S-matrix approach to 2 and 3 hadron interactions

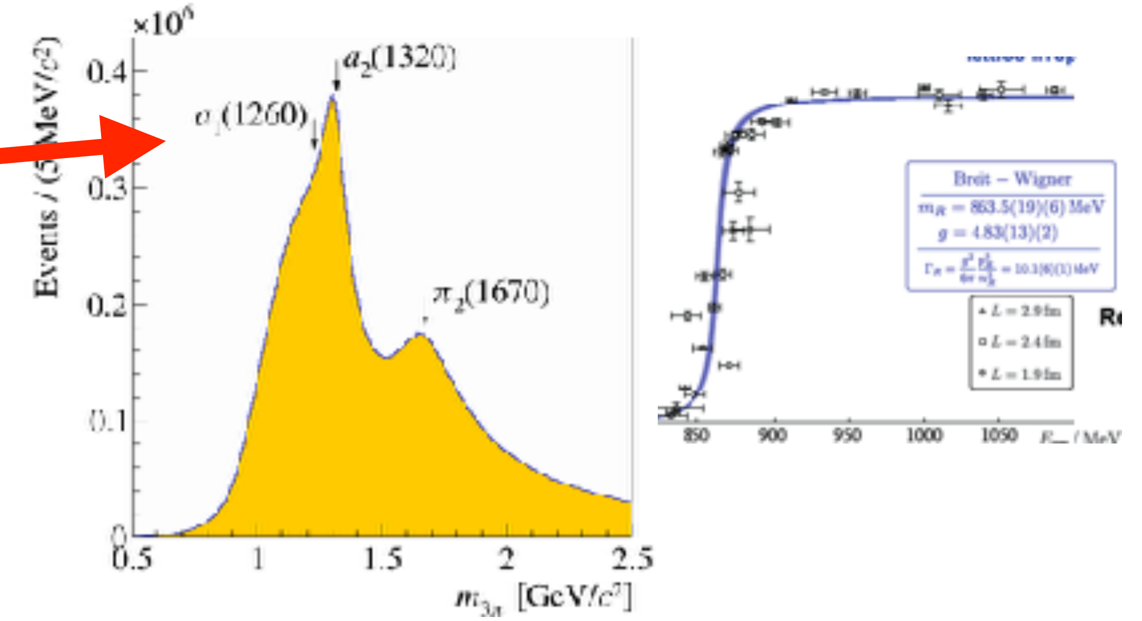
Adam Szczepaniak, Indiana University/Jefferson Lab

Outline

- Opportunities and challenges in hadron spectroscopy
- Amplitudes from S-matrix analysis: 2-to-2 scattering 1-to-3 decays : how virtual exchanges become real
- Three particle scattering : the framework

How Hadrons Emerge from QCD

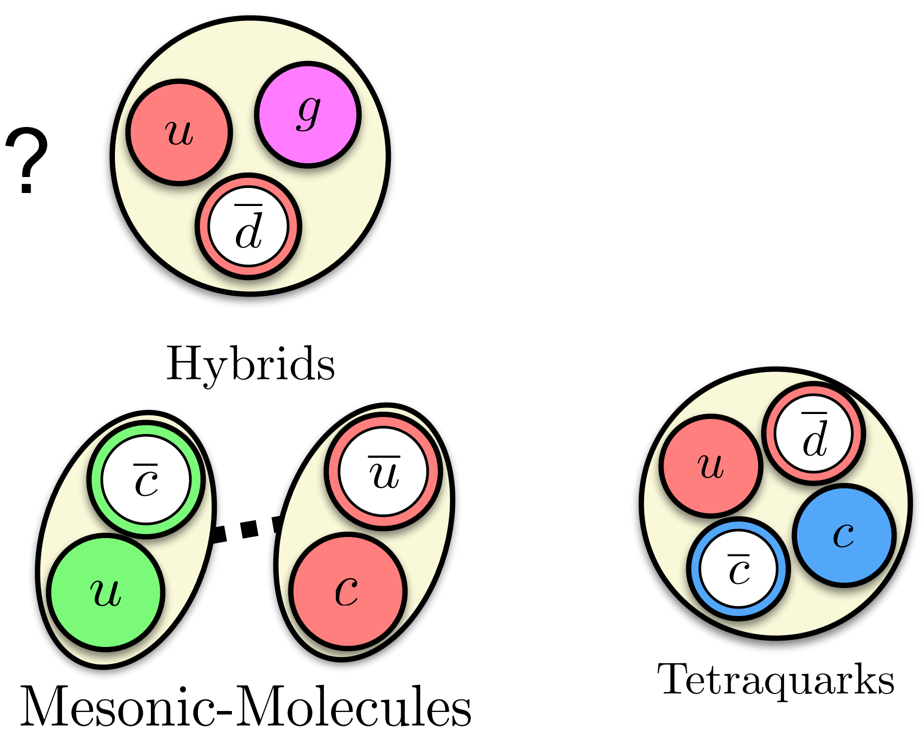
- Experimental or lattice signatures (real axis data: cross section bumps and dips, energy levels)



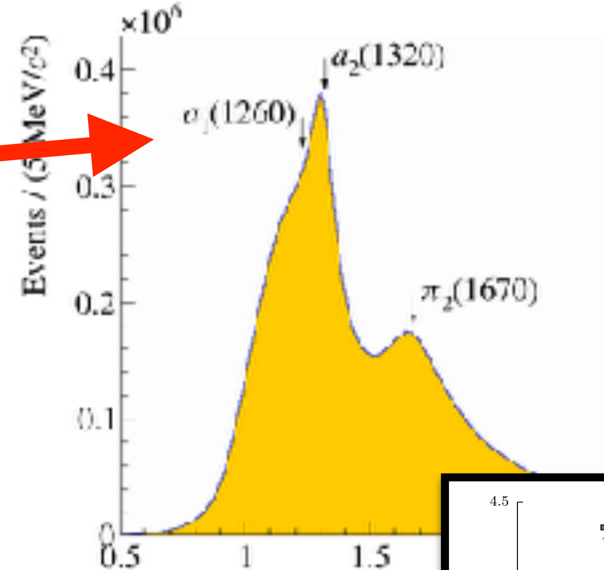
- Theoretical signatures (complex plane singularities: poles, cusps)

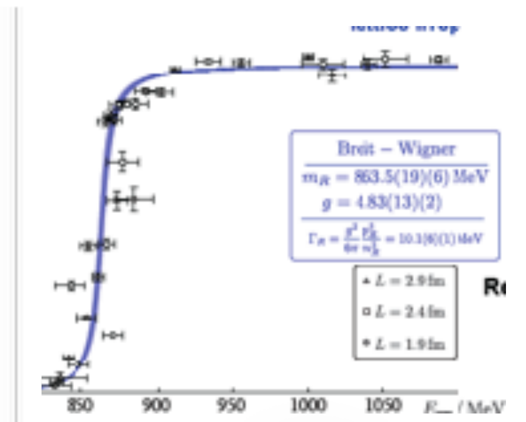
- What is the interpretation (constituent quarks, molecules, …) ?

Reaction amplitudes



How Hadrons Emerge from QCD





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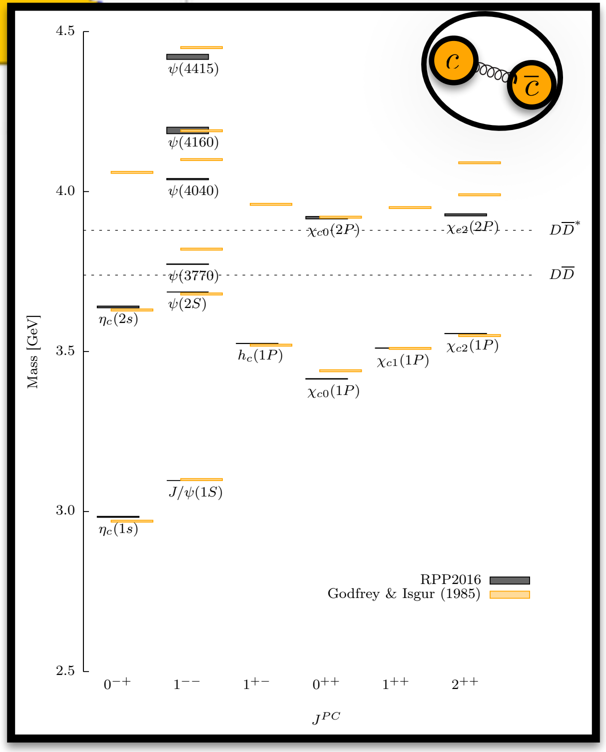

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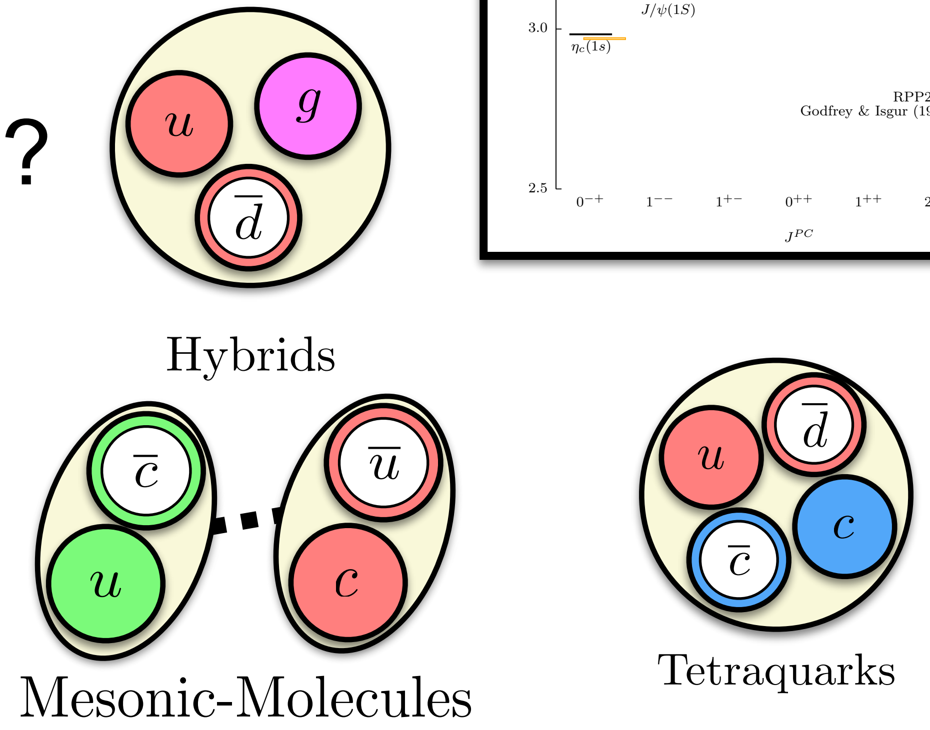
- What is the interpretation (constituent quarks, molecules, …) ?

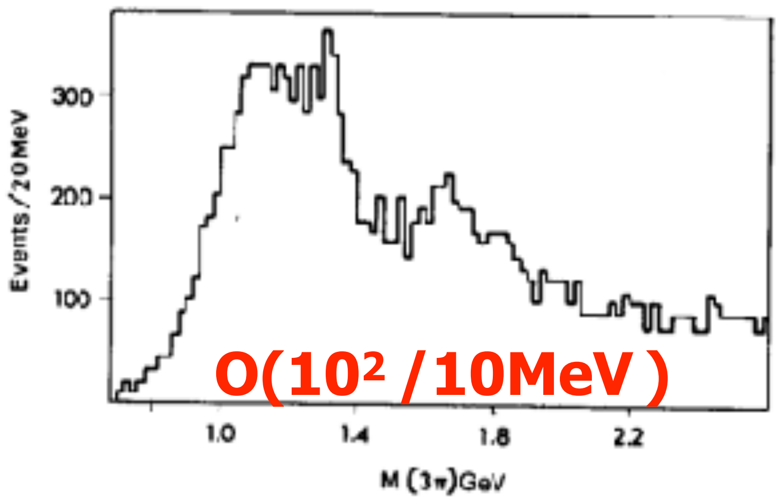


Reaction amplitudes

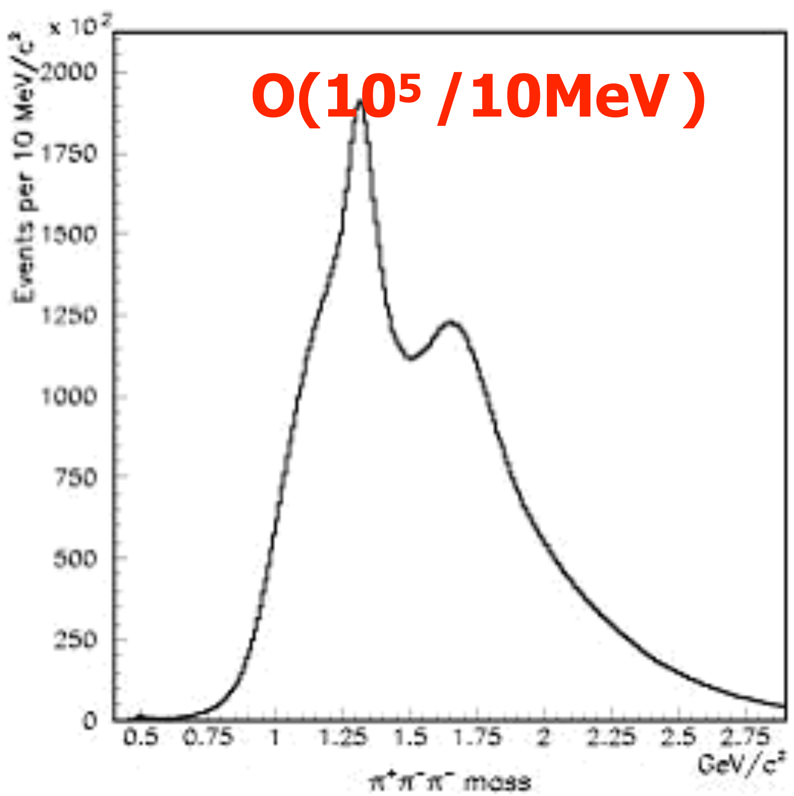
3pion challenge

evolution in statistics $\pi^- p \rightarrow \pi^-\pi^+\pi^- p$

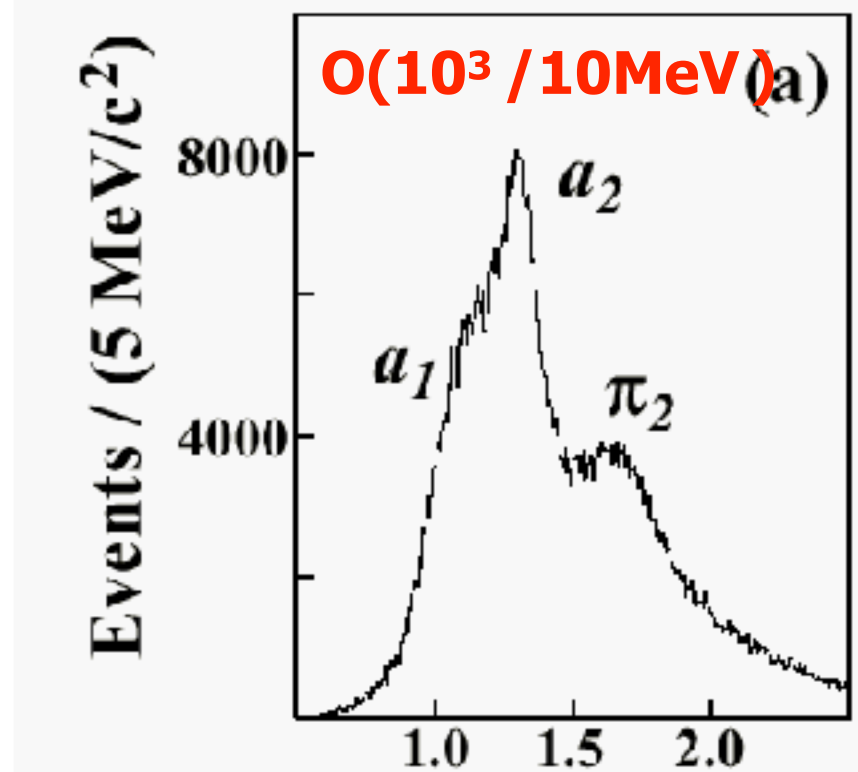
CERN ca. 1970



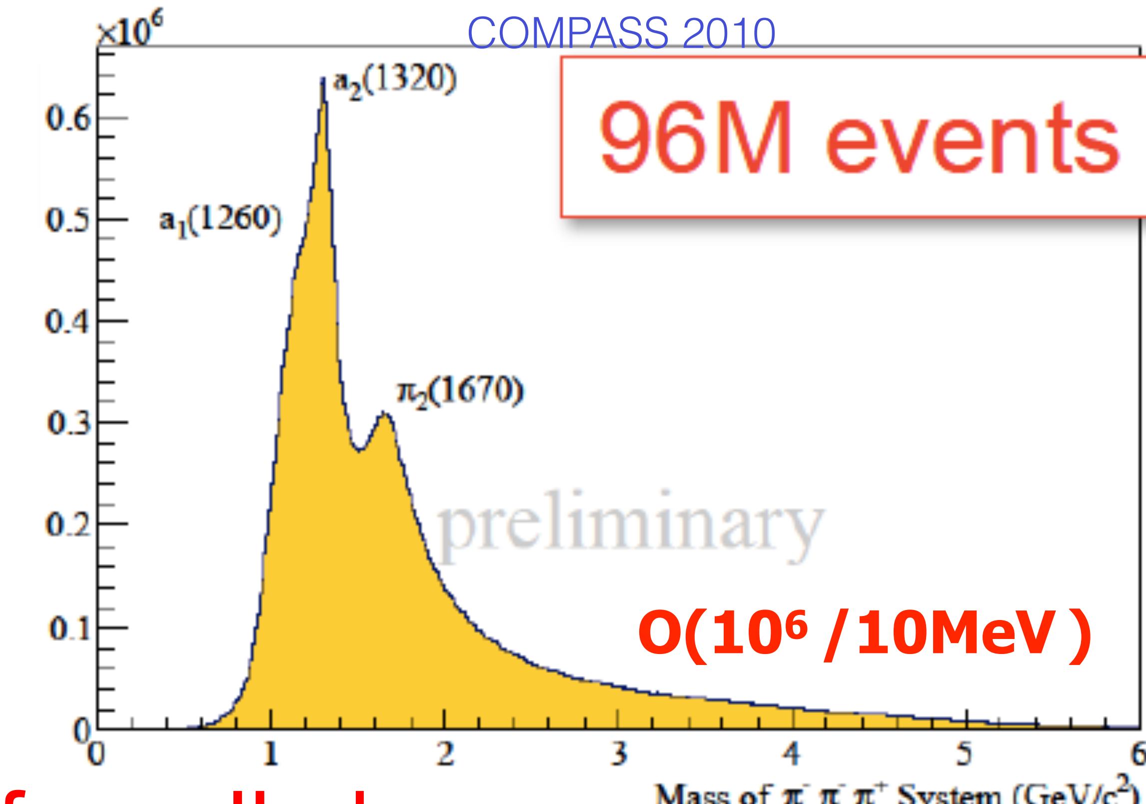
E852 (Full sample)



BNL (E852) ca 1995

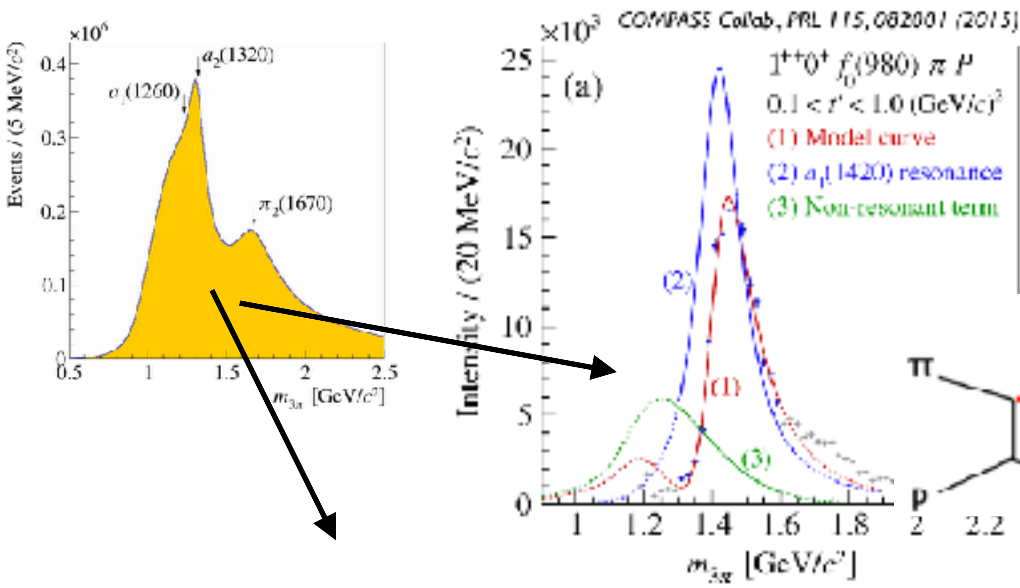


COMPASS 2010

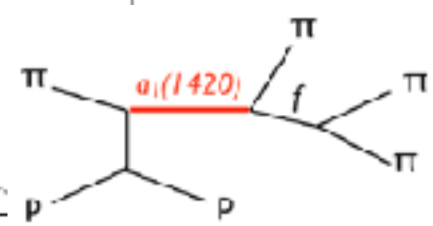


Comparable statistics expected from JLab

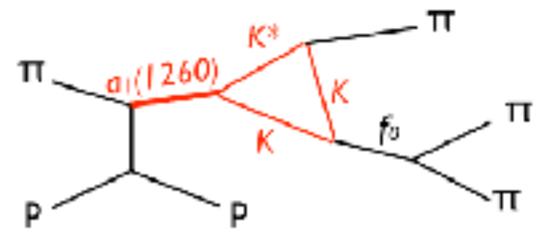
Signatures of new, unusual light resonances



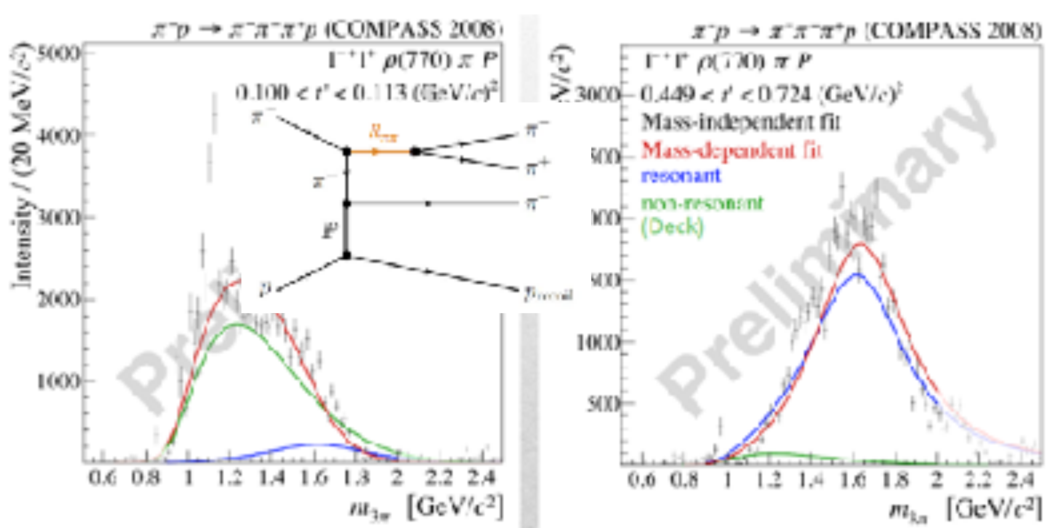
- High precision PWA of 3pi diffractive association yields a new $a_1(1460)$ incompatible with the quark model/Regge expectations.



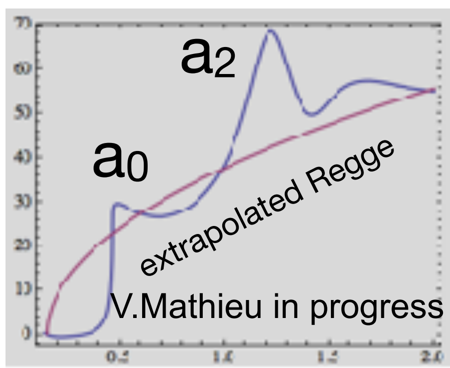
Or ?

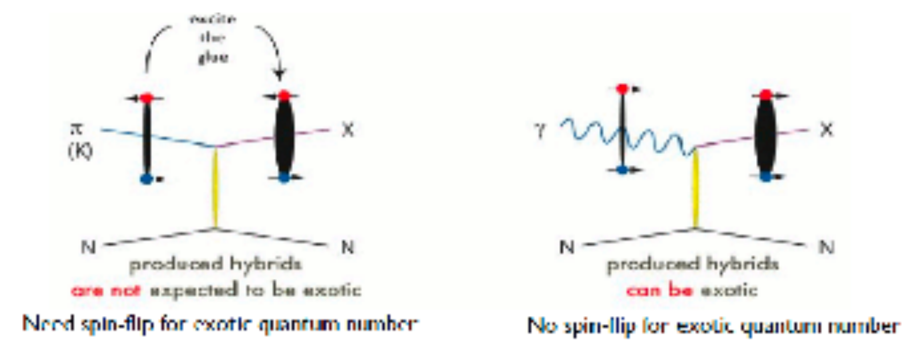


- At low-t exotic wave production compatible with one pion exchange

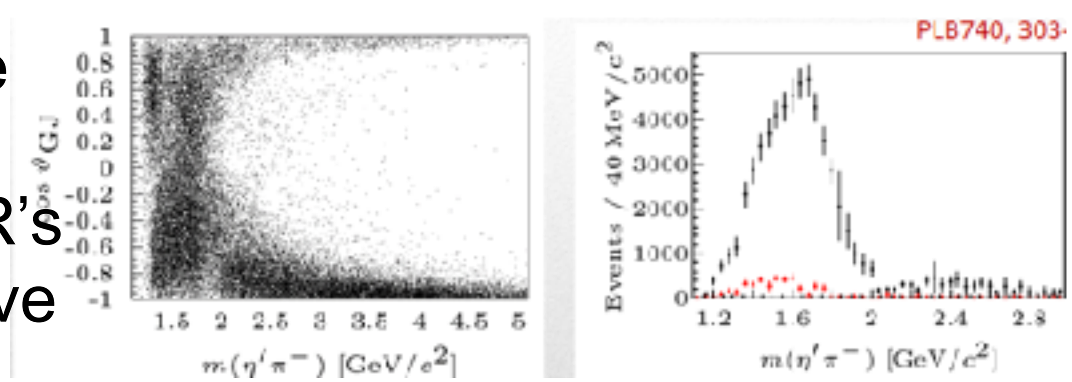


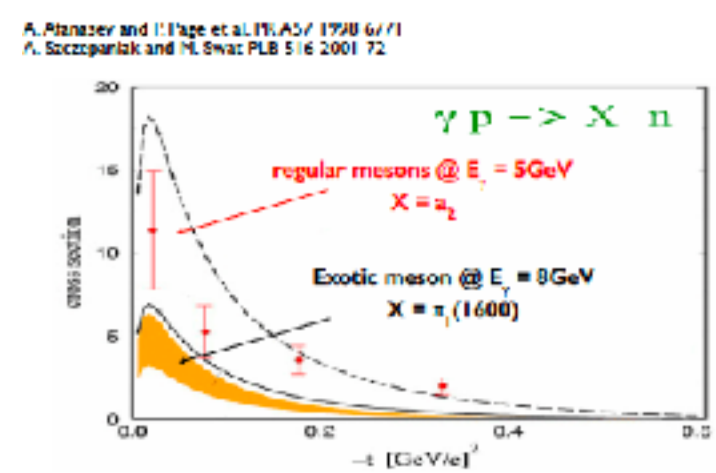
- In photoproduction exotic mesons be produced via pion exchange



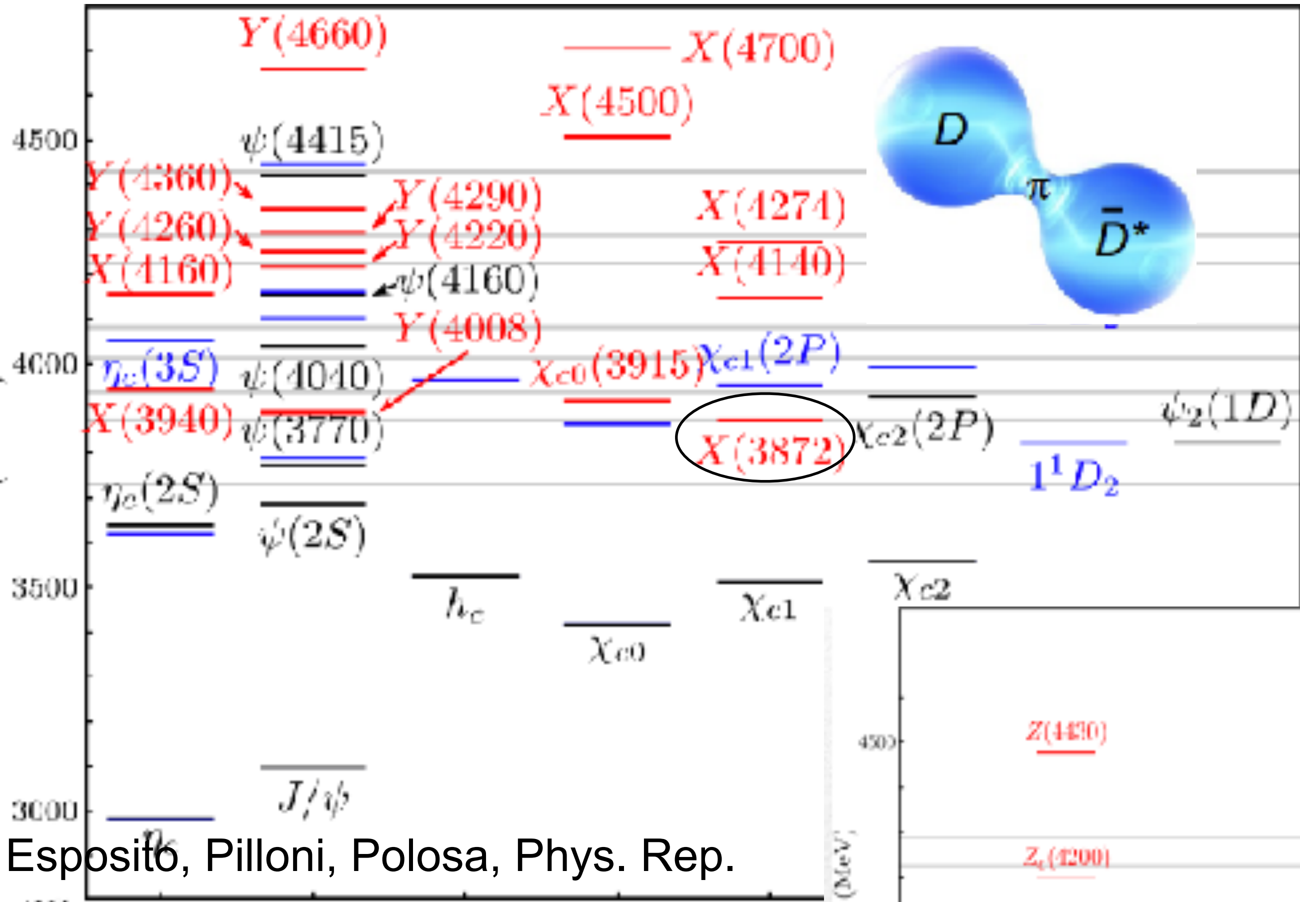


- Large exotic wave seen in $\eta^{(')}\pi$ production : FESR's to constrain P-wave

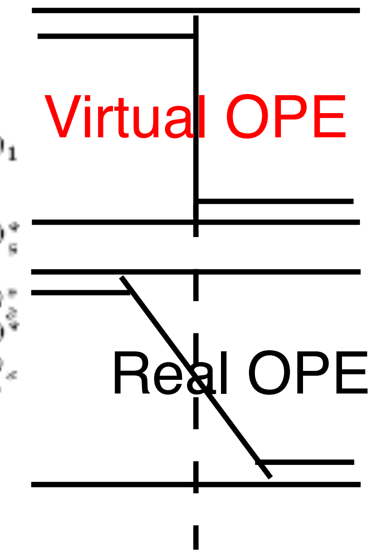


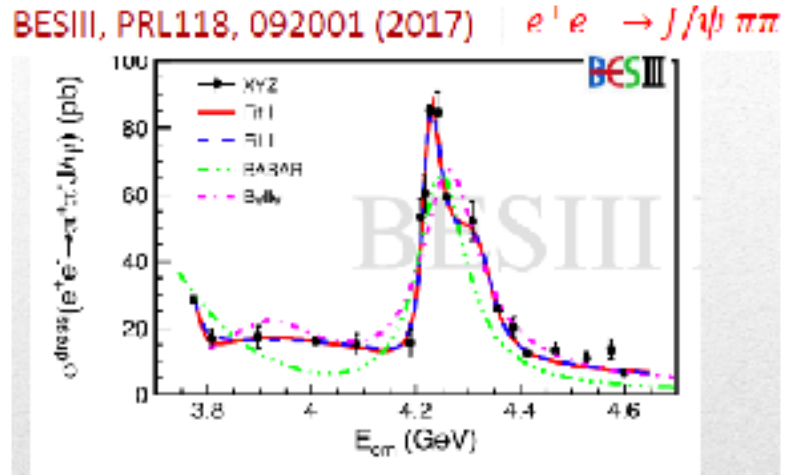


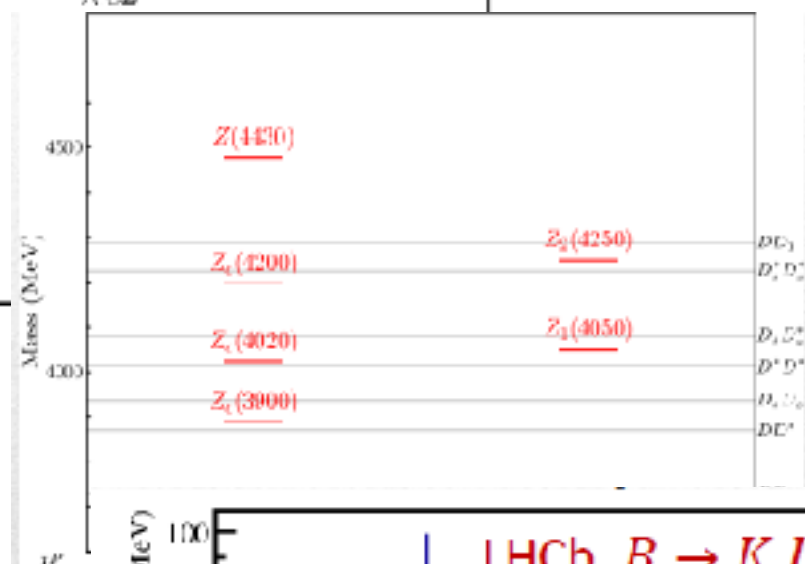
Signatures of unusual heavy quark resonances



Esposito, Pilloni, Polosa, Phys. Rep.





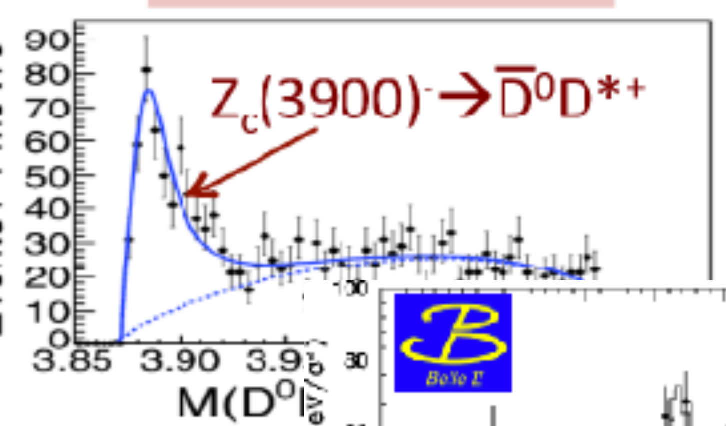


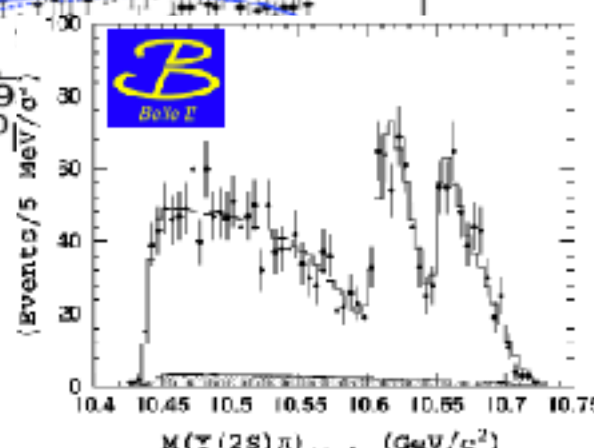
EMARK ON ENERGY PEAKS IN MESON SYSTEMS

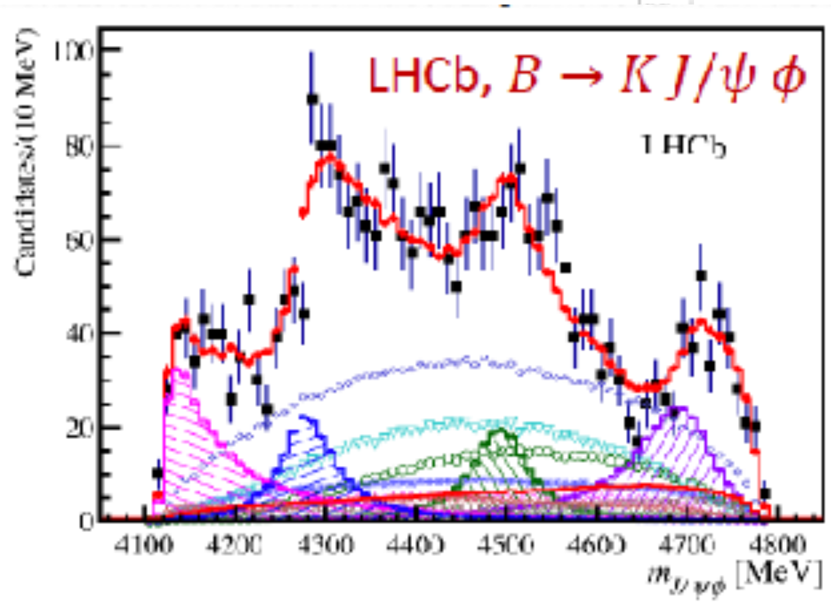
M. Nauenberg A. Pais

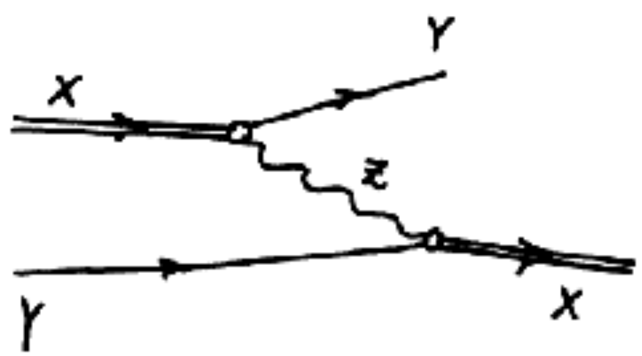
If the width of particle X is not very large we will stay close to the physical region. This almost singular behavior of A(s) for certain physical s causes the peaking effect to which we refer as an (X, Y, Z) peak.

BESIII PRL 112, 022001

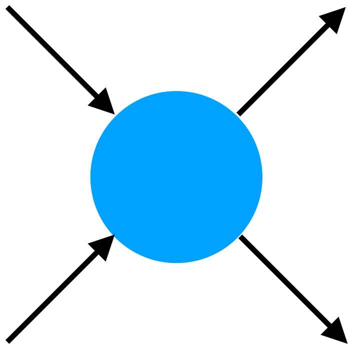






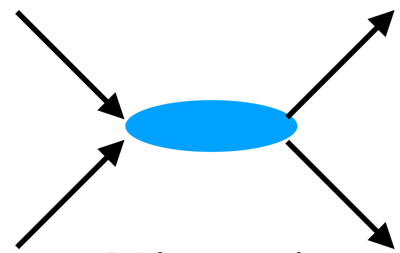


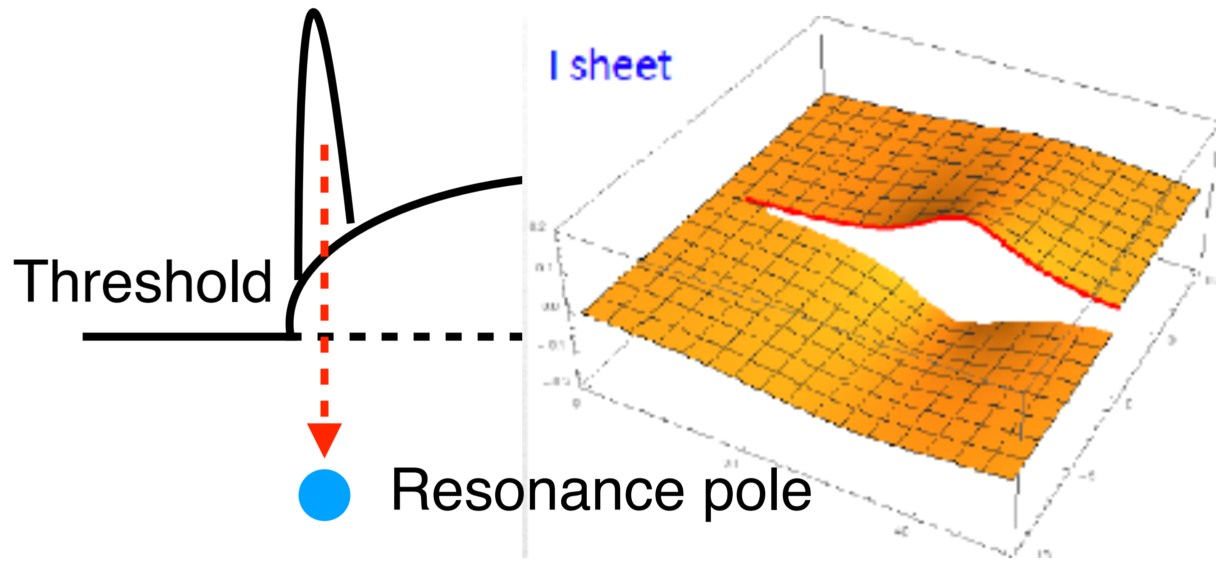
Amplitude signatures



$\bar{D}\,D^{*} \rightarrow \bar{D}\,D^{*}$

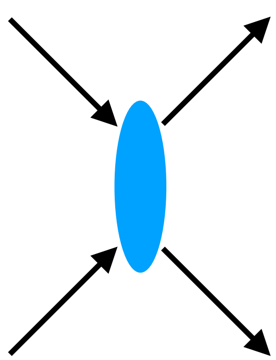
New particles in the QCD spectrum

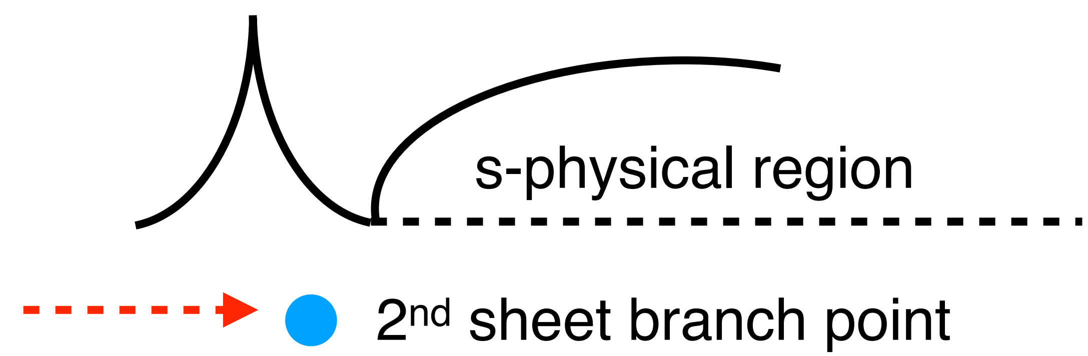




Other effects can be "generated" exchanges forces" (*)

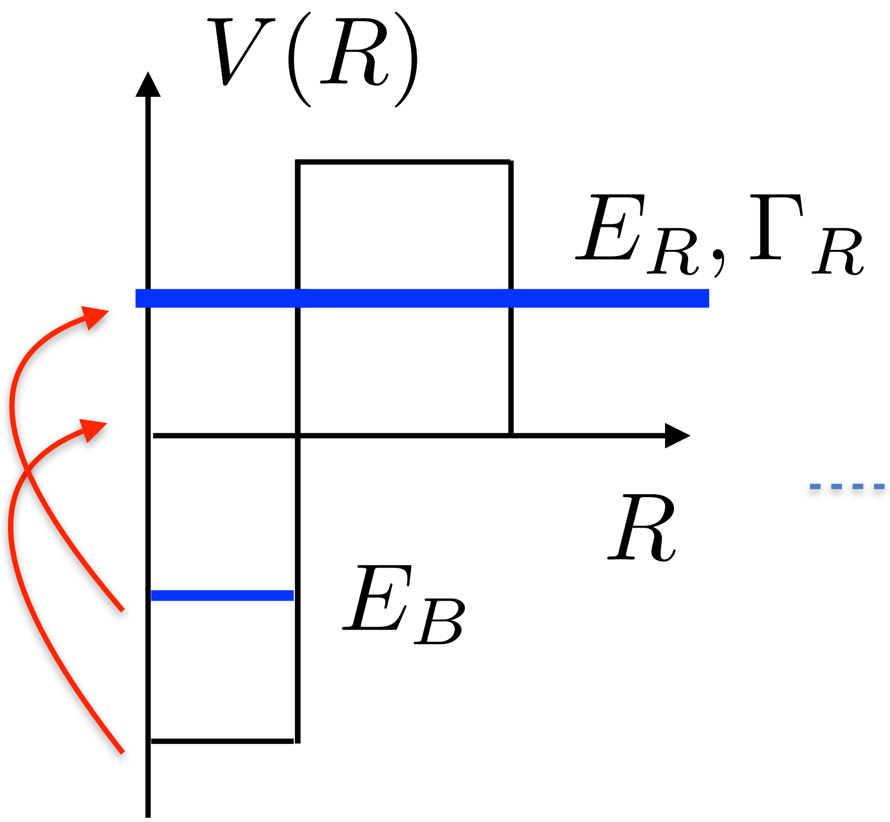
The "interesting stuff" happens on unphysical sheets. When singularities are close to the physical region rapid variations in amplitudes (cross sections) appear

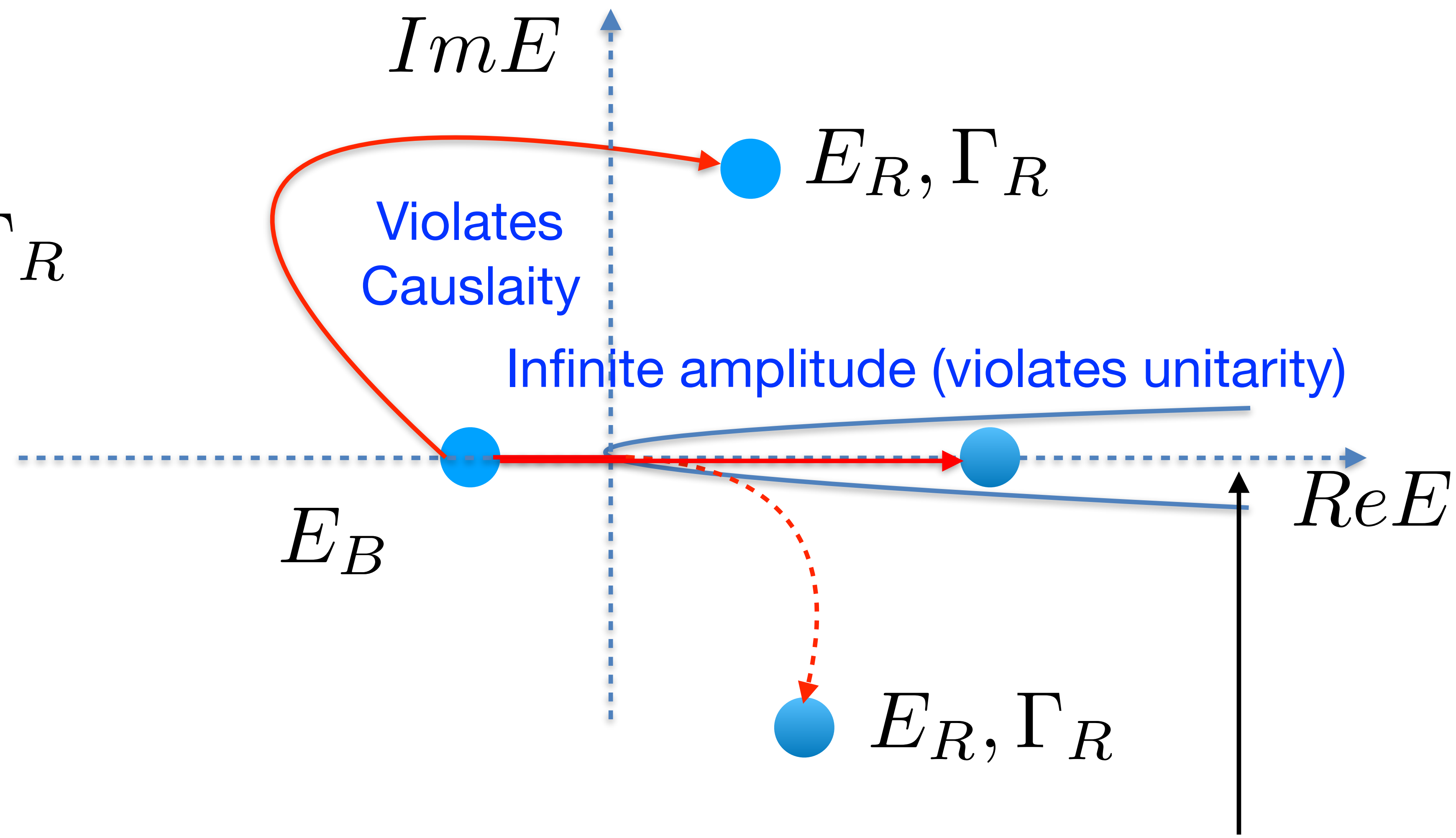




(*) (?) singularities because of confinement

Anatomy of resonances





When strength of interactions is reduces bound states become resonances

The only place for bound stets pole to migrate is onto an unphysical sheet connected to the open channel branch point

S-matrix formalism to 3 hadron interactions

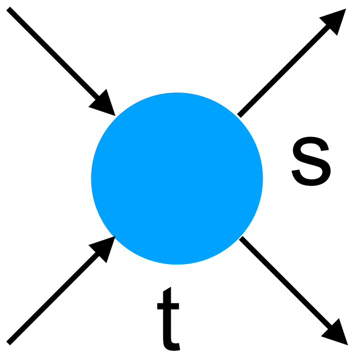
Properties of reaction amplitudes are determined by

Causality: Reaction amplitudes are smooth (analytical) functions of kinematical variables with singularities reflecting existence of constraints (laws)

Unitarity: Determines singularities.

Crossing: Dynamical relations: reaction amplitudes in the exchange channel (forces) are analogous to amplitude in the direct channel (resonance)

2-to-2 scattering



$$A(s,t) = \sum_l (2l+1) a_l(s) P_l(z_s)$$

Analysis of $a_l(s)$ model dependent: Testing hypothesis

- Dispersion realtion (Analiticity)

- Unitarity …

$$a_l(s) = B_l(s) + \frac{1}{\pi}\int_{tr} ds' \frac{Im a_l(s')}{s'-s} \longleftarrow |a(s'+i\epsilon)|^2 \rho(s')$$

…implies a relation between Re a_l and Im a_l

- B_l from cross channel interactions. Given B_l (e.g. for elastic ππ scattering it can be related to $a_l(s)$'s. Unitarity "converts" a dispersion relation to an integral equation (eg Roy eq.) for the partial waves). Solution is non unique — there are resonances/bound states not constrained by "exchange forces"

Real axis vs complex plane

To look for poles need to continue though the unitary cut

$$a_l(s) = B_l(s) + \frac{1}{\pi}\int_{tr} ds' \frac{Im a_l(s')}{s' - s}$$

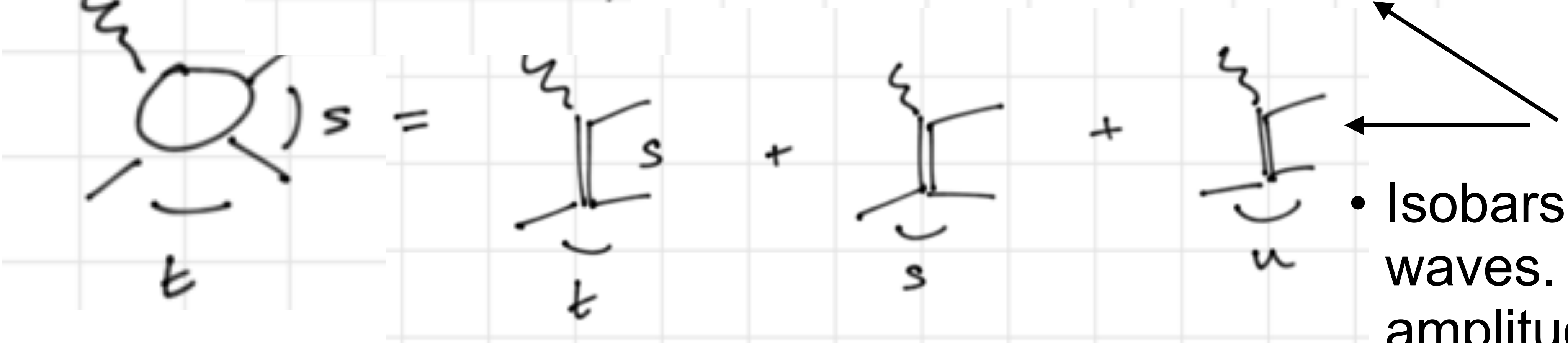
$$\downarrow$$

$$a_l^{II}(s) = a_l(s) + 2i Im a_l(s)$$

- (All) that is needed is an analytical continuation of the phase shift ($a_l(s)$ on the real axis)
- In a typical application of Roy eq., Im a_l does not correspond to limit of an analytical function. Thus what is quoted is not the “true pole” (but the probably a value close to it)

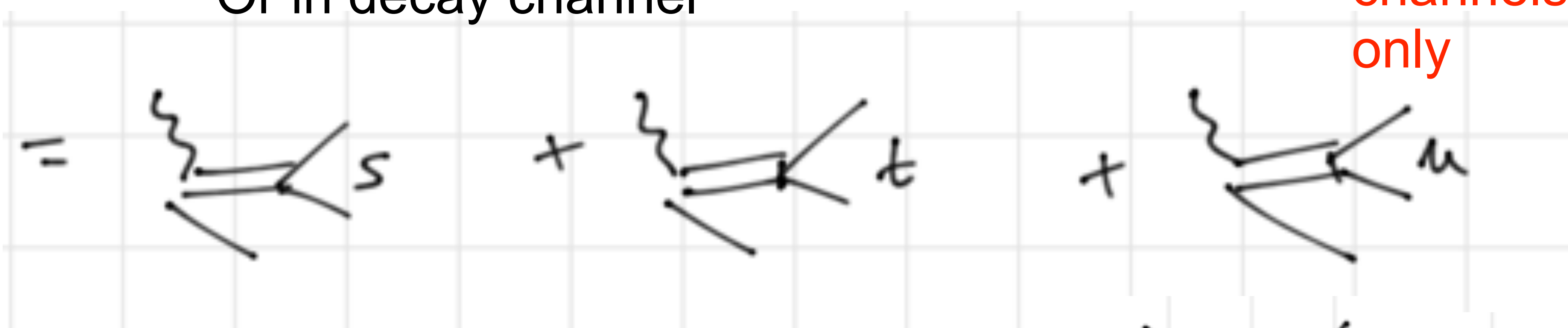
Beyond 2 particle production Khuri-Treiman (KT) model

$$A(s,t,M^2) = a(t,M^2) + a(s,M^2) + a(u,M^2)$$

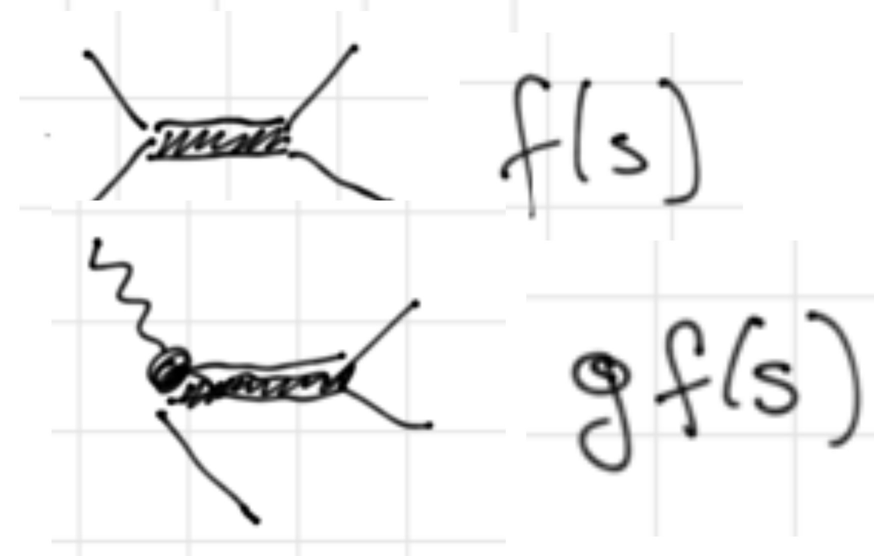


- Isobars are not partial waves. They are model amplitudes with direct channels branch points only

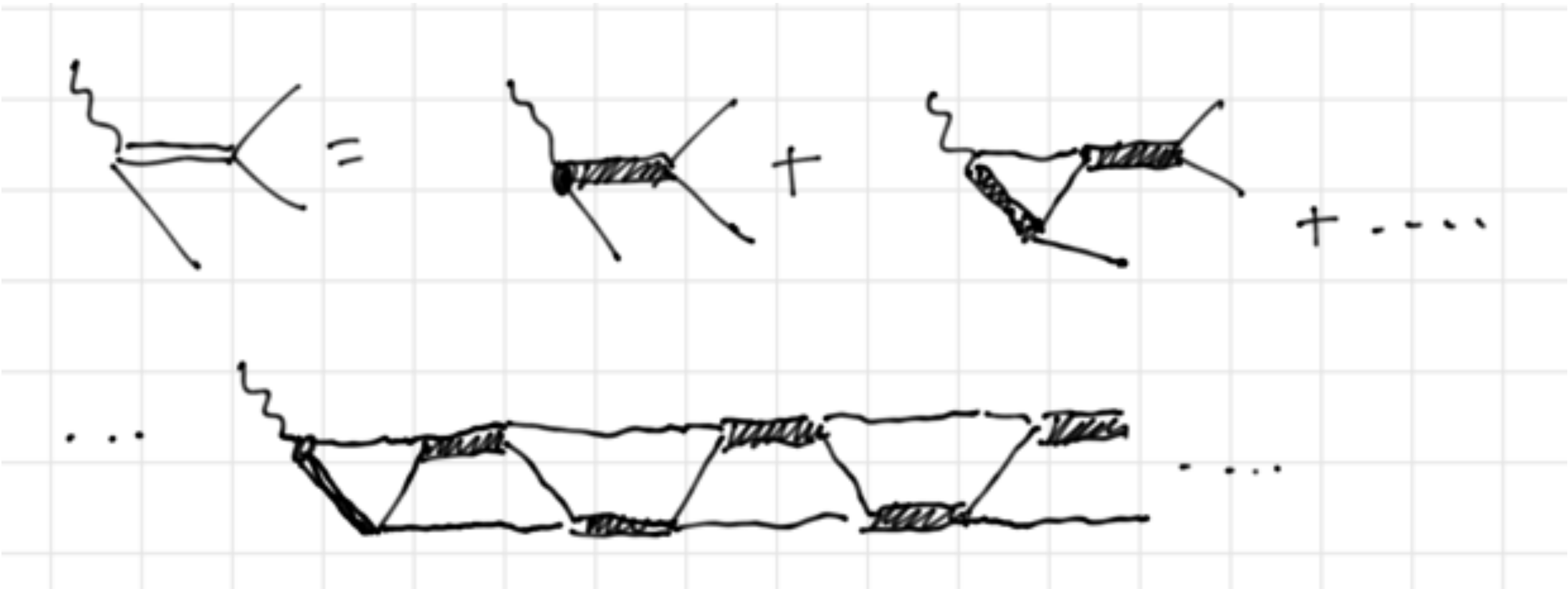
Or in decay channel



- Isobars to be determined in terms of 2-to-2 scattering amplitude and a coupling, g to the production channel (g=1)
- unitarity : t-channel process acts as a "driving" term for s-channel isobar



KT equations

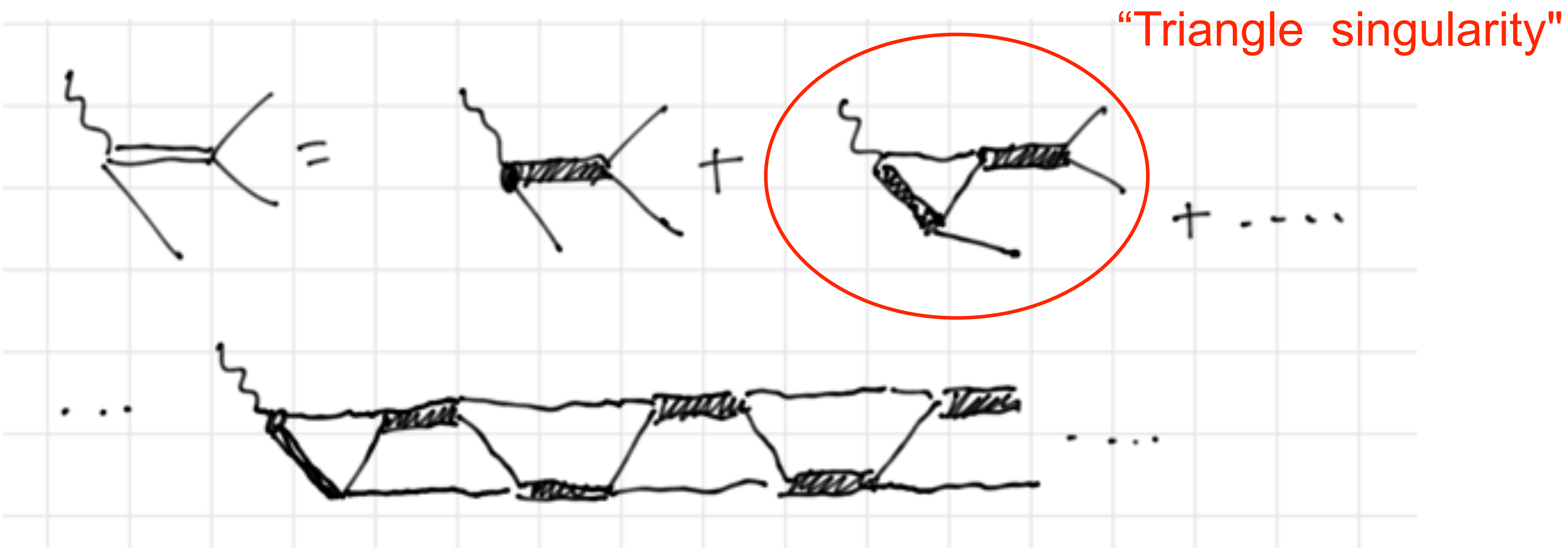


Isobar amplitudes have only the
normal threshold singularities

$$a(s, M^2) = f(s) + f(s) \int_{tr} \frac{ds'}{\pi} \frac{\rho(s')b(s', M^2)}{s' - s}$$

$$b(s, M^2) = \int_{-1}^{1} dz a(t(s, z, M^2), M^2)$$

KT equations



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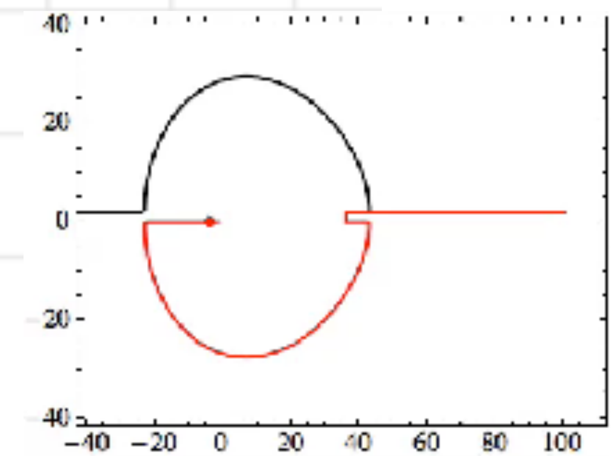
KT equations

"Triangle singularity"



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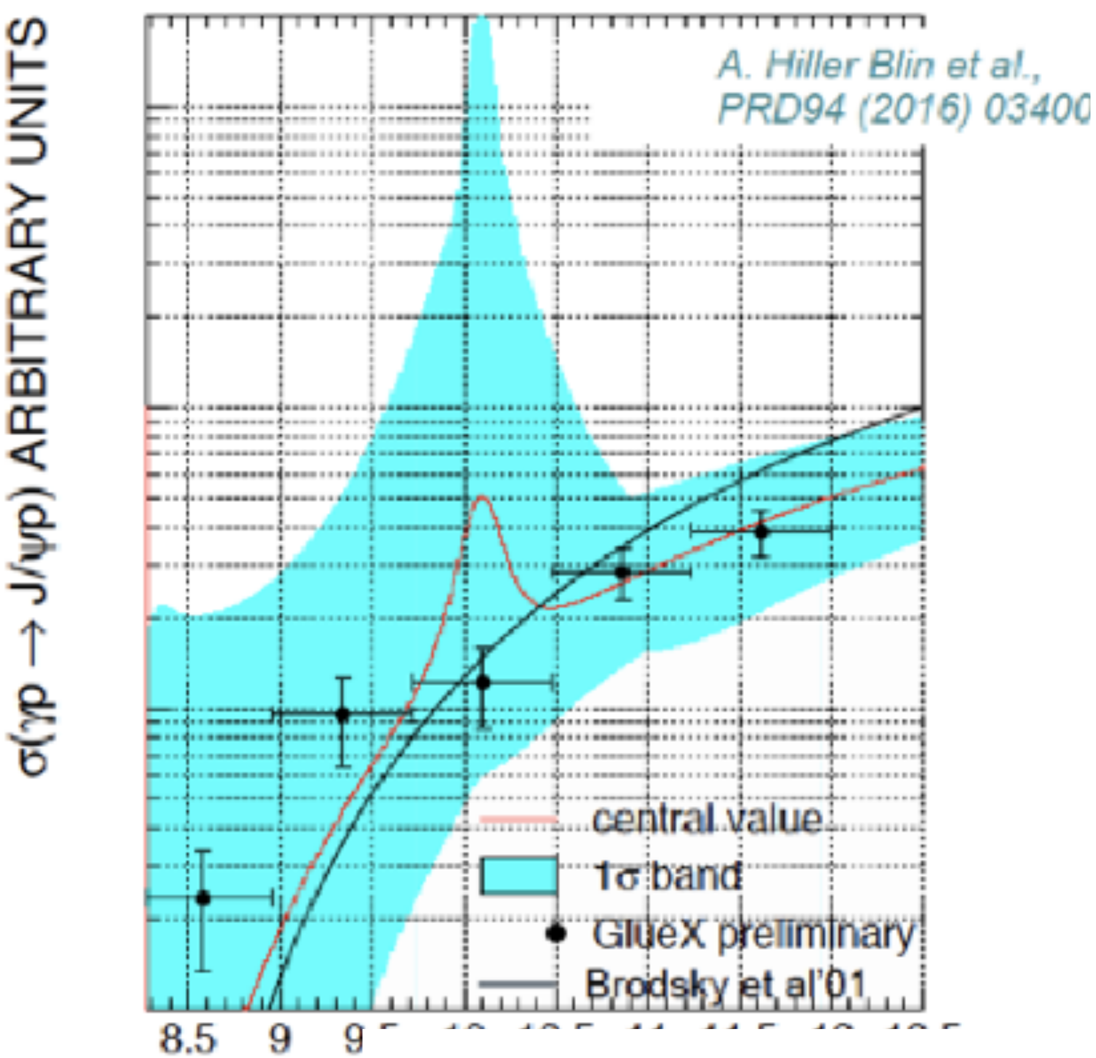
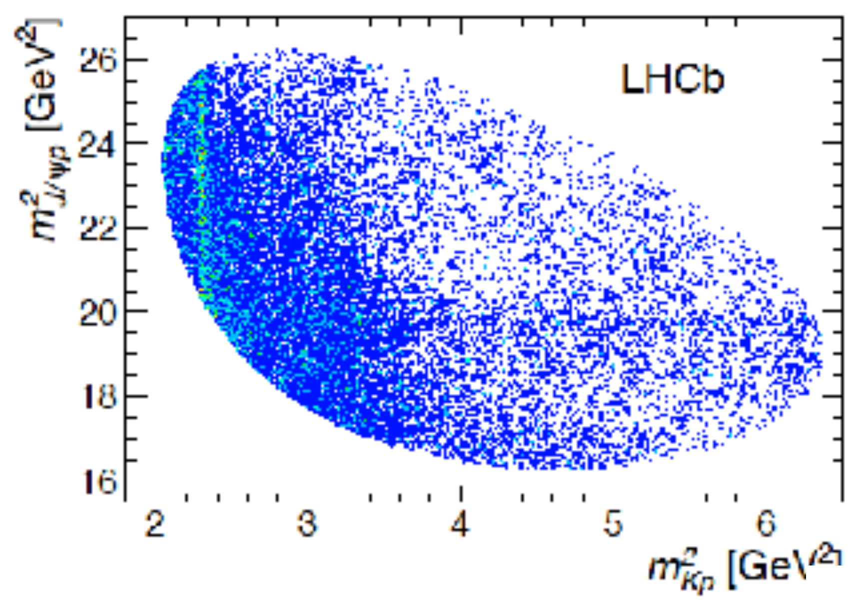
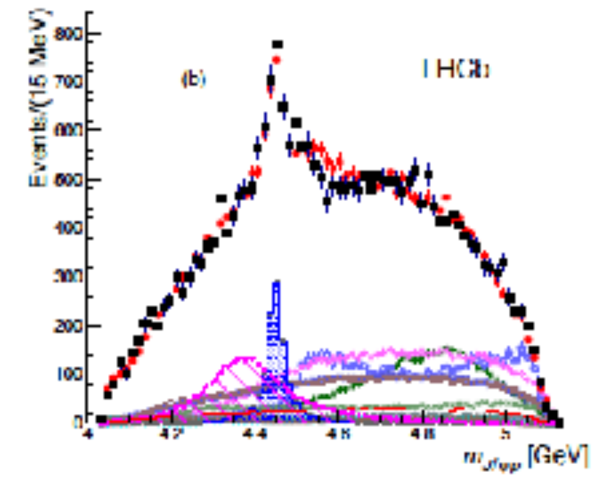
$$a(s,M^2) = f(s) + f(s)\int_{tr} \frac{ds'}{\pi}\frac{\rho(s')b(s',M^2)}{s'-s}$$

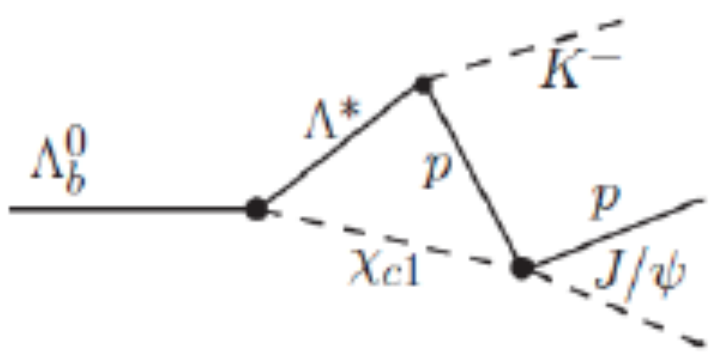
$$b(s,M^2) = \int_{-1}^{1} dz a(t(s,z,M^2),M^2)$$

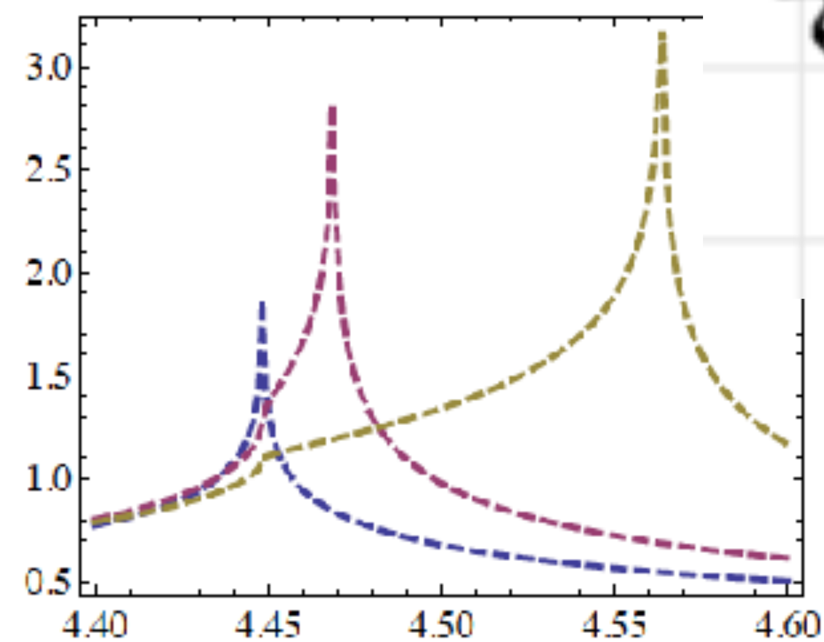
Pentaquark as a triangle singularity ?







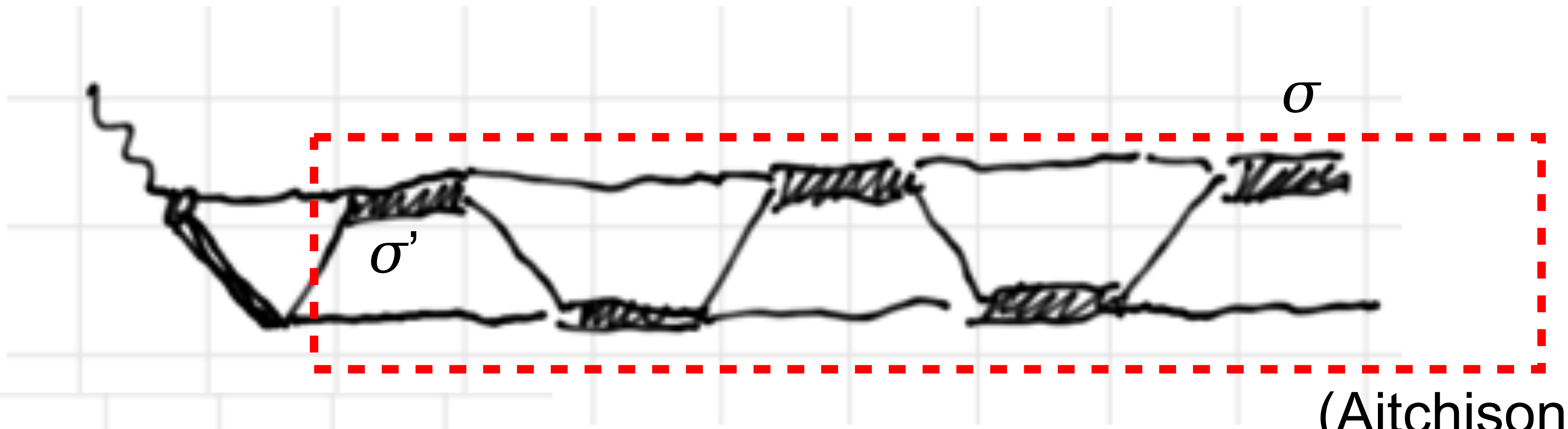




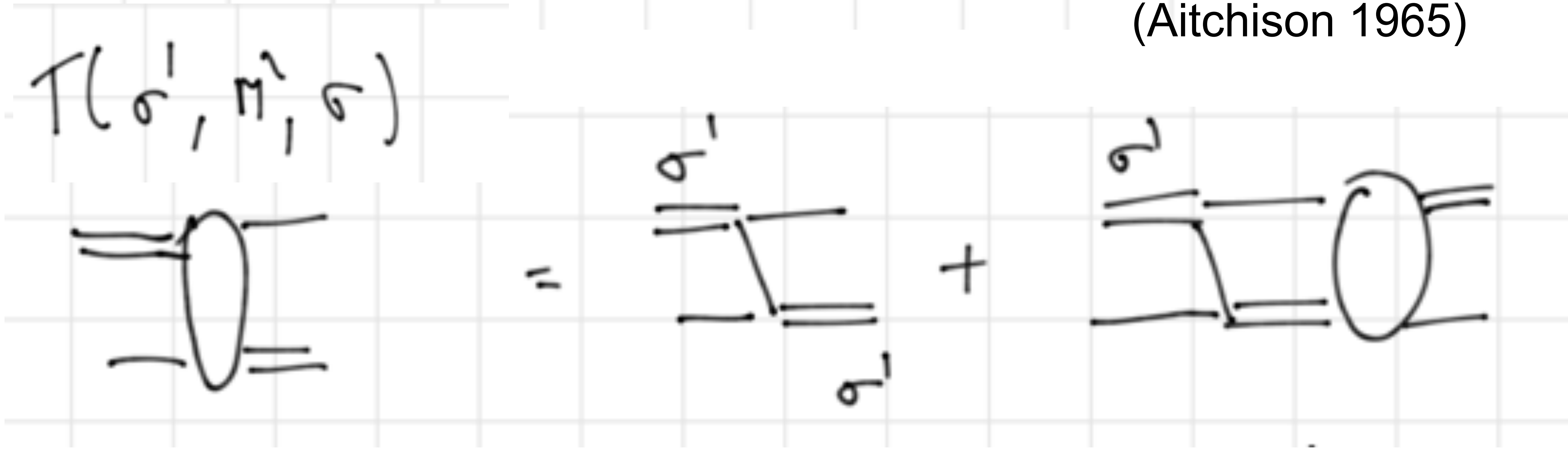
Axes Abs[$T(s)$], $\sqrt{s}$

Lines : blue (λ – 1.89 GeV), red (λ – 1.99 GeV), yellow (λ – 2.09 GeV)

3-to-3 amplitude from a 2-to-2 (1-to-3) KT model



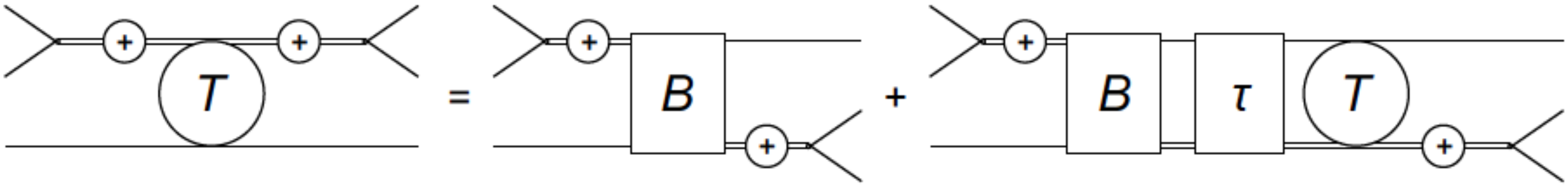
(Aitchison 1965)



$$T(\sigma', M^2, \sigma) = B(\sigma', M^2, \sigma)$$

$$+ \int d\sigma'' B(\sigma', M^2, \sigma'')\tau(\sigma'', M^2)T(\sigma'', M^2, \sigma)$$

Relation to other approaches



$$T(q,p;s) = B(q,p;s) - \int \frac{d^3 l}{(2\pi)^3} B(q,l;s) \frac{\tau(\sigma(l))}{2E(l)} T(l,p;s)$$

$$T_{3\to 3}(\sigma', s, \sigma) = \sum_{\sigma''} \left[\frac{1}{1 - \tau(s) B(s)} \right]_{\sigma', \sigma''} [B(s)]_{\sigma'', \sigma}$$

M. Mai et al. (JPAC)
I.Aitchison (Khuri-Treiman)
H.Hammer et al. (EFT)

→

“ϱ vs Chew-Madelstam” phase space (properly removes unphysical singularities from ϱ)

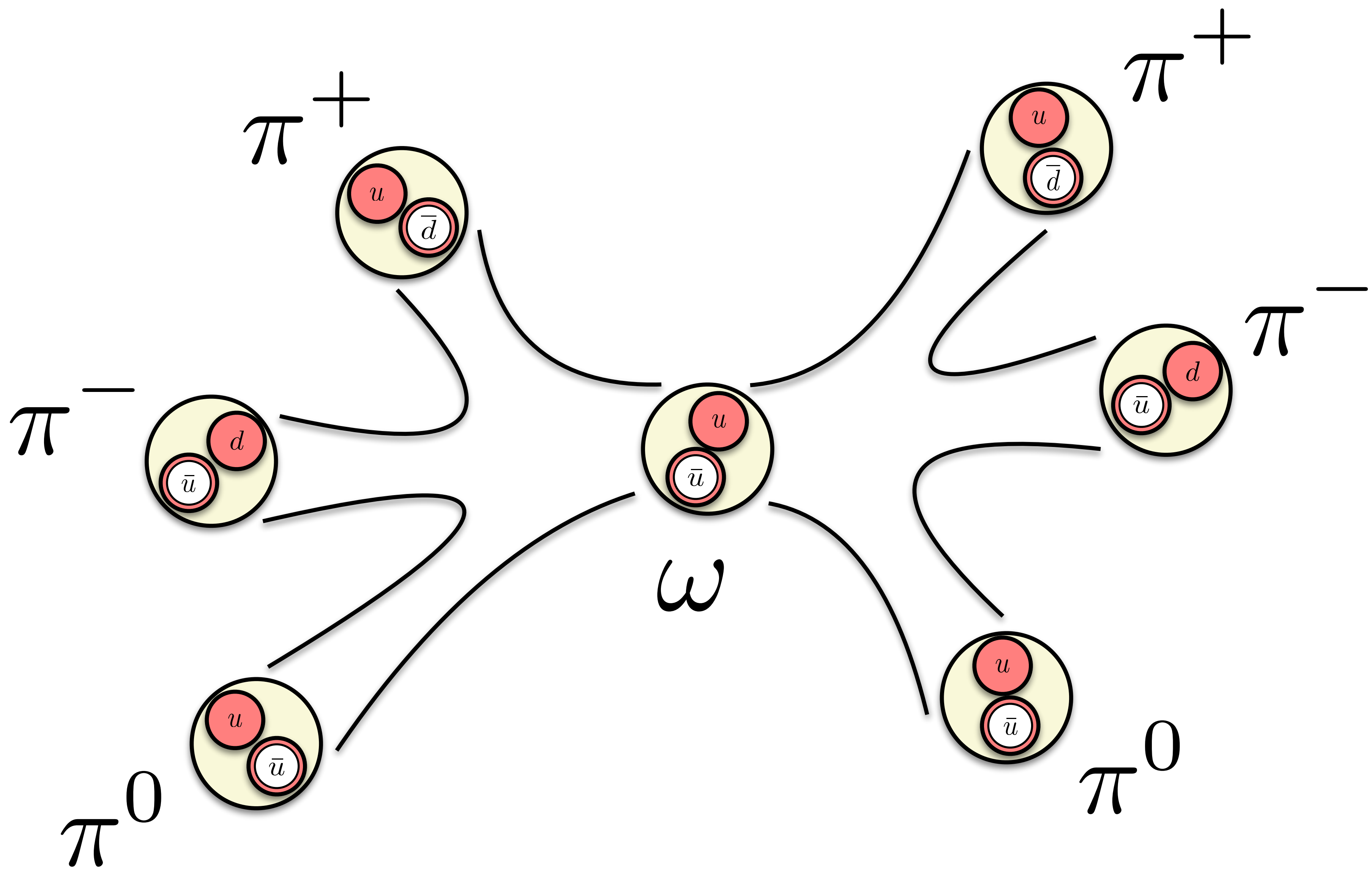
- Reproduces 3-to-3 unitarity on the real axis only
- Analyticity in sub-channel variables ?

$$T = \frac{1}{K^{-1}(s) - i\rho(s)}$$

$$T = \frac{1}{K^{-1}(s) - \frac{1}{\pi} \int_{tr} ds' \frac{\rho(s')}{s' - s}}$$

Next step is to derive the proper dispersive representation

3-to-3 scattering from dispersion relations

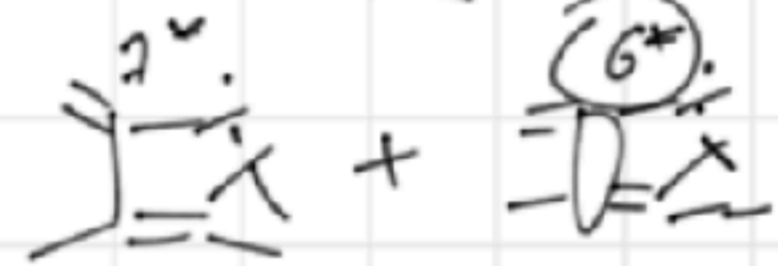


Special thanks to Andrew Jackura

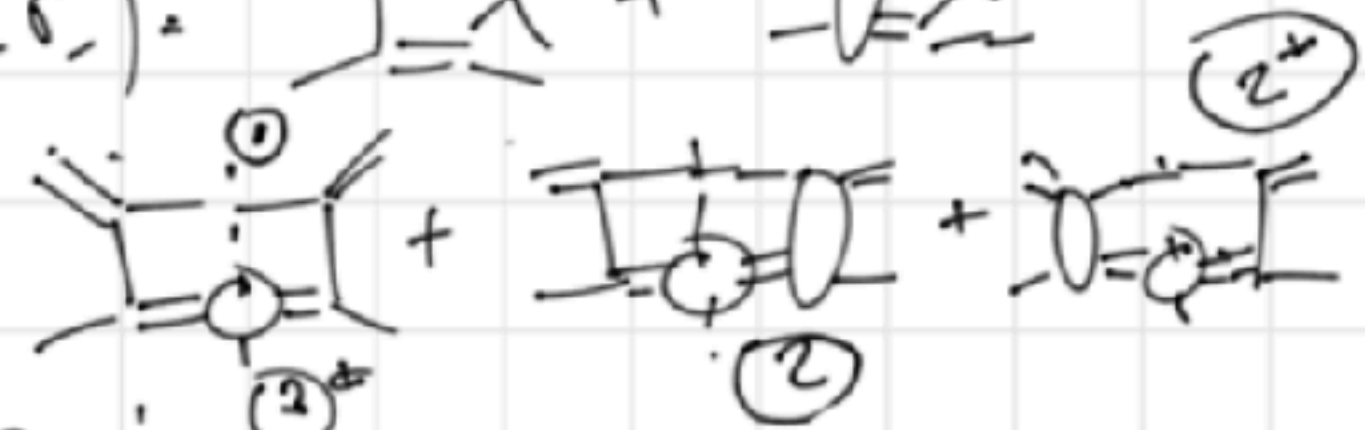
JPAC

$C_1(\sigma'_+ s_+ \sigma_+) - C_1(\sigma'_- s_+ \sigma_+) =$

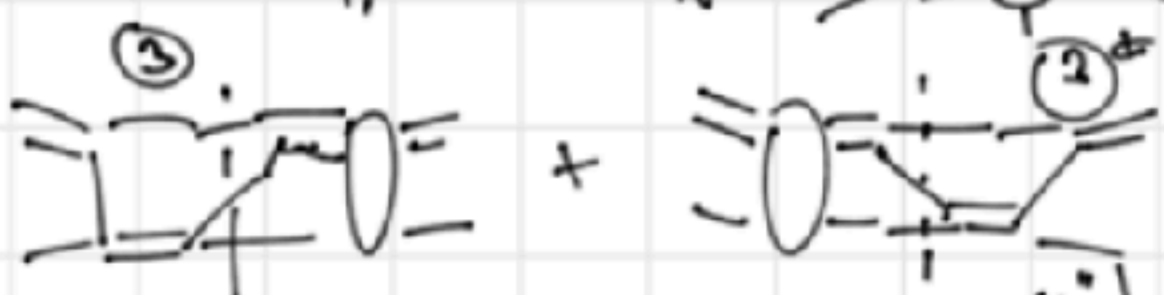
$C(\sigma'_- s_- \sigma_+) - C(\sigma'_- s_- \sigma_-) =$



$C_1(\sigma'_- s_+ \sigma_+) - C_1(\sigma'_- s_- \sigma_+) =$



$+$



$+$ Boxes

$(m_2+m_3)^2 < \sigma''_{23} < (\sqrt{s}-m_1)^2$

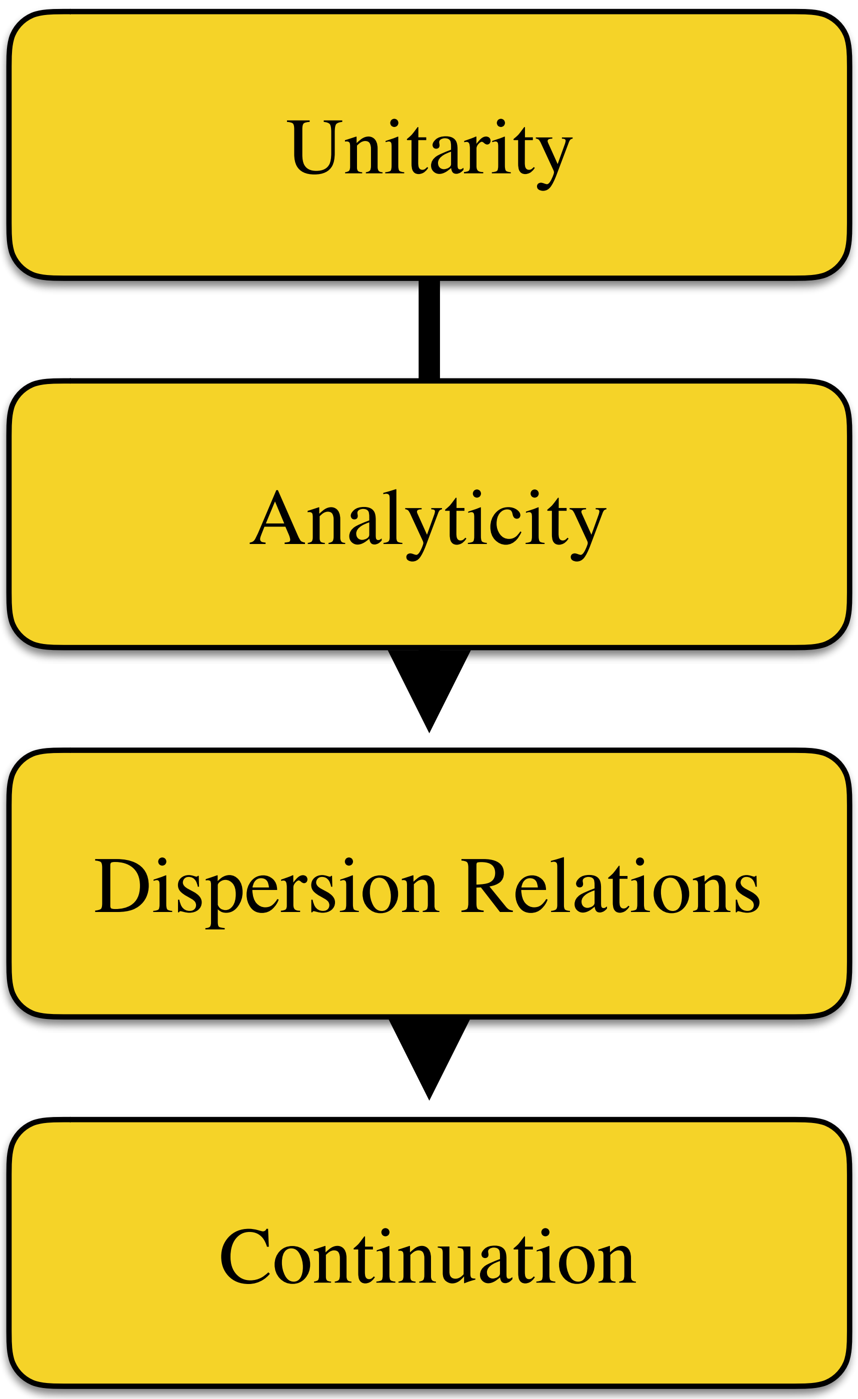
$$② = \frac{1}{2}\int d\sigma''_{23} \frac{\lambda^{1/2}(s,\sigma''_{23},m_1^2)}{4s} \int \frac{dz}{2} \left[\frac{1}{2\pi} \frac{N(\sigma'_{12})N(\sigma''_{23})}{\mu^2 - t(z,s,\sigma'_{12},\sigma''_{23}) + i\epsilon} \right] \ln\left(\frac{1}{D(\sigma''_{23})}\right) C_{3,1}(\sigma''_{23+}, s_+ \sigma_{12+})$$

s_- ; real

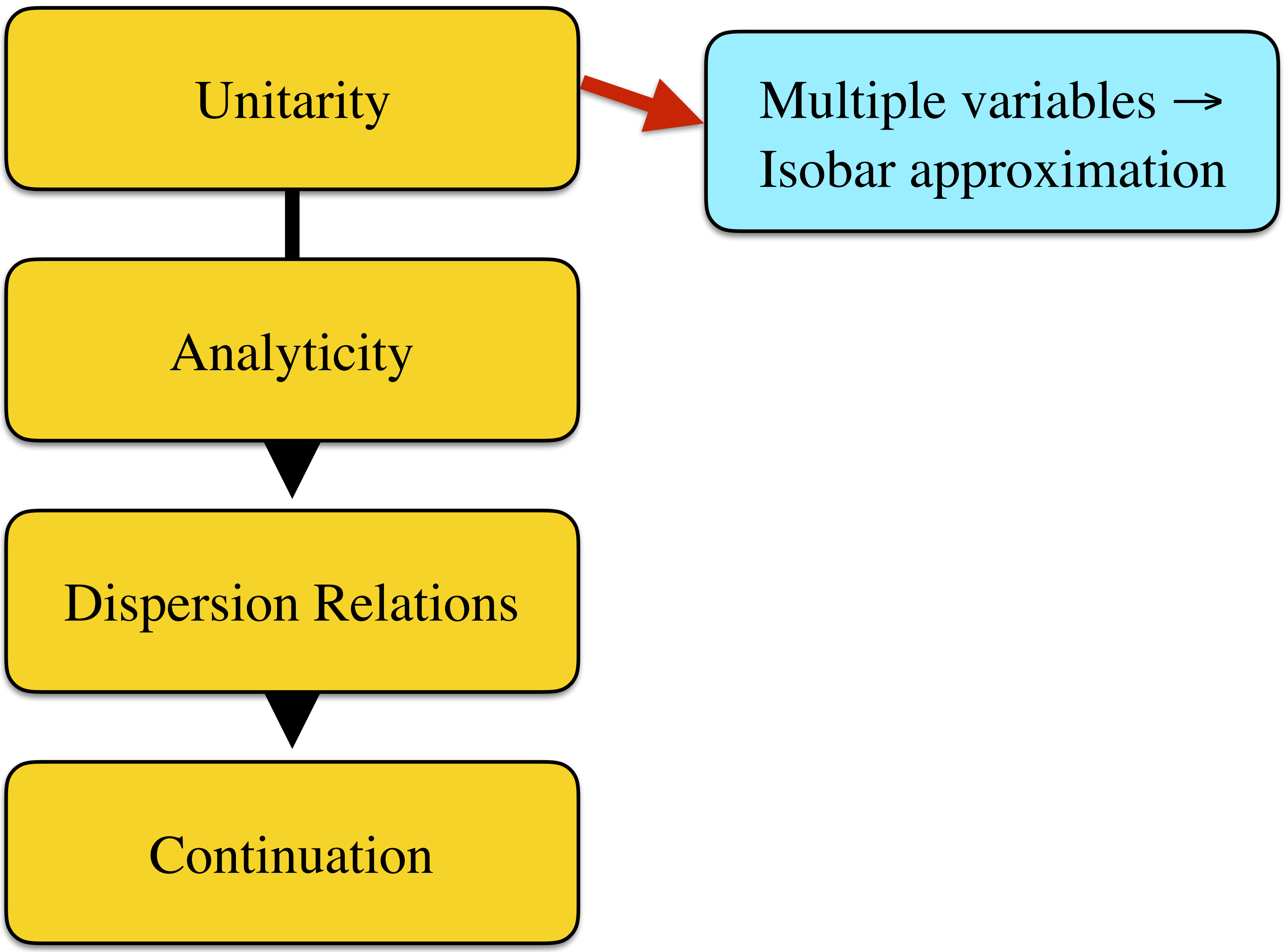
no singularities in σ'_{12-}

$$① = \frac{1}{2}\int d\sigma''_{23} \frac{\lambda^{1/2}(s,\sigma''_{23},m_1^2)}{4s} \int \frac{dz}{2} \left[\frac{1}{2\pi} \frac{N(\sigma'_{12})N(\sigma''_{23})}{\mu^2 - t(z,s,\sigma'_{12},\sigma''_{23}) + i\epsilon} \right] \ln\left(\frac{1}{D(\sigma''_{23})}\right) \left[\int \frac{dz}{2} \frac{N(\sigma''_{23})N(\sigma_{12})}{\mu^2 - t(z,s,\sigma''_{23},\sigma_{12}) - i\epsilon} \frac{1}{2\pi} \right]$$

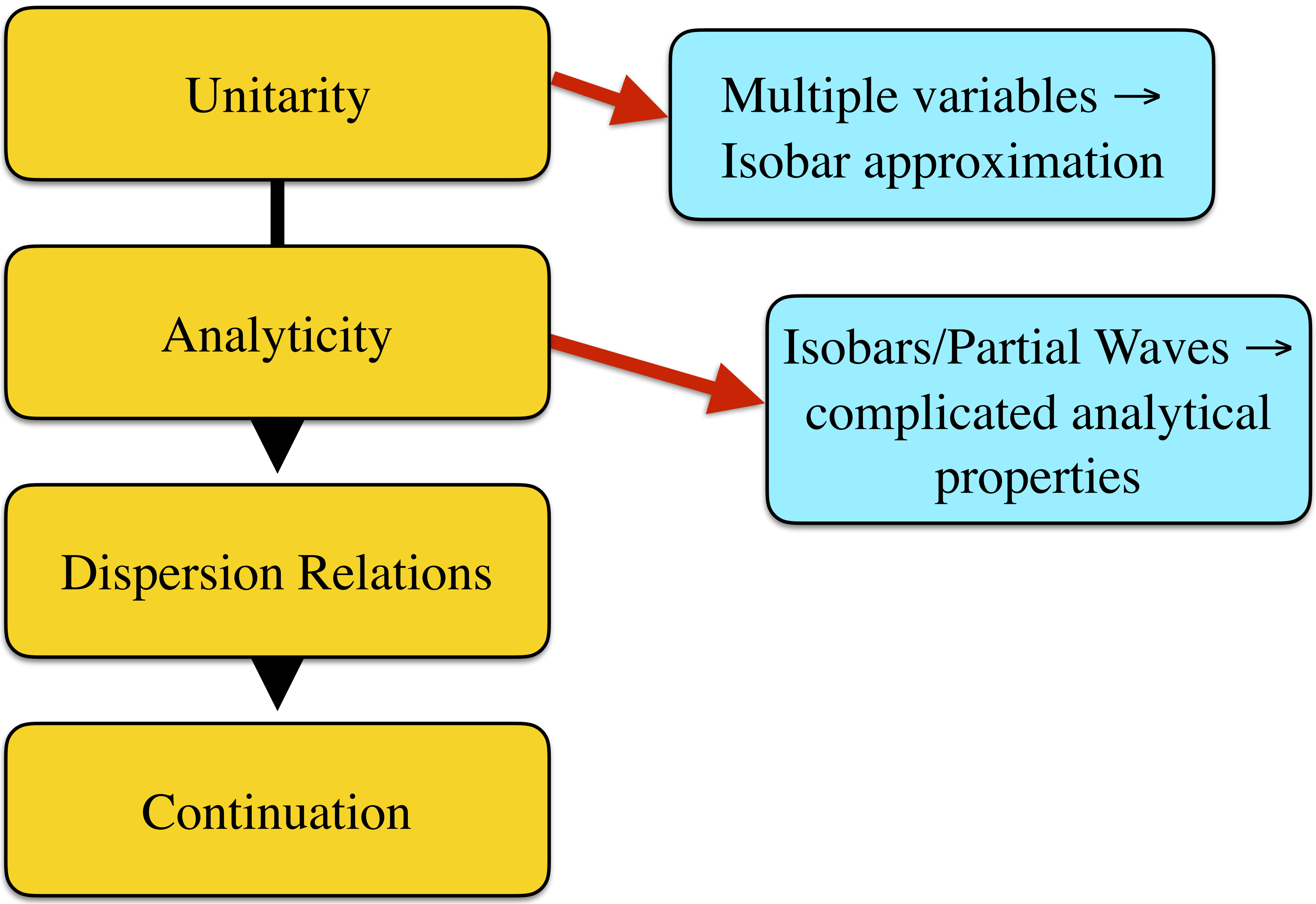
Workflow


Unitarity
Analyticity
Dispersion Relations
Continuation

Workflow


Unitarity
Multiple variables →
Isobar approximation
Analyticity
Dispersion Relations
Continuation

Workflow


Unitarity
Analyticity
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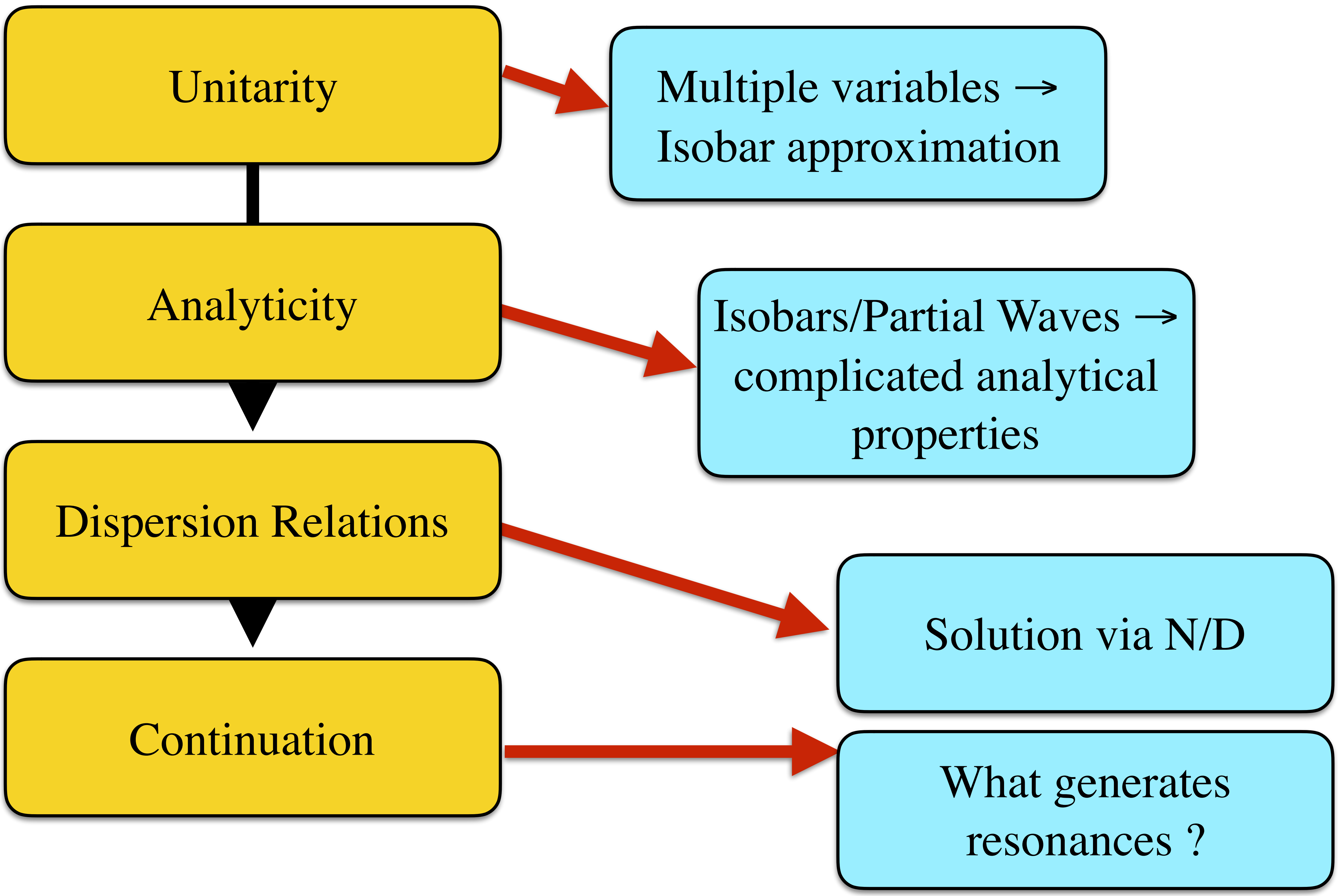
Multiple variables → Isobar approximation
Isobars/Partial Waves → complicated analytical properties

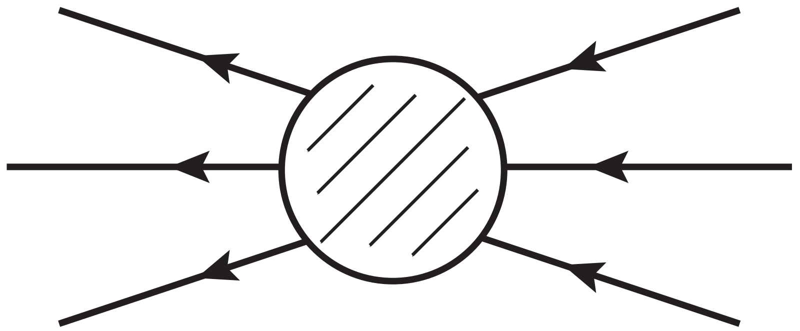
Workflow



Unitarity
Analyticity
Dispersion Relations
Continuation
Multiple variables → Isobar approximation
Isobars/Partial Waves → complicated analytical properties
Solution via N/D
What generates resonances ?

3-to-3 elastic scattering

Consider the elastic scattering of the **3→3** system *123→123*, where *1*, *2*, and *3* are distinguishable particles



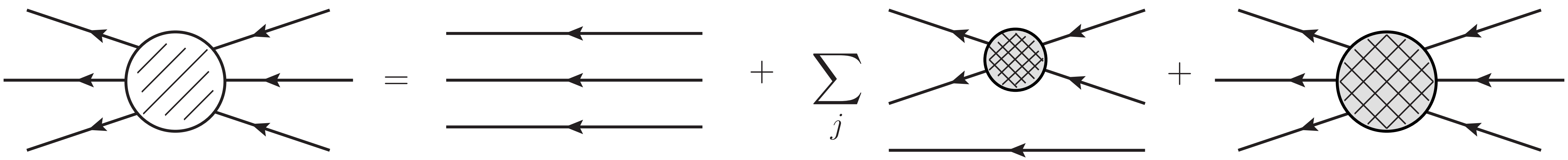
The S-matrix is decomposed as

$$
\begin{aligned}
\langle\{\mathbf{p}'\}|\, S \,|\{\mathbf{p}\}\rangle = {} & \langle\{\mathbf{p}'\}|\{\mathbf{p}\}\rangle \\
& + i\sum_j \widetilde{\delta}(p'_j - p_j)(2\pi)^4\delta^{(4)}(Q'_j - Q_j)\mathcal{F}_j(\{\mathbf{p}',\mathbf{p}\}_j) \\
& + i(2\pi)^4\delta^{(4)}(P' - P)\mathcal{A}(\{\mathbf{p}',\mathbf{p}\})
\end{aligned}
$$

Completely Disconnected

Disconnected

Connected



$\mathcal{F}_j(\{\mathbf{p}',\mathbf{p}\}_j)$ $\mathcal{A}(\{\mathbf{p}',\mathbf{p}\})$

Unitarity relations

$$1 = S^\dagger S \to 2Im(F+A) = (F+A)^\dagger P.S.(F+A)$$

Disconnected 2→2 Unitarity Relation

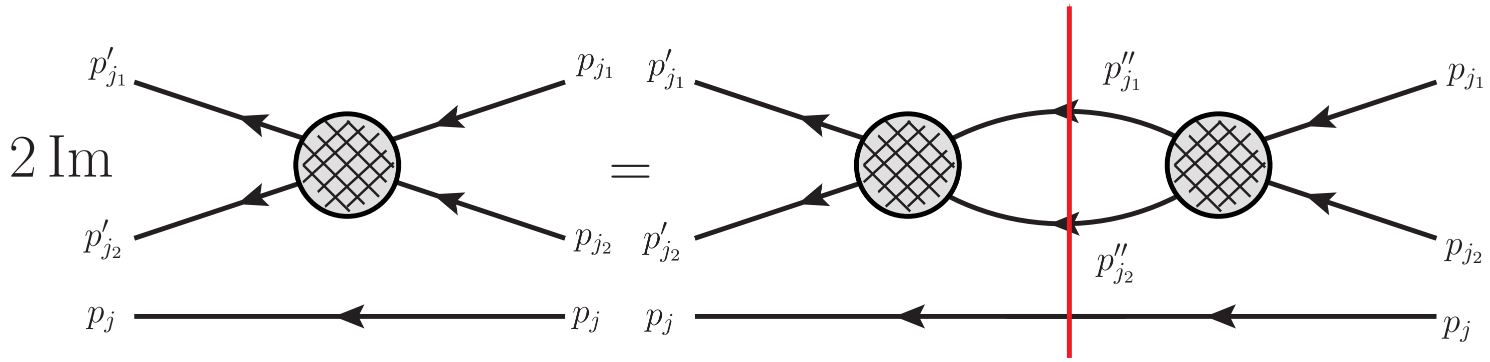
$$2\,\mathrm{Im}\;\mathcal{F}_j(\{\mathbf{p}',\mathbf{p}\}_j) = \rho_2(\sigma_j)\int d\Omega''_j\,\mathcal{F}^*_j(\{\mathbf{p}'',\mathbf{p}'\}_j)\mathcal{F}_j(\{\mathbf{p}'',\mathbf{p}\}_j)$$

Connected 3→3 Unitarity Relation

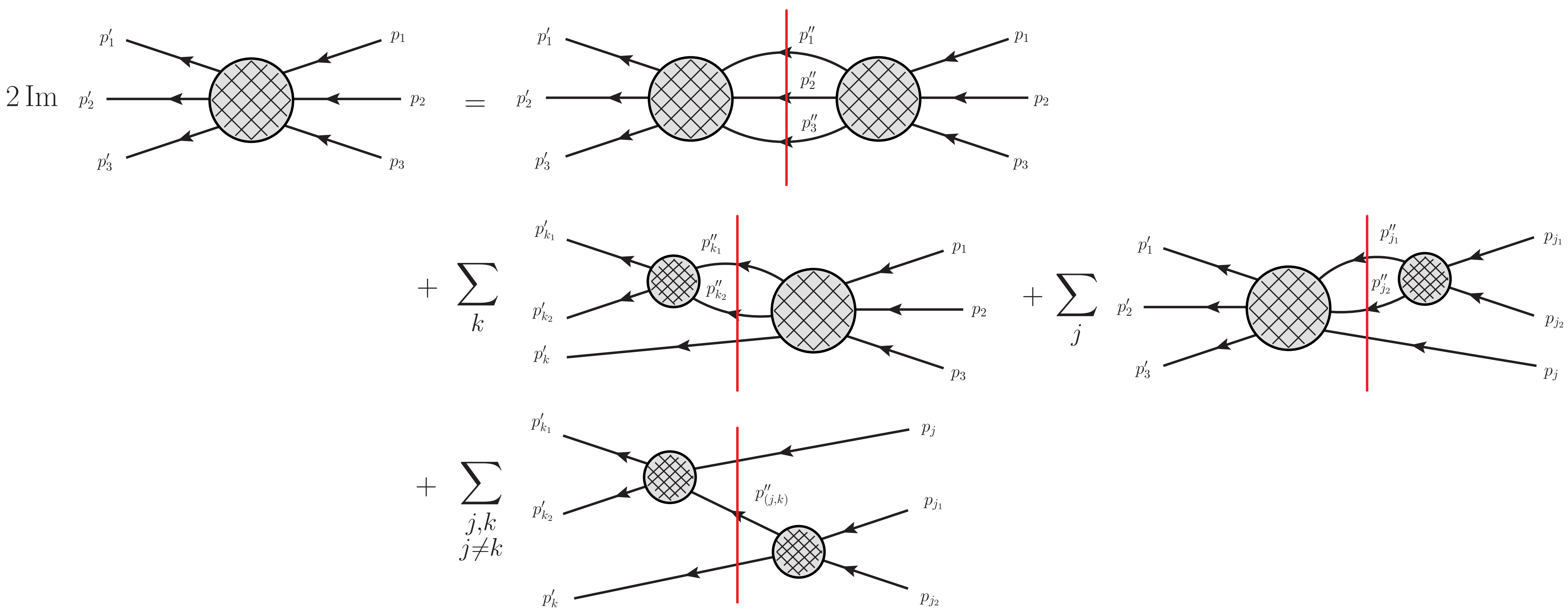
$$\begin{aligned}
2\,\mathrm{Im}\;\mathcal{A}(\{\mathbf{p}',\mathbf{p}\}) = &\int \widetilde{d}p''_1\widetilde{d}p''_2\widetilde{d}p''_3\,(2\pi)^4\delta^{(4)}(P''-P)\,\mathcal{A}^*(\{\mathbf{p}'',\mathbf{p}'\})\mathcal{A}(\{\mathbf{p}'',\mathbf{p}\}) \\
&+\sum_k \rho_2(\sigma'_k)\int d\Omega''_k\,\mathcal{F}^*_k(\{\mathbf{p}'',\mathbf{p}'\}_k)\mathcal{A}(\{\mathbf{p}'',\mathbf{p}\})\Theta(\sigma'_k-\sigma^{(k)}_{th}) \\
&+\sum_j \rho_2(\sigma_j)\int d\Omega''_j\,\mathcal{A}^*(\{\mathbf{p}'',\mathbf{p}'\})\mathcal{F}(\{\mathbf{p}'',\mathbf{p}\}_j)\Theta(\sigma_j-\sigma^{(j)}_{th}) \\
&+\sum_{\substack{j,k \\ j\neq k}} 2\pi\,\delta(u_{jk}-m^2_{(jk)})\,\mathcal{F}^*_k(\{\mathbf{p}'',\mathbf{p}'\}_k)\mathcal{F}_j(\{\mathbf{p}'',\mathbf{p}\}_j)
\end{aligned}$$

Unitarity relations

Disconnected 2→2 Unitarity Relation



Connected 3→3 Unitarity Relation



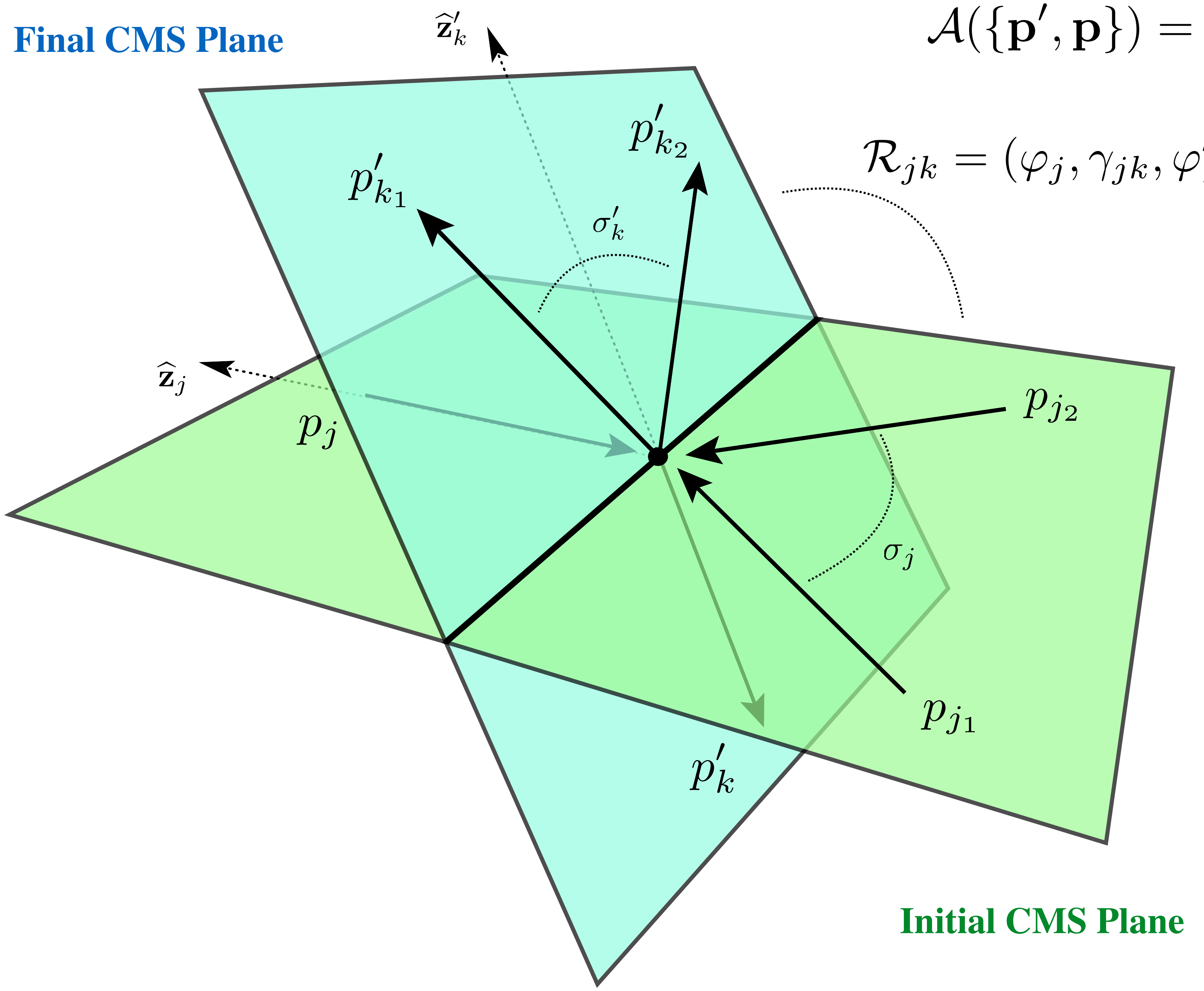
Too many variables …

3→3 amplitudes depend on 8 independent variables. One representation is

$$\mathcal{A}(\{\mathbf{p}', \mathbf{p}\}) = \sum_J \sum_{\lambda,\lambda'} \left(\frac{2J+1}{8\pi^2}\right) \mathcal{A}^J_{\lambda'\lambda}(\{E\}) \mathcal{D}^{(J)}_{\lambda'\lambda}(\mathcal{R})$$

Invariant energies — Euler angles

$$\{E\} = \{\sigma'_1, \sigma'_2, s, \sigma_1, \sigma_2\}$$



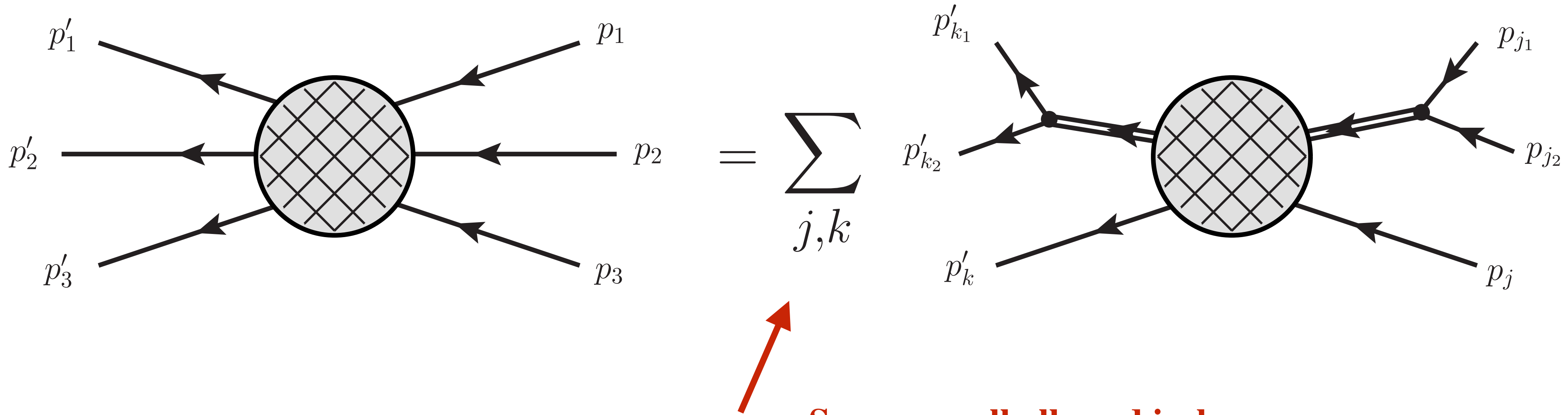
A choice of σ's selects an isobar e.g. σ'_1 is the inv. mass squared of (23) subsystem

Isobar-particle scattering

Assume that the amplitude can be expanded into *Isobar Amplitudes*

$$\mathcal{A}(\{\mathbf{p}', \mathbf{p}\}) = \sum_{j,k} \mathcal{A}_{kj}(\{\mathbf{p}', \mathbf{p}\}_{kj})$$

Two particles interact before interacting with spectator



Sum over all allowed isobars
Infinite sum analytically continued accounts for singularities in the other two σ variables

Isobar-particle scattering

Assume that the amplitude can be expanded into *Isobar Amplitudes*

$$\mathcal{A}(\{\mathbf{p}',\mathbf{p}\}) = \sum_{j,k} \mathcal{A}_{kj}(\{\mathbf{p}',\mathbf{p}\}_{kj})$$

act before
ctator

$$\mathcal{A} \to \sum_{J'\lambda' J\lambda'}^{J_{max}} A_{11}(\sigma_1', s, t_{11}, \sigma_1) Y^*_{J'\lambda'}(\Omega_1') Y_{J\lambda}(\Omega_1)$$

$$+ \sum_{\cdots}^{J_{max}} A_{21}(\sigma_2', s, t_{21}, \sigma_2) \cdots$$

Like in a KT model, adding truncated sums of particle-isobar amplitudes leads to a complicated sub-channel energy dependence (but with only normal threshold branch points)

Isobar-particel amplitude unitarity relations

These are coupled equations for each A_{ij}

Factorizes the sub-energy rescattering

$$\mathcal{A}_{kj}(\sigma'_k, s, t_{jk}, \sigma_j) = \frac{1}{D_k(\sigma'_k)} \widehat{\mathcal{A}}_{kj}(\sigma'_k, s, t_{jk}, \sigma_j) \frac{1}{D_j(\sigma_j)}$$

2→2 Rescattering

Still sub-energy dependence

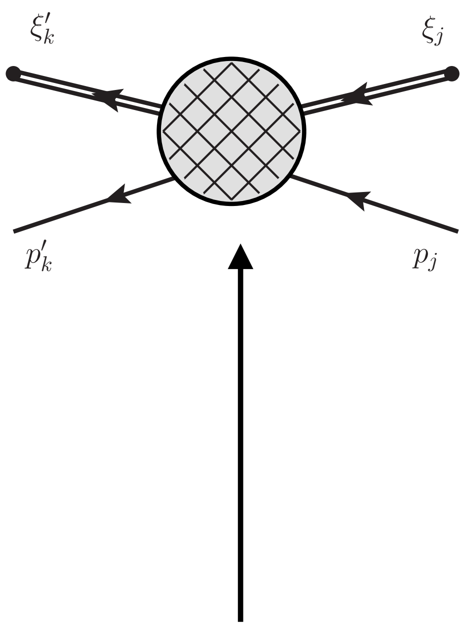
$$A_{11}(\sigma'_1, s, t_{11}, \sigma_1) = f_1(\sigma'_1)\hat{A}_{11}(\sigma'_1, s, t_{11}, \sigma_1) f_1(\sigma_1)$$

$$f_j(\sigma_j) = N_j(\sigma_j)/D_j(\sigma_j)$$

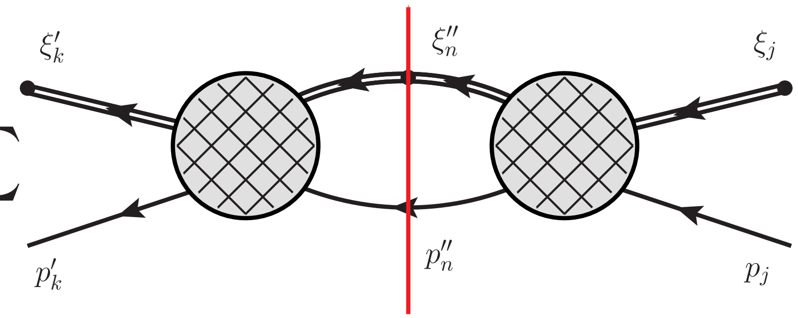
Isobar-particel amplitude unitarity relations

These are coupled equations for each A_{ij}

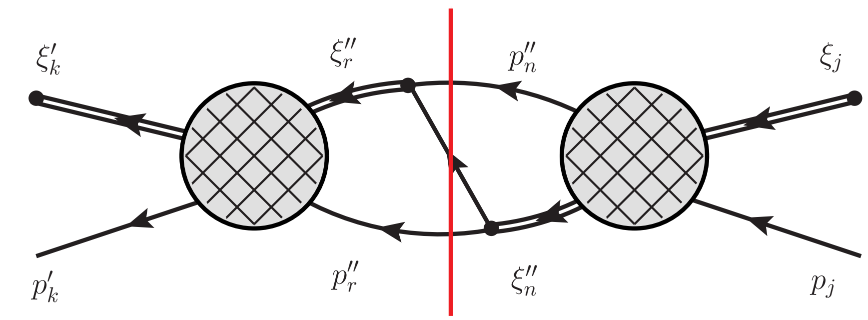
$2\,\mathrm{Im}$



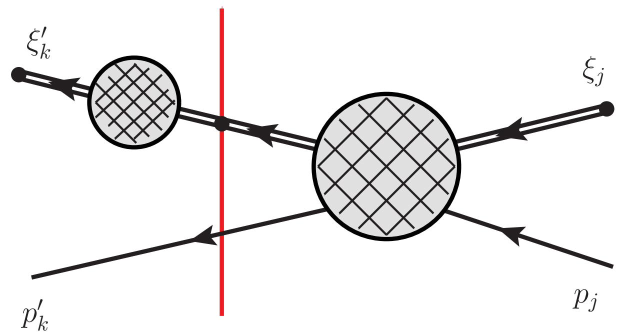
$= \sum_n$



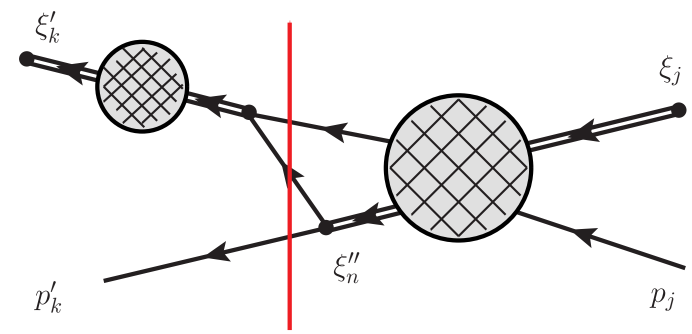
$+ \sum_{\substack{n,r \\ n\neq r}}$



$+$

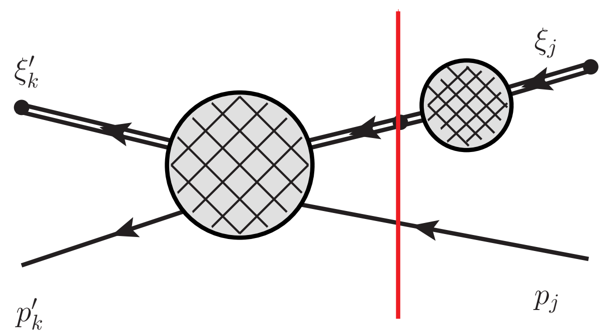


$+ \sum_{n\neq k}$

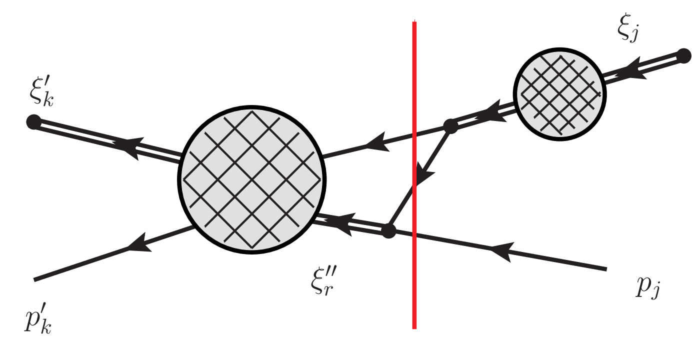


$$A_{11}(\sigma'_1, s, t_{11}, \sigma_1)$$
$$= f_1(\sigma'_1)\hat{A}_{11}(\sigma'_1, s, t_{11}, \sigma_1) f_1(\sigma_1)$$

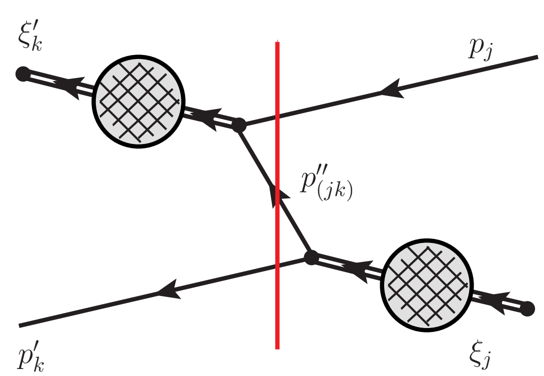
$+$



$+ \sum_{r\neq j}$



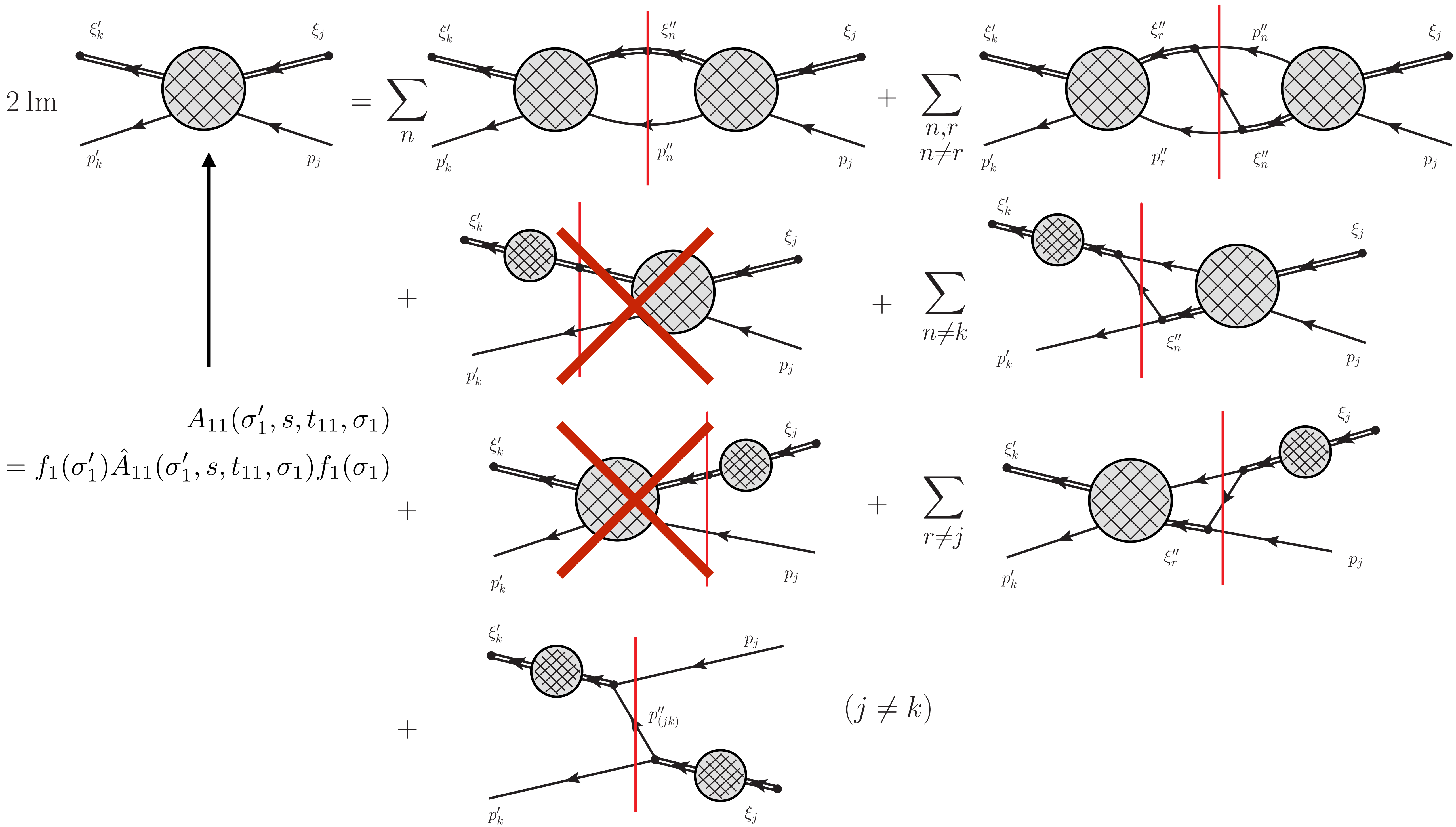
$+$



$(j \neq k)$

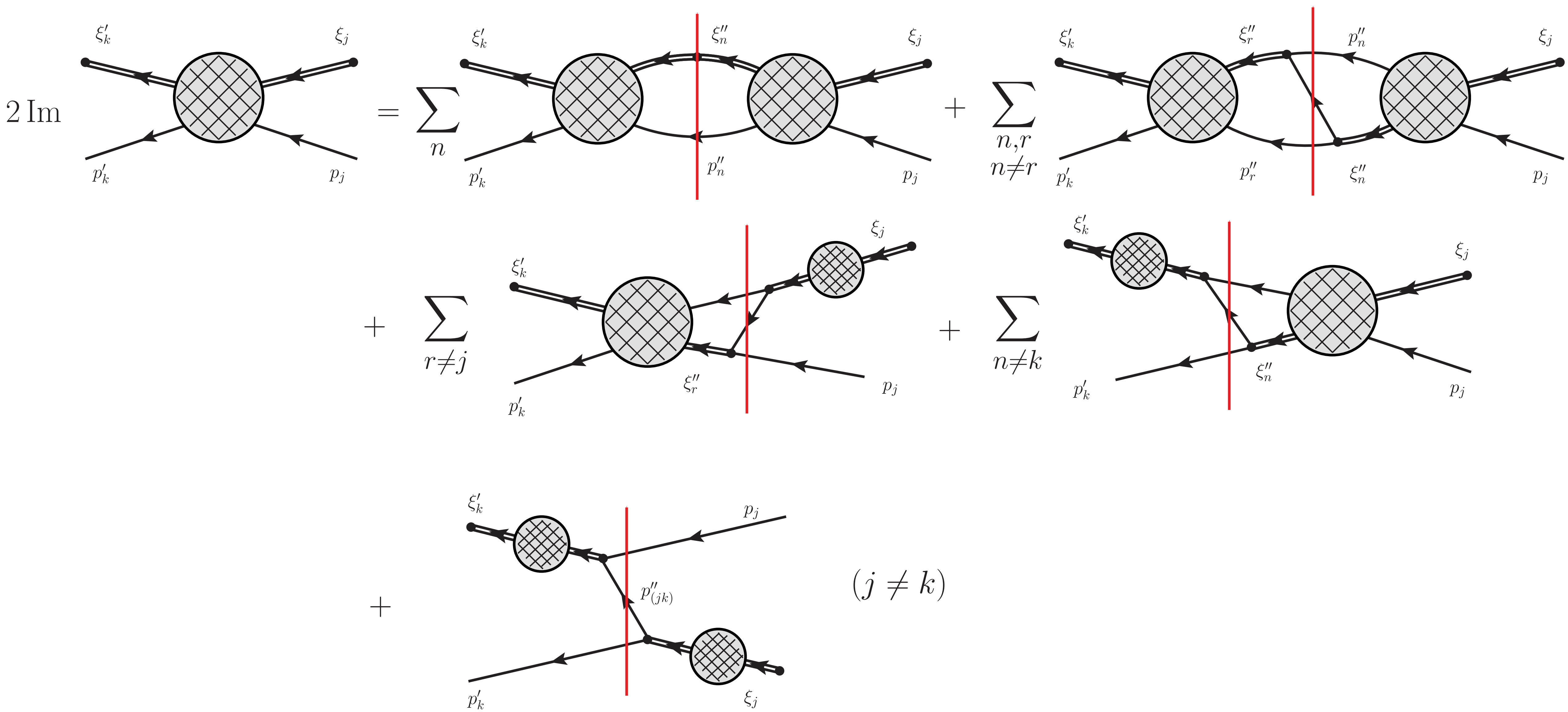
Isobar-particel amplitude unitarity relations

These are coupled equations for each A_{ij}



$$A_{11}(\sigma'_1, s, t_{11}, \sigma_1)$$
$$= f_1(\sigma'_1)\hat{A}_{11}(\sigma'_1, s, t_{11}, \sigma_1) f_1(\sigma_1)$$

Isobar amplitude unitarity relatons



Need to associate $Im\hat{A}_{ij}(\sigma'_i, s, u_{ij}, \sigma_j)$

with discontinuities across individual variables

From real axis (unitarity) to the complex plane

We can split the imaginary part into discontinuities across all variables

Need to be careful on which direction we approach the real axis from the complex planes

$$
\begin{aligned}
2i \ \mathrm{Im} \ \widehat{\mathcal{A}}_{kj}(\sigma'_k, s, t_{jk}, u_{jk}, \sigma_j) = & \ \Delta_{\sigma'_k} \widehat{\mathcal{A}}_{kj}(s_+, t_{jk+}, u_{jk+}, \sigma_{j+}) \\
& + \Delta_s \widehat{\mathcal{A}}_{kj}(\sigma'_{k-}, t_{jk+}, u_{jk+}, \sigma_{j+}) \\
& + \Delta_{t_{jk}} \widehat{\mathcal{A}}_{kj}(\sigma'_{k-}, s_-, u_{jk+}, \sigma_{j+}) \\
& + \Delta_{u_{jk}} \widehat{\mathcal{A}}_{kj}(\sigma'_{k-}, s_-, t_{jk-}, \sigma_{j+}) \\
& + \Delta_{\sigma_j} \widehat{\mathcal{A}}_{kj}(\sigma'_{k-}, s_-, t_{jk-}, u_{jk-})
\end{aligned}
$$

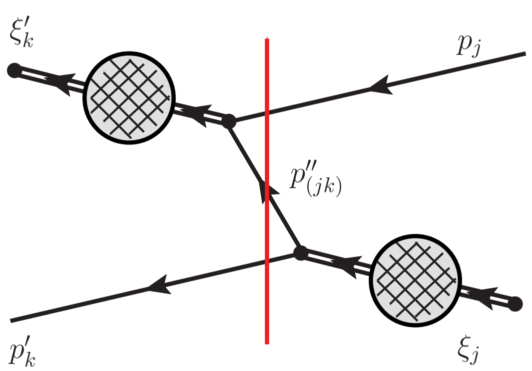
$$x_\pm = x \pm i\epsilon$$

From real axis (unitarity) to the complex plane

We can split the imaginary part into discontinuities across all variables

Need to be careful on which direction we approach the real axis from the complex planes

For $j \neq k$, have to worry about singularities in u_{jk} from One Particle Exchange (OPE)



$$(j \neq k) \quad \sim \delta(u_{jk} - m_{(jk)})$$

$$\ldots {}_{jk_+}, \sigma_{j_+})$$

$$\ldots, u_{jk_+}, \sigma_{j_+})$$

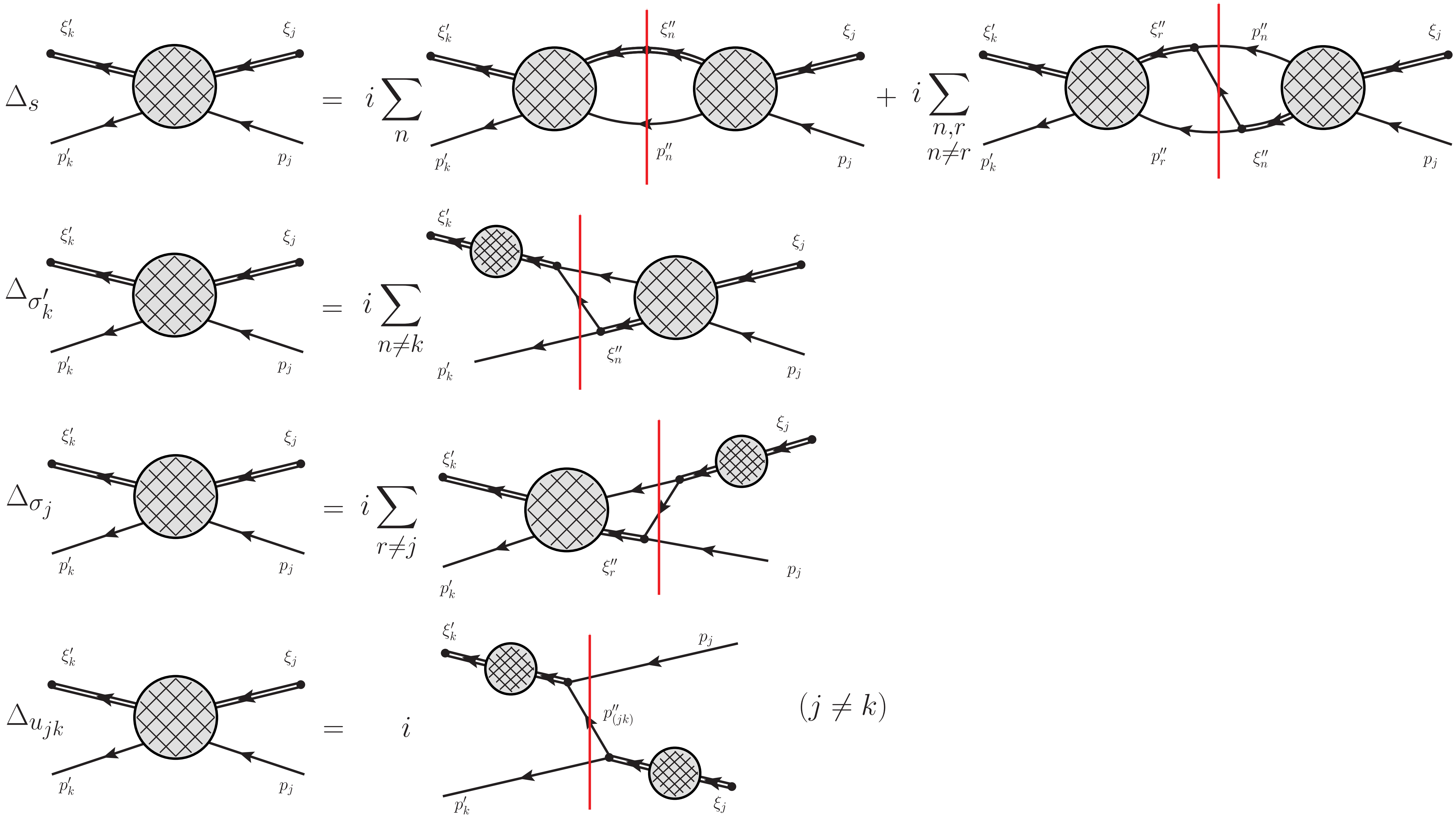
$$\ldots, u_{jk_+}, \sigma_{j_+})$$

$$\ldots, t_{jk_-}, \sigma_{j_+})$$

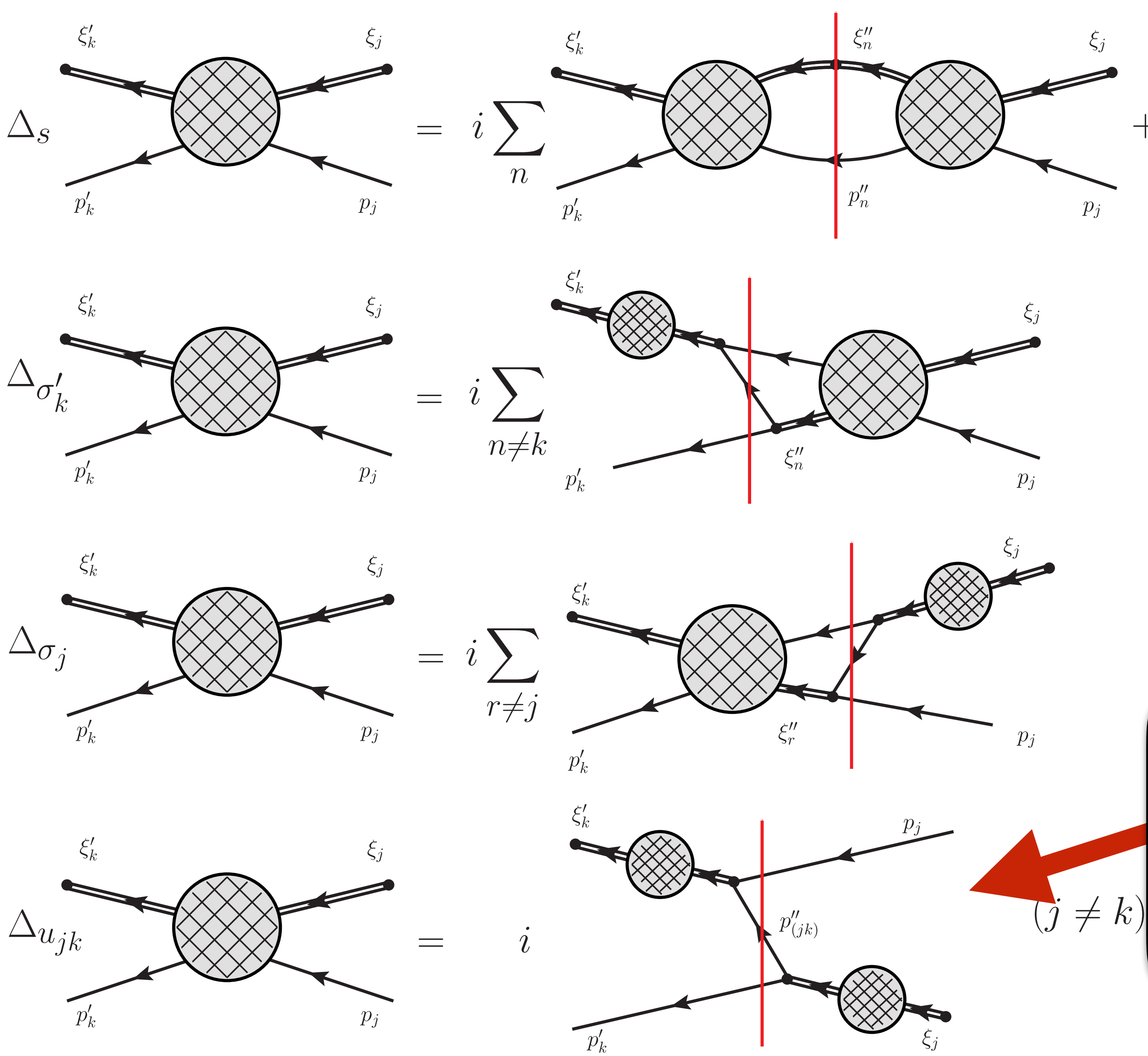
$$+ \Delta_{\sigma_j} \mathcal{A}_{kj}(\ldots_{k_-}, \ldots_-, t_{jk_-}, u_{jk_-})$$

$$x_\pm = x \pm i\epsilon$$

Single variable discontinuity reactions



Single variable discontinuity reactions



Will turn this into an s-channel cut via Partial Wave Projection

$$u_{jk} = u_{jk}(\sigma'_k, s, z_{jk}, \sigma_j)$$

Partial wave projection

We now want to consider partial wave projections of the amplitude

To simplify the expressions, let's consider the case for $J = 0$, and spin-0 isobars

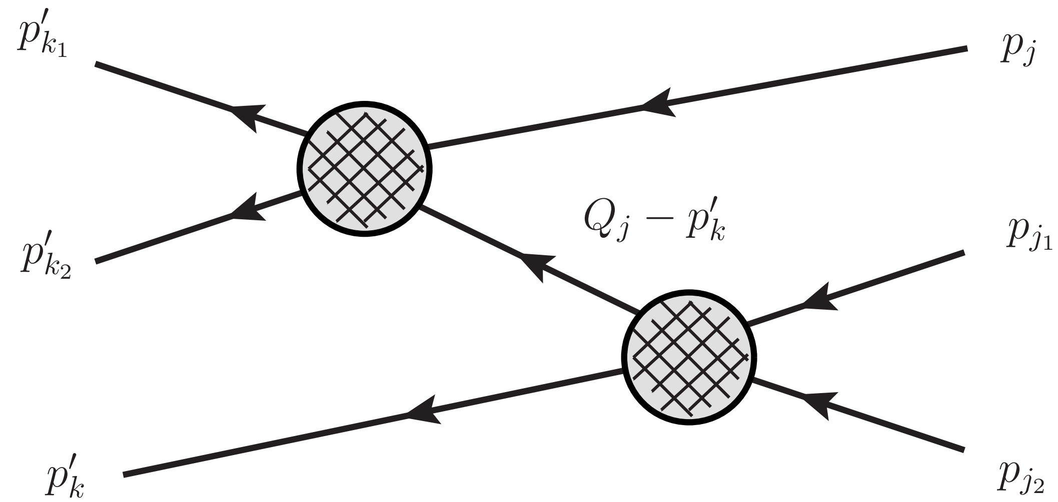
$$\mathcal{C}_{kj}(\sigma'_k, s, \sigma_j) = \int_{-1}^{+1} dz_{jk}\, \widehat{\mathcal{A}}_{kj}(\sigma'_k, s, t_{jk}(s, z_{jk}), \sigma_j)$$

We can proceed to project out the discontinuities

Note : The off-diagonal ($j \neq k$) amplitudes have a subtlety because of the OPE amplitude

One particle exchange

Partial wave projection of the OPE term gives an extra cut in the complex *s*-plane

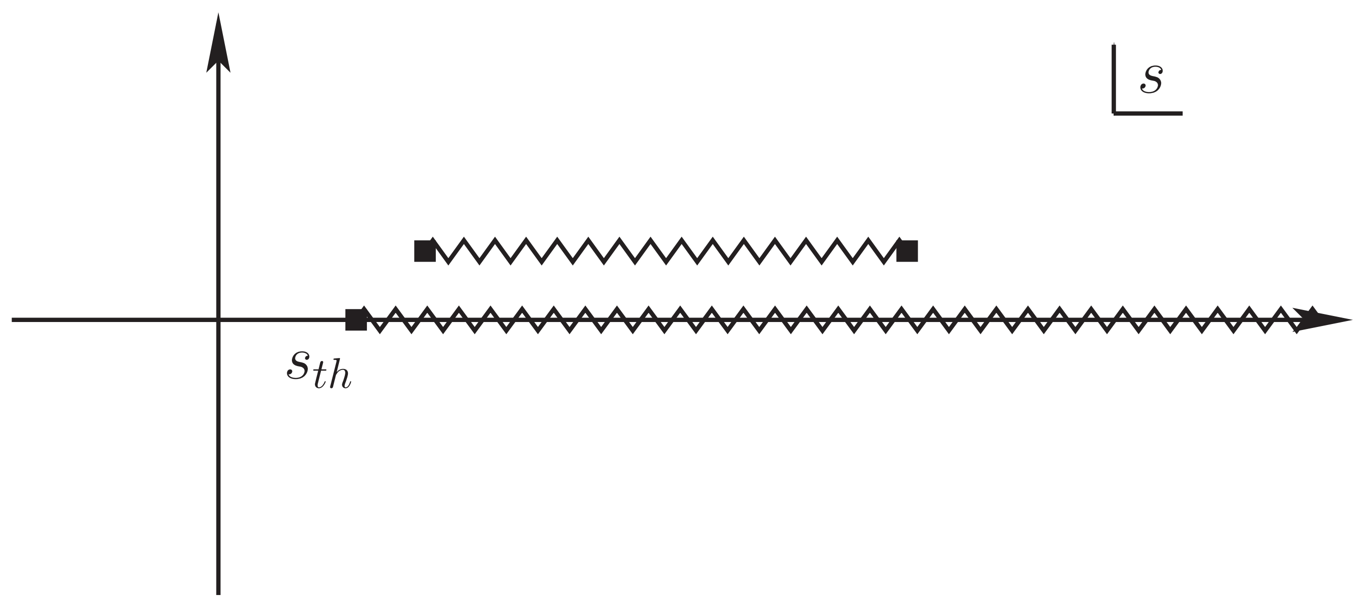


Exchange Mass

$$\int_{-1}^{+1} dz_{jk}\, \delta(u_{jk}(s, z_{jk}) - m^2_{(jk)})$$

$$\sim \frac{2s}{\lambda^{1/2}(s, \sigma_j, m_j^2)\lambda^{1/2}(s, \sigma'_k, m_k^2)} \Theta(s - s^{(+)})\Theta(s^{(-)} - s)$$

Non-zero in Dalitz region

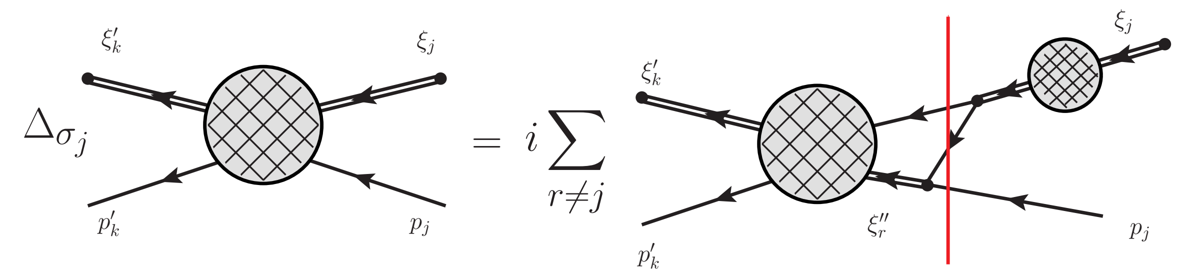


Triangle singularities

Kinematics may require deformation of dispersive contours

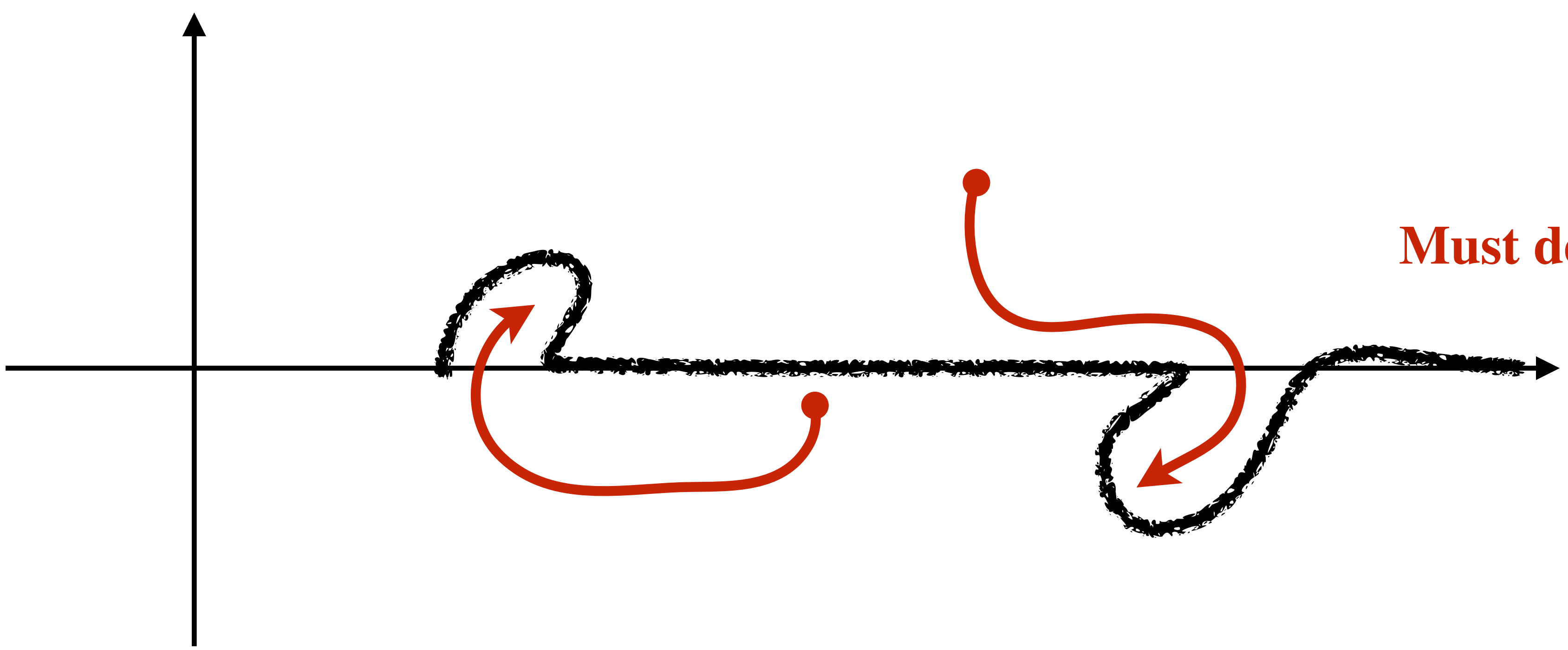
$$\Delta_{\sigma_1}\mathcal{C}_{31}(\sigma_3{'}_-, s_-, \sigma_{1+}) = i\rho_2(\sigma_{1+})N_1(\sigma_{1+})\int d\sigma_3''\, D_3^{-1}(\sigma_3'')\mathcal{C}_{33}(\sigma_{3-}, s_-, \sigma_3{'}_-)$$

Fix s, σ_3', investigate contour in σ_1



C has only normal threshold branch points

$$\mathcal{C}_{31}(\sigma_3{'}_-, s_-, \sigma_{1+}) = \frac{1}{\pi}\int_{\sigma_{th}^{(1)}}^{(\sqrt{s_-}-m_1)^2} d\widehat{\sigma}\,\frac{1}{\widehat{\sigma}-\sigma_{1+}}\rho_2(\widehat{\sigma})N_1(\widehat{\sigma})b(\widehat{\sigma}, s_-, \sigma_3{'}_-)$$



Must deform contour

Summary

- Existing 3 particle scattering amplitudes may have spurious singularities — analyticity not enforced.

- Proposed framework: unitarity + analyticity (only normal thresholds in particle-isobar amplitudes) → dispersion relations + short range inputs → N/D equations → amplitudes !