Lecture II

Theory: Two and Three Nucleon Systems

- Introduction to the Covariant Spectator Theory.
- How are the bound state and scattering equations obtained? What are the normalization conditions?

Outline

- Theoretical assumptions
- CS equations for two-body systems
- Equivalence of two-body BS and CS equations
- Effective theory; estimate of the bound state mass
- The one-body limit and cancellations
- Freeze-out of nucleon resonances in the CS theory
- Normalization condition for two-body relativistic bound states
- Three-body CS equation
- Conclusions
Theoretical Assumptions

- Elementary particles (those in the Lagrangian) produce poles in the scattering amplitude

\[ M \sim \frac{g^2}{m^2 - s} \]

- Nuclei are not elementary (comment: in some, very low energy EFT calculations, they may be treated as effective particles). No single Feynman diagram will have the bound state pole; it must be generated from an infinite sum of Feynman diagrams, much as the geometric series generates a pole:

\[ z + z^2 + \cdots = \frac{z}{1 - z} \]

Therefore we must sum up infinite series of diagrams in order to treat nuclear bound states.

- Nuclei arise from the NN (and NNN) interactions.

- Nucleon resonances are frozen out (i.e. they do not needed to be treated dynamically, but can be put into the interaction).

CS equations for 2-body systems

- scattering amplitudes: an infinite sum of interactions \( V \)

\[ \begin{array}{c}
\text{M} \quad = \quad \text{interaction} + \text{M} \\
\end{array} \]

- if a bound state exists, there is a pole in the scattering amplitude

\[ \begin{array}{c}
\text{M} \quad = \quad \text{interaction} + \text{R} \\
\text{residue: finite at the pole} \\
\end{array} \]

- equation for the bound state vertex functions: obtained from the scattering equation near the bound state pole

\[ \begin{array}{c}
\text{interaction} \quad = \quad \text{interaction} \\
\end{array} \]

- the bound state normalization condition follows from examination of the residue of the bound state pole
Equivalence of the two-body BS and CS equations

- In both cases, the two-body equation has the same form
  \[ M_{BS}(p', p; P) = V_{BS}(p', p; P) + \int V_{BS}(p', k; P) G_{BS}(k; P) M_{BS}(k, p; P) \]
  \[ M_{BS}(p', p; P) = V_{BS}(p', p; P) + \int M_{BS}(p', k; P) G_{BS}(k; P) V_{BS}(k, p; P) \]
  \[ M_{CS}(p', p; P) = V_{CS}(p', p; P) + \int V_{CS}(p', k; P) G_{CS}(k; P) M_{CS}(k, p; P) \]

  Equate the amplitudes, and determine the relation between the kernels
  \[ M_{BS} = V_{BS} \left[ 1 - G_{BS} V_{BS} \right]^{-1} = \left[ 1 - V_{CS} G_{CS} \right]^{-1} V_{CS} = M_{CS} \Rightarrow \]
  \[ V_{BS} = V_{CS} + V_{CS} \left[ G_{CS} - G_{BS} \right] V_{BS} \]
  or
  \[ V_{CS} = V_{BS} + V_{BS} \left[ G_{BS} - G_{CS} \right] V_{CS} \]

- The solutions of one equation are identical to the solutions of the other, provided the kernels are properly related

Equivalent summations of the generalized ladder sum

- To 6th order, the generalized ladder sum is

- In the BS theory, these terms require the following irreducible kernel:
  - 2nd order
  - 4th order
  - 6th order

- In the CS theory, the kernel is
Effective theory; estimate of the bound state mass

- Take an effective short range interaction (treated as a contact term)
- The bubble sum is

\[
M = \lambda + i\lambda B(s)\lambda + i^2\lambda B(s)\lambda B(s)\lambda + \cdots
\]

\[
= \lambda + i\lambda B(s)M + \cdots
\]

\[
= \frac{\lambda}{1 - i\lambda B(s)}
\]


- This means that all the Feynman diagrams in the series are the same size - the physics is non-perturbative.
- Bound states arise in field theory from the infinite sum of Feynman diagrams.

Estimate: bound state mass in 1+1 dimensions (1)

- Work in 1 time and 1 space dimensions \((p_0; p_z)\) to remove divergences; most results carry over to 1+3 dimensions
- The bubble in 1+1 dimensions is easy to calculate

\[
iB(P^2) = i(-i)^4 \int \frac{d^2k}{(2\pi)^2} \left( \frac{1}{m_1^2 - \left(\frac{1}{2} P + k\right)^2 - i\varepsilon} \right) \left( \frac{1}{m_2^2 - \left(\frac{1}{2} P - k\right)^2 - i\varepsilon} \right) = i \int \frac{d^2k}{(2\pi)^2} \left( \frac{1}{A_1} \right) \left( \frac{1}{A_2} \right)
\]

\[
= \frac{1}{4\pi} \int \frac{dx}{m_1^2 x + m_2^2 (1-x) - P^2 x (1-x) - k^2 - i\varepsilon}
\]

\[
= \frac{1}{2\pi \Delta} \left\{ \tan^{-1} \left( \frac{m_1^2 - m_2^2 + P^2}{\Delta} \right) - \tan^{-1} \left( \frac{m_1^2 - m_2^2 - P^2}{\Delta} \right) \right\}
\]

where \(\Delta^2 = (P^2 - (m_1 - m_2)^2)((m_1 + m_2)^2 - P^2)\)
Estimate: bound state mass in 1+1 dimensions (2)

Assume equal masses and weak binding: \( m_1 = m_2 = m \); \( P^2 = 4m^2 - \delta^2 \); \( m \gg \delta \); \( \Delta = 2m\delta \)

\[
iB(4m^2 - \delta^2) = -\frac{1}{2\pi\Delta} \left[ \tan^{-1}\left( \frac{m_1^2 - m_2^2 + P^2}{\Delta} \right) - \tan^{-1}\left( \frac{m_1^2 - m_2^2 - P^2}{\Delta} \right) \right] = -\frac{1}{4m\delta}
\]

The binding energy is approximately

\[
-\frac{\lambda}{4m\delta} \approx 1 \implies \delta = -\frac{\lambda}{4m}
\]

The contact term must be negative (attractive) for a bound state to exist.

exercise: work this out for 1+2 dimensions

The one-body limit

If \( m_1 \to \infty \), the equation should reduce to a one-body equation for \( m_2 \) with a potential independent of the coordinates of \( m_1 \). This is the one-body limit.

In scalar \( \phi^3 \) theory, the generalized ladder sum has this property to each order. The proof is in my textbook "Relativistic Quantum Mechanics and Field Theory". Diagrammatically, for the 2nd and 4th orders

\[
\lim_{m_1 \to \infty} \left\{ \begin{array}{c}
\text{1st order terms} \\
\text{2nd order terms} \\
\text{3rd order terms} \\
\text{4th order terms}
\end{array} \right\} \rightarrow \begin{array}{c}
\text{1st order terms} \\
\text{2nd order terms} \\
\text{3rd order terms} \\
\text{4th order terms}
\end{array}
\]

For scalar theories in the \( m_1 \to \infty \) limit, the OBE approximation in CS theory gives the exact result for the generalized ladder sum.

exercise: prove this
Cancellations: $\phi^4$ theory in 1+1 dimensions

- Study a simple example: $\phi^4$ theory with one interaction
- On shell scattering to 2nd order:

\[
\mathcal{M} = \langle s = 0 \rangle + \langle s = P^2 \rangle + \langle s = u \rangle
\]

\[
i(-i\lambda) = \lambda \quad \frac{1}{2}P + k
\]

\[
i(-i\lambda) = \lambda \quad \frac{1}{2}P - k
\]

\[
\text{bubble B(s)} \quad \text{crossed bubble B(u)}
\]

- $B(s)$ already evaluated previously:

\[
B(s) = \frac{\lambda^2}{2\pi \Delta} \left\{ \tan^{-1} \left( \frac{m_1^2 - m_2^2 + P^2}{\Delta} \right) - \tan^{-1} \left( \frac{m_1^2 - m_2^2 - P^2}{\Delta} \right) \right\}
\]

- where $\Delta^2 = (P^2 - (m_1 - m_2)^2)(m_1 + m_2)^2 - P^2$

Interesting limits

- $m_1 = m_2 = m; \quad P^2 = 4m^2 - \delta^2; \quad u = \delta^2, \quad m \gg \delta$

\[
B(s) \equiv -\frac{\lambda^2}{2\pi m \delta} \tan^{-1} \left( \frac{2m}{\delta} \right) = -\frac{\lambda^2}{2\pi m \delta} \left\{ \frac{\pi}{2} - \frac{\delta}{2m} \right\} \quad \text{and} \quad B(u) = -\frac{\lambda^2}{2\pi m \delta} \left\{ \frac{\pi}{2} \right\}
\]

Note the cancellation

- $m_1 \gg m_2 \gg \delta; \quad P^2 = (m_1 + m_2)^2 - \delta^2; \quad u = (m_1 - m_2)^2 + \delta^2$

\[
B(s) \equiv -\frac{\lambda^2}{4\pi \sqrt{m_1 m_2} \delta} \left\{ \tan^{-1} \left( \frac{m_1 (m_1 + m_2)}{\sqrt{m_1 m_2} \delta} \right) + \tan^{-1} \left( \frac{m_2 (m_1 + m_2)}{\sqrt{m_1 m_2} \delta} \right) \right\} = -\frac{\lambda^2}{4\pi \sqrt{m_1 m_2} \delta} \left\{ \frac{\pi - \sqrt{m_1 m_2}}{2\sqrt{m_1 m_2} \delta} \right\}
\]

\[
B(u) \equiv -\frac{\lambda^2}{4\pi \sqrt{m_1 m_2} \delta} \left\{ \tan^{-1} \left( \frac{m_1 (m_1 - m_2)}{\sqrt{m_1 m_2} \delta} \right) - \tan^{-1} \left( \frac{m_2 (m_1 - m_2)}{\sqrt{m_1 m_2} \delta} \right) \right\} = -\frac{\lambda^2}{4\pi \sqrt{m_1 m_2} \delta} \left\{ \frac{\sqrt{m_1 m_2}}{2m_1 m_2} \right\}
\]

\[
B(s) + B(u) \equiv -\frac{\lambda^2}{4\pi \sqrt{m_1 m_2} \delta} \left\{ \pi \right\}
\]

Note the cancellation

exercise: evaluate these bubbles in 1+2 dimensions
Evaluation of the CS bubble in 1+1 dimension (1)

★ The CS bubble has particle #1 on-shell; there is no crossed bubble
\[ \frac{1}{2} P + k \]
\[ \frac{1}{2} P - k \]

★ This can be written in the convenient form
\[
C(s) = i\lambda^2 \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{A_i - i\epsilon} \right\} \left\{ \frac{1}{A_2 - A_1 - i\epsilon} \right\} 
= i\lambda^2 \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{E_i^2 - \left(\frac{1}{2} P + k_0 \right)^2 - i\epsilon} \right\} \left\{ \frac{1}{m_i^2 - m_2^2 + 2P_0 - i\epsilon} \right\}
\]
\[
\left\{ \frac{(E_i - \frac{1}{2} P - k_0 - i\epsilon)(E_i + \frac{1}{2} P + k_0 - i\epsilon)}{(E_i + \frac{1}{2} P - k_0 - i\epsilon)(E_i + \frac{1}{2} P + k_0 - i\epsilon)} \right\}
\]

only pole in the lower half-plane and hence
this integral gives the exact CS result.

Evaluation of the CS bubble in 1+1 dimension (2)

★ This can be also be written
\[
C(s) = i\lambda^2 \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{A_i - i\epsilon} \right\} \left\{ \frac{1}{A_2 - A_1 - i\epsilon} \right\} = i\lambda^2 \int \frac{d^3k}{(2\pi)^3} \int dx \frac{1}{(A_i - i\epsilon)x + (A_2 - 2i\epsilon)(1-x)}^2
= -\frac{\lambda^2}{4\pi} \int dx \left( m_i^2 x + m_2^2 (1-x) - P^2 x(1-x) \right) = -\frac{\lambda^2}{2\pi\Delta} \left\{ \tan^{-1} \left( \frac{m_i^2 - m_2^2 + P^2}{\Delta} \right) + \frac{\pi}{2} \right\}
\]

★ Interesting limits (as before)
- \[ m_1 \gg m_2 \gg \delta; P^2 \approx (m_1 + m_2)^2 - \delta^2 \]
\[ C(s) \approx -\frac{\lambda^2}{4\pi\sqrt{m_1 m_2}} \left\{ \pi \right\} \equiv B(s) + B(u) \]

The correction \( \frac{\delta}{\sqrt{m_1 m_2}} \) is much smaller than the term cancelled by \( B(u) \).
- \[ m_1 = m_2 = m; \ P^2 = 4m^2 - \delta^2 \]
\[ C(s) \approx -\frac{\lambda^2}{4\pi m\delta} \left\{ \pi - \frac{\delta}{m} \right\} = B(s) \]

★ Conclusion: the CS equation (in the scalar case when \( m_1 \to \infty \)) builds in the cancellations.

research exercise: this bubble diverges in 1+2 dimensions; how can it be regularized?
Freeze-out of nucleon resonances (in the CS theory)

- Nucleon resonances can be excited when the mass of the off-shell nucleon becomes bigger than \((m + m_{\pi})^2\).

- However, in the CS theory, the mass of the off-shell nucleon is bounded from above. For two nucleon scattering at lab energy of \(W > 2m\) (with \(k\) the internal relative nucleon three-momentum),

\[
\rho = (W - k)^2 - m^2 = W^2 - 2W\sqrt{m^2 + k^2} < W(W - 2m)
\]

- Hence, nucleon resonances are not explicitly excited unless

\[
W > 2m + m_{\pi}
\]

- This is fundamentally different from Hamiltonian dynamics, where they are excited for all \(W\). The internal momentum must only be larger than a minimum value

\[
2E(k) > 2m + m_{\pi} \Rightarrow k^2 > mm_{\pi} + \frac{1}{4} m_{\pi}^2
\]

Resonances frozen out because “left hides right”

- Compare the “left-hand-side” of two resonance structures

- Under certain conditions they are indistinguishable

- in this case, the two functions agree on the left-hand side to 1%!

- LESSON:
  THE RIGHT-HAND NUCLEON RESONANCE STRUCTURE CANNOT BE INFERRED UNIQUELY FROM THE LEFT-HAND STRUCTURE

- Low energy NN scattering does not “see” the resonances
The normalization condition for the bound state vertex function also follows from the scattering equation. First find the nonlinear forms of the equation:

\[ M = V + \int V G M = V + \int M G M - \int \bar{M} G V G M \]

\[ \bar{M} = V + \int M G V \]

Then substitute the pole part of \( M \) and expand (away from the pole, \( \epsilon \to 0 \) and \( G = G \)):

\[ \frac{\Gamma \bar{\Gamma}}{M_B^2 - P^2} = V + \int \frac{\Gamma \bar{\Gamma}}{M_B^2 - P^2} \left[ G + \frac{\partial G}{\partial M_B^2} (P^2 - M_B^2) \right] \frac{\Gamma \bar{\Gamma}}{M_B^2 - P^2} \]

\[ -\int \int \frac{\Gamma \bar{\Gamma}}{M_B^2 - P^2} \left[ G V G + G \frac{\partial G}{\partial M_B^2} G + G V \frac{\partial G}{\partial M_B^2} (P^2 - M_B^2) \right] \frac{\Gamma \bar{\Gamma}}{M_B^2 - P^2} \]

Relativistic normalization condition (1)

The double poles give the bound state equation (again)

The single poles give the normalization condition:

\[ \frac{K \bar{K}}{M_B^2 - P^2} = \int \frac{\Gamma \bar{\Gamma}}{M_B^2 - P^2} \left\{ \frac{\partial G}{\partial M_B^2} (P^2 - M_B^2) \right\} \frac{\Gamma \bar{\Gamma}}{M_B^2 - P^2} \]

\[ -\int \int \frac{\Gamma \bar{\Gamma}}{M_B^2 - P^2} \left\{ \frac{\partial G}{\partial M_B^2} V G + G \frac{\partial V}{\partial M_B^2} G + G V \frac{\partial G}{\partial M_B^2} (P^2 - M_B^2) \right\} \frac{\Gamma \bar{\Gamma}}{M_B^2 - P^2} \]

\[ 1 = \int \Gamma \left\{ \frac{\partial G}{\partial M_B^2} \right\} \Gamma + \int \int \Gamma \left\{ \frac{\partial V}{\partial M_B^2} \right\} \Gamma \]

exercise: work through these details
Define three-body vertex functions for each possibility

Then three body Faddeev-like equations emerge automatically. For identical particles they are:

\[
\frac{\Gamma}{M} = 2 \frac{M}{\Gamma}
\]

this amplitude already known from the 2-body sector

CS equations for three-body systems*

Applications of the CS theory

*Gauge invariance can be treated exactly (lecture 4)

*Excellent fits to the NN data below 350 MeV (with \(\chi^2 \approx 1.06\) - lecture 3)

*Excellent description of the 3N binding energy with no explicit three body force (lecture 3)

*Excellent fit to all deuteron form factors to \(Q^2 \sim 6 \text{ GeV}^2\) with one free parameter in the current (lecture 4)

*Satisfactory description of \(\pi N\) scattering and various quark model calculations (not discussed)

*Exploratory study of \(d(e,e'p)n\) in Born approximation*

*To do (work in progress)
  * photodisintegration and electrodisintegration of 2 and 3 body nuclei

Conclusions

★ Few body nuclei are composite systems. They must be described non-perturbatively ⇒ integral equations for amplitudes in $p$ space.

★ The features of a relativistic description depend on the formalism. In Field form -- all generators are kinematic at the cost of negative energy states (twice as many degrees of freedom).

★ Physics depends on whether or not nucleon resonances are explicitly excited (recall: "left hides right").

★ A theoretically sound description of few-body reactions requires FSI and MEC and $NNN$ forces consistent with the two-body dynamics assumed. We will return to this in the subsequent lectures.

★ The CS theory can serve as a framework for the use of any method. Take nonrelativistic limit to interpret correspondence with relativistic theory.