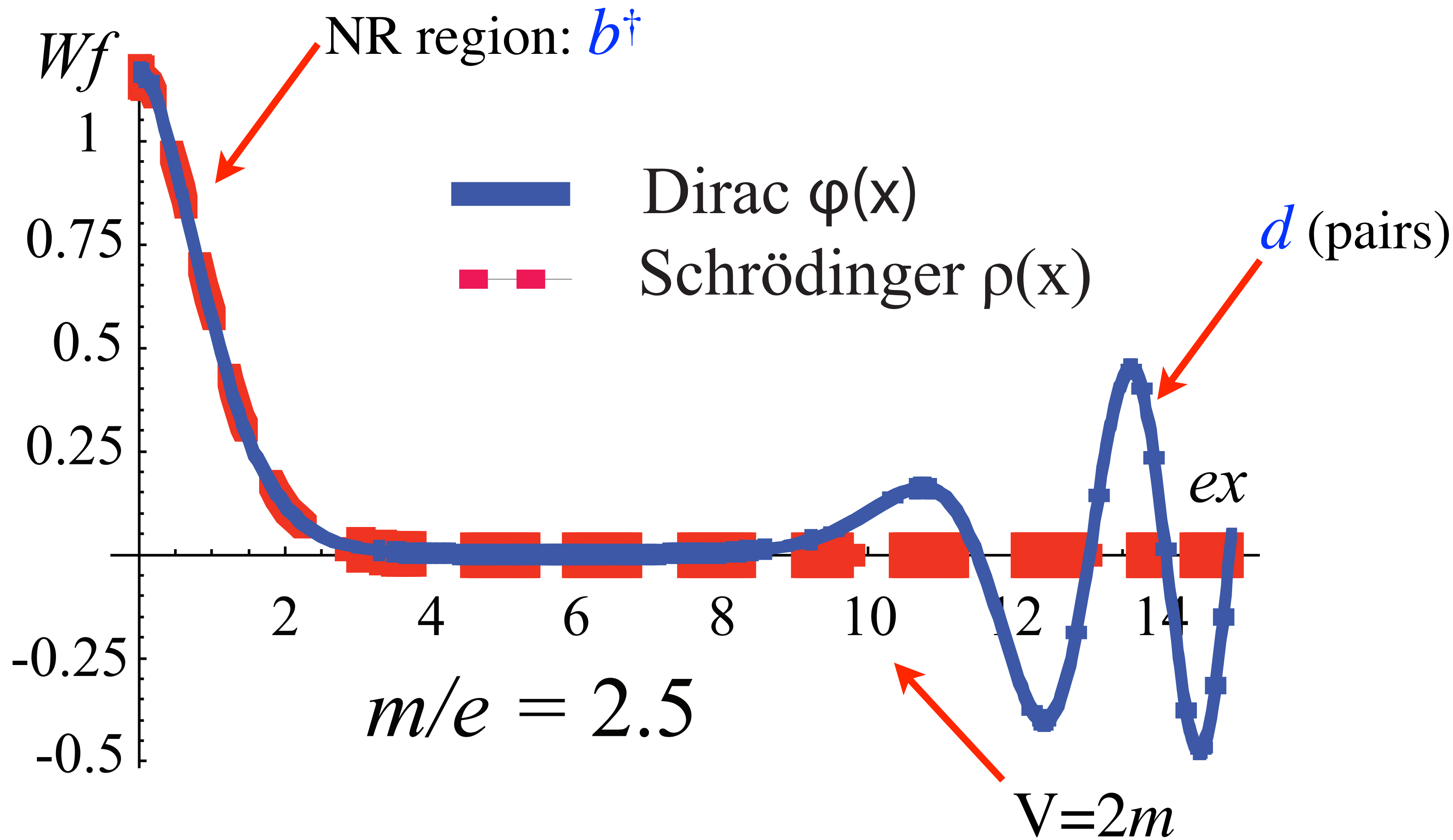
(Received June 6, 1932)

The relativity wave equations for the Dirac electron are transformed in a simple manner into a symmetric canonical form. This canonical form makes readily possible the investigation of the characteristics of the solutions of these relativity equations for simple potential fields. If the potential is a polynomial of any degree in x, a continuous energy spectrum characterizes the solutions. If the potential is a polynomial of any degree in $1/x$, the solutions possess a continuous energy spectrum when the energy is numerically greater than the rest-energy of the electron; values of the energy numerically less than the rest-energy are barred. When the potential is a polynomial of any degree in r, all values of the energy are allowed. For potentials which are polynomials in $1/r$ of degree higher than the first, the energy spectrum is again continuous. The quantization arising for the Coulomb potential is an exceptional case.

See also: E. C. Titchmarsh, Proc. London Math. Soc. (3) 11 (1961) 159 and 169; Quart. J. Math. Oxford (2), 12 (1961), 227.

Example of Dirac wave functions: $V(x) = \frac{1}{2}e^2|x|$ in D=1+1

$$|M \geq 0\rangle = \int \frac{dp}{2\pi 2E} \int dx \left[b_p^\dagger u^\dagger(p) e^{-ipx} + d_p v^\dagger(p) e^{ipx}\right] \left[\begin{array}{c} \varphi(x) \\ \chi(x) \end{array}\right] |\Omega\rangle$$



D. D. Dietrich, PH, M. Järvinen, arXiv 1212.4747

The "single particle" Dirac wave function contains pair contributions (duality)

Dirac states: $V(x) = \frac{1}{2}|x|$ in D=1+1

The Coulomb potential in D=1+1 is $V(x) = \frac{1}{2}e^2|x|$. We set $e = 1$ (scale).

The potential confines electrons, and repels positrons: $V(e^+) = -V(e^-)$
Any e^+ in the state is accelerated to large $|x|$.

To keep T+V ≃ constant, positrons have large momenta at high $|x|$:
$|p| \sim E_p \sim |x|/2$

Since we consider time independent solutions, there will also be decelerating positrons, moving towards $x = 0$.

The positron energy spectrum is continuous, whereas the electrons form bound states around $x = 0$, with discrete energies.

The relative size of the electron and positron components can be adjusted. However, positrons are completely absent only in the NR limit, $m \to \infty$.

Thus we can understand the observation made by Plesset in the 1930's.

Plane waves in bound states

In the parton picture, high energy quarks can be treated as free constituents.
They are momentum eigenstates, described by plane waves.
How does this fit into the bound state wave functions?

Consider a highly excited state (P=0): $M \to \infty,\ V(x) << M$

$\sigma = (M\text{-}V)^2 \approx M^2 - 2MV \to \infty$

$$\Phi(\sigma \to \infty) \sim \exp(\pm i\sigma/2) = e^{\pm iM^2} \exp(\mp ix\, M/2)$$

Thus oscillations of the wf at large σ gives a plane wave with $p = \pm M/2$

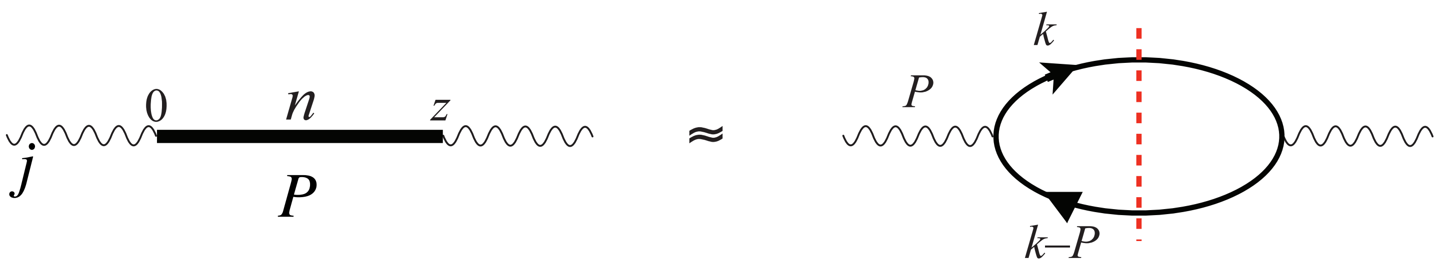
The operator expression for the state is in this limit:

$$|M, P=0\rangle = \frac{\sqrt{2\pi}}{2M}\big(b^\dagger_{M/2} d^\dagger_{-M/2} + b^\dagger_{-M/2} d^\dagger_{M/2}\big)|\Omega\rangle$$

As in the parton picture, only *"valence"* particles appear (no *b* or *d* operators).

Quark - Hadron duality

The wave functions of highly excited (large mass M) bound states are similar to free $f\bar{f}$ pairs (for $V(x) << M$). This determines their normalization:

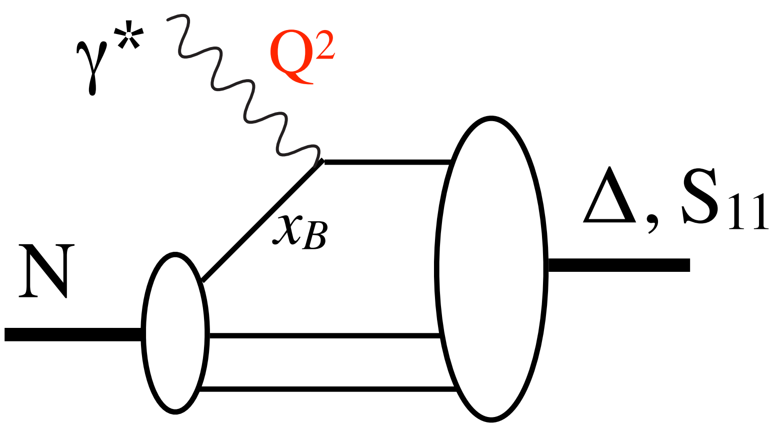


$$\Rightarrow \quad |\Phi_0(x=0)|^2 = |\Phi_1(x=0)|^2 = \pi/2$$

The same result for $j = S, P, V, A$ currents

D. D. Dietrich, PH, M. Järvinen, arXiv 1212.4747

The solutions are consistent with Bloom-Gilman duality: Plane wave partons in bound state wave function.



B-G Duality