Results: Energies below the pion production threshold

- New, high precision fits to np data below 350 MeV lab energy, and the relativistic properties of the deuteron and triton.
- What do these new results tell us about the nature of nuclear forces?

Outline

- CS theory with spin
- Structure of the deuteron wave functions
- Antisymmetrize the kernel
- Removal of spurious singularities
- One Boson Exchange (OBE) interaction with off-shell coupling
- Implications of off-shell couplings
- New model fits to the NN data
- Three-body binding energy and the role of off-shell couplings
- Changes in the phase shifts
- Scaling and rejecting certain data sets
- Conclusions
In the $n$ nucleon problem, make the following substitution for $n-1$ nucleon propagators (with $\overline{u}_a(p,s)u_a(p',s') = 2m\delta_{ss'}$):

$$S_{\alpha\beta}(p) = \frac{(m+p)_{\alpha\beta}}{m^2 - p^2 - i\epsilon} \implies 2\pi i \delta_+(m^2 - p^2) \sum_s u_a(p,s)\overline{u}_\beta(p,s)$$

The off-shell propagator is (in the CM with $k = P - p$):

$$S_{\alpha\beta}(k) = \frac{(m+\not{k})_{\alpha\beta}}{m^2 - k^2 - i\epsilon} \implies \left\{ \begin{array}{c} \frac{1}{2E(p)} \sum_s \left[ \frac{u_a(-p,s)\overline{u}_\beta(-p,s)}{(2E(p) - W)} \right] \\ \frac{\nu_a(p,s)\overline{\nu}_\beta(p,s)}{W} \end{array} \right.$$

Integration over all internal $p_0$'s places $n-1$ particles on their positive energy mass-shell. All 4-d integrations reduce to 3-d integrations.

Antisymmetrize for identical fermions; remove spurious singularities(!)

Mass of off-shell particle is $k^2 = (P - p)^2 - m^2 = W^2 - 2WE(p) < W(W - 2m)$. If $W < m + m_{\text{res}}$, then $k^2 < (m_{\text{res}})^2$ and nucleon resonances are frozen out (see last lecture).
Coupled equations with spin (1)

* The positive and negative energies give separate coupled channels

\[
M^{++} = V^{++} - \left\{ \mathcal{V}G M^{++} + \mathcal{V}G M^{+} \right\}
\]

\[
M^{-+} = V^{-+} - \left\{ \mathcal{V}G M^{++} + \mathcal{V}G M^{-+} \right\}
\]

where + and - refer to the \( u \) or \( v \) spinor matrix element of the off-shell particle 2

\[
V^{++}(k, p; P) = \bar{u}_\alpha(k, \lambda_1) \bar{u}_\beta(-k, \lambda_2) \mathcal{V}_{\alpha', \beta'}(k, p; P) u_\alpha(p, \lambda_1) u_\beta(-p, \lambda_2)
\]

\[
V^{-+}(k, p; P) = \bar{u}_\alpha(k, \lambda_1) \bar{v}_\beta(-k, \lambda_2) \mathcal{V}_{\alpha', \beta'}(k, p; P) v_\alpha(p, \lambda_1) v_\beta(-p, \lambda_2)
\]

Coupled equations with spin (2)

* In the nonrelativistic limit, the equations reduce in coordinate space to

\[
\left( \frac{\nabla^2}{m} + \varepsilon \right) \Psi^+(r) = V^{++}(r)\Psi^+(r) + V^{-+}(r)\Psi^-(r)
\]

\[
2m \Psi^-(r) = V^{-+}(r)\Psi^+(r) + V^{--}(r)\Psi^-(r)
\]

* These can be solved (even when dependent on spin operators—see Ref.~1*). For scalar \( V^{--} \), we have (\( V^{--} = (V^{++})^\dagger \))

\[
\Psi^-(r) = \frac{V^{-+}(r)}{2m - V^{--}(r)} \Psi^+(r)
\]

\[
\left( \frac{\nabla^2}{m} + \varepsilon \right) \Psi^+(r) = \left\{ \Psi^{++}(r) + \frac{V^{++}(r)(V^{++}(r))^\dagger}{2m - V^{--}(r)} \right\} \Psi^+(r)
\]

* if \( V^{--} < 2m \), this is a positive definite repulsive core

The relativistic deuteron wave function has one nucleon off-shell. This off-shell nucleon has both a positive energy spinor part \((u)\) and a negative energy spinor part \((v)\):

\[
\Psi_{\alpha,\lambda}(P, p) = u_{\alpha}(p_2, \lambda') \psi_{\alpha,\lambda}^+ (P, p) + v_{\alpha}(-p_2, \lambda') \psi_{\alpha,\lambda}^- (P, p)
\]

Four scalar wave functions are needed, 2 for each part:

\[
\psi_{\alpha,\lambda}^+ (P, p) = \int_{4\pi} [u(p) \sigma_1 \cdot \sigma_2 - \frac{i}{2} w(p) (3\sigma_1 \cdot \hat{p} \sigma_2 \cdot \hat{p} - \sigma_1 \cdot \sigma_2)]
\]

\[
\psi_{\alpha,\lambda}^- (P, p) = -\sqrt{\frac{1}{8} \frac{4\pi}{N}} [v(p) (\sigma_1 - \sigma_2) \cdot \hat{p} + \frac{1}{2} w(p) (\sigma_1 + \sigma_2) \cdot \hat{p}]
\]

The normalization condition becomes:

\[
1 = \int_0^\infty p^4 \left[ u^2 + w^2 + v^2 + u^2 \right] + \left( \frac{\partial V}{\partial M_0^2} \right)
\]

P-state probabilities
Deuteron wave functions (2)

- AV18: Argonne AV18 nonrelativistic model
- Model IIB: earlier model* used to predict the deuteron form factors
- WJC-1: new high precision fit described here.
  \[ P_s = 92.35\% \quad D/S = 0.0256 \]
  \[ P_d = 7.32\% \quad P_{vt} = 0.11\% \quad P_{vs} = 0.22\% \]
  agrees with experimental value

Antisymmetrize the Kernel

- The kernel must be explicitly antisymmetrized
  \[ \mathcal{V}_{\alpha\alpha',\beta\beta'}(k, p; P) \rightarrow \bar{\mathcal{V}}_{\alpha\alpha',\beta\beta'}(k, p; P) \]
  \[ = \frac{1}{2} \{ \mathcal{V}_{\alpha\alpha',\beta\beta'}(k, p; P) + (-1)^I \bar{\mathcal{V}}_{\beta\alpha',\alpha\beta'}(-k, p; P) \} \]
- Under interchange of Dirac and momentum indices,
  \[ \bar{\mathcal{V}}_{\alpha\alpha',\beta\beta'}(k, p; P) = (-1)^I \bar{\mathcal{V}}_{\beta\alpha',\alpha\beta'}(-k, p; P) \]
  corresponding to antisymmetry of \( I=1 \) states and symmetry of \( I=0 \) states, corresponding to full antisymmetry.
- Diagrammatically
  \[ \begin{array}{c}
  \begin{array}{c}
  \times \\
  \times \\
  \end{array}
  \end{array}
  = \frac{1}{2} \left\{ \begin{array}{c}
  \begin{array}{c}
  \times \\
  \times \\
  \end{array}
  \end{array} \pm \begin{array}{c}
  \begin{array}{c}
  \times \\
  \times \\
  \end{array}
  \end{array} \right\} \]
Implications of the antisymmetrization

☆ The direct OBE has no singularities
\[
\frac{1}{m_b^2 + (k - p)^2 - (E_k - E_p)^2} = \frac{1}{m_b^2 + 2(E_p E_k - m^2) - 2k \cdot p} \geq 0
\]

☆ However, the exchange OBE has singularities
\[
\frac{1}{m_b^2 + (k + p)^2 - (W - E_k - E_p)^2} = \frac{1}{m_b^2 + 2(E_p E_k - m^2) - (W - 2E_k)(W - 2E_p) + 2k \cdot p}
\]

If \( W < 2m + m_b \), these singularities are **spurious**, because they are cancelled if the kernel is calculated to ALL orders.

☆ So, imaginary part may be dropped (calculate the principal value), but how to handle the real part?

Removal of spurious singularities

☆ Exploit a great freedom: the kernel may be defined in any convenient way, with "corrections" included in higher order

☆ An elegant way to remove the singularities is to replace
\[
\frac{1}{m_b^2 + (k + p)^2 - (W - E_k - E_p)^2} \Rightarrow \frac{1}{m_b^2 + (k + p)^2 - (W - E_k - E_p)^2} = \frac{1}{m_b^2 + q^2}
\]
- preserves exchange symmetry exactly
- removes all singularities (does NOT work for coulomb scattering)
- does not change the direct term, or any results if either the initial or final state is on shell

**research exercise:** calculate the 4th order kernel in \( \phi^3 \) theory using this prescription, and study the cancellations
Look at the details for $S$-wave scattering!

The angular integral is

$$V_{ex}(k, p) = \int_{-1}^{1} \frac{dz}{m_b^2 + |q^2|} = \int_{-1}^{1} \frac{dz}{m_b^2 + |k^2 + p^2 + kpz - q_0^2|}$$

Locus of singularities and (in units of $m$)

$$V_{ex} = \frac{dz}{m_b^2 + q^2} + \frac{dz}{1 - m_b^2 + k^2 + p^2 + kpz - q_0^2}$$

$$F(p) = V(0.6, p)$$

The prescription smooths out the singularities and interpolates between them

Kernel is a sum of One Boson Exchange diagrams

CS Dynamics: OBE with off-shell couplings

<table>
<thead>
<tr>
<th>boson</th>
<th>$J^P(b)$</th>
<th>$\epsilon_b$</th>
<th>$\Lambda_1 \otimes \Lambda_2$</th>
<th>$\Lambda(k, p)$ or $\Lambda^\mu(k, p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_0$</td>
<td>$\sigma_1$</td>
<td>0$^-(s)$</td>
<td>$-\Lambda_1 \Lambda_2$</td>
<td>$\Lambda(k, p) = g_b + \gamma \left[ \theta(k) + \theta(p) \right]$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$\eta$</td>
<td>0$^-(p)$</td>
<td>$+\Lambda_1 \Lambda_2$</td>
<td>$\Lambda(k, p) = g_p \left[ \gamma^5 (1 - \gamma \cdot \gamma) \theta(k) + \gamma^5 \theta(p) \right]$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$\omega$</td>
<td>1$^-(\nu)$</td>
<td>$+\Lambda^\mu \Lambda_2 \Delta_{\mu\nu}$</td>
<td>$\Lambda(k, p) = g_{\nu} \left[ \gamma^\mu + \frac{\mathbf{k}<em>\nu}{2m} i\sigma^{\nu\mu} (k - p)</em>\nu \right] + \gamma \left[ \theta(k) \gamma^\mu + \gamma^\mu \theta(p) \right]$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$h_1$</td>
<td>1$^+(a)$</td>
<td>$+\Lambda^\mu \Lambda^\nu g_{\mu\nu}$</td>
<td>$\Lambda(k, p) = g_{\nu} \gamma^\mu \gamma^\nu$</td>
</tr>
</tbody>
</table>

$\theta(p) = \frac{m - \mathbf{p}}{2m}$ vanishes on-shell

$\Delta_{\mu\nu} = g_{\mu\nu} - q_\mu q_\nu / m^2_c$
**Implications of off-shell couplings**

* In basic connection is, diagrammatically

\[ \nu \frac{m - \kappa}{2m} \left( \frac{1}{m - \kappa} \right) g_{\sigma} + g_{\sigma} \left( \frac{1}{m - \kappa} \right) \frac{m - \kappa}{2m} \nu_{\sigma} = \frac{g_{\sigma} \nu_{\sigma}}{m} \]

* This can happen repeatedly

**Two precision fits to the 2007 data base**

**Comparison with other precision fits**

<table>
<thead>
<tr>
<th>models</th>
<th>Data set [ $\chi^2/N_{\text{data}}(N_{\text{data}})$ ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref  year  # para</td>
<td>1993</td>
</tr>
<tr>
<td>PWA93 1993 39</td>
<td>0.99 (2514)</td>
</tr>
<tr>
<td></td>
<td>1.09 (3010)</td>
</tr>
<tr>
<td>Nijm I 1993 41</td>
<td>1.03 (2514)</td>
</tr>
<tr>
<td>AV18 1995 40</td>
<td>1.06 (2526)</td>
</tr>
<tr>
<td>CD-Bonn 2000 43</td>
<td>1.02 (3058)</td>
</tr>
<tr>
<td>WJC-1 2007 27</td>
<td>1.03 (3010)</td>
</tr>
<tr>
<td>WJC-2 2007 15</td>
<td>1.09 (3010)</td>
</tr>
</tbody>
</table>

#’s in green are for fits to BOTH np and pp data
OBE parameters obtained from the fits

<table>
<thead>
<tr>
<th>( b )</th>
<th>( I )</th>
<th>( G_b = \frac{g_b^2}{4\pi} )</th>
<th>( m_b )</th>
<th>( \lambda_b ) or ( \nu_b )</th>
<th>( \kappa_b )</th>
<th>( \Lambda_b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi^0 )</td>
<td>1</td>
<td>14.608</td>
<td>14.038</td>
<td>134.9766</td>
<td>0.153</td>
<td>0.0</td>
</tr>
<tr>
<td>( \pi^\pm )</td>
<td>1</td>
<td>13.703</td>
<td>14.038*</td>
<td>139.5702</td>
<td>-0.312</td>
<td>0.0</td>
</tr>
<tr>
<td>( \eta )</td>
<td>0</td>
<td>10.684</td>
<td>4.386</td>
<td>604</td>
<td>547.51</td>
<td>0.622</td>
</tr>
<tr>
<td>( \sigma_0 )</td>
<td>0</td>
<td>2.307</td>
<td>4.486</td>
<td>429</td>
<td>478</td>
<td>-6.500</td>
</tr>
<tr>
<td>( \sigma_1 )</td>
<td>1</td>
<td>0.539</td>
<td>0.477</td>
<td>515</td>
<td>454</td>
<td>0.987</td>
</tr>
<tr>
<td>( \omega )</td>
<td>0</td>
<td>3.456</td>
<td>8.711</td>
<td>657</td>
<td>782.65</td>
<td>0.843</td>
</tr>
<tr>
<td>( \rho )</td>
<td>1</td>
<td>0.327</td>
<td>0.626</td>
<td>787</td>
<td>775.50</td>
<td>-1.263</td>
</tr>
<tr>
<td>( h_i )</td>
<td>0</td>
<td>0.0026</td>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a_i )</td>
<td>1</td>
<td>-0.436</td>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( \Lambda_N \) 1656 1739

left column: WJC-1 27 parameters
right column: WJC-2 15 parameters

Conclusions from the fits

★ Model WJC-1: 27 parameters:
- As good as any phase shift analysis or any fit to date; truly QUANTITATIVE
- 27 parameters is less than other fits (but only \( np \) data fit so far)
- OBE parameters are reasonable:
  - masses close to observed masses of actual mesons (within 50 MeV except for the \( \omega \), which is 126 MeV lower); \( \pi \) masses fixed at observed values
  - \( \sigma \) (0 and 1) have masses near the peak of the 2 pion continuum
  - \( \pi \) couplings are close to expected values; BUT \( g_0 > g_+ \) (!)
  - \( \omega \) and \( \rho \) are week; \( \eta \) is strong (compared to WJC-2)

★ Model WJC-2: ONLY 15 parameters
- EXCELLENT; as good as the Nijmegen phases
- OBE parameters are reasonable, and SATISFY constraints:
  - masses of \( \omega, \rho, \) and \( \eta \) (and \( \pi \)) fixed at observed values; \( \sigma \) (0 and 1) masses still near the peak of the 2 pion continuum
  - \( \pi^0 \) and \( \pi^\pm \) couplings equal; PURE \( pv \) coupling as required by chiral symmetry
  - No novel features (i.e. \( \kappa_\omega=0 \), \( \eta \) pure \( pv \), no off-shell coupling for \( \omega \)) EXCEPT off-shell couplings for \( \sigma \) (0 and 1) and \( \rho \)
Relativistic effects in $^3$H binding (1997 results)*

It turns out that the relativistic calculation of the three body binding energy is sensitive to a new, relativistic off-shell coupling (described by the parameter $\nu$). Non-zero $\nu$ is equivalent to effective three-body (and n-body forces).

The value of $\nu$ that gives the correct binding energy is close to the value that gives the best fit to the two-body data!


New results confirm the 1997 findings

Minimum $\chi^2/N_{\text{data}}$ for Model WJC-1 coincides with experimental triton binding energy of -8.48 MeV
Changes in the phase shifts

Nijmegen phases differ by several degrees from the WJC-1 phases. (Explains earlier problem fitting the data.)

Low $\chi^2$ implies excellent fits to data (of course)

Total cross sections fit over the entire energy range

New accurate differential cross sections
Scaling and rejection of data sets

Experimentalists may specify that data has a systematic error; it may be scaled (within the error) to agree with theory. The red Bonner data has been scaled (below).

This data initially unscaled

However, the Uppsala data (blue) is rejected; no scaling can change its incorrect shape.

Rejected data sets can be identified

- Nijmegen identifies a 3σ criterion. Data sets with $\chi^2$ too large or too small are rejected.
Conclusions

★ We have a simple (comparatively) covariant model of the NN kernel based on OBE that gives a quantitatively EXCELLENT description of the low energy NN data.

★ The OBE mechanism works very well, with only a few parameters needed.

★ ALL Poincaré transformations are kinematic -- i.e. exact.

★ Three body forces are incorporated as off-shell effects arising from two body interactions.

★ These models can be used for precision calculations of few body interactions.

★ The kernel provides a “bridge” between hadronic physics and QCD -- in the sense that the task of QCD is now to understand the kernel we have found.