Relativistic Description of Few-Nucleon Systems

LECTURES FOR THE 2007 PRAGUE SUMMER SCHOOL

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Plan: Each lecture will be accompanied by questions and problems, some requiring original research

Outline:

★ Lecture 1: Overview: discussion of relativistic methods
  • Two schools for the relativistic description of few nucleon systems will be described.
  • How do these approaches handle the problem of relativity and what are the advantages and disadvantages of each?

★ Lecture 2: Theory: two and three nucleon systems
  • Introduction to the Covariant Spectator Theory.
  • How are the bound state and scattering equations obtained, what are the normalization conditions?
Outline (continued)

★ Lecture 3: Results: energies below the pion production threshold
  • New, high precision fits to np data below 350 MeV lab energy, and the relativistic properties of the deuteron and triton.
  • What do these new results tell us about the nature of nuclear forces?

★ Lecture 4: Electromagnetic interactions: gauge invariance and effective current operators
  • General method for doing gauge invariant calculations in systems composed of composite particles.
  • What can be learned from high energy electron scattering experiments?

Lecture I:
Overview: Discussion of Relativistic Methods
Outline

★ Overview of relativistic methods: “Two schools”
★ Field dynamics (also referred, in these lectures, to as “field form”)
  • Relativistic interactions and equations in field theory
  • Introduction to Bethe-Salpeter (BS) and Covariant Spectator® (CS) equations
  • Description of bound states in field dynamics
★ Hamiltonian Dynamics
  • Basic theory in “instant form”
  • Comparison with field form
  • Poincaré transformations
  • Dirac’s forms of dynamics
  • The Bakamjian-Thomas construction
  • The mass operator
★ Cluster separability
★ Conclusions

First -- why use a relativistic theory?

★ NOT because
  • of size of \((v/c)^2\) corrections (although they may be large in some applications)
  • it is more accurate (it may not be)
  • it is "better" than EFT (it complements EFT)
★ Use a covariant theory for the following reasons
  • Intellectual: to preserve an exact symmetry (Poncare' invariance)
  • Practical: to calculate boosts and Lorentz kinematics consistently to all orders (essential when energies are of the order of 1 GeV)
★ Consistent: to use field theory for guidance in the construction of
  • forces (2⇌3 body consistency)
  • currents consistent with forces
★ Conceptual: for "phenomenological economy", and to understand the non relativistic limit:
  • spin 1/2 particles (Dirac equation)
  • interpretation of L•S forces (covariant scalar-vector theory of N matter)
  • efficient one boson exchange models of NN forces (?)
Overview of relativistic methods: Two “schools”

Relativity with a fixed number of particles

Hamiltonian dynamics

* On-shell particles
  + no negative energy states
  - loose locality and manifest covariance

Point form
Front form
Klink
Strikman
Miller
Salme

Instant form
Schiavilla
Arenhove

Field dynamics

* Off-shell particles
  + manifest covariance and locality
  - must include negative energy states

Equal Time (ET)
manifest covariance

BSLT*  PWM†
Spectator
Bethe
Salpeter

*Blankenbecler & Sugar, Logunov & Tavkhelidze
†Phillips, Wallace, and Mandelzweig

FIELD DYNAMICS
Relativistic interactions in field theory

Diagrammatic derivation for 2 body scattering:

- The exact scattering amplitude is the sum of all Feynman diagrams:

  \[ M = \text{ladder sum} + \text{crossed ladder} + \text{vertex correction} + \text{self energy} + \cdots \]

- Divide the sum into irreducible and 2-body reducible terms, and collect the irreducible terms into a kernel, which is iterated.

Field Theory: How are bound states described?

- What is a bound state in field theory?

  A bound state is a new particle (not in the Lagrangian) that arises because of the interactions. The vertex function \( \Gamma \) describes how it couples to the elementary particles in the Lagrangian:

  \[
  p_1 = \frac{1}{2} P + p \\
  p_2 = \frac{1}{2} P - p
  \]

  Notation:
  - \( P \) = total momentum (always conserved)
  - \( p \) = relative momentum

- The bound state produces a pole in the scattering amplitude which does not correspond to one of the elementary particles in the theory:

  \[ M = \Gamma(p') \frac{1}{(M_B^2 - P^2)} \Gamma(p) \]

- If the bound state is not elementary, no single Feynman diagram will have the bound state pole; it must be generated from an infinite sum of Feynman diagrams, much as the geometric series generates a pole at \( z = 1 \):

  \[ M = z + z^2 + z^3 + \cdots = \frac{z}{1 - z} \]
Relativistic scattering equations in field theory

- The scattering equation is
  \[ M(p', p; P) = V(p', p; P) + \int V(p', k; P) G(k; P) M(k, p; P) \]
  where \( V \) is the kernel (i.e. potential) and \( G \) is the propagator.

- If the kernel is phenomenological, this is field dynamics instead of field theory.

- The bound state equation follows by assuming the \( M \) matrix has a pole, and substituting
  \[ M(p', p; P) = \frac{\Gamma(p', M_\beta) \overline{\Gamma}(p, M_\beta)}{M_\beta^2 - p^2} = V(p', p; P) \]
  \[ + \int V(p', k; P) G(k; P) \frac{\Gamma(k, M_\beta) \overline{\Gamma}(p, M_\beta)}{M_\beta^2 - P^2} \]
  extracting the pole part gives the bound state equation uniquely
  \[ \Gamma(p', M_\beta) = \int V(p', k; M_\beta) G(k; M_\beta) \Gamma(k, M_\beta) \]

- This equation also insure that the non-pole parts of the scattering amplitude do not contribute near the pole (next lecture).

Two field dynamical equations

- The Bethe-Salpeter (BS) propagator depends on all four components of the relative momentum, \( \{k_0, k\} \). For two scalar particles it is
  \[ G_{BS}(k; P) = \frac{1}{(m_1^2 - p_1^2 - \Sigma(p_1^2) - i\epsilon)(m_2^2 - p_2^2 - \Sigma(p_2^2) - i\epsilon)} \]
  with \( p_1 = \frac{1}{2} P + k \) \( p_2 = \frac{1}{2} P - k \)

- The Covariant Spectator\(^\text{®} \) propagator depends on only three components of the relative momentum, \( k \). One particle is on-shell
  \[ G_{CS}(k; P) = \frac{2\pi i \delta\left(m_1^2 - \left(\frac{1}{2} P + k\right)^2\right)}{(m_2^2 - p_2^2 - \Sigma(p_2^2) - i\epsilon)} = \frac{2\pi i \delta(k_0 - E_1 + \frac{1}{2} P_0)}{2E_1 \left(E_2^2 - (P_0 - E_1)^2 - \Sigma(p_2^2) - i\epsilon\right)} \]

**exercise:** write the explicit form of these equations in a \( \phi^4 \) theory
These equations both have a connection to field theory

- The Bethe-Salpeter amplitude is a well defined field theoretic matrix element:
  \[ \Psi(x_1, x_2) = \left\langle 0 | T(\psi(x_2)\psi(x_1)) | d \right\rangle \]

- The Covariant Spectator© amplitude is also a well defined field theoretic amplitude:
  \[ \Psi(x_1) = \left\langle N | \psi(x_1) | d \right\rangle \]

- Equations for the Bethe-Salpeter and the Spectator* amplitudes can be derived from field theory
  
  - Both are manifestly covariant under all Poincaré transformations (advantage)
  - Both incorporate negative energy (antiparticle) states (disadvantage)

*O. W. Greenberg's "n-quantum approximation"

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**HAMILTONIAN DYNAMICS**

*B. D. Keister and W. N. Polyzou, Ad. in Nucl. Phys. 20, 225 (1991)*
Hamiltonian dynamics: basic theory (in “instant” form)

- Start with a Hilbert space of free particle states \((H_0 - E_i)\phi_i = 0\)
- Interactions described by the interaction Hamiltonian, \(H_I\)
  \[
  (H_0 - E_i)\Psi_i = H_I \Psi_i \Rightarrow \Psi_i = \phi_i + \frac{1}{H_0 - E_i} H_I \Psi_i
  \]
- Solve by iteration (perturbation theory)
  \[
  \Psi_i = \phi_i + \sum_{j \neq i} \phi_j \frac{1}{E_j - E_i} H_{ji} + \sum_{j \neq i} \phi_j \left( \frac{1}{E_j - E_i} H_{ji} \left( \frac{1}{E_j - E_i} H_{ji} \right) \right) + \cdots
  \]
- The scattering amplitude is then
  \[
  M_{ki} = \langle \phi_k | H_I | \Psi_i \rangle = H_{ki} + \sum_{j \neq i} H_{kj} \frac{1}{E_j - E_i} H_{ji} + \sum_{j \neq i} H_{kj} \left( \frac{1}{E_j - E_i} H_{ji} \left( \frac{1}{E_j - E_i} H_{ji} \right) \right) H_{ji} + \cdots
  \]

exercise: check these relations

Comparison with Field form: \(\phi^4\) theory in 1+1 dimensions

- Consider scattering from the 2nd order bubble
  \[
  \frac{1}{2} P + k = \frac{1}{2} P - k + \text{additional part for manifest covariance}
  \]
- In field theory (Feynman diagrams) this is
  \[
  B(s) = i \int \frac{d^2k}{(2\pi)^2} \frac{\lambda^2}{(m_1^2 + k_+^2 - (\frac{1}{2} P + k_0)^2 - i\epsilon)(m_2^2 + k_-^2 - (\frac{1}{2} P - k_0)^2 - i\epsilon)}
  \]
  \[
  = -\lambda^2 \int \frac{dk_z}{(2\pi)} \left( \frac{1}{4E_1E_2} \left( \frac{1}{E_1 + E_2 - P_0} + \frac{1}{E_1 + E_2 + 2P_0 - P_0} \right) \right)
  \]
- Conclusion 1: Manifest covariance obtained when BOTH positive and negative energy contributions are included.
- Conclusion 2: ONE Feynman diagram is the sum of ALL possible time-ordered graphs.

Hamiltonian part

additional part for manifest covariance
The Poincaré group and Dirac forms of dynamics

- The Poincaré group are unitary operators on the Hilbert space, with 10 generators: \( P_0, P, J, \) and \( K \), satisfying the following 45 CR’s:

\[
\begin{align*}
[J^0, J^i] &= i\epsilon^{ij} J^j, \\
[J^i, K^j] &= i\epsilon^{ij} K^j, \\
[J^i, P^j] &= i\epsilon^{ij} P^j, \\
[K^i, K^j] &= -i\epsilon^{ij} J^k \\
[K^i, P^j] &= -i\delta^{ij} P^0, \\
[P^i, P^j] &= 0, \\
[J^i, P^0] &= 0
\end{align*}
\]

- Forms of dynamics: The Poincaré group has three subgroups:
  - The instant-form is based on the subgroup
    \[
    \begin{align*}
    [J^i, J^j] &= i\epsilon^{ij} J^j, \\
    [J^i, P^j] &= i\epsilon^{ij} P^j, \\
    [P^i, P^j] &= 0
    \end{align*}
    \]
  - The point-form is based on the Lorentz subgroup
    \[
    \begin{align*}
    [J^i, J^j] &= i\epsilon^{ij} J^j, \\
    [J^i, K^j] &= i\epsilon^{ij} K^j, \\
    [K^i, K^j] &= -i\epsilon^{ij} J^k
    \end{align*}
    \]
  - The front-form is based on the subgroup constructed from 7 generators
    \[
    P^+ = P^0 + P^3, \quad P_\perp = \{P^i, P^j\}, \quad J^3, \quad K^3, \quad E_\perp = K_\perp - \frac{\hat{z} \times J_\perp}{2}
    \]

**Exercise:** prove that the commutation relations for these 7 generators close.

Definition of generators

- Finite transformations “generated” by the generators

\[
T \psi(z) = \exp(-i P_\mu \cdot x^\mu) \psi(z) = \exp(i P_z a_z) \psi(z) = \exp(a_z \nabla_z) \psi(z) = \psi(z) + a_z \frac{\partial}{\partial z} \psi(z) + \frac{1}{2} a_z^2 \frac{\partial^2}{\partial^2 z} \psi(z) + \cdots = \psi(z + a_z)
\]
Kinematic surfaces and generalized hamiltonian

★ Instant-form:
States with definite momentum and spin (eigenstates of \( P \) and \( J \)):
defined on a surface connected by translations and rotations (the
t=0 surface). \( P^0 \) and \( K \) are dynamical; evolving the states away
from the t=0 surface

★ Point-form:
States with definite four-velocity (eigenstates of \( J \) and \( K \)):
defined on a hyperboloid with \( x_\mu x^\mu = 1 \). The 4 components of \( P^\mu \) are
dynamical.

★ Front-form:
States defined on a light-front, \( x^- = t - z = 0 \). The dynamical
generators are \( P^- = P^0 - P^3 \), \( F_\perp = K_\perp + z \times J_\perp \)

Dirac Hamiltonian classifications

★ Plane forms
\[ t - a \, z = 0 \]
\[ -1 \leq a \leq 1 \]
\( a \leq 1 \): instant form
\( a = 1 \): front form

★ Hyperbolic forms
\[ t = \sqrt{(r^2 + a^2)} \]
\( a = 0 \): point form on the light cone
\( a = \infty \): instant form

Some of the Poincaré transformations are kinematic; others involve the dynamics
The Bakamjian-Thomas construction (in instant-form)*

★ The commutation relations can be automatically satisfied if the operators \( P, J, K \), and \( H = P^0 \) are replaced by \( P, r, s, \) and \( M \).

★ For a single particle, \( \alpha \), the generators are written in lower case:

\[
\begin{align*}
p_\alpha, & \quad j_\alpha = s_\alpha + r_\alpha \times p_\alpha, \quad h_\alpha = \sqrt{m_\alpha^2 + p_\alpha^2}, \quad k_\alpha = -\frac{1}{2} \left\{ h_\alpha, r_\alpha \right\} - \frac{p_\alpha \times s_\alpha}{m_\alpha + h_\alpha} \\
\end{align*}
\]

with inverse relations

\[
\begin{align*}
m_\alpha = \sqrt{h_\alpha^2 - p_\alpha^2}, \quad r_\alpha &= -\frac{1}{2} \left\{ h_\alpha^{-1}, k_\alpha \right\} - \frac{p_\alpha \times (h_\alpha j_\alpha - p_\alpha \times k_\alpha)}{m_\alpha h_\alpha (m_\alpha + h_\alpha)}, \\
\end{align*}
\]

\[
\begin{align*}
s_\alpha &= m_\alpha^{-1} (h_\alpha j_\alpha - p_\alpha \times k_\alpha) - \frac{p_\alpha (p_\alpha \cdot j_\alpha)}{m_\alpha (m_\alpha + h_\alpha)} \\
\end{align*}
\]

with non-zero commutators

\[
\begin{align*}
\left[ r^i_\alpha, p^j_\alpha \right] &= i \delta^{ij}, \quad \left[ s^i_\alpha, s^j_\alpha \right] = i \epsilon^{ijk} s^k_\alpha \\
\end{align*}
\]

*B. Bakamjian and H. L. Thomas, Phys. Rev. 92, 1300 (1953)

Bakamjian-Thomas for \( n > 1 \)

★ Proceed in 4 steps

1. **Construct** total \( P^\mu, J, \) and \( K \) by adding generators for each particle

\[
\begin{align*}
P^\mu &= \sum_\alpha p^\mu_\alpha, \quad J = \sum_\alpha j_\alpha, \quad K = \sum_\alpha k_\alpha \\
\end{align*}
\]

2. **Construct** the operators \( M_0, R \) and \( S \) (together with \( P \), already constructed) using the inverse relations (previous slide)

3. **Add** the interactions to \( M_0, M = M_0 + V \). Require that \( V \) commute with \( M_0, P, R \) and \( S \)

4. **Construct** the new generators \( H, J, \) and \( K \) as functions of \( M, P, R \) and \( S \). This completes the construction. All interactions are in the mass operator.

Exercise: think about this and work through the relations
The mass operator

To achieve manifest covariance without negative energy states, introduce the mass operator

\[ M = M_0 + U = \left( (H_0 + H_1)^2 - P^2 \right)^{1/2} \Rightarrow M^2 = M_0^2 + V \]

where \( V = H_0 H_1 + H_1 H_0 + H_1^2 \)

Following the steps we used with the hamiltonian, we have

\[ \left( M_0^2 - M_i^2 \right) \Psi_i = V \Psi_i \Rightarrow \Psi_i = \phi_i + \frac{1}{M_0^2 - M_i^2} V \Psi_i \]

Solving by iteration, the scattering amplitude becomes

\[ M_{kl} = \langle \phi_k | V | \Psi_l \rangle = V_{kl} + \sum_{j \neq l} V_{kj} \frac{1}{M_j^2 - M_i^2} V_{jl} + \sum_{j' \neq l} V_{kj'} \left( \frac{1}{M_j^2 - M_i^2} \right) V_{j'l} \cdots \]

where, for the second order bubble

\[ M_i^2 - M_i^2 = \left( E_+ + E_- \right)^2 - P_0^2 \]

This agrees with the covariant result.

Comparison with Field form: \( \phi^4 \) theory in 1+1 dimensions

Consider scattering from the 2nd order bubble

\[ \frac{1}{2} P + k \]

\[ \frac{1}{2} P - k \]

In field theory (Feynman diagrams) this is

\[ B(s) = i \int \frac{d^2k}{(2\pi)^2} \left( \frac{\lambda^2}{m_1^2 + k^2 + \left( \frac{1}{2} P + k_0 \right)^2 - i\epsilon} \right) \left( \frac{1}{m_2^2 + k^2 + \left( \frac{1}{2} P - k_0 \right)^2 - i\epsilon} \right) \]

\[ = -\lambda^2 \int \frac{dk_z}{(2\pi)} \left( \frac{1}{4E_+ E_-} \right) \left( \frac{1}{E_+ + E_- - P} + \frac{1}{E_+ + E_- + 2P - P} \right) \]

Conclusion 1: Manifest covariance obtained when BOTH positive and negative energy contributions are included.

Conclusion 2: ONE Feynman diagram is the sum of ALL possible time-ordered graphs.
Overview of relativistic methods: Two “schools”

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*Blankenbecler & Sugar, Logunov & Tavkhelidze |
†Phillips, Wallace, and Mandelzweig

Cluster separability -- 3-body example

- **Definition:** when one particle is far away, the interaction between the other two is the same as it would be without the third particle.

\[
\begin{align*}
1 & \quad r \rightarrow \infty \\
2 \quad 3 & \quad =
\end{align*}
\]

- If \( P = p_1 + p_2 + p_3 = 0 \), and \( p_1 \neq 0 \), then the 23 amplitude is in a moving frame. The boost depends on the mass of the 2-body system.

- Hamiltonian dynamics is off-energy shell, \( E_2 + E_3 \neq \sqrt{M_{23}^2 + p_1^2} \). The energies of particles and subsystems do not match the free particle energies, and under boosts the cluster property is not easy to implement.

- Field dynamics is off-mass shell, \( p_0 \neq \sqrt{m^2 + p^2} \). Energy is conserved so boosts and cluster properties are easily satisfied, but off-mass shell \( \Rightarrow \) negative energy states.

Research study: How is separability handled by the two schools; Can you support my claim that here off-mass shell techniques work better?
Conclusions and comparison

* Hamiltonian dynamics
  • Advantages:
    • Real quantum mechanics
    • No negative energy states
  • Disadvantages
    • More ambiguities; no direct connection to field theory
    • Difficulties with cluster separability (?)

* Field Dynamics
  • Advantages
    • Manifest covariance and cluster separability easily implemented
    • Close connection to field theory guides the construction of interactions and currents
  • Disadvantages
    • Not conventional quantum mechanics; a new approach (if you think that 1951 is new?) still requiring conceptual development
    • Singularities, and need to treat negative energy states is more work

Exercise: What’s your opinion?