Hadron Structure

David Richards

Jefferson Laboratory
Hadron Structure

How are
- charge and currents
- momentum
- spin and angular momentum

apportioned amongst the quarks and gluons that make up a hadron?

Encapsulated in
- electromagnetic form factors
- unpolarized structure functions and Transverse-momentum-dependent distributions (TMDs)
- polarized structure functions, Generalized Parton Distributions (GPDs), TMDs
Paradigm: Pion EM form factor

\[ \langle \pi(p_f) | V_\mu(0) | \pi(p_i) \rangle = (p_i + p_f)_\mu F(Q^2) \]

where

\[ V_\mu = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d \]

\[ -Q^2 = [E_\pi(p_f) - E_\pi(p_i)]^2 - (p_f - p_i)^2 \]
Pion Interpolating Operator

\[
\begin{align*}
\phi(x) &= \bar{d}(x)\gamma_5 u(x) \\
\phi^\dagger(x) &= -\bar{u}(x)\gamma_5 d(x) \\
V_\mu(x) &= e_u \bar{u}(x)\gamma_\mu u(x) + e_d \bar{d}(x)\gamma_\mu d(x).
\end{align*}
\]

Sequential-source propagator

\[
\Gamma_{\pi^+\mu\pi^+}(t_f, t; \vec{p}, \vec{q}) = \sum_{\vec{x}, \vec{y}} \langle 0 | \phi(\vec{x}, t_f) V_\mu(\vec{y}, t) \phi^\dagger(\vec{0}, 0) | 0 \rangle e^{-i\vec{p} \cdot \vec{x}} e^{-i\vec{q} \cdot \vec{y}},
\]

\[
V^\text{cont}_\mu = Z_V V^\text{lattice}_\mu; \quad Z_V = 1 \quad \text{for conserved current}
\]
Anatomy of a Matrix Element Calculation - II

Construction of three-point function

Introduce quark propagators

\[ U_{\alpha\beta}^{ij}(x, y) = \langle u_{\alpha}^i(x) \bar{u}_{\beta}^j(y) \rangle \]
\[ D_{\alpha\beta}^{ij}(x, y) = \langle d_{\alpha}^i(x) \bar{d}_{\beta}^j(y) \rangle, \]

Then U-contribution to three-point function given by

\[ \Gamma_{\pi+\mu+}^U = e_u \sum \left( e^{-i\vec{p} \cdot \vec{x} - i\vec{q} \cdot \vec{y}} \right) \text{Tr} \left\{ \gamma_5 U(x, y) \gamma_\mu U(y, 0) \gamma_5 D(0, x) \right\} \]

Quark propagator: \( G_{\alpha\beta}^{ij}(x, y) = \langle q_{\alpha}^i(x) q_{\beta}^j(y) \rangle \) satisfies

\[ M_{\alpha\gamma}(x, z) G_{\gamma\beta}^{kj}(z, y) = \delta_{ij} \delta_{\alpha\beta} \delta_{xy}; \quad G(y, x) = \gamma_5 G(x, y)^\dagger \gamma_5 \]

Introduce Sequential Quark Propagator \( H^u(y, 0; t_f, \vec{p}) = \sum_x e^{i\vec{p} \cdot \vec{x}} U(x, y) \gamma_5 D(x, 0) \gamma_5 \)

Satisfies: \( M(z, y) H^u(y, 0; t_f, \vec{p}) = \delta_{t_z, t_f} e^{i\vec{p} \cdot \vec{z}} \gamma_5 D(z, 0) \gamma_5 \)

Finally:

\[ \Gamma_{\pi+\mu+}^U = e_u \sum \left( e^{-i\vec{q} \cdot \vec{y}} \right) \text{Tr} \left\{ H^u(y, 0; t_f, \vec{p})^\dagger \gamma_5 \gamma_\mu U(y, 0) \gamma_5 \right\} \]
Anatomy of a Matrix Element Calculation - II

\[
\Gamma_{\pi^+\mu\pi^+}(t_f, t; \vec{p}, \vec{q}) = \sum_{\vec{x}, \vec{y}} \langle 0 | \phi(\vec{x}, t_f) V_\mu(\vec{y}, t) \phi^\dagger(\vec{0}, 0) | 0 \rangle e^{-i\vec{p} \cdot \vec{x}} e^{-i\vec{q} \cdot \vec{y}} ,
\]

Resolution of unity – insert states

\[
\langle 0 | \phi(0) | \pi, \vec{p} + \vec{q} \rangle \langle \pi, \vec{p} + \vec{q} | V_\mu(0) | \pi, \vec{p} \rangle \langle \pi, \vec{p} | \phi^\dagger | 0 \rangle e^{-E(\vec{p}(t-t_i))} e^{-E(\vec{p}+\vec{q})(t_f-t)}
\]

\[
\Gamma_{\pi^+\pi^+}(t, 0; \vec{p}) = \sum_{\vec{x}} \langle 0 | \phi(\vec{x}, t_f) \phi^\dagger(0) | 0 \rangle e^{-i\vec{p} \cdot \vec{x}}
\]

\[
\propto |\langle 0 | \phi(0) | \pi, \vec{p} \rangle|^2 e^{-E(\vec{p}) t}
\]

\[
3pt
\]

\[
2pt
\]
Pion Form Factor

\[ F(Q^2) = \frac{1}{1 + \frac{Q^2}{M_{\text{VMD}}^2}} \]

Charge radius \( \langle r^2 \rangle = 6 \left. \frac{dF(q^2)}{dq^2} \right|_{q^2=0} \)


\[ m_{\text{VMD}} = 1030(73) \text{ MeV} \]
\[ m_{\pi}/m_p = 758 / 1060 (\text{MeV}) \]
\[ m_{\text{VMD}} = 888(56) \text{ MeV} \]
\[ m_{\pi}/m_p = 318 / 956 (\text{MeV}) \]

Nguyen et al, 1102.3652
Two form factors

\[ \langle p_f \mid V_\mu \mid p_i \rangle = \bar{u}(p_f) \left[ \gamma_\mu F_1(q^2) + iq_\nu \frac{\sigma_{\mu\nu}}{2m_N} F_2(q^2) \right] u(p_i) \]

Related to familiar Sach’s electromagnetic form factors through

\[
\begin{align*}
G_E(Q^2) &= F_1(Q^2) - \frac{Q^2}{(2m_N)^2} F_2(Q^2) \\
G_M(Q^2) &= F_1(Q^2) + F_2(Q^2)
\end{align*}
\]

**Isovector**: difference between p and n or difference between u and d currents.
Electromagnetic Form Factors

Wilson-clover lattices from BMW


Hadron structure at nearly-physical quark masses

Large $Q^2$ behavior: Hall C at JLab to 15 GeV$^2$
Hadron Structure

M Constantinou, arXiv:1511.00214

- Governs beta-decay rate
- Important for proton-proton fusion rate in solar models
- Benchmark for lattice QCD calculations of hadron structure

Luxury of large statistical errors! $m_\pi L < 4$
Structure Functions - I

\[ Q^2 = -q^2 = (k' - k)^2 \]
\[ \nu = q \cdot P/M \]
\[ x = \frac{Q^2}{2M \nu} \]

Bjorken limit:
\[ Q^2 \to \infty, \nu \to \infty, x \text{ fixed} \]

The structure functions are defined in terms of the hadronic tensor:

\[ W_{\mu\nu} = \frac{1}{4\pi} \int dz e^{i\mathbf{q} \cdot \mathbf{z}} \langle N(p, S) \mid J_\mu(z) J_\mu(0) \mid N(p, S) \rangle \]

Yields two unpolarized structure functions \( F_1(x, Q^2) \) and \( F_2(x, Q^2) \), and two polarized structure functions \( g_1(x, Q^2) \) and \( g_2(x, Q^2) \)

Leading twist structure functions: product of currents at light-like \( z^2 \to 0 \)

In Euclidean lattice QCD, use OPE to write in terms of local operators whose matrix elements we can compute in Euclidean space
Structure Functions - II

Operators

\[ O_q^{\mu_1 \mu_2 \ldots \mu_n} = \bar{\psi}_q \gamma_5 \gamma^{\mu_1} i D^{\mu_2} \ldots D^{\mu_n} \psi_q \]

Matrix elements related to moments of structure functions

\[ \int_0^1 dx \, x^{n-1} F_2(x, Q^2) = \sum_{q=u,d} C_n (\mu^2/Q^2, g(\mu)) \langle x^n \rangle (\mu) \]

where

\[ \langle N(p) \mid O_q^{\mu_1 \ldots \mu_{n+1}} \mid N(p) \rangle = \langle x^n \rangle (\mu) [p_{\mu_1} \ldots p_{\mu_{n+1}}] \]

Perturbation theory

Non-perturbatively

\[ O^{\text{cont}} = Z O^{\text{latt}} \]
Axial-vector Charge

Luxury of large statistical errors! $m_\pi L < 4$

M Constantinou, arXiv:1511.00214
• Need to go to approach physical light-quark masses: chiral behavior
Quark Momentum Helicities

LHPC, 2010: DWF valence, Asqtad sea

- Need to go to approach physical light-quark masses: chiral behavior

Similar renormalization prescription

HBChPT

RBC/UKQCD 2010: DWF
Moments of Parton Distributions

We are computing moments

$$O_q^{\{\mu_1 \mu_2 \ldots \mu_n\}} = \bar{\psi}_q \gamma_5 \gamma^{\{\mu_1 , D^\mu_2 \ldots D^{\mu_n}\}} \psi_q$$

Do not have full Lorentz symmetry

$n >= 5$: operator mixing

Need to assume parametrization

$$x(u_v(x) - d_v(x)) = a x^b (1 - x)^c (1 + e \sqrt{x} + \gamma x)$$

Detmold, Melnitchouk, Thomas
3D Imaging of Nucleon

Wigner distributions

Parton Distribution Functions

Form Factors

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Different Regimes in Different Experiments

Form Factors
- transverse quark distribution in Coordinate space

Structure Functions
- longitudinal quark distribution in momentum space

GPDs
- Fully-correlated quark distribution in both coordinate and momentum space

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Generalized Parton Distributions (GPDs)


\[ J_q = \frac{1}{2} - J^G = \frac{1}{2} \int_{-1}^{1} x \, dx \left[ H^q(x, \xi, 0) + E^q(x, \xi, 0) \right] \]

\( \xi \) is skewness

Measured in Deeply Virtual Compton Scattering
Moments of GPD’s

- Matrix elements of light-cone correlation functions

\[ O(x) = \int \frac{d\lambda}{4\pi} e^{i\lambda x} \overline{\psi} \left( -\frac{\lambda}{2} n \right) nP e^{-ig \int_{\lambda/2}^{\lambda} d\alpha n \cdot A(\alpha n)} \psi \left( \frac{\lambda}{2} n \right) \]

- Expand \( O(x) \) around light-cone

\[ O_q^{\{\mu_1 \mu_2 \ldots \mu_n\}} = \overline{\psi}_q \gamma^{\{\mu_1 i} D^{\mu_2} \ldots D^{\mu_n\}} \psi_q \]

- **Off-forward** matrix element

\[ \langle P' | O_q^{\{\mu_1 \ldots \mu_n\}} | P \rangle \simeq \int dx x^{n-1} [H(x, \xi, t), E(x, \xi, t)] \]

\[ \rightarrow A_{ni}(t), B_{ni}(t), C_n(t), \bar{A}_{ni}(t), \bar{B}_{ni}(t), \bar{C}_n(t) \]

\[ \uparrow \]

LHPC, QCDSF, 2003

Co-efficient of \( \xi^i \)
GPDs and Orbital Angular Momentum

- Form factors of energy momentum tensor - quark and gluon angular momentum

\[
\frac{1}{2} = \sum_q J^q + J^g = \frac{1}{2} \left\{ \sum_q \left( A^q_{20}(t = 0) + B^q_{20}(t = 0) \right) + A^g_{20}(t = 0) + B^g_{20}(t = 0) \right\} \\
\sum_q \left( \frac{1}{2} \Delta \Sigma^q + L^q \right)
\]

X.D. Ji, PRL 78, 610 (1997)

Decomposition
- Gauge-invariant
- Renormalization-scale dependent
- Handle on Quark orbital angular momentum


Gluon operators - see later
Origin of Nucleon Spin

\[
J^q = \frac{1}{2} \left( A^q_{20}(t = 0) + B^q_{20}(t = 0) \right)
\]

\[
\Delta \Sigma^q / 2 = \bar{A}^q_{10}(t = 0) / 2
\]

\[
\frac{1}{2} = \frac{1}{2} \Delta \Sigma^{u+d} + L^{u+d} + J^g
\]

LHPC, Haegler et al.,

Total orbital angular momentum carried by quarks small
Orbital angular momentum carried by quark flavors substantial
Origin of Nucleon Spin - II

Ph. Hägler, MENU 2010

[JLab Hall A PRL '07; HERMES JHEP '08]

Ph. Hagler, Menu 20010

LHPC arXiv:1001.3620 (this work)
LHPC PRD '08 0705.4295
QCDSF (Ohtani et al.) 0710.1534
Goloskokov&Kroll EPJC '09 0809.4126
Wakamatsu 0908.0972
DiFeJaKr EPJC '05 hep-ph/0408173
(Myhrer&)Thomas PRL '08 0803.2775

\[ M_{\text{S at 4 GeV}^2} \]
Transverse Distribution - I

- t-dependence ↔ impact parameter

\[ A_{n0}^q(-\bar{\Delta}_x^2) = \int d^2 b_\perp \ e^{i\bar{\Delta}_x \cdot \bar{b}_\perp} \int_{-1}^{1} dx \ x^{n-1} q(x, \bar{b}_\perp) \]

Compare to phenomenological models

Decrease slope: decreasing transverse size as x 1

Burkardt
Lattice results consistent with narrowing of transverse size with increasing $x$

*Flattening of GFFs with increasing $n*$

Transverse momentum distributions (TMDs)

from experiment, e.g., SIDIS (semi-inclusive deep inelastic scattering)

**HERMES, COMPASS, JLab 6 GeV, JLab 12 GeV, ... , EIC**

Cf: measured in Drell-Yan, eg at RHIC-spin

**final state interactions!**
explain large asymmetries otherwise forbidden!
**signature of QCD!**

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Bernhard Musch
Transverse-Momentum Distributions


Introduce Momentum-space correlators

\[ \Phi_\Gamma = \int d(n \cdot k) \int \frac{d^4 l}{2(2\pi)^4} e^{-ik \cdot l} \tilde{\Phi}_\Gamma(l; P, S) \]

\[ = \int d(n \cdot k) \int \frac{d^4 l}{2(2\pi)^4} e^{-ik \cdot l} \langle P, S | \bar{q}(l) U q(0) | P, S \rangle \]

Choice of path - retain gauge invariance

Real world!: path runs to infinity

Lattice: equal time slice

continuum

\[ U \equiv \mathcal{P} \exp \left( -ig \int_0^\ell d\xi^\mu A_\mu(\xi) \right) \]

along path from 0 to \( \ell \)

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0  \( \ell \)  \( \infty \)
Flavor-Singlet Hadron Structure
Flavor-singlet Quantities

\[
\langle p_f \mid V_\mu \mid p_i \rangle = \bar{u}(p_f) \left[ \gamma_\mu F_1(q^2) + i q_\nu \frac{\sigma_\mu\nu}{2m_N} F_2(q^2) \right] u(p_i)
\]

\[
G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{(2m_N)^2} F_2(Q^2)
\]

\[
G_M(Q^2) = F_1(Q^2) + F_2(Q^2)
\]

**Isoscalar:** p and n separately, or u and d separated contribution.

\[
V_\mu = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s
\]

Strange-quark contribution to hadron structure.
Flavor-singlet: Disconnected Contributions

Parity-violating electron scattering

\[ G_{E/M}^{\gamma,p} = \frac{2}{3} G_{E,M}^{u} - \frac{1}{3} G_{E,M}^{d} - \frac{1}{3} G_{E,M}^{s} \]

\[ G_{E/M}^{Z,p} = (1 - \frac{8}{3} \sin^2 \theta_W) G_{E,M}^{u} - (1 - \frac{4}{3} \sin^2 \theta_W) G_{E,M}^{d} - (1 - \frac{4}{3} \sin^2 \theta_W) G_{E,M}^{s} \]

Expected to be small

Spin carried by s-quark

\[ \Delta s = -0.085(13)(8)(9) \]

HERMES: dominated by small x
Disconnected contributions

Three-point correlator looks like

\[ \Gamma_{N \mu N}^{\text{disc}}(t_f, t, 0; \vec{p}, \vec{q}) = \sum_{\vec{x}, \vec{y}} \langle 0 | N(\vec{x}, t_f) \bar{s}(\vec{y}, t) \Gamma s(\vec{y}, t) \bar{N}(0, 0) | 0 \rangle e^{-i\vec{p} \cdot \vec{x}} e^{-i\vec{q} \cdot \vec{y}} \]

\[ = \sum_{\vec{x}} \langle 0 | N(\vec{x}, t_f) \left( \sum_{\vec{y}} \bar{s}(\vec{y}, t) \Gamma s(\vec{y}, t) e^{-i\vec{q} \cdot \vec{y}} \right) \bar{N}(0, 0) | 0 \rangle e^{-i\vec{p} \cdot \vec{x}} \]

Need efficient means of evaluating

\[ \sum_{\vec{y}} \text{Tr}[M^{-1}(\vec{y}, t; \vec{y}, t) \Gamma] \]

Straightforward way: introduce noise vectors such that

\[ < \eta_i > = 0; \quad < \eta_i \eta_j > = \delta_{ij} \]

Solve \( MX = \eta \): then \( < M^{-1}_{ij} > = < \eta_j X_i > \)

Error both from Gauge Noise and from Stochastic noise

Noise-reduction methods
- Partitioning ("dilution") - sources have support on, say, 8 timeslices
- Hopping parameter expansion
- Different stochastic sources
Isotropically generated Clover Gauge Generation for Hadron Structure at ORNL and at BlueWaters.

**Proton**

\[ \frac{G_{E,M}^{(2/3u^{-1/3}d)}}{G_{E,M}^{(2/3u^{-1/3}d)}} \] (disconnected)

\[ \frac{G_{E,M}^{(2/3u^{-1/3}d)}}{G_{E,M}^{(2/3u^{-1/3}d)}} \] (connected)

**LHPC preliminary**

\[ m_\pi \approx 317 \text{ MeV} \]

\[ t_{\text{src–snk}} \approx 1.14 \text{ fm} \]
Sea Quark Contributions


Synergy with computer scientists - precision calculation of sea quark contributions now possible
Quark and gluons mix under renormalization

\[
\frac{\partial}{\partial \ln \mu^2} \begin{pmatrix} q^S \\ g \end{pmatrix} = \frac{\alpha_s(\mu^2)}{2\pi} \begin{pmatrix} P_{qq} & 2n_fP_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q^S \\ g \end{pmatrix}
\]

The local operators mix as follows:

\[
O^{qS}_{\mu_1\cdots\mu_N} = \frac{1}{2^N} \overline{\psi} \gamma_{[\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_N]} (1 \pm \gamma_5) \psi
\]

\[
O^{gS}_{\mu_1\cdots\mu_N} = \sum_{\rho} \text{Tr} \left[ F_{[\mu_1\rho} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_{N-1}}} F_{\rho\mu_N]} \right]
\]
Flavor-separated and Gluon Contributions

Complete calculation of flavor-separated and gluonic contributions to nucleon spin

Deka et al, arXiv:1312.4816

\[
T_{\mu\nu} = \frac{1}{4} \bar{\psi} \gamma(\mu D_\nu) \psi + G_{\mu\alpha} G^{\nu\alpha} - \frac{1}{4} \delta_{\mu\nu} G^2; \langle P \mid T_{\mu\nu} \mid P \rangle = P_\mu P_\nu / M
\]
Formulation of LQCD in Euclidean space precludes direct calculation of light-cone correlation functions

→ LQCD computes Moments of parton distributions

New ideas: calculations of QUASI-distributions in *infinite-momentum frame*

\[
\tilde{q}(x, \mu, P_z) = \int \frac{dz}{4\pi} e^{-izk} \times \left\langle \bar{\psi}(z) \gamma_z e^{ig \int_0^z A_z(z')dz'} \psi(0) \left| \vec{P} \right\rangle \right.
\]

"Equal time" correlator

First lattice calculations of Quasi Distributions


\[
\tilde{q}(x, \mu, P_z) = \int \frac{dy}{|y|} Z \left( \frac{x}{y}, \frac{\mu}{P_z} \right) q(y, \mu) + O \left( \frac{\Lambda_{QCD}^2}{P_z^2}, \frac{M_N^2}{P_z^2} \right) + \ldots
\]

MSTW
CJ12
Lattice

\[\Delta u - \Delta d\]

\[\bar{q}(x) \rightarrow q(x)\]

smallest \(x \approx 1/a\)

12 GeV; Future EIC

Violation of Gottfried sum rule \(\bar{d}(x) > \bar{u}(x)\)
Summary

• Lattice Calculations of the simplest quantities are now appearing at physical values of the quark masses

• *High-precision calculations of local matrix elements* - *relevant for searches for new physics in, e.g. UCN.*
  – To directly explore x distributions, there are now a slew of new ideas… Ji et al, Qiu et al.

• Major effort underway in US in generating lattices designed for hadron structure calculations.