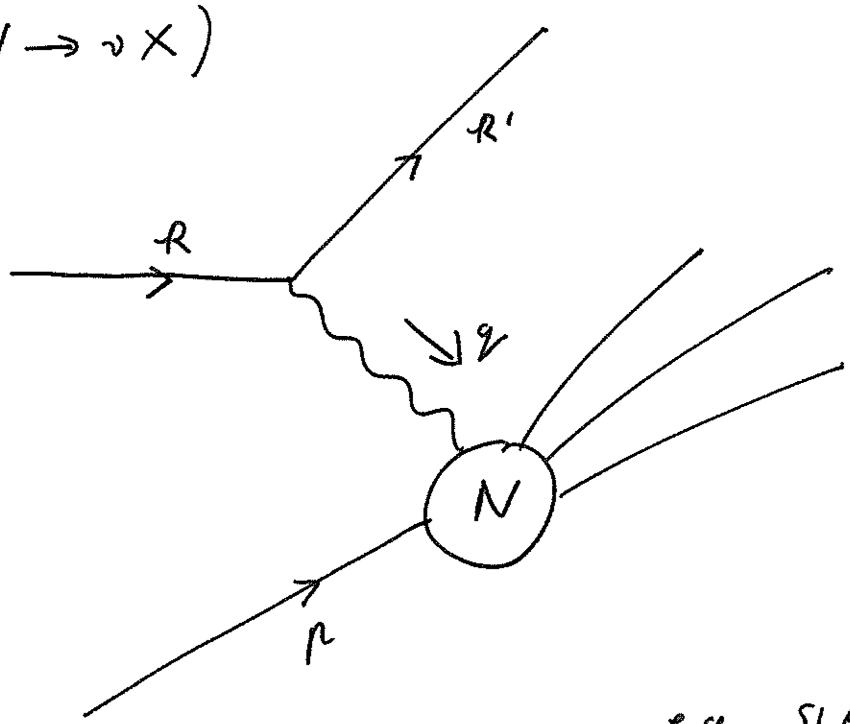
Deep Inelastic Scattering

$(eN \to eX)$

$(\nu N \to \nu X)$



X → anything

e.g. SLAC { N is fixed target, e^- beam.

HERA { p, e beam → higher c.of.m energy

In c.of.m of nucleon

$$p = (M, \underline{0})$$

$$k = (E, 0, 0, E) \qquad q = k' - k$$

$$k' = (E', E'\sin\theta, 0, E'\cos\theta)$$

Defn:

$$Q^2 = -q^2 \geqslant 0$$

$$\nu = p \cdot q$$

$$\nu = +M(E - E')$$

$$Q^2 = 4EE'\sin^2\tfrac{1}{2}\theta$$

For case of e.m. interaction we have interaction Lagrangian

$$\mathcal{L}_{int} = -e J_\mu^{e.m} A^\mu,$$

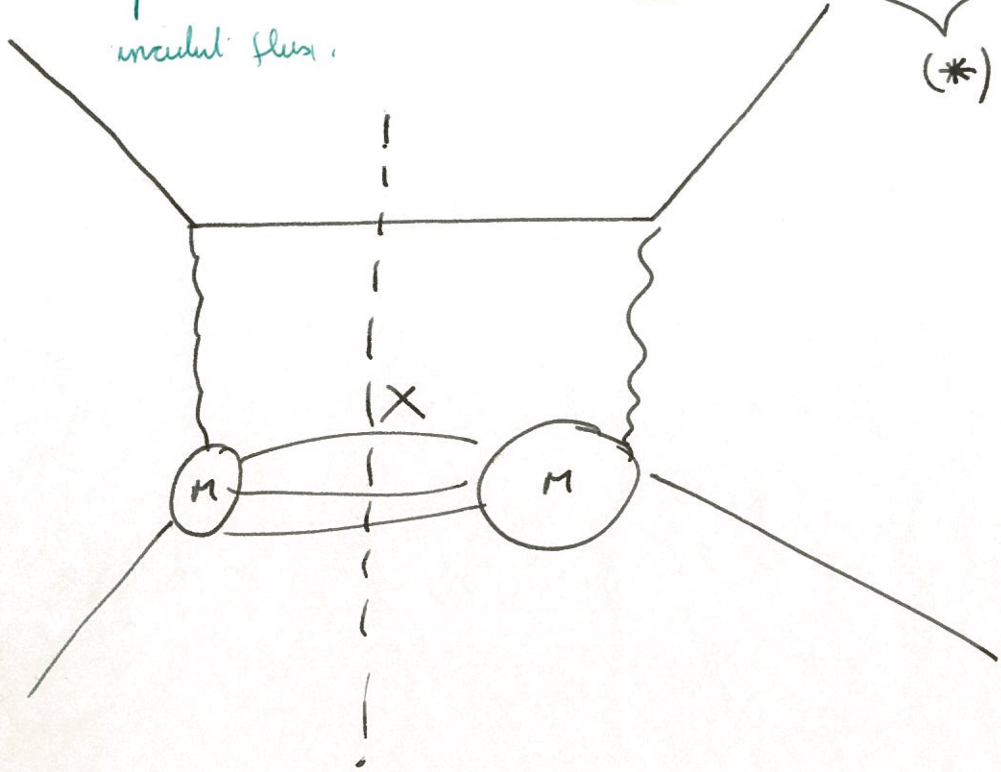
Then interaction matrix element

current code in e.m. interaction

Feynman propagator

$$i\mathcal{T} = (ie)^2 \langle \ell(k'), r | J^{\text{lept}}_\mu | \ell(k), s \rangle D^{\mu\nu}(q^2) \langle X | J^{\text{had.}}_\nu | N(p), \sigma \rangle$$

$$\& \; d\sigma = \underset{\uparrow \text{ incident flux.}}{\frac{1}{2E(k)}} \; \frac{d^3k'}{(2\pi)^3\, 2E(k')} \; \underbrace{\sum_X (2\pi)^4 \, \delta^{(4)}(p_X + k' - p - k) \, |\mathcal{T}|^2}_{(*)}$$



Now
$$(*) = (2\pi)^4 \sum_X \bar{u}_r(k') \gamma_\mu u_s(k) \, \bar{u}_s(k) \gamma_\nu u_r(k')$$
$$\left(\frac{e^2}{q^2}\right)^2 \langle X | J^\mu | N(p) \rangle \langle N(p) | J^{\nu\dagger} | X \rangle$$
$$\delta^4(p_X + k' - k - p)$$

Polarization sums $\sum_r u_r(k)\bar{u}_r(k) = \not{k} + m$

Sum over final polarizations, average over initial polarizations

$$\to (2\pi)^4 \cdot \frac{1}{4} \, \mathrm{Tr}\{\gamma_\mu(\not{k} + m_e)\gamma_\nu(\not{k}' + m_e)\} \cdot \left(\frac{e^2}{q^2}\right)^2$$
$$\delta^{(4)}(p_X + k' - k - p) \, \langle N(p) | J^{\nu\dagger} | X \rangle \langle X | J^\mu | N(p) \rangle$$

We ~~denote~~ write this in form

$$L_{\mu\nu}\cdot\left(\frac{e}{q^2}\right)^2\cdot W^{\mu\nu}$$

$$L_{\mu\nu} = \pi\,\mathrm{Tr}\{\gamma_\mu \not k\, \gamma_\nu \not k'\}\, e^2$$

J is real for e.m. current

$$W_{\mu\nu} = \frac{1}{4\pi}\sum_X (2\pi)^4\,\delta^4(p_X + \underbrace{k'-k}_{-q} - p)\sum_{pol}\langle N(p)|J^\dagger_\mu|X\rangle\langle X|J_\nu|N(p)\rangle$$

convention

$L_{\mu\nu}$ is easy

$$L_{\mu\nu} = 4\pi\{k_\mu k'_\nu + k'_\mu k_\nu - k\cdot k'\, g_{\mu\nu}\}e^2$$

Interesting question ("basis" of this course) is study of $W_{\mu\nu}$

Try and rewrite $W_{\mu\nu}$

$$(2\pi)^4\,\delta^{(4)}(p_X - q - p) = \int d^4y\; e^{-iy\cdot(p_X-q-p)}$$

$$W_{\mu\nu} = \frac{1}{4\pi}\sum_X\int d^4y\; e^{-iy\cdot(p_X-q-p)}\,\langle N(p)|J^\dagger_\mu(0)|X\rangle\,\langle X|J_\nu(0)|N(p)\rangle$$

$$= \frac{1}{4\pi}\int d^4y\,\langle N(p)|J^\dagger_\mu(y)\,J_\nu(0)|N(p)\rangle\, e^{iq\cdot y}$$

We can replace product of currents by commutator (since other term is from region ~~p_X+q~~ $-p_X - q + p = 0$)

i.e. $$W_{\mu\nu} = \frac{1}{4\pi}\int d^4y\; e^{iq\cdot y}\,\langle N(p)|[J^\dagger_\mu(y), J_\nu(0)]|N(p)\rangle$$

What is the most general form allowed for $W_{\mu\nu}$?

Linear combination of $p_\mu p_\nu$, $p_\mu q_\nu$, $p_\nu q_\mu$, $q_\mu q_\nu$, $g_{\mu\nu}$

or $\varepsilon_{\mu\nu\rho\sigma}\, q^\rho p^\sigma$

e.m. is pure vector

e.m. conservation $\Rightarrow$ $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$

$$\therefore\ W_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1 + \left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right) W_2$$

So D.I.S. of electron is described in terms of two structure functions.

Before we introduced two variables $Q^2 = -q^2$, $\nu = p\cdot q$.

Def: $\omega = \dfrac{2\nu}{Q^2}$ $(\omega \geqslant 1)$

Then experimentally find $(Q^2 \gtrsim 10\ \mathrm{GeV}^2)$

$$\left.\begin{aligned} W_1(\nu, Q^2) &= F_1(\omega) \\ \nu W_2(\nu, Q^2) &= F_2(\omega) \end{aligned}\right\}\ \text{structure } f^{=}$$

So F_1, F_2 are dimensionless $f^{=}$ of dimensionless variable.

This is Bjorken scaling!

$\Rightarrow$ Nucleon is composed of (free) point-like objects (quarks)

<u>Another rel=</u> Callan-Gross

$F_1 = \frac{1}{2}\,\omega F_2 \longrightarrow$ characteristic of spin $\frac{1}{2}$ objects

Another relation $F_1 = \frac{1}{2}\omega F_2$ $\rightarrow$ characteristic of spin-$\frac{1}{2}$ constituents

Naïve Parton Model

~~Assume that partons~~

Work in c. of m. frame of ~~lxx~~ interacting system.

Hadron consists of only weakly interacting partons, with momentum $\parallel$ momentum of parent hadron (ultra-relativistic approach).

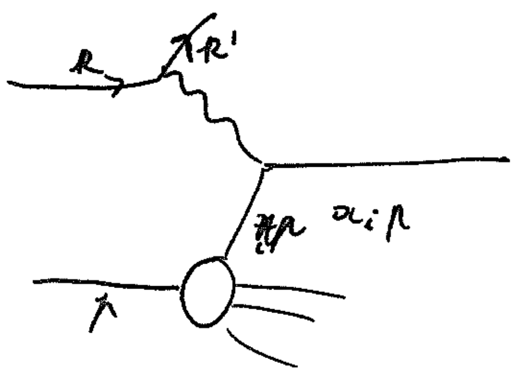
i.e. $\underset{\sim}{p}_i = x_i \underset{\sim}{p}$

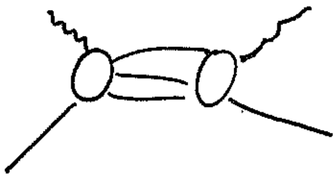
Then $p_i^0 = \sqrt{\mu_i^2 + x_i \underset{\sim}{p}_i^2}$

neglect

$$\Rightarrow \quad \underline{\underline{p_i^\mu = x_i p^\mu}}$$

So the picture we have for scattering is as follows



and $W_{\mu\nu} = \frac{1}{4\pi} \sum_X$  —— (†)

Our previous rewriting of $W_{\mu\nu}$

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4y \, e^{iq\cdot y} \langle N(p) | [J_\mu^+(y), J_\nu(0)] | N(p) \rangle$$

was just in some sense a rewriting of the optical theorem

Write scattering matrix $S = 1 + iT$

Then unitarity of S-matrix $\Rightarrow$ $S^+S = I$

$$\Rightarrow \quad (T - T^+) = iT^+T$$

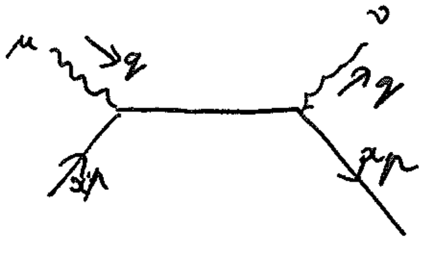
Momentum conservation yields $\langle f | T | i \rangle = (2\pi)^4 \delta^{(4)}(P_f - P_i) J_{fi}$

So $J_{fi} - J_{if}^* = i \sum_n (2\pi)^4 \delta^{(4)}(P_n - P_i) J_{nf}^* J_{ni}$

This is precisely (†).

In particular $2 \, \mathrm{Im} \, J_{ii} = i \sum_n (2\pi)^4 \delta^{(4)}(P_n - P_i) |J_{ni}|^2$

Thus we can ~~write~~ unpublished write $\boxed{W_{\mu\nu} = 2 \cdot \frac{1}{4\pi} \, \mathrm{Im} \, T_{\mu\nu} \qquad T_{\mu\nu} = i \int d^4y \, e^{iq\cdot y} \langle N | T J(y) J(0) | }$

$W_{\mu\nu} = 2 \, \mathrm{Im}$  $\cdot \frac{1}{4\pi}$

Now $\frac{1}{x + i\varepsilon} = P\left(\frac{1}{x}\right) + i\pi\,\delta(x)$

Thus $W_{\mu\nu} = \frac{q_i^2}{4\pi} \cdot 2\pi \cdot \frac{1}{2} \, \mathrm{Tr}\left[x_i \not{p} \, \gamma_\nu \, (x_i \not{p} + \not{q}) \, \gamma_\mu \right] \delta((x_i p + q)^2)$ ← charge of quark

where we neglect the quark mass.

$$= q_i^2 \, x \, \delta[2x_i p.q - Q^2] \{ x_i(p_\mu p_\nu + p_\nu p_\mu) + (p_\nu q_\mu + p_\mu q_\nu) - p.q \, g_{\mu\nu} \}.$$

Note that $\delta[2x_i p.q - Q^2] = \frac{1}{2p.q} \delta[x_i - \frac{1}{\omega}]$

So $$W_{\mu\nu} = \frac{1}{\nu} \delta[x_i - \frac{1}{\omega}] \{ x_i^2 p_\mu p_\nu + \frac{x_i}{2} \nu (-g_{\mu\nu}) + \cdots \}.$$

In order to pick out W_2, look at co-eff of $p_\mu p_\nu$

$$\Rightarrow \quad W_2^i(\nu, Q^2) = \frac{1}{\nu} \delta(x_i - \frac{1}{\omega}) . x_i^2 q_i^2$$

Note, however, that the cross-section has been normalised to 1 parton per unit volume

i.e. $\langle p_i | p_i' \rangle = 2p_i^0 (2\pi)^3 \delta(p_i - p_i')$.

where $2p_i^0 = 2x_i p^0$

Correct normalisation has one nucleon per unit volume $2p^0$

$F_2 = \nu W_2$

So $$W_2^N(\nu, Q^2) = \sum_i q_i^2 \frac{1}{\nu} \delta(x_i - \frac{1}{\omega}) x_i$$

and $$\underline{\underline{F_1 = \frac{1}{2} \omega F_2}}$$

fractional momentum carried by quark in ∞-momentum frame is just $\frac{1}{\omega}$.

So we have both scaling and Callan-Gross relation.

Let probability that parton i have fractional momentum of nucleon between x and $x+dx$ be $f_i(x)\,dx$

$\omega = \frac{2\nu}{Q^2}$

Thus $F_2(x,Q^2) = \sum_i q_i^2 \, x \, f_i(x, Q^2)$ where $x = \frac{1}{\omega}$

spin averaged.

In particular

$$F_2^p(x,Q^2) = \left[\frac{4}{9}\left(u_p(x) + \bar{u}_p(x)\right) + \frac{1}{9}\left(d_p(x) + \bar{d}_p(x)\right)\right] x$$
$$+ \cdots$$

where u, d are the structure function of u, d quarks in (say) proton

<u>Note</u> isospin symmetry $\Rightarrow$
$$u_p(x) = d_n(x)$$
$$d_p(x) = u_n(x)$$

Conventionally we use distⁿ of partons in proton rather than neutron, and drop the hadron index.

Importance of structure functions is that they are <u>universal</u> & process independent

Mon 11

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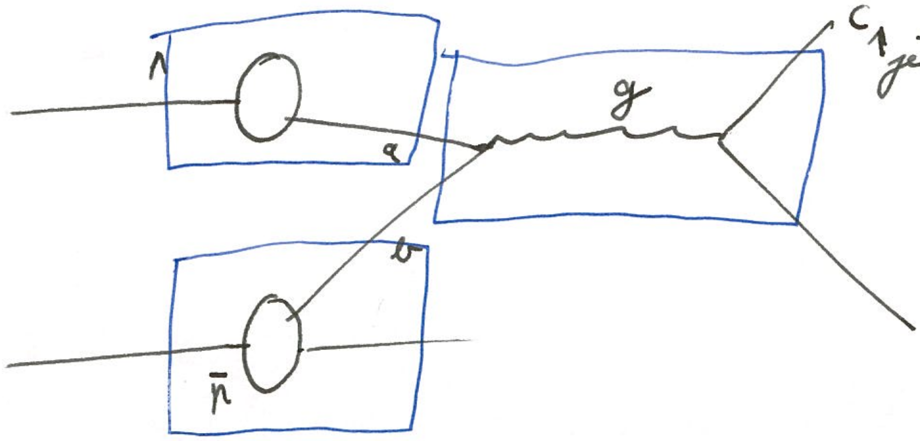
Consider jet production at $p\bar{p}$ collider



jet with momentum p_T

$$E_c \cdot \frac{d^3\sigma}{d^3p_c} = \sum_{a,b,d} \int_0^1 dx_a \int_0^1 dx_b \; f_{a/p}(x_a)\, f_{b/\bar{p}}(x_b)$$

$$\frac{1}{\pi} \frac{d\sigma}{d\hat{t}} (ab \to cd)$$

The f's are the same structure functions that occur in DIS scattering (up to renormalization / factorization prescription)

O.P.E.

$$T_{\mu\nu} = i \int d^4 z \, e^{iq\cdot z} \langle N | T J_\mu(z) J_\nu(0) | N \rangle \quad \text{——} (*)$$

We wish to study $T_{\mu\nu}$ in the Bjorken limit

light-like

Point is light-cone behaviour

$$Q^2 \to \infty$$
$$p\cdot q \to \infty$$
$$x = \frac{Q^2}{2p\cdot q} \quad \text{fixed.}$$

All momentum scales large — we expect asymptotic freedom to be valid

Work in infinite momentum frame

$$p = (P, 0, 0, P) \qquad P \approx \nu^{\frac{1}{2}}$$
$$q = (\tfrac{\nu}{2P}, \sqrt{Q^2}, 0, -\tfrac{\nu}{2P})$$

Defn $\quad q^{\pm} = q^0 \pm q^3 \qquad x\cdot y = \frac{1}{2}(x_+ y_- + x_- y_+) - x_\perp \cdot y_\perp$

Then $q\cdot z \sim \frac{1}{2} z_+ \frac{\nu}{P} - \sqrt{Q^2}\, z_\perp$

So as $\nu, Q^2 \to \infty$, dominant contribution is from regime

$$z_+ \, ; \, z_\perp \sim \frac{1}{\nu^{\frac{1}{2}}}$$

$$\text{i.e. } \underline{\underline{z^2 \to 0}}$$

Thus the study of $(*)$ in the limit $q^2 \to \infty$ is the study of the operator product as $z^2 \to 0$

<u>Wilson's O.P.E.</u>

end 5/5/93

Idea of operator product expansion (OPE)

Consider just time-ordered two point function for free scalar fields

$$T\phi(x)\phi(0) = \Delta_F(x)\, I + :\phi(x)\phi(0):$$

local operator

Feynman propagator

I is just the unit operator (local!)

$:\phi(x)\phi(0):$ is regular as $x \to 0$.

$$\Delta_F(x) = \frac{i}{(2\pi)^4}\int d^4q\; e^{-iq.x}\,\frac{1}{q^2+i\varepsilon}$$

$$\sim \frac{1}{x^2-i\varepsilon} \quad \text{as} \quad x \to 0$$

by dimensional arguments

So we have written $T\phi(x)\phi(0)$ as a singular "co-efficient function" $\Delta_F(x)$ ⊗ "~~local~~ operator regular as $x \to 0$".

In general we can write

$$TA(x)B(y) = \sum_n C^{(n)}(x,y)N^{(n)}(x,y)$$

where $C^{(n)}(x,y)$ is singular as $x \to y$
$N^{(n)}(x,y)$ is regular as $x \to y$

by (e.g.) doing a Wick expansion on

$$TA(x)B(y) = TA^0(x)B^0(y)\, e^{i\int d^4z\,\mathcal{L}_{int}(z)}$$

If we are interested in singularity behaviour then we write

$$T\,A(x)\,B(0) = \sum_n C^{(n)}(x)\; O^{(n)}(0)$$

↑ let $x \to y$.

where $O^{(n)}$ is a local operator and $C^{(n)}$ are some singular co-efficient functions. ← short distance behaviour of co-efficient functions ⟺ asymptotic freedom.

By sandwiching the OPE between external states

$$\Rightarrow \quad \langle f|\,T\,A(x)\,B(0)\,|i\rangle = \sum_n C^{(n)}(x)\,\langle f|\,O^{(n)}(0)\,|i\rangle$$

So the co-efficients are clearly independent of the external states

OPE for DIS

$$T_{\mu\nu} = i \int d^4 z \, e^{iq\cdot z} \langle N(p) | T J_\mu^+ (z) J_\nu(0) | N(p) \rangle$$

Let $t_{\mu\nu} = T J_\mu^+(z) J_\nu(0)$

Write OPE $\longrightarrow$ $t_{\mu\nu} = \sum_i C_i(z) N_i(z)$

where co-eff. f$^{\text{s}}$ in general singular as $z^2 \to 0$, and N_i regular.

As we are interested in singular behaviour, let us rewrite

$$t_{\mu\nu} = \sum_{(n,d,i)} C^{(n,d,i)}_{\mu\nu\mu_1\ldots\mu_n}(z) \, O^{(n,d,i)\mu_1\ldots\mu_n}(0)$$

symm. tracels.

where we have chosen a basis of operators of definite spin

$n \to$ spin
$d \to$ dimension
$i \to$ other label, suppress for time being

Then
$$\boxed{T_{\mu\nu} \sim \sum_{n,d} i \int d^4 z \, e^{iq\cdot z} C^{(n,d)}_{\mu\nu\mu_1\ldots\mu_n}(z) \times \langle N(p) | O^{(n,d)\mu_1\ldots\mu_n}(0) | N(p) \rangle}$$

matrix element

only a functn of p.

$$\langle N(p)|\, O^{(n,d)\mu_1\ldots\mu_n}\, |N(p)\rangle = \bar{A}^{(n,d)} \{ p^{\mu_1} \ldots p^{\mu_n} + g^{\mu_i\mu_j} \ldots \}$$

removed by traceless-condition (actually suppressed by p^2).

→ left hand side has dimension $(d-2)$

Note that $\bar{A}^{(n,d)}$ has dimension M^{d-n-2}

because $\langle p|p'\rangle = 2p^0 (2\pi)^3 \delta^{(3)}(p-p')$

co-efficient function

$$i \int d^4 z\, e^{iq\cdot z}\, C^{(n,d)}_{\mu\nu\mu_1\ldots\mu_n}(z)$$

only a functn of q.

ψ has dim. $M^{\frac{3}{2}}$

Recall that $T_{\mu\nu}$ is dimensionless

Therefore this co-eff. $f^{\underline{n}}$ has dimension $\cancel{(q^2)^{-(d-2)/2}}$ $\underline{\underline{M^{2-d}}}$

Most general form for C is

$$C^{(n,d)}_{\mu\nu\mu_1\ldots\mu_n}(z) \sim \{ - g_{\mu\nu} z_{\mu_1} \ldots z_{\mu_n}\, a^{(n,d)}(z^2)$$

$$+ g_{\mu_1\mu} g_{\mu_2\nu} z_{\mu_3} \ldots z_{\mu_n}\, b^{(n,d)}(z^2)$$

$$+ \ldots \}$$

$g_{\mu\nu}$

There are only two relevant terms.

$\propto p_\mu p_\nu$

$g_{\mu_1\mu_2} \to p^2$ when contracted with matrix element.

note e.g. $z_\mu z_\nu z_{\mu_1} \ldots z_{\mu_n} \to q_\mu q_\nu \to$

To evaluate F.T. note that

$$z_\mu \to -i\frac{\partial}{\partial q_\mu} \to (-2i)^n q_\mu \frac{\partial}{\partial q^2}$$

Thus $i\int d^4z\, e^{iq\cdot z}\, C^{(n,d)}_{\mu\nu\mu_1\ldots\mu_n}(z)$

$$\sim \Big\{ -g_{\mu\nu} q_{\mu_1}\ldots q_{\mu_n} \overbrace{(-2i)^n \left(\frac{\partial}{\partial q^2}\right)^n i\int d^4z\, e^{iq\cdot z}\, a^{(n,d)}(z^2)}^{\text{function of } q^2 \text{ only.}}$$

$$+ g_{\mu\mu_1} g_{\nu\mu_2} q_{\mu_3}\ldots q_{\mu_n} (-2i)^{n-2}\left(\frac{\partial}{\partial q^2}\right)^{n-2} i\int d^4z\, e^{iq\cdot z}\, b^{(n,d)}(z^2)$$

$$+ \ldots\ldots\ldots \Big\}$$

Now write in terms of dimensionless ~~...~~ "reduced" co-efficients

has dimension $\downarrow M^{2-d}$.

$$\sim \Big\{ -g_{\mu\nu} q_{\mu_1}\ldots q_{\mu_n} (-2i)^n (q^2)^{-(d+n-2)/2}\, \tilde{a}^{(n,d)}(q^2)$$

$$+ g_{\mu\mu_1} g_{\nu\mu_2} q_{\mu_3}\ldots q_{\mu_n} (-2i)^{n-2} (q^2)^{-(n+d-4)/2}\, \tilde{b}^{(n,d)}(q^2)$$

$$+ \ldots \Big\}.$$

Combining this with our expression for the hadronic matrix element yields

$$T_{\mu\nu} \sim \sum_{n,d} \bar{A}^{(n,d)} \{ p^{\mu_1}\ldots p^{\mu_n}\ \cancel{g^{\mu_1\mu_2}\ldots} \}$$

$$\times \Big\{ -g_{\mu\nu} q_{\mu_1}\ldots q_{\mu_n} (-2i)^n (q^2)^{-(d+n-2)/2}\, \tilde{a}^{(n,d)}(q^2)$$

$$+ g_{\mu\mu_1} g_{\nu\mu_2} q_{\mu_3}\ldots q_{\mu_n} (-2i)^{n-2} (q^2)^{-(n+d-4)/2}\, \tilde{b}^{(n,d)}(q^2)$$

$$+ \cancel{g_{\mu_1\mu_2}}\ldots \longrightarrow p^2 \text{ terms}$$

i.e. $$T_{\mu\nu} \sim \sum_{n,d} \bar{A}^{(n,d)} \Big\{ -g_{\mu\nu} \underset{W_1}{(2p.q)^n} (-q^2)^{-n} (q^2)^{-\frac{1}{2}(d-n-2)} i^n \tilde{a}^{(n,d)}(q^2)$$

$$- p_\mu p_\nu \underset{W_2}{\overset{\frac{1}{x}}{\left(\frac{2p.q}{-q^2}\right)}}^{n-1} \frac{1}{2p.q} i^{n-2} \tilde{b}^{(n,d)} (q^2)^{-\frac{1}{2}(d-n-2)} \Big\}$$

~~So the leading contribution to the structure functions comes from those operators for which "d−n" = "twist" is a minimum.~~

~~Higher twist operators are suppressed by powers of q^2~~

Leading twist operators for DIS are twist two

$$O^F_{\mu_1\ldots\mu_n} = \frac{1}{2} \bar{\Psi} T^{\text{flavour}} \gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2} \ldots \overleftrightarrow{D}_{\mu_n} \Psi \, \frac{i^{n-1}}{\cancel{(n-1)!}}$$

Next time will ~~describe calculation~~ relate moments of structure f^n to matrix elements of these operators.

OPE for $T J_\mu J_\nu$

Leading cont^n comes from region operators of lowest leading twist

↑ note this is the Euclidean kinematic region.

end 10/5/93

Lepage & MacKenzie.

$$U_\mu = e^{igaA_\mu}$$

fix in some gauge.

Then $\langle U \rangle = 1 + igaA_\mu - \frac{g^2a^2}{2} A_\mu^2 + \dots$

expect $\langle U \rangle = 1$ large cont$^{=}$

Idea work with $\tilde{U}_\mu = \frac{U_\mu}{U_0}$ where $U_0 = \langle \mathrm{Tr}\, U \rangle$.

Thus $\langle \mathrm{Tr}\, \tilde{U} \rangle = 1$.

"Tadpole improvement"

In this spirit

$$\tilde{S}_{gluon} = \frac{1}{2\tilde{g}^2} \mathrm{Tr}\, \tilde{U}_{plaq}$$

$$= \frac{1}{2\tilde{g}^2 U_0^4} \mathrm{Tr}\, U_{plaq}.$$

This is the same as our ~~ye~~ original action ~~if~~ we write

$$\tilde{g}^2 U_0^4 = g^2$$

$$\tilde{g}^2 = \frac{g^2}{U_0^4} \simeq \frac{g^2}{\frac{1}{3}\mathrm{Tr}\, U_{plaq}}$$

So we use $\tilde{g}$ rather than g as our <u>expansion parameter</u>.

Expansion co-effs for DIS operators

Point out twist / q^2 behavior of operators last time

$$T_{\mu\nu}(q^2) = i \int d^4z \, e^{iq\cdot z} \langle N(p) | T J^+_\mu(z) J_\nu(0) | N(p) \rangle$$

Our analysis of the OPE of last time showed that we could study the behavior of co-effi. between <u>any</u> external states.

$\therefore$ just consider the OPE between free quark states of $t_{\mu\nu}$

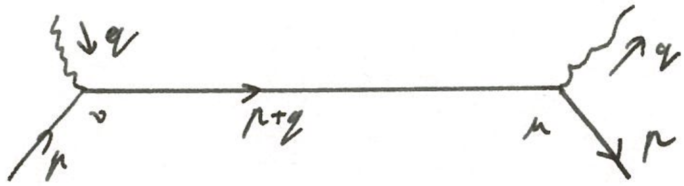
$$t_{\mu\nu}(q^2) = i \int d^4z \, e^{iq\cdot z} \, T J^+_\mu(z) J_\nu(z)$$

i.e. we shall study

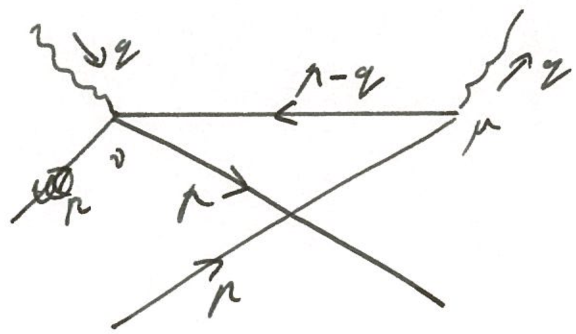
$$M_{\mu\nu}(q^2) = \frac{1}{2} \sum_\sigma i \int d^4z \, e^{iq\cdot z} \langle p,\sigma | T J^+_\mu(z) J_\nu(0) | p,\sigma \rangle$$

Two diagrams contribute

(a) $\rightarrow$



(b)



(b) is related to (a) by simple crossing symmetry $q \rightarrow -q$, $\mu \leftrightarrow \nu$

Then (a) $\to$ $\frac{1}{2} i \, \mathrm{Tr}\,[\not p \gamma_\mu \, i(\not p + \not q)\gamma_\nu] \, \{(p+q)^2\}^{-1}$

$$\to \quad -2\{p_\mu(p_\nu+q_\nu)+p_\nu(p_\mu+q_\mu)-p.q\, g_{\mu\nu}\}\ \{2\nu+q^2\}^{-1}$$

where ~~as~~ $\nu = p.q$

As before, set $\omega = \frac{2\nu}{-q^2}$

Then (a) $\to$ $-\frac{2}{q^2}\{p_\mu(p_\nu+q_\nu)+p_\nu(p_\mu+q_\mu)-p.q\, g_{\mu\nu}\}[1-\omega]^{-1}$

Then (b) is related by crossing

(b) $\to$ $-\frac{2}{q^2}\{p_\nu(p_\mu-q_\mu)+p_\mu(p_\nu-q_\nu)+p.q\, g_{\mu\nu}\}[1+\omega]^{-1}$

$\therefore$ (a) + (b) $\to$ $-g_{\mu\nu}\,\omega\,\{(1-\omega)^{-1}-(1+\omega)^{-1}\}$

$$-\frac{2}{q^2}\{2p_\mu p_\nu(1-\omega)+2p_\nu p_\mu(1+\omega)^{-1}\}.$$

+ cross terms!

<u>Exercise</u> show that the cross-terms appear correctly!

$$\to \quad -g_{\mu\nu}\, 2\omega \sum_{n=1,3} \omega^n + \frac{2\cdot 2p_\mu p_\nu}{(-q^2)}\sum_{n=0,2,4} 2\omega^n \qquad (*)$$

Our expection is that we can write ~~these~~ this matrix element in terms of a series in (twist two) operators

$$O^{\mu_1\ldots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1}\bar\Psi\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\ldots\overleftrightarrow{D}_{\mu_n}\psi$$

where symmetrization & removal of traces is understood. Note here that $\left(\frac{1}{2}\right)^{n-1} \Leftrightarrow$ derivative acting both ways etc

$iD_\mu \to p_\mu$.

Now $\frac{1}{2}\sum_{\tau=1}^{2}\langle p,\tau|O^{\mu_1\ldots\mu_n}|p,\tau\rangle = p^{\mu_1}\ldots p^{\mu_2}$

do this on board

Therefore we can identify powers of p in the expansion (*) with terms in the OPE

So $(*) \rightarrow \quad -g_{\mu\nu}\, 2. \sum_{n=2,4} \mu.\, \frac{2^n (p.q)^n}{(-q^2)^n} + 2.4\, \frac{p_\mu p_\nu}{(-q^2)} \sum_{n=0,2,..} \frac{2^n (p.q)^n}{(-q^2)^n}$

$$\rightarrow \quad -g_{\mu\nu}.2. \sum_{n=2,4,} \frac{2^n}{(-q^2)^n} q_{\mu_1} \ldots q_{\mu_n} \left\{\frac{1}{2}\sum_{r=1}^{2} \langle p,r | O^{\mu_1 \ldots \mu_n} | p,r\rangle\right\}$$

$$+ 2 \sum_{n=2,4,} (g_{\mu\mu_1} g_{\nu\mu_2} q_{\mu_3} \ldots q_{\mu_n})\, 2^n \left(\frac{1}{-q^2}\right)^{n-1} \left\{\frac{1}{2}\sum_{r=1}^{2} \langle p,r | O^{\mu_1 \ldots \mu_n} | p,r\rangle\right\}$$

Comparing with our original expression we find

$$t_{\mu\nu}(q^2) = \quad -g_{\mu\nu} . 2 \sum_{n=2,4,..} \frac{2^n}{(-q^2)^n} q_{\mu_1} . q_{\mu_n} \; O_V^{\mu_1 \ldots \mu_n}$$

$$+ 2 \sum_{n=2,4,} g_{\mu\mu_1} g_{\nu\mu_2} q_{\mu_3} \ldots q_{\mu_n} \; 2^n \left(-\frac{1}{q^2}\right)^{n-1} O_V^{\mu_1 \ldots \mu_n}$$

+ extra terms determined by Lorentz structure

<u>This is tree - level operator product expansion</u>

θ in general $\quad \langle N(p) | \, O_V^{\mu_1 \ldots \mu_n} | N(p)\rangle = \bar{A}^n p^{\mu_1} \ldots p^{\mu_n}$

So $\; T_{\mu\nu}(q^2) = \; i \int d^4 z \, e^{iq.z} \langle N(p) | \, T J_\mu^+(z) J_\nu(0) | N(p)\rangle$

$$\rightarrow \quad -g_{\mu\nu} . 2 \sum_{n=2,4..} \underbrace{\frac{2^n}{(-q^2)^n} \; (p.q)^n}_{\text{Tree level}} \; \bar{A}^n$$

$$+ 2 \sum_{n=2,4,6,...} p_\mu p_\nu \, (p.q)^{n-2} \; 2^n \left(-\frac{1}{q^2}\right)^{n-1} \bar{A}^n$$

Now we have to relate $T_{\mu\nu}(q^2;\nu)$ to our structure functions,

& in general $T_{\mu\nu}(q^2,\omega) = -g_{\mu\nu}\, 2 \sum_{n=2,4.} \omega^n\, C^{n,1}(q^2)\, \bar{A}^n$ (are 1 to lowest order)

$$+ 2 \sum_{n=2,4.} p_\mu p_\nu \cdot \frac{2\omega^{n-1}}{\nu} \bar{A}^n\, C^{n,2}(q^2)$$

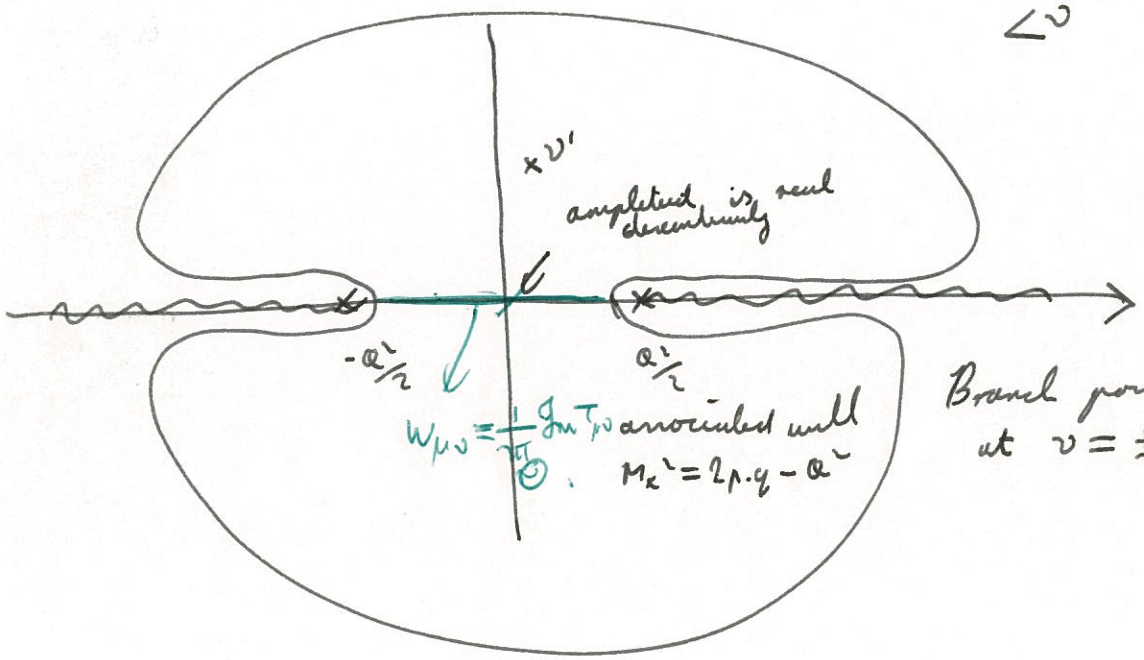
$$W_{\mu\nu} = 2 \cdot \frac{1}{4\pi}\, \mathrm{Im}\, T_{\mu\nu}$$

Write $W_{\mu\nu} = -(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) W_1 + (p_\mu - \frac{p\cdot q}{q^2} q_\mu)(p_\nu - \frac{p\cdot q}{q^2} q_\nu) W_2$

The ~~physical~~ regime, in which (4) is convergent, is $0 < \omega < 1$

The physical regime is $1 < \omega < \infty$ (i.e. $\frac{Q^2}{2} < \nu < \infty$)

We can relate the two through a dispersion relation



Branch point at $\nu = \pm Q^2/2$

$$T_i(q^2;\nu) = \frac{1}{2\pi i} \int \frac{d\nu'}{\nu'-\nu}\, T_i(q^2,\nu')$$

$$= \frac{1}{2\pi i}\left\{ \int_{Q^2/2}^{\infty} d\nu' \,\mathrm{Disc}\, \frac{T_i(q^2,\nu')}{\nu'-\nu} - \int_{-\infty}^{-Q^2/2} d\nu'\, \mathrm{Disc}\, \frac{T_i(q^2,\nu')}{\nu'-\nu} \right\}$$

i.e. $T_i(q^2,\nu)_{disc} = 2i\,\mathrm{Im}\,T_i(q^2,\nu)$

So $$T_i(q^2,\nu) = \frac{1}{\pi}\left\{ \int_{Q^2/2}^{\infty} \frac{d\nu'}{\nu'-\nu}\,\mathrm{Im}\,T_i(q^2,\nu') - \int_{-\infty}^{-Q^2/2} \frac{d\nu'}{\nu'-\nu}\,\mathrm{Im}\,T_i(q^2,\nu') \right\}$$

i.e. $$T_i(q^2,\nu) = \frac{1}{\pi}\left\{ \int_{Q^2/2}^{\infty} \frac{d\nu'}{\nu'-\nu}\,\mathrm{Im}\,T_i(q^2,\nu') + \int_{Q^2/2}^{\infty} \frac{d\nu'}{\nu'+\nu}\,\mathrm{Im}\,T_i(q^2,+\nu') \right\}$$

$T_i(q^2,\nu') = T_i(q^2,-\nu')$.

i.e. $$T_i(q^2,\nu) = \frac{1}{\pi}\int_{Q^2/2}^{\infty} \frac{2\nu'\,d\nu'}{\nu'^2-\nu^2}\,\mathrm{Im}\,T_i(q^2,\nu')$$

Let $x = \dfrac{Q^2}{2\nu}$

$$T_i(Q^2,x) = \frac{1}{\pi}\,2\int_0^1 \frac{1}{x'}\,\frac{dx'}{(1-x'^2/x^2)}\,\mathrm{Im}\,T_i(Q^2,x')$$

But $$W_{\mu\nu} = \frac{1}{2\pi}\,\mathrm{Im}\,T_{\mu\nu}$$

i.e. $$T_i(Q^2,x) = 4\int_0^1 \frac{1}{x'}\,\frac{dx'}{1-x'^2/x^2}\,W_i(Q^2,x')$$

i.e. $$T_i(Q^2,x) = 4\sum_{n=0,2,4} x^{-n}\int_0^1 dx'\,(x')^{n-1}\,W_i(Q^2,x')$$

So we can relate the moments of the structure functions to the ~~expansion~~ operators O^ν through

$\frac{2p \cdot q}{Q^2} \longrightarrow$ obviously converges at $|\nu| \to \infty$.

We can compare this with

$$T_{\mu\nu}(Q^2, x) = -2 g_{\mu\nu} \sum_{n=2,4} x^{-n} \, C^{n,1}(Q^2, \mu) \, \bar{A}^n$$

$$+ \, 2 \sum_{n=2,4} p_\mu p_\nu \, \frac{2 x^{-n+1}}{\nu} \, \bar{A}^n \, C^{n,2}(Q^2, \mu)$$

and we find

$$\left. \begin{aligned} 2 \int_0^1 F_1(x)\, x^{n-1}\, dx &= C^{n,1} \, \bar{A}^n \\ 2 \int_0^1 \frac{1}{2x} F_2(x)\, x^{n-1}\, dx &= C^{n,2} \, \bar{A}^n \end{aligned} \right\}$$

assoc. with average over spins.

The moments of the structure functions are related to the matrix elements of the local operators $O_V^{\mu_1 \ldots \mu_n}$

This also gives the Callan-Gross relation

$$\underline{\underline{F_1 = \frac{1}{2x} F_2}}$$