What are the Low-Q and Large-x Boundaries of Collinear QCD Factorization Theorems?

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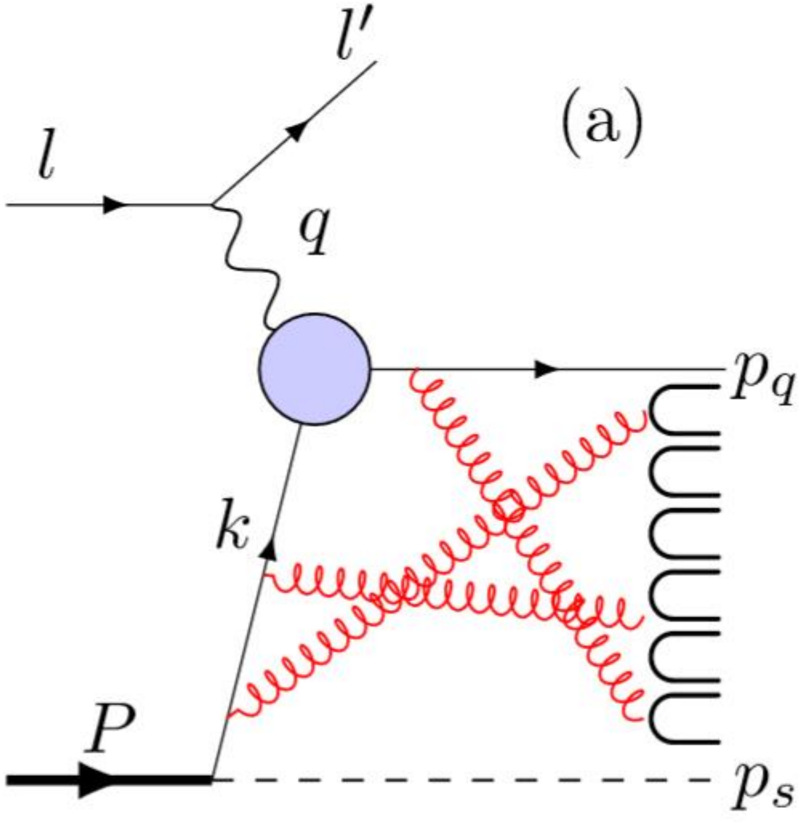
Introduction

- QCD complexities
  - Non-Abelian
  - Confinement
- Can only be solved analytically in the simplest of cases.
- Use Factorization theorems to simplify the calculation.



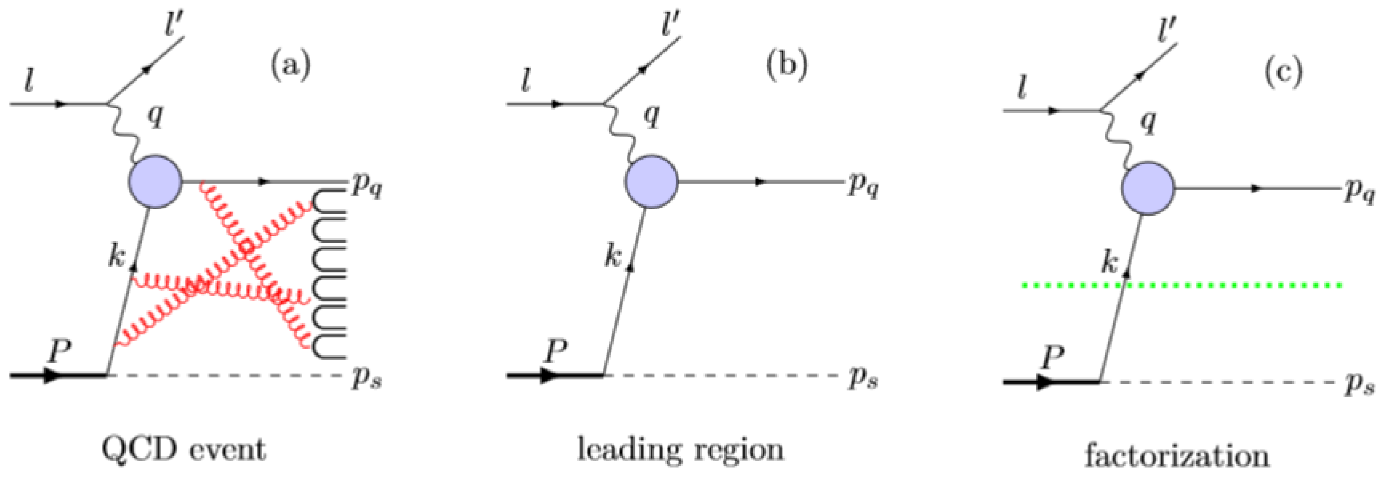
QCD event

Introduction

- Factorization:
  - Method of disentangling the physics at different space-time scales by taking the asymptotically large limit of some physical energy
- Useful in QCD:
  - Asymptotic freedom allows short-distance processes to be calculated using perturbative calculations
    - Factorize to separate perturbative part from non-perturbative part

Introduction

- Example: Collinear Factorization in Deep Inelastic Scattering (DIS)
  - Assume that $Q \gg m$ where $Q = \sqrt{-q^2}$ and m is a generic mass scale on the order of a hadron mass



Introduction

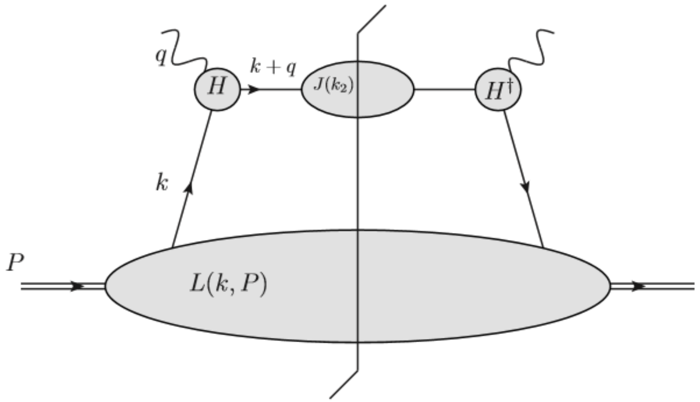
- Want to explore physics at lower Q (~ a few GeV) and larger x_{bj} ($\gtrsim 0.5$)
  - Interplay of perturbative and nonperturbative
- For example DIS at moderately low momentum transfer (Q ~ 1 – 2 GeV)
  - $Q \gg m$ is not an accurate assumption
  - But $\alpha_s/\pi \lesssim 0.1$ so can still use perturbative calculations.

Introduction

- Proposed techniques for extending QCD factorization to lower energies and/or larger x_{bj}:
  - Target mass corrections (Georgi and Politzer, 1976)
  - Large Bjorken-x corrections from re-summation (Sterman, 1987)
  - Higher twist operators (Jaffe and Soldate, 1982)
- Questions arise:
  - Which method would give the most accurate approximation?
  - Are there other corrections that should be included?

Introduction

- What can we do to test how effective these techniques really are?
  - Problem: Non-Abelian nature of QCD leaves “blobs” that cannot be calculated without making approximations



  - There is no reason these techniques can only be applied to QCD.
  - They should work for most re-normalizable Quantum Field Theories (QFT)

Introduction

- Use a simple QFT that requires no approximations
  - Perform an exact calculation in this QFT
  - Perform the same calculation after applying a factorization theorem to the QFT
  - Compare results numerically

Outline

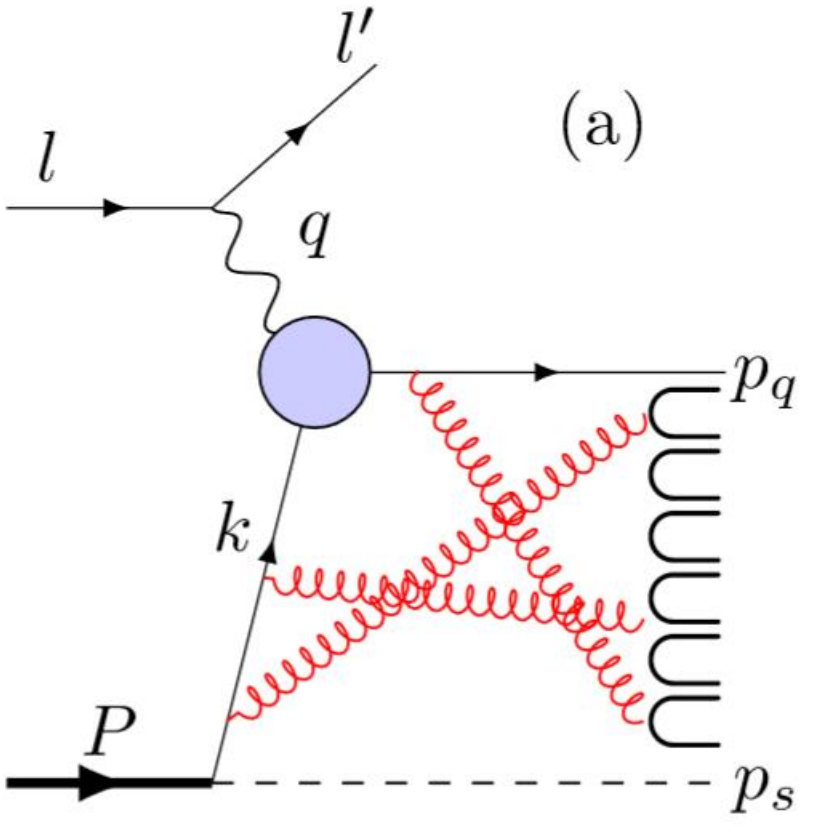
- Define Simple QFT
- Review of standard notation in inclusive DIS
- Exact Calculation of Structure Functions in the Simple QFT
- Collinear Factorization Calculation of Structure Functions in the Simple QFT
- Analyze numerical differences between the Exact and Approximate results
- Summary of findings

Simple Model Definition

- Interaction Lagrangian Density:
  - $\mathcal{L}_{\text{int}} = -\lambda\,\overline{\Psi}_N\,\psi_q\,\phi \;+\; \text{h.c.}$
    - Ψ_N: Spin-1/2 “Nucleon” Field with mass M
    - ψ_q : Spin-1/2 “Quark” Field with mass m_q
    - ϕ : Scalar “Diquark” Field with mass m_s
  - The nucleon and quark couple to photon while the scalar does not.

Standard Notation in Inclusive DIS

- Inclusive DIS process
- $e(l) + N(P) \rightarrow e(l') + X(p_x)$
  - l and l' are the initial and final lepton four-momenta
  - P is the four-momentum of the nucleon
  - $p_x = p_q + p_s$ is the four-momentum of the inclusive hadronic state



QCD event

Standard Notation in Inclusive DIS

- Using Breit frame where
  - Nucleon momentum in +z direction
  - Photon momentum in –z direction
- Using light-front coordinates
  - Four-vector:

$$v^\mu = (v^+, v^-, \boldsymbol{v}_\mathrm{T})$$

  - "±" components:

$$v^\pm = (v^0 \pm v^z)/\sqrt{2}$$

  - Transverse component:

$$\boldsymbol{v}_\mathrm{T}$$

Standard Notation in Inclusive DIS

- Momenta
  - Nucleon

$$P = \left(\frac{Q}{x_n \sqrt{2}}, \frac{x_n M^2}{Q\sqrt{2}}, \mathbf{0}_T \right)$$

  - Photon

$$q = l - l' \qquad q = \left(-\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \mathbf{0}_T \right)$$

  - Internal Parton

$$k = (k^+, k^-, \boldsymbol{k}_T)$$

  - Final Parton

$$k + q$$

- Where

$$Q \equiv \sqrt{-q^2}$$

Nachtmann x

$$x_n \equiv -\frac{q^+}{P^+} = \frac{2x_{bj}}{1 + \sqrt{1 + 4x_{bj}^2 M^2/Q^2}}$$

Bjorken x

$$x_{bj} = \frac{Q^2}{2P \cdot q}$$

Standard Notation in Inclusive DIS

- The DIS cross section can be written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x_{\mathrm{n}}\,\mathrm{d}Q^2} = \frac{4\alpha}{\Phi Q^4} L_{\mu\nu} W^{\mu\nu}$$

  - Where
    - α is the electromagnetic fine structure constant
    - Φ is a flux factor
    - $L_{\mu\nu}$ is the leptonic tensor given by

$$L_{\mu\nu} = 2\left(\ell_\mu \ell'_\nu + \ell'_\mu \ell_\nu - g_{\mu\nu} \ell \cdot \ell'\right)$$

    - $W^{\mu\nu}$ is the hadronic tensor, which in terms of structure functions F_1 and F_2 is given by

$$W^{\mu\nu}(P,q) = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1\left(x_{\mathrm{n}}, Q^2\right) + \left(P^\mu - q^\mu \frac{P \cdot q}{q^2}\right)\left(P^\nu - q^\nu \frac{P \cdot q}{q^2}\right) \frac{F_2\left(x_{\mathrm{n}}, Q^2\right)}{P \cdot q}$$

Standard Notation in Inclusive DIS

- Define Projection Tensors for the Structure Functions

$$\mathrm{P}_1^{\mu\nu} W_{\mu\nu}(P,q) = F_1\left(x_{\mathrm{n}}, Q^2\right), \qquad \mathrm{P}_2^{\mu\nu} W_{\mu\nu}(P,q) = F_2\left(x_{\mathrm{n}}, Q^2\right)$$

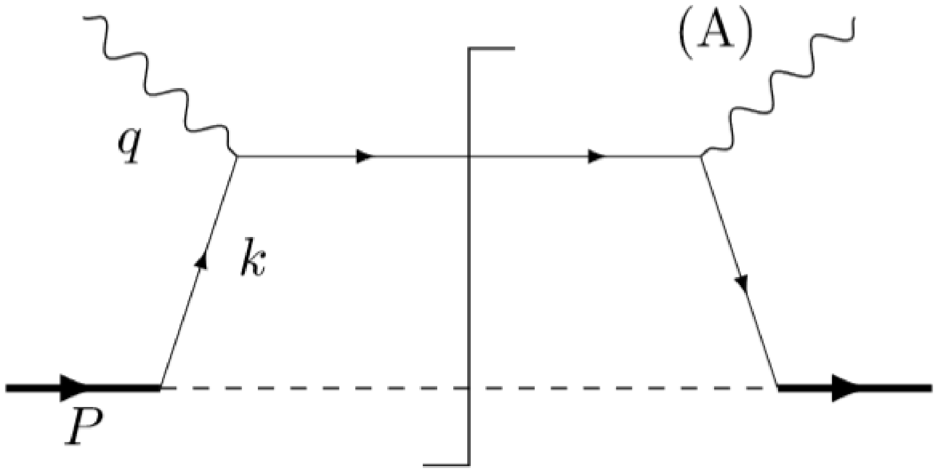
$$\begin{aligned} \mathrm{P}_1^{\mu\nu} &= -\frac{1}{2}\mathrm{P}_g^{\mu\nu} + \frac{2Q^2 x_{\mathrm{n}}^2}{(M_H^2 x_{\mathrm{n}}^2 + Q^2)^2}\mathrm{P}_{\mathrm{PP}}^{\mu\nu}, \\ \mathrm{P}_2^{\mu\nu} &= \frac{12 Q^4 x_{\mathrm{n}}^3\left(Q^2 - M_H^2 x_{\mathrm{n}}^2\right)}{\left(Q^2 + M_H^2 x_{\mathrm{n}}^2\right)^4}\left(\mathrm{P}_{\mathrm{PP}}^{\mu\nu} + \frac{\left(M_H^2 x_{\mathrm{n}}^2 + Q^2\right)^2}{12 Q^2 x_{\mathrm{n}}^2}\mathrm{P}_g^{\mu\nu}\right) \end{aligned}$$

  - Where

$$\mathrm{P}_g^{\mu\nu} = g^{\mu\nu}, \qquad \mathrm{P}_{PP}^{\mu\nu} = P^\mu P^\nu.$$

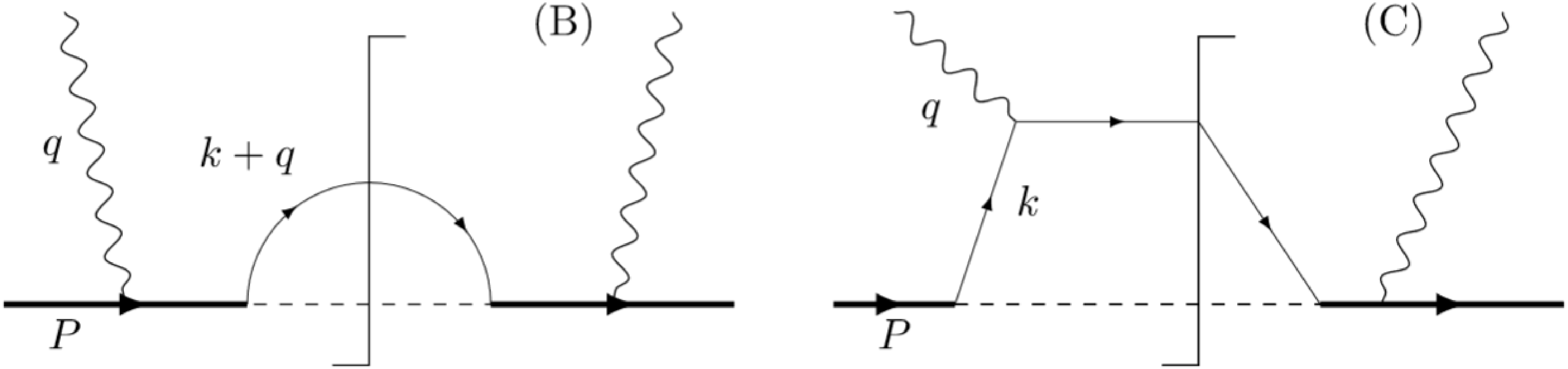
Exact Kinematics

- Familiar DIS Handbag Diagram



Exact Kinematics

- For electromagnetic gauge invariance these diagrams must also be included.



Exact Kinematics

- To demonstrate the calculations it is convenient to organize the hadronic tensor by separating the integrand into factors as follows:

$$W^{\mu\nu}(P,q) = \sum_{j\in \text{ graphs}} \int \frac{\mathrm{d}k^+ \mathrm{d}k^- \mathrm{d}^2\boldsymbol{k}_\mathrm{T}}{(2\pi)^4} \, [\mathrm{Jac}] \, \mathrm{T}_j^{\mu\nu} \, [\mathrm{Prop}]_j \, \delta(k^- - k^-_{\mathrm{sol}}) \, \delta(k^+ - k^+_{\mathrm{sol}})$$

  - Where
    - j refers to Figures A, B, and C
    - $[\mathrm{Prop}]_j$ is the denominators of the internal propagators in Figure j
    - $T_j^{\mu\nu}$ is the appropriate Dirac trace for Figure j
    - $[\mathrm{Jac}]$ is the appropriate jacobian factor to isolate k^- and k^+ in the arguments of the delta functions

Exact Kinematics

- The arguments of the delta functions give the quadratic system

$$(q+k)^2 - m_q^2 = 0\,,$$
$$(P-k)^2 - m_s^2 = 0\,.$$

  - Solving this system for $k^+ \equiv \xi P^+$ and k^- yields two solutions for k^-
  - Only one solution is physically realistic (0 if Q is taken to infinity)
  - The correct solution to the system is

$$k^- = k^-_{\text{sol}} \equiv \frac{\sqrt{\Delta} - Q^2(1-x_{\text{n}}) - x_{\text{n}}\left(m_s^2 - m_q^2 - M^2(1-x_{\text{n}})\right)}{2\sqrt{2}\,Q\,(1-x_{\text{n}})}\,,$$
$$k^+ = k^+_{\text{sol}} \equiv \frac{k_{\text{T}}^2 + m_q^2 + Q(Q+\sqrt{2}k^-)}{\sqrt{2}(Q+\sqrt{2}k^-)}\,,$$

  - Where $\Delta = \left[Q^2(1-x_{\text{n}}) - x_{\text{n}}\left(M^2(1-x_{\text{n}}) + m_q^2 - m_s^2\right)\right]^2$

$$- 4x_{\text{n}}(1-x_{\text{n}})\left[k_{\text{T}}^2(Q^2 + x_{\text{n}}M^2) - Q^2M^2(1-x_{\text{n}}) + Q^2m_s^2 + x_{\text{n}}M^2m_q^2\right]$$

Exact Kinematics

- The Jacobian factor is:

$$[\mathrm{Jac}] = \frac{x_{\mathrm{n}} Q\ (2k^- + \sqrt{2}Q)}{4(1-x_{\mathrm{n}})k^- Q^2(\sqrt{2}k^- + 2Q) + 2\sqrt{2}[Q^4(1-x_{\mathrm{n}}) - (k_{\mathrm{T}}^2 + m_q^2)x_{\mathrm{n}}(Q^2 + x_{\mathrm{n}}\, M^2)]}$$

- The propagator factors are:

$$[\mathrm{Prop}]_{\mathrm{A}} = \frac{1}{(k^2 - m_q^2)^2},$$

$$[\mathrm{Prop}]_{\mathrm{B}} = \frac{1}{\left((P+q)^2 - M^2\right)^2} = \frac{x_{\mathrm{n}}^2}{\left(Q^2(1-x_{\mathrm{n}}) - M^2 x_{\mathrm{n}}^2\right)^2},$$

$$[\mathrm{Prop}]_{\mathrm{C}} = \frac{1}{(k^2 - m_q^2)} \frac{x_{\mathrm{n}}}{\left(Q^2(1-x_{\mathrm{n}}) - M^2 x_{\mathrm{n}}^2\right)}.$$

Exact Kinematics

- The Dirac traces are:

$$\mathrm{T}_{\mathrm{A}}^{\mu\nu} = \mathrm{Tr}\left[(\not{P} + M)(\not{k} + m_q)\gamma^{\mu}(\not{k} + \not{q} + m_q)\gamma^{\nu}(\not{k} + m_q)\right],$$

$$\mathrm{T}_{\mathrm{B}}^{\mu\nu} = \mathrm{Tr}\left[(\not{P} + M)\gamma^{\mu}(\not{P} + \not{q} + M)(\not{k} + \not{q} + m_q)(\not{P} + \not{q} + M)\gamma^{\nu}\right],$$

$$\mathrm{T}_{\mathrm{C}}^{\mu\nu} = 2\,\mathrm{Tr}\left[(\not{P} + M)(\not{k} + m_q)\gamma^{\mu}(\not{k} + \not{q} + m_q)(\not{P} + \not{q} + M)\gamma^{\nu}\right],$$

  - Factor of 2 is for the Hermitian conjugate of Figure C.

- Define the projected quantities:

$$\mathrm{T}_j^{g} = \mathrm{P}_g^{\mu\nu}\,\mathrm{T}_{j\,\mu\nu}\,, \qquad \mathrm{T}_j^{PP} = \mathrm{P}_{PP}^{\mu\nu}\,\mathrm{T}_{j\,\mu\nu}$$

Exact Kinematics

- The $P_g^{\mu\nu}$ projections with traces evaluated are:

$$
\begin{aligned}
T_A^g =& -8\left[2(P\cdot k + m_q M)\, k\cdot q + (k^2 - 3m_q^2)\, P\cdot k - 2Mm_q^3 + (m_q^2 - k^2)\, P\cdot q\right],\\
T_B^g =& \quad 8\left[2M^3 m_q + P\cdot k\,(2M^2 - Q^2) - 2(M^2 + Mm_q)\, Q^2\right.\\
&\left. + 2k\cdot q\,(M^2 - P\cdot q) + [2(M^2 + Mm_q) + Q^2]\, P\cdot q\right],\\
T_C^g =& -16\left[-2(P\cdot k)^2 + k^2 M^2 + (M^2 - m_q M)\, k\cdot q - M^2 m_q^2 + 2Mm_q Q^2\right.\\
&\left. + (m_q^2 - Mm_q) P\cdot q - 2P\cdot k\,(k\cdot q + Mm_q - Q^2 + P\cdot q)\right],
\end{aligned}
$$

Exact Kinematics

- The $P_{PP}^{\mu\nu}$ projections with traces evaluated are:

$$
\begin{aligned}
\mathrm{T}_{\mathrm{A}}^{PP} = \;& 4\Big[4(P\cdot k)^3 + 4(P\cdot k)^2(Mm_q + P\cdot q) \\
& -M\,P\cdot k\,(3k^2M + 2M\,k\cdot q - 3Mm_q^2 - 4m_q\,P\cdot q) \\
& -M^3m_q(k^2 + 2k\cdot q - m_q^2) - M^2(k^2 - m_q^2)\,P\cdot q\Big], \\
\mathrm{T}_{\mathrm{B}}^{PP} = \;& 4M^2\Big[P\cdot k\,(4M^2 + Q^2) + 4M^2(k\cdot q + Mm_q) - Q^2(4M^2 + Mm_q) \\
& +[2k\cdot q + 4(M^2 + Mm_q) - Q^2]\,P\cdot q\Big], \\
\mathrm{T}_{\mathrm{C}}^{PP} = \;& 8M\Big[4M(P\cdot k)^2 + M\,P\cdot k\,(2k\cdot q + 4Mm_q - Q^2) \\
& -M^2[2M(k^2 + k\cdot q - m_q^2) + m_qQ^2] \\
& -[k^2M - (2M + m_q)(2P\cdot k + Mm_q)]\,P\cdot q\Big].
\end{aligned}
$$

Exact Kinematics

- Define the nucleon structure functions as:

$$F_1\left(x_{\mathrm{n}}, Q^2\right) = \int \frac{\mathrm{d}^2\boldsymbol{k}_{\mathrm{T}}}{(2\pi)^2} \mathcal{F}_1(x_{\mathrm{n}}, Q^2, k_{\mathrm{T}}^2),$$
$$F_2\left(x_{\mathrm{n}}, Q^2\right) = \int \frac{\mathrm{d}^2\boldsymbol{k}_{\mathrm{T}}}{(2\pi)^2} 2x_{\mathrm{n}}\, \mathcal{F}_2(x_{\mathrm{n}}, Q^2, k_{\mathrm{T}}^2)$$

- Where

$$\mathcal{F}_1\left(x_{\mathrm{n}}, Q^2, k_{\mathrm{T}}^2\right) = \frac{1}{(2\pi)^2}[\mathrm{Jac}]\sum_j \left(-\frac{1}{2}\mathrm{T}_j^g + \frac{2Q^2x_{\mathrm{n}}^2}{(M^2x_{\mathrm{n}}^2+Q^2)^2}\mathrm{T}_j^{PP}\right)[\mathrm{Prop}]_j,$$

$$2x_{\mathrm{n}}\mathcal{F}_2\left(x_{\mathrm{n}}, Q^2, k_{\mathrm{T}}^2\right) = \frac{1}{(2\pi)^2}\frac{12Q^4x_{\mathrm{n}}^3(Q^2-M^2x_{\mathrm{n}}^2)}{(Q^2+M^2x_{\mathrm{n}}^2)^4} \times[\mathrm{Jac}]\sum_j \left(\mathrm{T}_j^{PP} - \frac{(M^2x_{\mathrm{n}}^2+Q^2)^2}{12Q^2x_{\mathrm{n}}^2}\mathrm{T}_j^g\right)[\mathrm{Prop}]_j.$$

Exact Kinematics

- The exact kinematics impose an upper bound on k_T.
- Start from calculation of W in the center-of-mass frame:

$$W = p_q^0 + p_s^0\Big|_{\mathrm{c.m.}} = \sqrt{m_q^2 + k_{\mathrm{T}}^2 + k_z^2} + \sqrt{m_s^2 + k_{\mathrm{T}}^2 + k_z^2}\Big|_{\mathrm{c.m.}}$$

- W in the Breit frame :

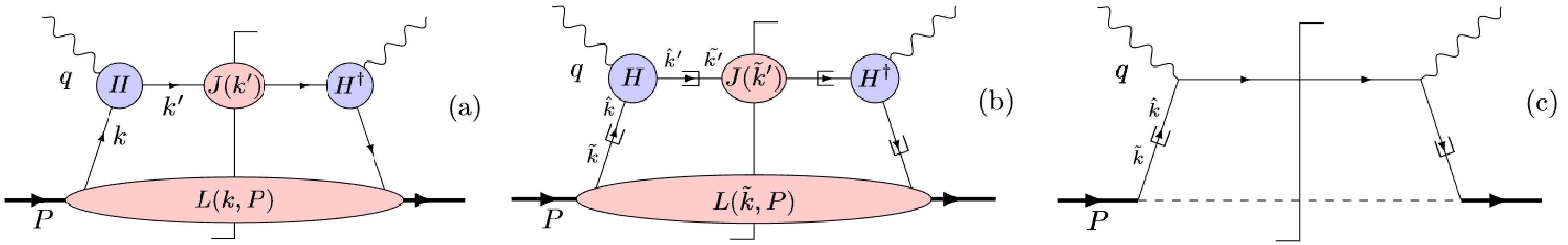
$$W^2 = (P+q)^2 = (p_q + p_s)^2 = M^2 + \frac{Q^2(1-x_{\mathrm{bj}})}{x_{\mathrm{bj}}}$$

- Set the two equations for W equal to each other, and solve for k_T with $k_z = 0$

$$k_{\mathrm{Tmax}} = \sqrt{\frac{\left[x_{\mathrm{bj}}(M^2 - (m_q + m_s)^2) + Q^2(1-x_{\mathrm{bj}})\right]\left[x_{\mathrm{bj}}\left(M^2 - (m_q - m_s)^2\right) + Q^2(1-x_{\mathrm{bj}})\right]}{4x_{\mathrm{bj}}\left[Q^2(1-x_{\mathrm{bj}}) + M^2 x_{\mathrm{bj}}\right]}}$$

Collinear Factorization

- Collinear Factorization



- Un-approximated hadronic tensor

$$W^{\mu\nu}(P,q) = \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\left[H^{\mu}(k,k')\, J(k')\, H^{\nu\dagger}(k,k')\, L(k,P)\right]$$

Collinear Factorization

- Internal quark momentum
  - Power Counting at low transverse momentum ($k_T \sim O(\boldsymbol{m}_T)$)
    - k^2 and $k'^2 \sim O(m^2)$
    - $k^+ \sim O(Q)$
  - Therefore

$$k \sim \left(O\left(Q\right), O\left(\frac{m^2}{Q}\right), O\left(\boldsymbol{m}_{\mathrm{T}}\right) \right)$$

Collinear Factorization

- Hard Factor $\left(H(k,k')\right)$
  - $k \cdot q = k^+ q^- + O(m^2)$
    - $k \to \hat{k} \equiv \left(\hat{k}^+, 0,0\right)$
    - $k' \to \widehat{k'} = \hat{k} + \mathrm{q}$
    - $H(k,k_2) \to H\left(\hat{k}, \widehat{k'}\right)$
    - $\delta\left({\hat{k}'}^2 - m_q^2\right) \to \hat{k}^+ = x_n P^+ = x_{bj} P^+$
- Lower Factor $(L(k,P))$
  - Contains propagator, large component of k can be approximated but the small components must be kept exact
    - $k \to \tilde{k} \equiv \left(\hat{k}^+, k^-, \boldsymbol{k}_T\right)$
    - $L(k,P) \to L\left(\tilde{k}, P\right)$

Collinear Factorization

- Jet Factor ($J(k')$)
  - Power counting for k' is:

$$k' \sim \left(O\left(Q\right), O\left(Q\right), O\left(\boldsymbol{m}_{\mathrm{T}}\right)\right)$$

  - Consider a frame labeled "*", where the outgoing transverse momentum vanishes $k'^{*}_{T} = 0$,

$$k'^{*} = \left(k^{+} + q^{+} - \frac{k_{\mathrm{T}}^{2}}{2(q^{-} + k^{-})}, q^{-} + k^{-}, \mathbf{0}_{\mathrm{T}}\right)$$

  - The outgoing parton's virtuality is:

$$k'^{*2} = 2\left(k^{+} + q^{+}\right)\left(k^{-} + q^{-}\right) - k_{\mathrm{T}}^{2}$$
$$\sim 2\left(k^{+} + q^{+}\right) q^{-} - k_{\mathrm{T}}^{2} + O\left(\frac{m^{3}}{Q}\right)$$

  - Make the approximation $k' \to \widetilde{k}' \equiv (l^{+}, q^{-}, \mathbf{0}_{T})$ where $l^{+} \equiv k^{+} - x_{n}P^{+} + \frac{k_{T}^{2}}{2q^{-}}$
  - Change integration variables from k^{+} to l^{+}
    - $J(k') \to J(l^{+})$

Collinear Factorization

- Hadronic Tensor now becomes

$$W^{\mu\nu}(P,q) = \int \frac{\mathrm{d}l^{+}\mathrm{d}k^{-}\mathrm{d}^{2}\boldsymbol{k}_{\mathrm{T}}}{(2\pi)^{4}} \operatorname{Tr}\left[H^{\mu}(Q^{2})J(l^{+})H^{\dagger\nu}(Q^{2})L(\tilde{k},P)\right] + O\left(\frac{m^{2}}{Q^{2}}\right)W^{\mu\nu}$$

- Separate the integrations into factors for the jet and target

$$W^{\mu\nu}(P,q) = \operatorname{Tr}\left[H^{\mu}(Q^{2})\left(\int \frac{\mathrm{d}l^{+}}{2\pi}J(l^{+})\right)H^{\nu\dagger}(Q^{2})\left(\int \frac{\mathrm{d}k^{-}\mathrm{d}^{2}\boldsymbol{k}_{\mathrm{T}}}{(2\pi)^{3}}L(\tilde{k},P)\right)\right] + O\left(\frac{m^{2}}{Q^{2}}\right)W^{\mu\nu}$$

Collinear Factorization

- To complete the factorization
  - Express the Jet and Lower factors in a basis of Dirac matrices.

$$J(l^+) = \gamma_\mu \Delta^\mu(l^+) + \Delta_S(l^+) + \gamma_5 \Delta_P(l^+) + \gamma_5 \gamma_\mu \Delta_A^\mu(l^+) + \sigma_{\mu\nu}\Delta_T^{\mu\nu}(l^+),$$

$$L(\tilde{k}, P) = \gamma_\mu \Phi^\mu(\tilde{k}, P) + \Phi_S(\tilde{k}, P) + \gamma_5 \Phi_P(\tilde{k}, P) + \gamma_5 \gamma_\mu \Phi_A^\mu(\tilde{k}, P) + \sigma_{\mu\nu}\Phi_T^{\mu\nu}(\tilde{k}, P)$$

    - Focusing on spin and azimuthally independent cross sections, only the vector part of those expressions contributes
  - Remembering that $Q \gg m$

$$\begin{aligned} J(l^+) &= \gamma^+ \Delta^-(l^+) + O\left(\frac{m^2}{Q^2}\right) J + (\text{spin dep.}) \\ &= \frac{\not{\tilde{k}}'}{4q^-} \mathrm{Tr}\left[\gamma^- J(l^+)\right] + O\left(\frac{m^2}{Q^2}\right) J + (\text{spin dep.}), \\ L(\tilde{k}, P) &= \gamma^- \Phi^+(\tilde{k}, P) + O\left(\frac{m^2}{Q^2}\right) L + (\text{spin dep.}) \\ &= \frac{\not{\tilde{k}}}{4x_n P^+} \mathrm{Tr}\left[\gamma^+ L(\tilde{k}, P)\right] + O\left(\frac{m^2}{Q^2}\right) L + (\text{spin dep.}), \end{aligned}$$

Collinear Factorization

- Now we have

$$W^{\mu\nu}(P,q) = \frac{1}{2Q^2}\,\mathrm{Tr}\left[H^{\mu}(Q^2)\tilde{k}'H^{\dagger\nu}(Q^2)\tilde{k}\right]\left(\int\frac{\mathrm{d}l^+}{4\pi}\mathrm{Tr}\left[\frac{\gamma^-}{2}J(l^+)\right]\right)$$
$$\times\left(\int\frac{\mathrm{d}k^-\mathrm{d}^2\boldsymbol{k}_\mathrm{T}}{(2\pi)^3}\mathrm{Tr}\left[\frac{\gamma^+}{2}L(\tilde{k},P)\right]\right)+O\left(\frac{m^2}{Q^2}\right)W^{\mu\nu}.$$

- Perform the l^+ integration to obtain the desired factorized structure

$$W^{\mu\nu}(P,q) = \underbrace{\frac{1}{2Q^2}\,\mathrm{Tr}\left[H^{\mu}(Q^2)\tilde{k}'H^{\dagger\nu}(Q^2)\tilde{k}\right]}_{\mathcal{H}^{\mu\nu}(Q^2)}\underbrace{\left(\int\frac{\mathrm{d}k^-\mathrm{d}^2\boldsymbol{k}_\mathrm{T}}{(2\pi)^3}\mathrm{Tr}\left[\frac{\gamma^+}{2}L(\tilde{k},P)\right]\right)}_{f(x_\mathrm{bj})}$$
$$+\,O\left(\frac{m^2}{Q^2}\right)W^{\mu\nu}.$$

Collinear Factorization

- For a specific structure function

$$F_i(x_{\mathrm{bj}}, Q^2) = \mathcal{H}_i(Q^2)\, f(x_{\mathrm{bj}}) + O\left(\frac{m^2}{Q^2}\right), \qquad i = 1, 2,$$

  - Where

$$\mathcal{H}_i(Q^2) \equiv \mathrm{P}_i^{\mu\nu}\, \frac{1}{2Q^2} \mathrm{Tr}\left[H_\mu(Q^2)\, \not{\hat{k}}'\, H_\nu^\dagger(Q^2)\, \not{\hat{k}} \right]$$

Collinear Factorization

- Factorization of the simple QFT
  - At the large Q limit, Figures B and C are suppressed by powers of m/Q
  - Only need to factorize Figure A
  - The hard functions are

$$H(Q^2)^\mu = \gamma^\mu, \qquad H^\dagger(Q^2)^\nu = \gamma^\nu$$

  - The projected hard functions are

$$\begin{aligned}
\mathcal{H}_1(Q^2) &= 1, \\
\mathcal{H}_2(Q^2) &= \frac{2Q^2 x_{\mathrm{bj}}\left(Q^2 - M^2 x_{\mathrm{bj}}^2\right)}{\left(Q^2 + M^2 x_{\mathrm{bj}}^2\right)^2} \\
&= 2x_{\mathrm{bj}}\left(1 + O\left(\frac{M^2 x_{\mathrm{bj}}^2}{Q^2}\right)\right)
\end{aligned}$$

Collinear Factorization

- The lower part is given by:

$$f(x_{\mathrm{bj}}) = \int \frac{\mathrm{d}k^- \mathrm{d}^2\boldsymbol{k}_{\mathrm{T}}}{(2\pi)^3} \left(\frac{1}{\tilde{k}^2 - m_q^2}\right)^2 \mathrm{Tr}\left[\frac{\gamma^+}{2}(\tilde{\not{k}} + m_q)(\not{P} + M)(\tilde{\not{k}} + m_q)\right] \times (2\pi)\,\delta_+\left((P - \tilde{k})^2 - m_s^2\right) .$$

- Integrating over k^- yields:

$$k^- = -\frac{x_{\mathrm{bj}}\left[k_{\mathrm{T}}^2 + m_s^2 + (x_{\mathrm{bj}} - 1)M^2\right]}{\sqrt{2}Q(1 - x_{\mathrm{bj}})}$$

- The parton virtuality is:

$$\tilde{k}^2 = -\frac{k_{\mathrm{T}}^2 + x_{\mathrm{bj}}\left[m_s^2 + (x_{\mathrm{bj}} - 1)M^2\right]}{1 - x_{\mathrm{bj}}}$$

- The k_T-unintegrated functions $\mathcal{F}_{1,2}$ (equivalent to what was defined in the exact case are:

$$\mathcal{F}_1(x_{\mathrm{bj}}, Q^2, k_{\mathrm{T}}^2) = \mathcal{F}_2(x_{\mathrm{bj}}, Q^2, k_{\mathrm{T}}^2) = \frac{1}{(2\pi)^2} \frac{(1 - x_{\mathrm{bj}})\left[k_{\mathrm{T}}^2 + (m_q + x_{\mathrm{bj}}M)^2\right]}{\left[k_{\mathrm{T}}^2 + x_{\mathrm{bj}}m_s^2 + (1 - x_{\mathrm{bj}})\,m_q^2 + x_{\mathrm{bj}}(x_{\mathrm{bj}} - 1)M^2\right]^2}$$

Collinear Factorization

- Expanding exact solutions in powers of $1/Q$

$$\xi = x_{\rm bj}\left[1+\frac{k_{\rm T}^2+m_q^2-x_{\rm bj}^2M^2}{Q^2} - \frac{x_{\rm bj}^3M^2\left(k_T^2+m_q^2\right)+x_{\rm bj}\left(k_T^2+m_q^2\right)\left(k_T^2+m_s^2-M^2\right)-2M^4x_{\rm bj}^4(x_{\rm bj}-1)}{Q^4\left(x_{\rm bj}-1\right)}\right] + O\left(\frac{m^6}{Q^6}\right),$$

$$k^- = -\frac{x_{\rm n}}{Q\sqrt{2}}\left[\frac{k_{\rm T}^2+m_s^2+(x_{\rm n}-1)M^2}{1-x_{\rm n}} - \frac{x_{\rm n}\left(k_{\rm T}^2+m_q^2\right)\left(k_{\rm T}^2+m_s^2\right)}{Q^2(x_{\rm n}-1)^2}\right] + O\left(m\cdot\frac{m^5}{Q^5}\right),$$

$$k^2 = -\frac{k_{\rm T}^2+x_{\rm n}\left[m_s^2+(x_{\rm n}-1)M^2\right]}{1-x_{\rm n}} - \frac{x_{\rm n}\left(k_{\rm T}^2+m_q^2\right)\left(k_{\rm T}^2+\left[m_s+(x_{\rm n}-1)M\right]\left[m_s-(x_{\rm n}-1)M\right]\right)}{Q^2(x_{\rm n}-1)^2} + O\left(m^2\cdot\frac{m^4}{Q^4}\right).$$

Comparison Between the Exact Calculation and the Standard Approximation

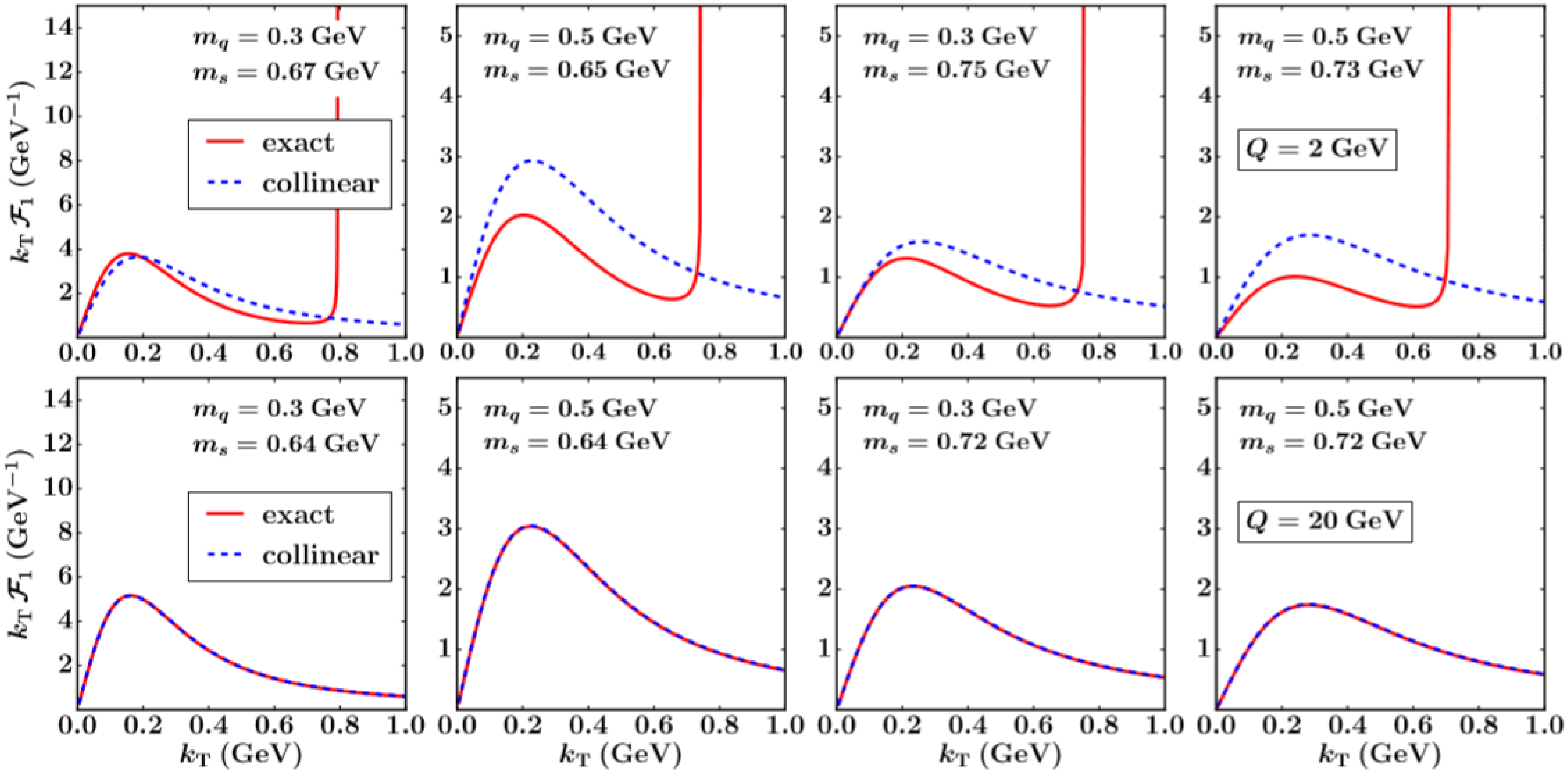
- Want to choose a set of masses that mimics QCD
  - For M, use the proton mass (0.938 GeV)
  - Choose values of m_q and m_s such that $|k|$ is on the order of a few MeV and the k_T distribution peaks at not more than 300 MeV
    - m_q should be on the order of a few MeV
    - m_s is chosen on a case by case basis:
      - In QCD, the remnant mass would grow with Q. The mass used here should behave similarly.
      - The mass in the quark-diquark rest frame is constrained

$$M - m_q < m_s \leq W(x_{\mathrm{bj}}, Q) - m_q$$

      - Solve $v \equiv \sqrt{-k^2}$ at $k_T = 0$ for m_s.

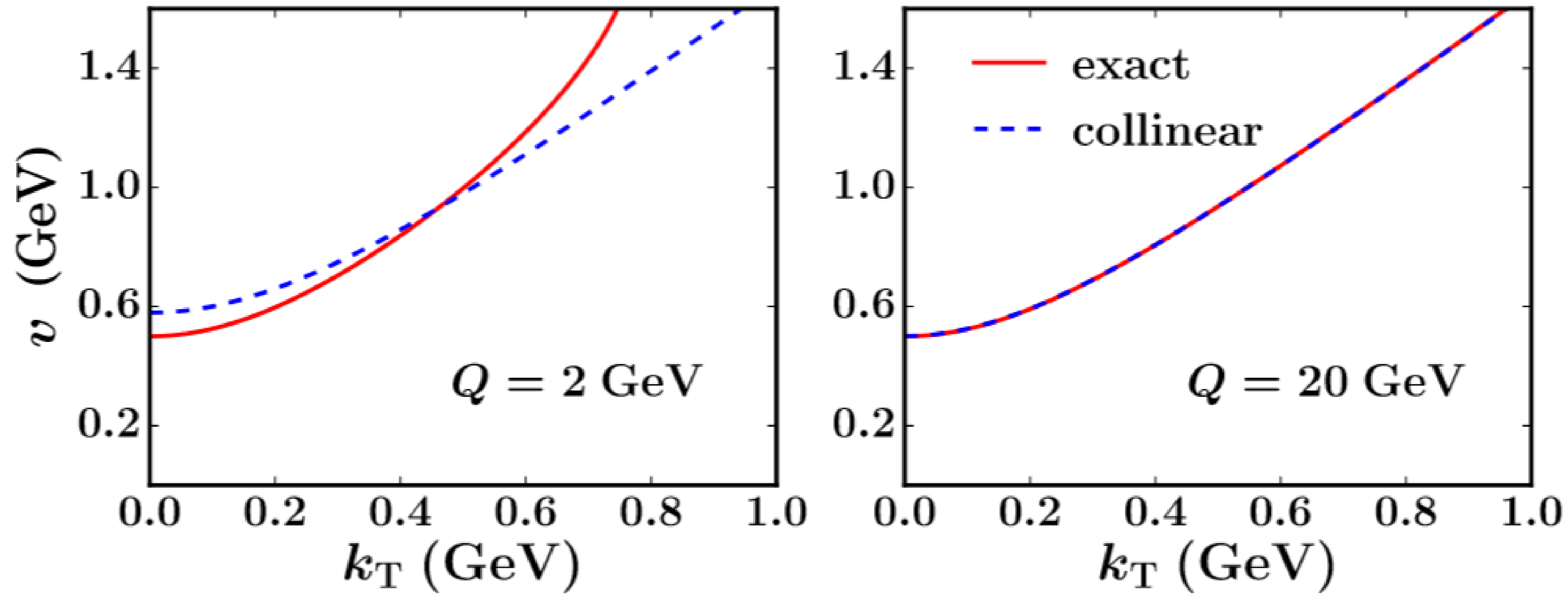
Comparison Between the Exact Calculation and the Standard Approximation

- Plots of exact and approximate $k_T\mathcal{F}_1$ for $x_{bj} = 0.6$



Comparison Between the Exact Calculation and the Standard Approximation

- Plot $v \equiv \sqrt{-k^2}$ vs. k_T ($x_{bj} = 0.6$, $m_q = 0.3$ GeV, and m_s corresponding to $v(k_T = 0) = 0.5$ GeV)



Comparison Between the Exact Calculation and the Standard Approximation

- Integrated Structure Functions
  - Exact: $$I(x_{\mathrm{bj}}, Q) \equiv \int_0^{k_{\mathrm{Tmax}}} \mathrm{d}k_{\mathrm{T}}\, k_{\mathrm{T}}\, \mathcal{F}_1^{\mathrm{exact}}(x_{\mathrm{bj}}, Q, k_{\mathrm{T}})$$
  - Approximate: $$\widehat{I}(x_{\mathrm{bj}}, Q, k_{\mathrm{cut}}) \equiv \int_0^{k_{\mathrm{cut}}} \mathrm{d}k_{\mathrm{T}}\, k_{\mathrm{T}}\, \mathcal{F}_1^{\mathrm{approx}}(x_{\mathrm{bj}}, Q, k_{\mathrm{T}})$$

	$Q = 2$ GeV				$Q = 20$ GeV			
m_q (GeV)	0.3	0.5	0.3	0.5	0.3	0.5	0.3	0.5
m_s (GeV)	0.67	0.65	0.75	0.73	0.64	0.64	0.72	0.72
$I/\widehat{I}(k_{\mathrm{Tmax}})$	0.88	0.64	0.76	0.57	1.00	1.00	1.00	1.00
$I/\widehat{I}(Q)$	0.67	0.45	0.49	0.35	0.90	0.88	0.86	0.85

Purely Kinematic TMCs

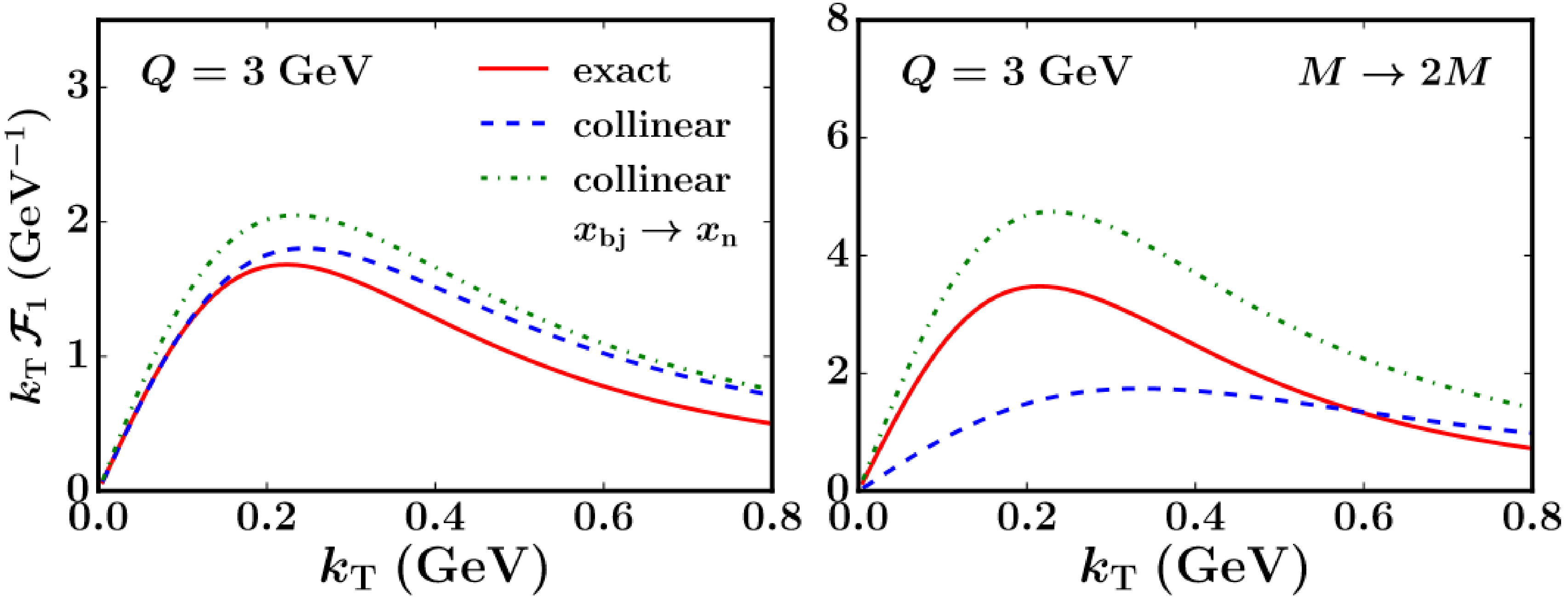
- Our analysis provides a means of clearly defining purely kinematic TMCs.
- Expand exact solutions in powers of m/Q, but keep only powers of M/Q (assume powers of k_T/Q, m_q/Q, and m_s/Q are still negligible):

$$\begin{aligned}
\xi &\rightarrow \xi_{\mathrm{TMC}} \equiv x_{\mathrm{bj}}\left[1-\frac{x_{\mathrm{bj}}^2 M^2}{Q^2}+\frac{2M^4 x_{\mathrm{bj}}^4}{Q^4}+\cdots\right]=x_{\mathrm{n}} \\
k^- &\rightarrow k^-_{\mathrm{TMC}} \equiv -\frac{x_{\mathrm{n}}\left[k_{\mathrm{T}}^2+m_s^2+(x_{\mathrm{n}}-1)M^2\right]}{\sqrt{2}Q(1-x_{\mathrm{n}})}, \\
k^2 &\rightarrow k^2_{\mathrm{TMC}} \equiv -\frac{k_{\mathrm{T}}^2+x_{\mathrm{n}}\left[m_s^2+(x_{\mathrm{n}}-1)M^2\right]}{1-x_{\mathrm{n}}}.
\end{aligned}$$

- This is equivalent to inserting x_n in place of x_{bj} in the collinear factorized equations for these quantities.
- Define purely kinematic TMCs as those corrections obtained from this substitution

Purely Kinematic TMCs

- Plots of $k_T \mathcal{F}_1$ (exact, approximate, and approximate with $x_{bj} \to x_n$) ($x_{bj} = 0.6$, $m_q = 0.3$ GeV, and m_s corresponding to $v(k_T = 0) = 0.5$ GeV)



Summary of Findings

- Analysis using the simple QFT demonstrates that the most accurate QCD factorization theorem for low-Q and large-x_{bj} would need to account for corrections due to parton mass, parton transverse momentum, and parton virtuality as well as the target mass.
- This type of analysis using a simple QFT can be used as a testing ground for any factorization theorem
- From this analysis, we can define purely kinematical TMCs as corrections that result from substituting x_n in place of x_{bj} in the collinear factorized formula.