Explore hadron’s structure using ab initio lattice QCD calculations

Jianwei Qiu
Theory Center, Jefferson Lab

Based on work done with T. Ishikawa, Y.-Q. Ma, S. Yoshida, … and work by many others, …
From QED to QCD, …

From Atomic Structure to Nano-Science:

✧ Tiny, but heavy, nucleus:
  *less than 1 trillionth in volume of an atom*
  slow-moving mass and charge centers
✧ Light, but fast, electrons:
  *quantum probability, …*
✧ Massless, charge neutral photons:
  *localized charges, …*

Atom:

Molecule:
“Water”

Crystal:
Rare-Earth metal

Nanomaterial:
Carbon-based Fullerene

Infinite opportunities to create & improve … !
From QED to QCD, …

- From Hadron Structure to Femto-Science:
  - Proton:
    - Extremely light and fast quarks: No still picture of the structure, … fluctuation, quantum probability, …
    - Massless, but charged gluons: non-local charge, …
    - Heavy quarks: “localized” charges, …
  - Nuclei – “Molecule”
    - Short-range correlation
  - XYZ – “Nuclei”
    - D^0 – D^0 “molecule”
    - diquark-diantiquark
    - q̅q-glueon “hybrid”
  - “Light-flavor”
  - “Heavy-flavor”

New frontier of hadron physics … !
Outline of the rest of my talk

- How to quantify hadron structure in QCD?
- How to “see” hadron structure in experiment?
- How to calculate hadron structure in QCD?
- How to explore hadron structure using lattice QCD calculations?
- Summary and outlook
Hadron structure in QCD

What do we need to know for the structure?

- **In theory:** \( \langle P, S | O(\bar{\psi}, \psi, A^\mu) | P, S \rangle \) – Hadronic matrix elements with all possible operators: \( O(\bar{\psi}, \psi, A^\mu) \)

- **In fact:** None of these matrix elements is a direct physical observable in QCD – color confinement!

- **In practice:** Accessible hadron structure = hadron matrix elements of quarks and gluons, which
  1) can be related to physical cross sections of hadrons and leptons with controllable approximation; and/or
  2) can be calculated in lattice QCD

Single-parton structure “seen” by a short-distance probe:

- **5D structure:**
  1) \( \int d^2b_T f(x, k_T, \mu) \) – TMDs: 2D confined motion!
  2) \( \int d^2k_T F(x, b_T, \mu) \) – GPDs: 2D spatial imaging!
  3) \( \int d^2k_T d^2b_T f(x, \mu) \) – PDFs: Number density!
Hadron structure in QCD

What do we need to know for the structure?

- In theory: \[ \langle P, S | O(\bar{\psi}, \psi, A^\mu) | P, S \rangle \] – Hadronic matrix elements with all possible operators: \[ O(\bar{\psi}, \psi, A^\mu) \]

- In fact: None of these matrix elements is a direct physical observable in QCD – color confinement!

- In practice: Accessible hadron structure = hadron matrix elements of quarks and gluons, which
  1) can be related to physical cross sections of hadrons and leptons with controllable approximation; and/or
  2) can be calculated in lattice QCD

Multi-parton correlations:

\[ \sigma(Q, \bar{s}) \propto \frac{p, \bar{s}}{t \sim 1/Q} + \frac{1}{2} \left( \frac{\langle k_\perp \rangle}{Q} \right)^n \] – Expansion

Quantum interference \[ \rightarrow \] 3-parton matrix element – not a probability!
“See” hadron structure in experiments

- One hadron:

$$\sigma_{\text{DIS}}^{\text{tot}} : \text{Hard-part Probe}$$

- Two hadrons:

$$\sigma_{\text{DY}}^{\text{tot}} : \text{Parton-distribution Structure}$$

Power corrections Approximation

Predictive power:

Ability to calculate the “probes” + Universal Parton Distributions, …
Global QCD analyses – a successful story

- World data with “Q” > 2 GeV
- Factorization:
  \[
  F_2(x_B, Q^2) = \sum_f C_f(x_B/x, \mu^2/Q^2) \otimes f(x, \mu^2)
  \]
- DIS:
  \[
  \frac{d\sigma}{dy dp_T^2} = \sum_{f'f} f(x) \otimes \frac{d\hat{\sigma}_{f'f'}}{dy dp_T^2} \otimes f'(x')
  \]
- H-H:
  \[
  \frac{\partial f(x, \mu^2)}{\partial \ln \mu^2} = \sum_{f'} P_{ff'}(x/x') \otimes f'(x', \mu^2)
  \]
- DGLAP Evolution:
  \[
  \frac{d\hat{\sigma}_{f'f}}{dy dp_T^2} = \sum_{f''} P_{f'f''}(x/x'') \otimes f''(x'', \mu^2)
  \]

Universal PDFs
Global QCD analyses – a successful story

- World data with “Q” > 2 GeV
  + Factorization:

  **DIS:** \( F_2(x_B, Q^2) = \sum_f C_f(x_B/x, \mu^2/Q^2) \otimes f(x, \mu^2) \)

  **H-H:** \( \frac{d\sigma}{dydp_T^2} = \sum f f' f(x) \otimes \frac{d\sigma f f'}{dydp_T^2} \otimes f'(x') \)

- DGLAP Evolution:

  \( \frac{\partial f(x, \mu^2)}{\partial \ln \mu^2} = \sum f' P_{ff'}(x/x') \otimes f'(x', \mu^2) \)

- The “BIG” question(s)

  Why these PDFs behave as what have been extracted from the fits?

  What have been tested is the evolution from \( \mu_1 \) to \( \mu_2 \)
  But, does not explain why they have the shape to start with!

  Can QCD calculate and predict the shape of PDFs at the input scale, and other parton correlation functions?
Operator definition of PDFs, …

- **Definition – from QCD factorization:**

  \[
  \Phi^{[U]}(x; P, \mu) = \int \frac{d\xi^-}{(2\pi)} e^{i k \cdot \xi} \langle P | \bar{\psi}(0) U(0, \xi) \psi(\xi) | P \rangle_{\xi^+ = 0, \xi_T = 0} + \text{UVCT}(\mu)
  \]

  - Depends on the choice of the gauge link:
  
    \[ U(0, \xi) = e^{-ig \int_0^\xi ds^\mu A_\mu} \]

  **PDFs are well defined in QCD, but, can’t be calculated perturbatively**

- **Transverse momentum dependent PDFs (TMDs):**

  \[
  \Phi^{[U]}(x, k_T; P, \mu) = \int \frac{d\xi^- d^2\xi_T}{(2\pi)^3} e^{i k \cdot \xi} \langle P | \bar{\psi}(0) U(0, \xi) \psi(\xi) | P \rangle_{\xi^+ = 0} + \text{UVCT}(\mu)
  \]

  - General gauge link:
Lattice QCD

- **Hadron masses:** Predictions with limited inputs

- **Lattice “time” is Euclidean:** $\tau = i t$

  Cannot calculate PDFs, TMDs, ..., directly, whose operators are time-dependent
PDFs from lattice QCD

- Moments of PDFs – matrix elements of local operators

\[ \langle x^n(\mu^2) \rangle_q \equiv \int_0^1 dx \, x^n \, q(x, \mu^2) \]

- Works, but, hard and limited moments:

Dolgov et al., hep-lat/0201021
Gockeler et al., hep-ph/0410187

Limited moments – hard to get the full x-dependent distributions!
PDFs from lattice QCD

- How to get \( x \)-dependent PDFs with a limited moments?
  - Assume a smooth functional form with some parameters
  - Fix the parameters with the lattice calculated moments

\[
xq(x) = a x^b (1 - x)^c (1 + \epsilon \sqrt{x} + \gamma x)
\]


Cannot distinguish valence quark contribution from sea quarks
From quasi-PDFs to PDFs

- “Quasi” quark distribution (spin-averaged):

\[ \tilde{q}(x, \mu^2, P_z) \equiv \int \frac{d\xi_z}{4\pi} e^{-ixP_z\xi_z} \langle P|\bar{\psi}(\xi_z)\gamma_z \exp \left\{ -ig \int_0^{\xi_z} d\eta_z A_z(\eta_z) \right\} \psi(0)|P\rangle + \text{UVCT}(\mu^2) \]

Quasi-PDFs \(=\) PDFs

- Proposed matching:

\[ \tilde{q}(x, \mu^2, P_z) = \int_x^1 \frac{dy}{y} Z \left( \frac{x}{y}, \frac{\mu}{P_z} \right) q(y, \mu^2) + \mathcal{O}\left( \frac{\Lambda^2}{P_z^2}, \frac{M^2}{P_z^2} \right) \]

Quasi-PDFs \(\Rightarrow\) Normal PDFs when \(P_z \rightarrow \infty\)?

- Excellent idea and great potential:

IDEA: Calculate something \(=\) PDFs, but, carry all the information of PDFs

CHALLENGES:

✧ Quasi-PDFs could be calculated using the lattice QCD method
✧ Extract PDFs from what you can calculate, …
“Quasi-PDFs” have no parton interpretation

- Normal PDFs conserve parton momentum:

\[ M = \sum_q \left[ \int_0^1 dx \, x f_q(x) + \int_0^1 dx \, x f_\bar{q}(x) \right] + \int_0^1 dx \, x f_g(x) \]

\[ = \sum_q \int_{-\infty}^{\infty} dx \, x f_q(x) + \frac{1}{2} \int_{-\infty}^{\infty} dx \, x f_g(x) \]

\[ = \frac{1}{2(P^+)^2} \langle P | T^{++}(0) | P \rangle = \text{constant} \]

- “Quasi-PDFs” do not conserve “parton” momentum:

\[ \widetilde{M} = \sum_q \left[ \int_0^{\infty} \tilde{d}x \, \tilde{x} f_q(\tilde{x}) + \int_0^{\infty} \tilde{d}x \, \tilde{x} f_\bar{q}(\tilde{x}) \right] + \int_0^{\infty} \tilde{d}x \, \tilde{x} f_g(\tilde{x}) \]

\[ = \sum_q \int_{-\infty}^{\infty} \tilde{d}x \, \tilde{x} f_q(\tilde{x}) + \frac{1}{2} \int_{-\infty}^{\infty} \tilde{d}x \, \tilde{x} f_g(\tilde{x}) \]

\[ = \frac{1}{2(P^z)^2} \langle P | [T^{zz}(0) - g^{zz}(\ldots)] | P \rangle \neq \text{constant} \]

Note: “Quasi-PDFs” are not boost invariant
Exploratory study:

Quasi-Quark Distribution with different $P_z$

Predicted quark distribution along with global fitted one

Matching – taking into account:

Target mass: $\left(\frac{M_N}{P_z}\right)^2$
High twist: $a + b/P_z^2$
Pseudo-PDFs

- **Pseudo-PDFs = generalization of PDFs:**
  - **Definition:** $\xi^2 < 0$
    \[
    \mathcal{M}^\alpha(\nu = p \cdot \xi, \xi^2) \equiv \langle p | \overline{\psi}(0) \gamma^\alpha \Phi_\nu(0, \xi, \nu \cdot A) \psi(\xi) | p \rangle \\
    \approx 2p^\alpha \mathcal{M}_p(\nu, \xi^2) + \xi^\alpha (p^2 / \nu) \mathcal{M}_\xi(\nu, \xi^2) \approx 2p^\alpha \mathcal{M}_p(\nu, \xi^2)
    \]
    \[
    \mathcal{P}(x, \xi^2) \equiv \int \frac{d\nu}{2\pi} e^{ix\nu} \frac{1}{2p^+} \mathcal{M}^+(\nu, \xi^2)
    \]
  - **Interpretation:**
    Off-light-cone extension of PDFs: $f(x) = \mathcal{P}(x, \xi^2 = 0)$

- **Quasi-PDFs:**
  - $\xi^\mu = (0, 0_\perp, \xi_z)$
  - $\tilde{q}(x, \mu^2, p_z) = \int \frac{d\nu}{2\pi} e^{ix\nu} \frac{1}{2p^+_z} \mathcal{M}^z(\nu = p_z \xi_z, -\xi_z^2)$

- **TMDs:**
  - $\xi^\mu = (0^+, \xi^-, \xi_\perp)$
  - $\mathcal{P}(x, -\xi^2_\perp) \equiv \int d^2k_\perp e^{ik_\perp \cdot \xi_\perp} \mathcal{F}(x, k^2_\perp)$

No longer Lorentz invariant

TMDs with a straight gauge link
Pseudo-PDFs

- Pseudo-PDFs:

  - Lattice calculation with $\alpha = 0$:
    \[
    \mathcal{M}^\alpha(\nu = p \cdot \xi, \xi^2) \equiv \langle p|\overline{\psi}(0)\gamma^\alpha \Phi_\nu(0, \xi, \nu \cdot A)\psi(\xi)|p\rangle
    \equiv 2p^\alpha \mathcal{M}_p(\nu, \xi^2) + \xi^\alpha (p^2/\nu) \mathcal{M}_\xi(\nu, \xi^2) \approx 2p^\alpha \mathcal{M}_p(\nu, \xi^2)
    \]
  
  - Model quasi-PDFs: with $\xi^\mu = (0, 0_\perp, \xi_z)$

- Numerical results:

![Numerical results graph]

Orginos, et al, 2017
1706.05373

Remove UV!
Go beyond quasi- and pseudo-PDFs?

- A pQCD factorization approach:
  - Recall: Collinear factorization of DIS cross section – single hadron

\[ \sigma_{\text{DIS}}(x, Q^2; \sqrt{s}) \propto 2 \sum_f C_f(x, \frac{Q^2}{\mu^2}, \sqrt{s}) \otimes f_i(x, \mu^2) + O \left[ \frac{1}{Q^2} \right] \]

- Renormalizable + factorizable + lattice calculable “cross section”:

\[ \tilde{\sigma}_{\text{Lattice}}(\tilde{x}, Q, P \sim \sqrt{s}) \approx \sum_f \int \frac{dx}{x} C_f \left( \frac{x}{\tilde{x}}, \frac{\mu^2}{Q^2}, \alpha_s(\mu); P \right) f(x, \mu^2) + O \left[ \frac{1}{Q^2} \right] \]

Key: controllable hard scale!
What is lattice “cross section”?

*Single hadron matrix elements, with the following properties:*

- **Lattice calculable:**
  Calculable using lattice QCD with an Euclidean time

- **UV Renormalizable:**
  Ensure a well-defined continuum limit, UV & IR finite!

- **CO Factorizable:**
  Share the same perturbative collinear divergences with PDFs
  Factorizable to PDFs with IR-safe hard coefficients
  with controllable power corrections

**Key requirement:**

*A controllable large “momentum” scale – conjugate to hadron momentum*

to define the “collision” dynamics of the “cross section”
to ensure the necessary condition for the factorization
Current correlators:

- Coordinate space:
  \[ T_{jj}(p, s, \xi) = \lim_{\xi^0 \to 0^+} \langle p, s | T \{ j_\Gamma(\xi^0, \vec{\xi}) j_\Gamma(0) \} | p, s \rangle \]

- Factorization:

Complementarity and advantages:

- Complementary to existing approaches for extracting PDFs,
- Quasi-PDFs and pseudo-PDFs are special cases,
- Have tremendous potentials:

  Neutron PDFs, … (no free neutron target!)
  Meson PDFs, such as pion, …
  More direct access to gluons – gluonic current, …
A little bit more details, ...

- **Lattice cross sections – definition:**
  \[ \sigma_n(\xi^2, \omega, P^2) = \langle P \mid T\{\mathcal{O}_n(\xi)\} \mid P \rangle \quad \omega = P \cdot \xi \]

  where the operator is defined as
  \[ \mathcal{O}_{j_1 j_2}(\xi) \equiv \xi^{d_{j_1} + d_{j_2} - 2} Z_{j_1}^{-1} Z_{j_2}^{-1} j_1(\xi) j_2(0) \]

  with
  \[ d_j : \text{ Dimension of the current} \]
  \[ Z_j : \text{ Renormalization constant of the current} \]

- **Lattice cross sections – requirements:**
  1. is calculable in lattice QCD with an Euclidean time
  2. has a well-defined continuum limit as the lattice spacing, \( a \to 0 \) and
  3. has the same and factorizable logarithmic CO divergences as PDFs

- **Lattice cross sections – two-current correlations:**
  \[ j_S(\xi) = \xi^2 Z_S^{-1} [\overline{\psi}_q \psi_q](\xi), \quad j_V(\xi) = \xi Z_V^{-1} [\overline{\psi}_q \gamma \cdot \xi \psi_q](\xi), \]
  \[ j_{V'}(\xi) = \xi Z_{V'}^{-1} [\overline{\psi}_q \gamma \cdot \xi \psi_{q'}](\xi), \quad j_G(\xi) = \xi^3 Z_G^{-1} [-\frac{1}{4} F_{\mu \nu}^c F_{\mu \nu}^c](\xi) , \ldots \]

- **Lattice cross sections – quasi- and pseudo-PDFs:**
  \[ \mathcal{O}_q(\xi) = Z_q^{-1}(\xi^2) \overline{\psi}_q(\xi) \gamma \cdot \xi \Phi(\xi, 0) \psi_q(0) \]
  \[ \Phi(\xi, 0) = \mathcal{P} e^{-ig \int_0^1 \xi \cdot A(\lambda \xi) d\lambda} \]
Lattice cross sections – definition:

\[ \sigma_n(\xi^2, \omega, P^2) = \langle P | T\{O_n(\xi)\} | P \rangle \]

where the operator is defined as

\[ O_{j_1j_2}(\xi) \equiv \xi^{d_{j_1} + d_{j_2} - 2} Z_{j_1}^{-1} Z_{j_2}^{-1} j_1(\xi) j_2(0) \]

with

- \( d_j \): Dimension of the current
- \( Z_j \): Renormalization constant of the current

Lattice cross sections – requirements:

- is calculable in lattice QCD with an Euclidean time
- has a well-defined continuum limit as the lattice spacing, \( a \to 0 \) and
- has the same and factorizable logarithmic CO divergences as PDFs

Identify good lattice cross sections:

\[ \sigma_{E_{\text{Lat}}}(\xi_z, 1/a, P_z) \overset{Z}{\leftrightarrow} \sigma_E(\xi_z, \tilde{\mu}^2, P_z) \]

\[ \sigma_M(\xi_z, \tilde{\mu}^2, P_z) \overset{C}{\rightarrow} f_i(x, \mu^2), \]

A “new” collaboration between lattice QCD and perturbative QCD!
Renormalization – summary

- Take care by construction:
  Construct operators by using renormalizable, or conserved currents

- Renormalization of quasi- and pseudo-PDFs:

  Quasi-quark distributions is multiplicatively renormalizable

  \[
  \tilde{q}_i^R(\xi_z, \mu^2, p_z) = e^{-C_i|\xi_z|}Z_{w_i}^{-1}Z_{v_i}^{-1}\tilde{q}_i(\xi_z, \mu^2, p_z)
  \]

  Three classes of elementary divergent diagrams:

  Pseudo-quark distributions takes care of the UV renormalization by

  \[
  \mathcal{P}(x, \xi^2) \equiv \int \frac{d\nu}{2\pi} e^{ix\nu} \mathcal{M}_{p=p^0}(\nu, \xi^2)/\mathcal{M}_{p=p^0}(0, \xi^2)
  \]

Different matching

Ishikawa, Ma, Qiu and Yoshida
arXiv: 1701.03108
Renormalization – quasi-quark

- Coordinate-space definition:
  \[ \tilde{F}_{q/p}(\xi_z, \vec{\mu}^2, p_z) = \frac{e^{ip_z\xi_z}}{p_z} \langle h(p)|\overline{\psi}_q(\xi_z) \frac{\gamma_z}{2} \Phi_{n_z}^{(f)}(\{\xi_z, 0\}) \psi_q(0)|h(p)\rangle \]

- Why the proof is hard:
  - Because of $z$-direction dependence, Lorentz symmetry is broken, hard to exhaust all possible UV divergences
  - Renormalization of composite operator is needed

- Broken Lorentz symmetry:
  Both 3D and 4D loop-integration can generate UV divergences

- UV: 4-D integration
  \[ \int \frac{d^dl}{(2\pi)^d} \frac{1}{l^2(p - l)^2} \]

- UV: 3-D integration
  \[ \int \frac{d^3\vec{l}}{\vec{l}^2} = \int \frac{d^3\vec{l}}{\vec{l}^2 - l_z^2} \]
Renormalization – quasi-quark

- **Quasi-quark at one-loop:**

\[ n_z^2 = -1 \]

\[ \frac{1}{2} \gamma_z \]

- **Fig. 1(a):**

\[ M_{1a} = \frac{e^{i p_z z}}{p_z} \frac{1}{N_c} \text{Tr}[T^a T^a] \int_a^{\xi_z-2a} \int_{r_1+a}^{r_2} \int_{r_1+a}^{r_2} \int d^4 l e^{i l_z (r_2-r_1)} \left( \frac{-i g_{\mu \nu}}{l^2} \right) \times \left( -i g_s n_\mu \right) \left( -i g_s n_\nu \right) \text{Tr} \left[ \frac{1}{2} \gamma^z \right] = \frac{\alpha_s C_F}{4i\pi^3} \int_a^{\xi_z-2a} \int_{r_1+a}^{r_2} \int_{r_1+a}^{r_2} d^4 l e^{i l_z (r_2-r_1)} \frac{1}{l^2} \]

\[ M_{1a} \overset{\text{div}}{=} -\frac{\alpha_s C_F}{\pi} \frac{|\xi_z|}{a} + \frac{\alpha_s C_F}{\pi} \ln \frac{|\xi_z|}{a} \]

- **Cutoff “a” between fields**
- **Conclusion independent of regulator**
- **3D-integration:**

\[ d^4 l = d^3 l \, dl_z \]

\[ \int \frac{d^3 l}{l^2} = \int \frac{d^3 l}{l^2 - l_z^2} \]

\[ = \int d^3 l \left( \frac{1}{l^2} + \frac{l_z^2}{(l^2 - l_z^2)^2} \right) \]

\[ \int dl_z e^{i l_z (r_2-r_1)} = 2\pi \delta(r_2-r_1) \]

- **1st term vanishes for** \( r_1 \neq r_2 \)
Renormalization – quasi-quark

- Quasi-quark at one-loop:

- Complete one-loop contribution:

\[ M^{(1)\, \text{div}} = M_{1a} + 2 \times M_{1b} + 2 \times \frac{1}{2} M_{1c} + M_{1d} \]

\[ = \frac{\alpha_s C_F}{\pi} \left( -\frac{|\xi_z|}{a} + 2 \ln \frac{|\xi_z|}{a} - \frac{1}{4\epsilon} \right) \]

- At one-loop, all 3D integrations are finite
- Divergence only come from the region when all momentum components go to infinity

\[ \text{Localized UV divergence in all directions!} \]

Very different from the UV behavior of normal PDFs: \((1, \lambda^2, \lambda), \lambda \rightarrow \infty\)
Power counting and divergent sub-diagrams:

(a) \(-1/\alpha, \ln(1/\alpha)\):

\[
\begin{align*}
\int_0 r_2 r &= \int_0 r_1 r_2 r + \int_0 r_1 r_2 r + \ldots \\
&\quad (a)
\end{align*}
\]

(b) \(-\ln(1/\alpha)\):

\[
\begin{align*}
\int_0 r_2 r &= \int_0 r_1 r_2 r + \int_0 r_2 r + \ldots \\
&\quad (b)
\end{align*}
\]

(c) \(-\ln(1/\alpha)\):

\[
\begin{align*}
\int_0 r &= \int_0 r + \int_0 r + \ldots \\
&\quad (c)
\end{align*}
\]

Happen only when all loop momenta go to infinity – localized!

Example of convergent sub-diagrams:
Renormalization – quasi-quark

- **Power divergence:**

\[
0 \quad r \quad 0 \quad r_i \quad r \quad 0 \quad r_i \quad r_2 \quad r \quad + \ldots
\]

\[
1 + c \int_0^r dr_1 + c^2 \int_0^r dr_1 \int_0^r dr_2 + \ldots
\]

\[
= P e^{c \int_0^r dr'} = e^{cr},
\]

- It is allowed to introduce an overall factor \( e^{-c|\xi_z|} \) to remove all power UV divergences

- **Interpretation:**

  - Mass renormalization of test particle

- **Log divergence in from gauge link:**

  - Besides power divergence, there are also logarithmic UV divergences
  - It is known that these divergences can be removed by a “wave function” renormalization of the test particle, \( Z_{\text{wq}}^{-1} \)
Log divergence from gluon-gauge link vertex:

- Logarithmic UV: can be absorbed by the coupling constant renormalization of QCD.

UV from vertex correction:

- The most dangerous UV diagram, may mix with other operators
- Locality of UV divergence: no dependence on $r_2 - r_1$ or $p$
- UV divergence is proportional to quark-gaugelink vertex at lowest order, with a constant coefficient
- A constant counter term is able to remove this UV divergence.

Renormalization to all orders:

- Using bookkeeping forests subtraction method, the net effect is to introduce a constant multiplicative renormalization factor $Z_{vq}^{-1}$ for the quark-gaugelink vertex.
Does the renormalized lattice cross section and quasi-PDFs share the same CO properties with PDFs?

Can we extract PDFs from lattice cross section and/or renormalized quasi-PDFs reliably?

\[ \gamma^+ \delta(x - k^+ / p^+) \]
**Factorization**

- **Factorized formula for lattice cross section:**

  \[
  \sigma_n(\omega, \xi^2, P^2) = \sum_a \int_{-1}^{1} \frac{dx}{x} f_a(x, \mu^2) \times K_n^a(x\omega, \xi^2, x^2 P^2, \mu^2) + O(\xi^2 \Lambda_{QCD}^2)
  \]

  with

  \[f_a(x, \mu^2) = -f_a(-x, \mu^2)\]

- **Steps needed to prove:**

  Let \(\xi^2\) be small but not vanishing, apply OPE to the operator,

  \[
  \sigma_n(\omega, \xi^2, P^2) = \sum_{J=0} \sum_a W_n^{(J,a)}(\xi^2, \mu^2) \xi^{\nu_1} \cdots \xi^{\nu_J} \times \langle P|\mathcal{O}^{(J,a)}_{\nu_1 \cdots \nu_J}(\mu^2)|P\rangle
  \]

  with

  \[\langle P|\mathcal{O}^{(J,a)}_{\nu_1 \cdots \nu_J}(\mu^2)|P\rangle = 2A^{(J,a)}(\mu^2) \times (P_{\nu_1} \cdots P_{\nu_J} - \text{traces})\]

  With reduced matrix element:

  \[A^{(J,a)}(\mu^2) = \langle P|\mathcal{O}^{(J,a)}(\mu^2)|P\rangle\]

  \[
  \sigma_n(\omega, \xi^2, P^2) = \sum_{J=0} \sum_a W_n^{(J,a)}(\xi^2, \mu^2) 2A^{(J,a)}(\mu^2) \times \Sigma_J(\omega, P^2 \xi^2)
  \]

  with

  \[\Sigma_J(\omega, P^2 \xi^2) \equiv \xi^{\nu_1} \cdots \xi^{\nu_J} (P_{\nu_1} \cdots P_{\nu_J} - \text{traces})\]

  \[= \sum_{i=0}^{[J/2]} C_i^{J-i}(\omega)^{J-2i} (-P^2 \xi^2 / 4)^i\]

  **No approximation yet!**
Factorization

- Approximation – leading power/twist:

\[ A^{(J,a)}(\mu^2) = \frac{1}{S_a} \int_{-1}^{1} dx x^{J-1} f_a(x, \mu^2) \]

With symmetry factor:

\[ S_a = 1, 2 \text{ for } a = q, g \]

\[ \sigma_n(\omega, \xi^2, P^2) = \sum_a \int_{-1}^{1} \frac{dx}{x} f_a(x, \mu^2) \times K_n^a(x\omega, \xi^2, x^2P^2, \mu^2) + O(\xi^2 \Lambda_{QCD}^2) \]

with

\[ K_n^a = \sum_{J=1}^{J_a} \frac{2}{S_a} W_n^{(J,a)}(\xi^2, \mu^2) \Sigma_J(x\omega, x^2P^2\xi^2) \]

Note: our proof of factorization is valid only when \( |\omega| \ll 1 \) and \( |p^2\xi^2| \ll 1 \)

- Extrapolate into large \( \omega \) region:

★ Validity of OPE guarantees that \( \sigma_n \) is an analytic function of \( \omega \), so as its Taylor series of \( \omega \) around \( \omega = 0 \), defined above

★ If we fix \( \xi \) to be short-distance, while we increase \( \omega \) by adjusting \( p \), we can’t introduce any new perturbative divergence

★ That is, \( \sigma_n \) remains to be an analytic function of \( \omega \) unless \( \omega = \infty \)

Factorization holds for any finite value of \( \omega \) and \( p^2\xi^2 \), if \( \xi \) is short-distance
Coefficient/matching functions

- Matching coefficients for current-current correlators:

\[
K_n^a = \sum_{J=1}^{2} \frac{2}{S_a} W_n^{(J,a)}(\xi^2, \mu^2) \Sigma_J(x\omega, x^2 P^2 \xi^2) \quad \Rightarrow \quad \text{Need} \quad W_n^{(J,a)}(\xi^2, \mu^2)
\]

a) Calculate \(K_n^a(x\omega, \xi^2, 0, \mu)\) - coefficient in CO factorization with \(p^2=0\)

b) Expand \(K_n^a(x\omega, \xi^2, 0, \mu)\) in power series of \(x\omega\)

c) Extract \(W_n^{(J,a)}(\xi^2, \mu^2)\) with \(\Sigma_J(x\omega, 0) = (x\omega)^J\)

- LO matching:

\[k^\mu = x p^\mu\]
\[p^2 = 0\]

Fig. (a)

\[K_q^{(0)}(x\omega, \xi^2, 0, \mu) = \frac{1}{2} \text{Tr}[k^\xi] e^{i\xi^\cdot k} = 2 x \omega e^{i x \omega}
\]

\[W_q^{(J,q)} = i^{J-1} / (J - 1) \]

\[M_b = \frac{i \xi^4}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{\text{Tr}[k^\xi] e^{i\xi^\cdot (k-l)}}{l^2 + i \varepsilon} = \frac{i}{\pi^2} x \omega e^{i x \omega} \quad M_c = M_b^*
\]

\[\tilde{K}_{S/V/\bar{V}}^{(0)}(x\bar{\omega}, q^2, 0, \mu) = -2i \frac{x^2 \bar{\omega}^2}{1 - x^2 \bar{\omega}^2}
\]

Fig. (b, c)

Flavor change current
No crossing diagram
Connection to quasi- and pseudo-PDFs

- **Momentum-space version – Fourier transform:**

\[
\tilde{\sigma}_n(\tilde{\omega}, q^2, P^2) \equiv \int \frac{d^4 \xi}{\xi^4} e^{iq \cdot \xi} \sigma_n(P \cdot \xi, \xi^2, P^2)
\]

With \(\tilde{\omega} \equiv \frac{2P \cdot q}{q^2} = \frac{1}{x_B}\), and valid for \(\tilde{\omega}^2 < 1\)

\[
\tilde{K}^a_n = \int \frac{d^4 \xi}{\xi^4} e^{iq \cdot \xi} K^a_n(xP \cdot \xi, \xi^2, x^2 P^2, \mu)
\]

Care is needed for the physical region when \(\tilde{\omega}^2 > 1\)

Contribution from large \(\tilde{\omega}^2 > 1\) region – poles and cuts

- **Comparison with other approaches:**

With \(K^q(0)\)

\[
\int \frac{d\omega}{\omega} \frac{e^{-ix\omega}}{4\pi} \sigma_q(\omega, \xi^2, P^2) \approx f_q(x, \mu)
\]

modulo \(O(\alpha_s)\) corrections and higher twist corrections.

With \(\xi_0 = 0\), the integral over \(\omega = -\tilde{\xi} \cdot \tilde{P} = -|\tilde{\xi}| |\tilde{P}| \cos \theta\)

- **Quasi-PDFs:** \(\xi_0 = 0, \ \tilde{p} = p_z, \ \tilde{\xi} = \xi_z\) with fixed \(p_z\)

- **Pseudo-PDFs:** \(\xi_0 = 0, \ \tilde{p} = p_z, \ \tilde{\xi} = \xi_z\) with fixed \(\xi_z\)
One-loop example: quark \( \rightarrow \) quark

Expand the factorization formula:

\[
\tilde{f}_{q/q}(\tilde{x}, \tilde{\mu}^2, P_z) \approx \sum_j \int_0^1 \frac{dx}{x} C_{ij}(\frac{\tilde{x}}{x}, \tilde{\mu}^2, P_z) f_{j/h}(x, \mu^2)
\]

To order \( \alpha_s \):

\[
\tilde{f}^{(1)}_{q/q}(\tilde{x}) = f^{(0)}_{q/q}(x) \otimes C^{(1)}_{q/q}(\tilde{x}/x) + f^{(1)}_{q/q}(x) \otimes C^{(0)}_{q/q}(\tilde{x}/x)
\]

Feynman diagrams:

Same diagrams for both

\( \tilde{f}_{q/q} \) and \( f_{q/q} \)

But, in different gauge:

\( n_z \cdot A = 0 \) for \( \tilde{f}_{q/q} \)
\( n \cdot A = 0 \) for \( f_{q/q} \)

Gluon propagator in \( n_z \cdot A = 0 \):

\[
\tilde{d}^{\alpha\beta}(l) = -g^{\alpha\beta} + \frac{l^\alpha n^\beta_z + n^\alpha_z l^\beta}{l_z} - \frac{n_z^2 l^\alpha l^\beta}{l_z^2}
\]

with \( n_z^2 = -1 \)
One-loop “quasi-quark” distribution in a quark

Real + virtual contribution:

\[
\hat{f}_{q/q}^{(1)}(\bar{x}, \mu^2, P_z) = C_F \frac{\alpha_s}{2\pi} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \int_0^{\mu^2} \frac{dl_\perp^2}{l_\perp^2 + \epsilon} \int_0^{+\infty} \frac{dl_z}{P_z} \left[ \delta(1 - \bar{x} - y) - \delta(1 - \bar{x}) \right] \left\{ \frac{1}{y} \left( 1 - y + \frac{1-\epsilon}{2} y^2 \right) \right. \\
\times \left[ \frac{y}{\sqrt{\lambda^2 + y^2}} + \frac{1-y}{\sqrt{\lambda^2 + (1-y)^2}} \right] + \frac{(1-y)\lambda^2}{2y\sqrt{\lambda^2 + y^2}} + \frac{\lambda^2}{2y\sqrt{\lambda^2 + (1-y)^2}} + \frac{1-\epsilon}{2} \frac{(1-y)\lambda^2}{[\lambda^2 + (1-y)^2]^{3/2}} \right\}
\]

where \( y = l_z/P_z, \lambda^2 = l_\perp^2/P_z^2, C_F = (N_c^2 - 1)/(2N_c) \)

Cancelation of CO divergence:

\[
\frac{y}{\sqrt{\lambda^2 + y^2}} + \frac{1-y}{\sqrt{\lambda^2 + (1-y)^2}} = 2\theta(0 < y < 1) - \left[ \text{Sgn}(y) \frac{\sqrt{\lambda^2 + y^2} - |y|}{\sqrt{\lambda^2 + y^2}} + \text{Sgn}(1-y) \frac{\sqrt{\lambda^2 + (1-y)^2} - |1-y|}{\sqrt{\lambda^2 + (1-y)^2}} \right]
\]

Only the first term is CO divergent for \( 0 < y < 1 \), which is the same as the divergence of the normal quark distribution – necessary!

UV renormalization:

Different treatment for the upper limit of \( l_\perp^2 \) integration - “scheme”

Here, a UV cutoff is used – other scheme is discussed in the paper.
One-loop coefficient functions

- **MS scheme for** $f_{q/q}(x, \mu^2)$:

$$C_{q/q}^{(1)}(t, \tilde{\mu}^2, \mu^2, P_z) = f_{q/q}^{(1)}(t, \tilde{\mu}^2, P_z) - f_{q/q}^{(1)}(t, \mu^2)$$

$$C_{q/q}^{(1)}(t) \frac{\alpha_s}{2\pi} = \left[ \frac{1 + t^2 \ln \frac{\tilde{\mu}^2}{\mu^2} + 1 - t}{1 - t} \right] + \left[ \frac{t \Lambda_{1-t}}{(1-t)^2} + \frac{\Lambda_t}{1-t} + \frac{\text{Sgn}(t) \Lambda_t}{\Lambda_t + |t|} \right]$$

$$- \frac{1 + t^2}{1 - t} \left[ \text{Sgn}(t) \ln \left( 1 + \frac{\Lambda_t}{2|t|} \right) + \text{Sgn}(1-t) \ln \left( 1 + \frac{\Lambda_{1-t}}{2|1-t|} \right) \right]_N$$

where $\Lambda_t = \sqrt{\tilde{\mu}^2/P_z^2 + t^2} - |t|$, $\text{Sgn}(t) = 1$ if $t \geq 0$, and $-1$ otherwise.

- **Generalized “+” description:** $t = \tilde{x}/x$

$$\int_{-\infty}^{+\infty} dt \left[ g(t) \right]_N h(t) = \int_{-\infty}^{+\infty} dt \left[ g(t) \right] \left[ h(t) - h(1) \right]$$

**For a testing function** $h(t)$

Explicit verification of the CO factorization at one-loop

- Note: $\Lambda_t \rightarrow O \left( \frac{\tilde{\mu}}{P_z} \right)$ as $P_z \rightarrow \infty$ the linear power UV divergence!
Summary and outlook

- "lattice cross sections" = single hadron matrix elements calculable in Lattice QCD, renormalizable + factorizable in QCD
  Going beyond the quasi-PDFs

- Extract PDFs by global analysis of data on "Lattice x-sections". Same should work for other distributions (TMDs, GPDs)

\[
\tilde{\sigma}_{\text{Lat}}^{E}(\tilde{x}, \frac{1}{a}, P_z) \approx \sum_i \int_0^1 \frac{dx}{x} f_i/h(x, \mu^2) \tilde{C}_i\left(\frac{x}{\tilde{x}}, \frac{1}{a}, \mu^2, P_z\right)
\]

- Conservation of difficulties – complementarity:
  High energy scattering experiments
  – less sensitive to large x parton distribution/correlation
  “Lattice factorizable cross sections”
  – more suited for large x PDFs, but limited to large x for now

- Lattice QCD can be used to study hadron structure, but, more works are needed!

Thank you!
BACKUP SLIDES
Renormalization

- **Gluon-to-quark at one-loop:**

\[ M_{2a} \propto \int_0^{\xi_z} dr_1 \int_{r_1}^{\xi_z} dr_2 \int d^4l \, e^{-\bar{u}_z \xi_z \frac{l_z}{l^2}} \]

\[ = \frac{\xi_z^2}{2} \int dl_z \, e^{-\bar{u}_z \xi_z \frac{l_z}{l^2}} \int d^3\bar{l} \left( \frac{1}{l^2} + \frac{l_z^2}{(l^2 - l_z^2)l^2} \right) \]

- UV divergence from 3-D \( \propto \delta' (\xi_z) \), vanishes for finite \( \xi_z \)

- **Caution for momentum-space version:**

  Finite-term:

  \[ \frac{\xi_z^2}{2} \int dl_z \, e^{-\bar{u}_z \xi_z \frac{l_z}{l^2}} \int d^3\bar{l} \frac{l_z^2}{(l^2 - l_z^2)l^2} \]

  \[ \propto \frac{\xi_z^2}{2} \int dl_z \, e^{-\bar{u}_z \xi_z \frac{l_z^3}{|l_z|}} \frac{l_z^3}{|l_z|} = \frac{2i}{\xi_z} \]

  - Divergent as \( \xi_z \to 0 \)
  - Result in bad large \( \tilde{x} \) behavior in momentum space