Outline

• Opportunities and challenges in hadron spectroscopy

• Amplitudes from S-matrix analysis: 2-to-2 scattering 1-to-3 decays : how virtual exchanges become real

• Three particle scattering : the framework
How Hadrons Emerge from QCD

- Experimental or lattice signatures (real axis data: cross section bumps and dips, energy levels)
- Theoretical signatures (complex plane singularities: poles, cusps)
- What is the interpretation (constituent quarks, molecules, …) ?

Reaction amplitudes
How Hadrons Emerge from QCD

- Experimental or lattice signatures (real axis data: cross section bumps and dips, energy levels)

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Reaction amplitudes
3pion challenge

evolution in statistics \( \pi^- p \rightarrow \pi^- \pi^+ \pi^- p \)

CERN ca. 1970

\( \mathcal{O}(10^2/10\text{MeV}) \)

BNL (E852) ca 1995

\( \mathcal{O}(10^3/10\text{MeV}) \)

COMPASS 2010

\( \mathcal{O}(10^5/10\text{MeV}) \)

\( \mathcal{O}(10^6/10\text{MeV}) \)

96M events

Comparable statistics expected from JLab
Signatures of new, unusual light resonances

- High precision PWA of 3π diffractive association yields a new $a_1(1460)$ incompatible with the quark model/Regge expectations.

- At low-t exotic wave production compatible with one pion exchange

- In photoproduction exotic mesons be produced via pion exchange

- Large exotic wave seen in $\eta(\gamma)\pi$ production: FESR’s to constrain P-wave
Signatures of unusual heavy quark resonances


Virtual OPE

Real OPE

EMARK ON ENERGY PEAKS IN MESON SYSTEMS

M. Nauenberg | A. Pais

If the width of particle X is not very large we will stay close to the physical region. This almost singular behavior of \( A(s) \) for certain physical \( s \) causes the peaking effect to which we refer as an \((X,Y,Z)\) peak.
Amplitude signatures

New particles in the QCD spectrum

$\bar{D} D^* \rightarrow \bar{D} D^*$

$X(3872)$

Threshold

Resonance pole

The “interesting stuff” happens on unphysical sheets. When singularities are close to the physical region rapid variations in amplitudes (cross sections) appear

Other effects can be “generated” exchanges forces” (*)

(*) (?) singularities because of confinement

s-physical region

$2^{nd}$ sheet branch point
Anatomy of resonances

When strength of interactions is reduced, bound states become resonances.

The only place for bound states pole to migrate is onto an unphysical sheet connected to the open channel branch point.

Violates Causality

Infinite amplitude (violates unitarity)
Properties of reaction amplitudes are determined by

**Causality**: Reaction amplitudes are smooth (analytical) functions of kinematical variables with singularities reflecting existence of constraints (laws)

**Unitarity**: Determines singularities.

**Crossing**: Dynamical relations: reaction amplitudes in the exchange channel (forces) are analogous to amplitude in the direct channel (resonance)
### 2-to-2 scattering

\[
A(s, t) = \sum_l (2l + 1) a_l(s) P_l(z_s)
\]

Analysis of \(a_l(s)\) model dependent: Testing hypothesis

- Dispersion relation (Analyticity)

\[
a_l(s) = B_l(s) + \frac{1}{\pi} \int_{tr} ds' \frac{Im a_l(s')}{{s'} - s}
\]

\[
|a(s' + i\epsilon)|^2 \rho(s') \rightarrow \text{implies a relation between Re } a_l \text{ and Im } a_l
\]

- Unitarity …

\[
B_l \text{ from cross channel interactions. Given } B_l \text{ (e.g. for elastic } \pi\pi \text{ scattering it can be related to } a_l(s)'s. \text{ Unitarity “converts” a dispersion relation to an integral equation (e.g Roy eq.) for the partial waves). Solution is non unique — there are resonances/bound states not constrained by “exchange forces”}
\]
Real axis vs complex plane

To look for poles need to continue though the unitary cut

\[ a_l(s) = B_l(s) + \frac{1}{\pi} \int_{tr} ds' \frac{Ima_l(s')}{s' - s} \]

\[ a^{II}_l(s) = a_l(s) + 2iIma_l(s) \]

• (All) that is needed is an analytical continuation of the phase shift \(a_l(s)\) on the real axis

• In a typical application of Roy eq., Im \(a_l\) does not correspond to limit of an analytical function. Thus what is quoted is not the “true pole” (but the probably a value close to it)
Beyond 2 particle production Khuri-Treiman (KT) model

\[ A(s, t, M^2) = \alpha(t, M^2) + \alpha(s, M^2) + \alpha(u, M^2) \]

Or in decay channel

- Isobars are not partial waves. They are model amplitudes with direct channels branch points only.

- Isobars to be determined in terms of 2-to-2 scattering amplitude and a coupling, \( g \) to the production channel (\( g = 1 \)).

- Unitarity: \( t \)-channel process acts as a "driving" term for \( s \)-channel isobar.
KT equations

Isobar amplitudes have only the normal threshold singularities

\[ a(s, M^2) = f(s) + f(s) \int_{tr} \frac{ds'}{\pi} \frac{\rho(s') b(s', M^2)}{s' - s} \]

\[ b(s, M^2) = \int_{-1}^{1} dz a(t(s, z, M^2), M^2) \]
KT equations

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a(s, M^2) = f(s) + f(s) \int_{tr} \frac{ds'}{\pi} \frac{\rho(s') b(s', M^2)}{s' - s}
\]

\[
b(s, M^2) = \int_{-1}^{1} dz a(t(s, z, M^2), M^2)
\]
Pentaquark as a triangle singularity?
3-to-3 amplitude from a 2-to-2 (1-to-3) KT model

\[ T(\sigma', M^2, \sigma) = B(\sigma', M^2, \sigma) \]

\[ + \int d\sigma'' B(\sigma', M^2, \sigma'') \tau(\sigma'', M^2) T(\sigma'', M^2, \sigma) \]

(Aitchison 1965)
Relation to other approaches

M. Mai et al. (JPAC)
I. Aitchison (Khuri-Treiman)
H. Hammer et al. (EFT)

“\( \varrho \) vs Chew-Madelstam” phase space (properly removes unphysical singularities from \( \varrho \) )

- Reproduces 3-to-3 unitarity on the real axis only
- Analyticity in sub-channel variables?

Next step is to derive the proper dispersive representation

\[
T(q, p; s) = B(q, p; s) - \int \frac{\alpha - i}{(2\pi)^3} B(q, l; s) \frac{\tau(\sigma(l))}{2E(l)} T(l, p; s)
\]

\[
T_{3\to3}(\sigma', s, \sigma) = \sum_{\sigma''} \left[ \frac{1}{1 - \tau(s)B(s)} \right]_{\sigma', \sigma''} [B(s)]_{\sigma''', \sigma}
\]

\[
T = \frac{1}{K^{-1}(s) - i\rho(s)}
\]

\[
T = \frac{1}{K^{-1}(s) - \frac{1}{\pi} \int_{tr} ds' \frac{\rho(s')}{s' - s}}
\]
3-to-3 scattering from dispersion relations

Special thanks to Andrew Jackura
Workflow

Unitarity

Analyticity

Dispersion Relations

Continuation
Workflow

1. Unitarity
2. Analyticity
3. Dispersion Relations
4. Continuation
5. Multiple variables → Isobar approximation
Workflow

1. **Unitarity**
   - Multiple variables → Isobar approximation

2. **Analyticity**
   - Isobars/Partial Waves → complicated analytical properties

3. **Dispersion Relations**

4. **Continuation**
Workflow

1. **Unitarity**
   - Multiple variables → Isobar approximation

2. **Analyticity**
   - Isobars/Partial Waves → complicated analytical properties

3. **Dispersion Relations**
   - Solution via N/D

4. **Continuation**
   - What generates resonances?
Consider the elastic scattering of the $3 \rightarrow 3$ system $123 \rightarrow 123$, where 1, 2, and 3 are distinguishable particles.

The S-matrix is decomposed as

$$\langle \{p'\} | S | \{p\} \rangle = \langle \{p'\} | \{p\} \rangle + i \sum_j \delta(p'_j - p_j)(2\pi)^4 \delta(4)(Q'_j - Q_j) F_j(\{p', p\}_j)$$

$$+ i(2\pi)^4 \delta(4)(P' - P) A(\{p', p\})$$
Unitarity relations

1 = \mathbf{S}^\dagger \mathbf{S} \rightarrow 2 \text{Im}(\mathbf{F} + \mathbf{A}) = (\mathbf{F} + \mathbf{A})^\dagger \mathbf{P.S.}(\mathbf{F} + \mathbf{A})

Disconnected 2→2 Unitarity Relation

\[ 2 \text{Im} \, \mathcal{F}_j(\{\mathbf{p}', \mathbf{p}\}_j) = \rho_2(\sigma_j) \int d\Omega''_j \, \mathcal{F}^*_j(\{\mathbf{p}'', \mathbf{p}'\}_j) \mathcal{F}_j(\{\mathbf{p}'', \mathbf{p}\}_j) \]

Connected 3→3 Unitarity Relation

\[ 2 \text{Im} \, \mathcal{A}(\{\mathbf{p}', \mathbf{p}\}) = \int \tilde{d}p'' \tilde{d}p'' \tilde{d}p_3'' (2\pi)^4 \delta^{(4)}(P'' - P) \mathcal{A}^*(\{\mathbf{p}'', \mathbf{p}'\}) \mathcal{A}(\{\mathbf{p}'', \mathbf{p}\}) \]
\[ + \sum_k \rho_2(\sigma'_k) \int d\Omega''_k \, \mathcal{F}^*_k(\{\mathbf{p}'', \mathbf{p}'\}_k) \mathcal{A}(\{\mathbf{p}'', \mathbf{p}\}) \Theta(\sigma'_k - \sigma_{th}^{(k)}) \]
\[ + \sum_j \rho_2(\sigma_j) \int d\Omega''_j \, \mathcal{A}^*(\{\mathbf{p}'', \mathbf{p}'\}_j) \mathcal{F}(\{\mathbf{p}'', \mathbf{p}\}_j) \Theta(\sigma_j - \sigma_{th}^{(j)}) \]
\[ + \sum_{j,k, j \neq k} 2\pi \delta(u_{jk} - m^2_{(jk)}) \mathcal{F}^*_k(\{\mathbf{p}'', \mathbf{p}'\}_k) \mathcal{F}_j(\{\mathbf{p}'', \mathbf{p}\}_j) \]
Unitarity relations

Disconnected 2→2 Unitarity Relation

\[ 2 \text{Im} p'_{j_1} p_{j_1} p'_{j_2} p_{j_2} p'_3 p_3 = \]

Connected 3→3 Unitarity Relation

\[ 2 \text{Im} p'_1 p_2 p'_3 p_3 + \sum_k p'_k p_{k_1} p'_{k_2} p_{k_2} p'_k p_{k_3} + \sum_{j,k} p'_j p'_{(j,k)} p_{j_1} p'_{j_2} p'_{j_3} p_{j_3} \]
Too many variables ...

3→3 amplitudes depend on 8 independent variables. One representation is

\[ A(p', p) = \sum_J \sum_{\lambda, \lambda'} \left( \frac{2J + 1}{8\pi^2} \right) A_{\lambda', \lambda}(\{E\}) D_{\lambda', \lambda}(R) \]

\[ R_{jk} = (\varphi_j, \gamma_{jk}, \varphi'_k) \]

\{E\} = \{\sigma'_1, \sigma'_2, s, \sigma_1, \sigma_2\}

A choice of \(\sigma\)'s selects an isobar e.g. \(\sigma'_1\) is the inv. mass squared of (23) subsystem.
Isobar-particle scattering

Assume that the amplitude can be expanded into *Isobar Amplitudes*

\[
\mathcal{A}(\{p', p\}) = \sum_{j, k} A_{k, j}(\{p', p\}_{k, j})
\]

Two particles interact before interacting with spectator

Two particles interact before interacting with spectator

Sum over all allowed isobars

Infinite sum analytically continued accounts for singularities in the other two \(\sigma\) variables
Assume that the amplitude can be expanded into *Isobar Amplitudes*

\[
\mathcal{A}(\{p', p\}) = \sum_{i, k} \mathcal{A}_{kij}(\{p', p\})_{kj}
\]

\[
\mathcal{A} \rightarrow \sum_{J' \lambda' J \lambda}^{J_{\text{max}}} A_{11}(\sigma'_1, s, t_{11}, \sigma_1)Y_{j', \lambda'}^{*}(\Omega'_1)Y_{j \lambda}(\Omega_1)
\]

\[
+ \sum_{J_{\text{max}}}^{J_{\text{max}}} A_{21}(\sigma'_2, s, t_{21}, \sigma_2) \cdots
\]

Like in a KT model, adding truncated sums of particle-isobar amplitudes leads to a complicated sub-channel energy dependence (but with only normal threshold branch points)
Isobar-particle amplitude unitarity relations

These are coupled equations for each $A_{ij}$

Factorizes the sub-energy rescattering

$$A_{k,j}(\sigma'_k, s, t_{jk}, \sigma_j) = \frac{1}{D_k(\sigma'_k)} \hat{A}_{k,j}(\sigma'_k, s, t_{jk}, \sigma_j) \frac{1}{D_j(\sigma_j)}$$

$2\rightarrow2$ Rescattering

Still sub-energy dependence

$$A_{11}(\sigma'_1, s, t_{11}, \sigma_1) = f_1(\sigma'_1) \hat{A}_{11}(\sigma'_1, s, t_{11}, \sigma_1) f_1(\sigma_1)$$

$$f_j(\sigma_j) = N_j(\sigma_j) / D_j(\sigma_j)$$
Isobar-particel amplitude unitarity relations

These are coupled equations for each $A_{ij}$

$2 \text{Im} \left[ A_{11}(\sigma'_1, s, t_{11}, \sigma_1) \right] = f_1(\sigma'_1) \hat{A}_{11}(\sigma'_1, s, t_{11}, \sigma_1) f_1(\sigma_1) + \sum_{n \neq r}^{n,r} \sum_{n \neq k}^{n,k} + \sum_{n \neq k}^{n \neq j} + \sum_{n \neq k}^{n \neq j} + \sum_{j \neq k}^{j \neq k}$

$A_{11}(\sigma'_1, s, t_{11}, \sigma_1) = f_1(\sigma'_1) \hat{A}_{11}(\sigma'_1, s, t_{11}, \sigma_1) f_1(\sigma_1)$
Isobar-particle amplitude unitarity relations

These are coupled equations for each $A_{ij}$

$$2 \text{Im} \left( A_{11}(\sigma_1', s, t_{11}, \sigma_1) \right) = f_1(\sigma_1') \hat{A}_{11}(\sigma_1', s, t_{11}, \sigma_1) f_1(\sigma_1)$$

$$(j \neq k)$$
Isobar amplitude unitarity relations

\[ 2 \text{Im} \hat{A}_{i,j}(\sigma'_i, s, u_{ij}, \sigma_j) \]

with discontinuities across individual variables
We can split the imaginary part into discontinuities across all variables.

Need to be careful on which direction we approach the real axis from the complex planes.

\[2i \text{ Im } \hat{A}_{kj}(\sigma'_k, s, t_{jk}, u_{jk}, \sigma_j) = \Delta_{\sigma'_k} \hat{A}_{kj}(s+, t_{jk}+, u_{jk}+, \sigma_j+) + \Delta_s \hat{A}_{kj}(\sigma'_k-, t_{jk}+, u_{jk}+, \sigma_j+) \]
\[+ \Delta_{t_{jk}} \hat{A}_{kj}(\sigma'_k-, s-, u_{jk}+, \sigma_j+) + \Delta_{u_{jk}} \hat{A}_{kj}(\sigma'_k-, s-, t_{jk}-, \sigma_j+) \]
\[+ \Delta_{\sigma_j} \hat{A}_{kj}(\sigma'_k-, s-, t_{jk}-, u_{jk}-)\]

\[x_\pm = x \pm i\epsilon\]
From real axis (unitarity) to the complex plane

We can split the imaginary part into discontinuities across all variables.

Need to be careful on which direction we approach the real axis from the complex planes.

For $j \neq k$, have to worry about singularities in $u_{jk}$ from One Particle Exchange (OPE):

\[ (j \neq k) \sim \delta(u_{jk} - m_{jk}) \]

\[ x_\pm = x \pm i\epsilon \]
Single variable discontinuity reactions

\[ \Delta_s = i \sum_n \xi_k \left( \begin{array}{c} n \neq r \\ p_k \end{array} \right) \xi_j \]

\[ \Delta_{\sigma_k} = i \sum_{n \neq k} \xi_k \left( \begin{array}{c} n \neq j \\ p_k \end{array} \right) \xi_j \]

\[ \Delta_{\sigma_j} = i \sum_{r \neq j} \xi_k \left( \begin{array}{c} p_k \end{array} \right) \xi_j \]

\[ \Delta_{u_{jk}} = i \left( \begin{array}{c} j \neq k \\ p_k \end{array} \right) \xi_k \left( \begin{array}{c} \xi_j \end{array} \right) \]

\[ \Delta_{\sigma_k} = i \sum_{n \neq r} \xi_k \left( \begin{array}{c} n \neq r \\ p_k \end{array} \right) \xi_j \]

\[ \Delta_{\sigma_j} = i \sum_{r \neq j} \xi_k \left( \begin{array}{c} r \neq j \\ p_k \end{array} \right) \xi_j \]

\[ \Delta_{u_{jk}} = i \left( \begin{array}{c} j \neq k \\ p_k \end{array} \right) \xi_k \left( \begin{array}{c} \xi_j \end{array} \right) \]
Single variable discontinuity reactions

\[ \Delta_s = i \sum_n \xi_k^{n} \xi_j^{n} \]

\[ \Delta_{\sigma_k'} = i \sum_{n \neq k} \xi_k^{n} \xi_j^{n} \]

\[ \Delta_{\sigma_j} = i \sum_{r \neq j} \xi_k^{r} \xi_j^{r} \]

\[ \Delta_{u_{jk}} = i \]

Will turn this into an s-channel cut via Partial Wave Projection

\[ u_{jk} = u_{jk}(\sigma'_k, s, z_{jk}, \sigma_j) \]
We now want to consider partial wave projections of the amplitude

To simplify the expressions, let’s consider the case for $J = 0$, and spin-0 isobars

$$C_{kj}(\sigma'_k, s, \sigma_j) = \int_{-1}^{+1} dz_{jk} \, \hat{A}_{kj}(\sigma'_k, s, t_{jk}(s, z_{jk}), \sigma_j)$$

We can proceed to project out the discontinuities

**Note**: The off-diagonal ($j \neq k$) amplitudes have a subtlety because of the OPE amplitude
Partial wave projection of the OPE term gives an extra cut in the complex $s$-plane

$$
\int_{-1}^{+1} dz_{jk} \delta(u_{jk}(s, z_{jk}) - m^2_{(jk)})
$$

$$
\sim \frac{2s}{\lambda^{1/2}(s, \sigma_j, m_j^2)\lambda^{1/2}(s, \sigma'_k, m_k^2)} \Theta(s - s^{(+)})\Theta(s^{(-)} - s)
$$

Exchange Mass

Non-zero in Dalitz region

$\Theta$ function cuts the complex plane at $+1$ and $-1$.
Triangle singularities

Kinematics may require deformation of dispersive contours

\[ \Delta_{\sigma_1} C_{31}(\sigma'_3, s_-, \sigma_{1+}) = i \rho_2(\sigma_{1+}) N_1(\sigma_{1+}) \int d\sigma'_3 D_3^{-1}(\sigma''_3) C_{33}(\sigma'_3, s_-, \sigma'_3) \]

Fix \( s, \sigma'_3 \), investigate contour in \( \sigma_1 \)

C has only normal threshold branch points

\[ C_{31}(\sigma'_3, s_-, \sigma_{1+}) = \frac{1}{\pi} \int_{\sigma_{th}^{(1)}} d\tilde{\sigma} \frac{1}{\tilde{\sigma} - \sigma_{1+}} \rho_2(\tilde{\sigma}) N_1(\tilde{\sigma}) b(\tilde{\sigma}, s_-, \sigma'_3) \]

Must deform contour
Summary

• Existing 3 particle scattering amplitudes may have spurious singularities — analyticity not enforced.

• Proposed framework: unitarity + analyticity (only normal thresholds in particle-isobar amplitudes) → dispersion relations + short range inputs → N/D equations → amplitudes!