What are the Low-Q and Large-x Boundaries of Collinear QCD Factorization Theorems?

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Introduction

- QCD complexities
  - Non-Abelian
  - Confinement
- Can only be solved analytically in the simplest of cases.
- Use Factorization theorems to simplify the calculation.
Introduction

- **Factorization:**
  - Method of disentangling the physics at different space-time scales by taking the asymptotically large limit of some physical energy

- **Useful in QCD:**
  - Asymptotic freedom allows short-distance processes to be calculated using perturbative calculations
    - Factorize to separate perturbative part from non-perturbative part
Introduction

Example: Collinear Factorization in Deep Inelastic Scattering (DIS)

Assume that $Q \gg m$ where $Q = \sqrt{-q^2}$ and $m$ is a generic mass scale on the order of a hadron mass.
Introductions

- Want to explore physics at lower $Q$ (~ a few GeV) and larger $x_{bj}$ ($\gtrsim 0.5$)
  - Interplay of perturbative and nonperturbative
- For example DIS at moderately low momentum transfer ($Q \sim 1 - 2$ GeV)
  - $Q \gg m$ is not an accurate assumption
  - But $\alpha_s/\pi \lesssim 0.1$ so can still use perturbative calculations.
Introduction

- Proposed techniques for extending QCD factorization to lower energies and/or larger $x_{bj}$:
  - Target mass corrections (Georgi and Politzer, 1976)
  - Large Bjorken-$x$ corrections from re-summation (Sterman, 1987)
  - Higher twist operators (Jaffe and Soldate, 1982)

- Questions arise:
  - Which method would give the most accurate approximation?
  - Are there other corrections that should be included?
Introduction

- What can we do to test how effective these techniques really are?
  - Problem: Non-Abelian nature of QCD leaves “blobs” that cannot be calculated without making approximations
  
  ![Diagram](image)

- There is no reason these techniques can only be applied to QCD.
- They should work for most re-normalizable Quantum Field Theories (QFT)
Introduction

- Use a simple QFT that requires no approximations
  - Perform an exact calculation in this QFT
  - Perform the same calculation after applying a factorization theorem to the QFT
  - Compare results numerically
Outline

- Define Simple QFT
- Review of standard notation in inclusive DIS
- Exact Calculation of Structure Functions in the Simple QFT
- Collinear Factorization Calculation of Structure Functions in the Simple QFT
- Analyze numerical differences between the Exact and Approximate results
- Summary of findings
Simple Model Definition

Interaction Lagrangian Density:

\[ \mathcal{L}_{\text{int}} = -\lambda \bar{\Psi}_N \psi_q \phi + \text{h.c.} \]

- \( \Psi_N \) : Spin-1/2 “Nucleon” Field with mass \( M \)
- \( \psi_q \) : Spin-1/2 “Quark” Field with mass \( m_q \)
- \( \phi \) : Scalar “Diquark” Field with mass \( m_s \)

The nucleon and quark couple to photon while the scalar does not.
Inclusive DIS process

\[ e(l) + N(P) \rightarrow e(l') + X(p_x) \]

- \( l \) and \( l' \) are the initial and final lepton four-momenta
- \( P \) is the four-momentum of the nucleon
- \( p_x = p_q + p_s \) is the four-momentum of the inclusive hadronic state
Standard Notation in Inclusive DIS

- Using Breit frame where
  - Nucleon momentum in +z direction
  - Photon momentum in -z direction

- Using light-front coordinates
  - Four-vector:
    \[ v^\mu = (v^+, v^-, \mathbf{v}_T) \]
  - “±” components:
    \[ v^{\pm} = (v^0 \pm v^z) / \sqrt{2} \]
  - Transverse component:
    \[ \mathbf{v}_T \]
Standard Notation in Inclusive DIS

- **Momenta**
  - **Nucleon**
    \[ P = \left( \frac{Q}{x_n \sqrt{2}}, \frac{x_n M^2}{Q \sqrt{2}}, 0_T \right) \]
  - **Photon**
    \[ q = l - l' \quad q = \left( -\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, 0_T \right) \]
  - **Internal Parton**
    \[ k = (k^+, k^-, k_T) \]
  - **Final Parton**
    \[ k + q \]

- **Where**
  \[ Q \equiv \sqrt{-q^2} \]
  \[ x_n \equiv -\frac{q^+}{P^+} = \frac{2x_{\text{bj}}}{1 + \sqrt{1 + 4x_{\text{bj}}^2 M^2/Q^2}} \]
  \[ x_{\text{bj}} = \frac{Q^2}{2P \cdot q} \]
The DIS cross section can be written as

\[ \frac{d\sigma}{dx_n\, dQ^2} = \frac{4\alpha}{\Phi Q^4} L_{\mu\nu} W^{\mu\nu} \]

Where

- \( \alpha \) is the electromagnetic fine structure constant
- \( \Phi \) is a flux factor
- \( L_{\mu\nu} \) is the leptonic tensor given by
  \[ L_{\mu\nu} = 2 (\ell_\mu \ell'_\nu + \ell'_\mu \ell_\nu - g_{\mu\nu} \ell \cdot \ell') \]
- \( W^{\mu\nu} \) is the hadronic tensor, which in terms of structure functions \( F_1 \) and \( F_2 \) is given by

\[ W^{\mu\nu}(P, q) = \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) F_1 (x_n, Q^2) + \left( P^\mu - q^\mu \frac{P \cdot q}{q^2} \right) \left( P^\nu - q^\nu \frac{P \cdot q}{q^2} \right) \frac{F_2 (x_n, Q^2)}{P \cdot q} \]
Define Projection Tensors for the Structure Functions

\[ P_1^{\mu\nu} W_{\mu\nu}(P, q) = F_1(x_n, Q^2), \quad P_2^{\mu\nu} W_{\mu\nu}(P, q) = F_2(x_n, Q^2) \]

\[ P_1^{\mu\nu} = -\frac{1}{2} P_g^{\mu\nu} + \frac{2Q^2 x_n^2}{(M_H^2 x_n^2 + Q^2)^2} P_{PP}^{\mu\nu}, \]

\[ P_2^{\mu\nu} = \frac{12Q^4 x_n^3 (Q^2 - M_H^2 x_n^2)}{(Q^2 + M_H^2 x_n^2)^4} \left( P_{PP}^{\mu\nu} + \frac{(M_H^2 x_n^2 + Q^2)^2}{12Q^2 x_n^2} P_g^{\mu\nu} \right) \]

Where

\[ P_g^{\mu\nu} = g^{\mu\nu}, \quad P_{PP}^{\mu\nu} = P^{\mu} P^{\nu}. \]
Exact Kinematics

- Familiar DIS Handbag Diagram
Exact Kinematics

- For electromagnetic gauge invariance these diagrams must also be included.
To demonstrate the calculations it is convenient to organize the hadronic tensor by separating the integrand into factors as follows:

\[ W^{\mu\nu}(P,q) = \sum_{j \in \text{graphs}} \int \frac{dk^+dk^-d^2k_T}{(2\pi)^4} [\text{Jac}] T_j^{\mu\nu} [\text{Prop}]_j \delta(k^- - k_{sol}^-) \delta(k^+ - k_{sol}^+) \]

Where

- \( j \) refers to Figures A, B, and C
- \([\text{Prop}]_j\) is the denominators of the internal propagators in Figure \( j \)
- \( T_j^{\mu\nu} \) is the appropriate Dirac trace for Figure \( j \)
- \([\text{Jac}]\) is the appropriate jacobian factor to isolate \( k^- \) and \( k^+ \) in the arguments of the delta functions
The arguments of the delta functions give the quadratic system
\[(q + k)^2 - m_q^2 = 0,\]
\[(P - k)^2 - m_s^2 = 0.\]
Solving this system for \(k^+ \equiv \xi P^+\) and \(k^-\) yields two solutions for \(k^-\).
Only one solution is physically realistic (0 if \(Q\) is taken to infinity).
The correct solution to the system is
\[k^- = k_{sol}^- \equiv \frac{\sqrt{\Delta} - Q^2(1 - x_n) - x_n(m_s^2 - m_q^2 - M^2(1 - x_n))}{2\sqrt{2} Q (1 - x_n)},\]
\[k^+ = k_{sol}^+ \equiv \frac{k_T^2 + m_s^2 + Q(Q + \sqrt{2}k^-)}{\sqrt{2}(Q + \sqrt{2}k^-)},\]
Where \[\Delta = [Q^2(1 - x_n) - x_n(M_s^2(1 - x_n) + m_q^2 - m_s^2)]^2 - 4x_n(1 - x_n)[k_T^2(Q^2 + x_nM_s^2) - Q^2M^2(1 - x_n) + Q^2m_q^2 + x_nM^2m_q^2].\]
Exact Kinematics

- The Jacobian factor is:

\[
[Jac] = \frac{x_n Q (2k^- + \sqrt{2}Q)}{4(1 - x_n)k^- Q^2(\sqrt{2}k^- + 2Q) + 2\sqrt{2}[Q^4(1 - x_n) - (k_T^2 + m_q^2)x_n(Q^2 + x_n M^2)]}
\]

- The propagator factors are:

\[
[Prop]_A = \frac{1}{(k^2 - m_q^2)^2},
\]

\[
[Prop]_B = \frac{1}{((P + q)^2 - M^2)^2} = \frac{x_n^2}{(Q^2(1 - x_n) - M^2 x_n^2)^2},
\]

\[
[Prop]_C = \frac{x_n}{(k^2 - m_q^2)(Q^2(1 - x_n) - M^2 x_n^2)}.
\]
The Dirac traces are:

\[
T_A^{\mu\nu} = \text{Tr} \left[ (\not{p} + M)(\not{k} + m_q)\gamma^\mu(\not{k} + \not{q} + m_q)\gamma^\nu(\not{k} + m_q) \right],
\]

\[
T_B^{\mu\nu} = \text{Tr} \left[ (\not{p} + M)\gamma^\mu(\not{p} + \not{q} + M)(\not{k} + \not{q} + m_q)(\not{p} + \not{q} + M)\gamma^\nu \right],
\]

\[
T_C^{\mu\nu} = 2 \text{Tr} \left[ (\not{p} + M)(\not{k} + m_q)\gamma^\mu(\not{k} + \not{q} + m_q)(\not{p} + \not{q} + M)\gamma^\nu \right],
\]

- Factor of 2 is for the Hermitian conjugate of Figure C.

Define the projected quantities:

\[
T^g_j = P_{g}^{\mu\nu} T_{j \mu\nu}, \quad T^{PP}_j = P_{PP}^{\mu\nu} T_{j \mu\nu}
\]
Exact Kinematics

The $p_{\mu\nu}^{\mu}$ projections with traces evaluated are:

\[ T^g_A = -8 \left[ 2(P \cdot k + m_q M) k \cdot q + (k^2 - 3m_q^2) P \cdot k - 2Mm_q^3 + (m_q^2 - k^2) P \cdot q \right], \]

\[ T^g_B = 8 \left[ 2M^3m_q + P \cdot k(2M^2 - Q^2) - 2(M^2 + Mm_q)Q^2 \right. \]
\[ \left. + 2k \cdot q (M^2 - P \cdot q) + [2(2M^2 + Mm_q) + Q^2] P \cdot q \right], \]

\[ T^g_C = -16 \left[ -2(P \cdot k)^2 + k^2 M^2 + (M^2 - m_q M) k \cdot q - M^2m_q^2 + 2Mm_qQ^2 \right. \]
\[ \left. + (m_q^2 - Mm_q) P \cdot q - 2P \cdot k (k \cdot q + Mm_q - Q^2 + P \cdot q) \right], \]
The $P_{PP}^{\mu\nu}$ projections with traces evaluated are:

$$
T_A^{PP} = 4 \left[ 4(P \cdot k)^3 + 4(P \cdot k)^2(Mm_q + P \cdot q) \\
- M P \cdot k (3k^2M + 2M k \cdot q - 3Mm_q^2 - 4m_q P \cdot q) \\
- M^3 m_q (k^2 + 2k \cdot q - m_q^2) - M^2 (k^2 - m_q^2) P \cdot q \right],
$$

$$
T_B^{PP} = 4M^2 \left[ P \cdot k (4M^2 + Q^2) + 4M^2 (k \cdot q + Mm_q) - Q^2 (4M^2 + Mm_q) \\
+ [2k \cdot q + 4(M^2 + Mm_q) - Q^2] P \cdot q \right],
$$

$$
T_C^{PP} = 8M \left[ 4M (P \cdot k)^2 + M P \cdot k (2k \cdot q + 4Mm_q - Q^2) \\
- M^2 [2M (k^2 + k \cdot q - m_q^2) + m_q Q^2] \\
- [k^2 M - (2M + m_q)(2P \cdot k + Mm_q)] P \cdot q \right].
$$
Define the nucleon structure functions as:

\[ F_1 (x_n, Q^2) = \int \frac{d^2 k_T}{(2\pi)^2} \mathcal{F}_1 (x_n, Q^2, k_T^2), \]

\[ F_2 (x_n, Q^2) = \int \frac{d^2 k_T}{(2\pi)^2} 2x_n \mathcal{F}_2 (x_n, Q^2, k_T^2) \]

Where

\[ \mathcal{F}_1 (x_n, Q^2, k_T^2) = \frac{1}{(2\pi)^2} \text{[Jac]} \sum_j \left( -\frac{1}{2} T^q_j + \frac{2Q^2x_n^2}{(M^2x_n^2 + Q^2)^2} T^{PP} \right) [\text{Prop}]_j, \]

\[ 2x_n \mathcal{F}_2 (x_n, Q^2, k_T^2) = \frac{1}{(2\pi)^2} \frac{12Q^4x_n^3(Q^2 - M^2x_n^2)}{(Q^2 + M^2x_n^2)^4} \times \text{[Jac]} \sum_j \left( T^{PP}_j - \frac{(M^2x_n^2 + Q^2)^2}{12Q^2x_n^2} T^q_j \right) [\text{Prop}]_j. \]
Exact Kinematics

- The exact kinematics impose an upper bound on $k_T$.
- Start from calculation of $W$ in the center-of-mass frame:
  \[ W = p_q^0 + p_s^0 \bigg|_{\text{c.m.}} = \sqrt{m_q^2 + k_T^2 + k_z^2} + \sqrt{m_s^2 + k_T^2 + k_z^2} \bigg|_{\text{c.m.}} \]
- $W$ in the Breit frame:
  \[ W^2 = (P + q)^2 = (p_q + p_s)^2 = M^2 + \frac{Q^2(1 - x_{bj})}{x_{bj}} \]
- Set the two equations for $W$ equal to each other, and solve for $k_T$ with $k_z = 0$
  \[ k_{T_{\text{max}}} = \sqrt{\frac{x_{bj}(M^2 - (m_q + m_s)^2) + Q^2(1 - x_{bj})[x_{bj}(M^2 - (m_q - m_s)^2) + Q^2(1 - x_{bj})]}{4x_{bj}[Q^2(1 - x_{bj}) + M^2x_{bj}]}}. \]
Collinear Factorization

- Collinear Factorization

- Un-approximated hadronic tensor

\[ W^{\mu\nu}(P, q) = \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ H^\mu(k, k') J(k') H^{\nu\dagger}(k, k') L(k, P) \right] \]
Collinear Factorization

- Internal quark momentum
  - Power Counting at low transverse momentum ($k_T \sim O(m_T)$)
    - $k^2$ and $k^2 \sim O(m^2)$
    - $k^+ \sim O(Q)$
  - Therefore
    \[
    k \sim \left( O(Q), O\left(\frac{m^2}{Q}\right), O(m_T) \right)
    \]
Collinear Factorization

- **Hard Factor** \(H(k, k')\)
  - \(k \cdot q = k^+ q^- + O(m^2)\)
  - \(k \to \vec{k} \equiv (k^+, 0, 0)\)
  - \(k' \to \vec{k}' = \vec{k} + q\)
  - \(H(k, k_2) \to H(\vec{k}, \vec{k}'')\)
  - \(\delta \left( \vec{k}'^2 - m_q^2 \right) \to \vec{k}^+ = x_n P^+ = x_{bj} P^+\)

- **Lower Factor** \(L(k, P)\)
  - Contains propagator, large component of \(k\) can be approximated but the small components must be kept exact
  - \(k \to \vec{k} \equiv (k^+, k^-, k_T)\)
  - \(L(k, P) \to L(\vec{k}, P)\)
Collinear Factorization

- Jet Factor ($f(k')$)
  - Power counting for $k'$ is:
    \[
    k' \sim (O(Q), O(Q), O(m_T))
    \]
  - Consider a frame labeled "***", where the outgoing transverse momentum vanishes $k'^*_T = 0$,
    \[
    k'^* = \left( k^+ + q^+ - \frac{k_T^2}{2(q^- + k^-)}, q^- + k^- , 0_T \right)
    \]
  - The outgoing parton's virtuality is:
    \[
    k'^*^2 = 2 (k^+ + q^+) (k^- + q^-) - k_T^2
    \]
    \[
    \sim 2 (k^+ + q^+) q^- - k_T^2 + O \left( \frac{m_T^4}{Q} \right)
    \]
  - Make the approximation $k' \rightarrow \vec{k}' \equiv (l^+, q^-, 0_T)$ where $l^+ \equiv k^+ - x_n P^+ + \frac{k_T^2}{2q^-}$
  - Change integration variables from $k^+$ to $l^+$
    - $f(k') \rightarrow f(l^+)$
Collinear Factorization

- Hadronic Tensor now becomes

\[ W^{\mu\nu}(P, q) = \int \frac{dl^+dk^-d^2k_T}{(2\pi)^4} \text{Tr} \left[ H^\mu(Q^2)J(l^+)H^\nu(Q^2)L(\tilde{k}, P) \right] + O\left(\frac{m^2}{Q^2}\right) W^{\mu\nu} \]

- Separate the integrations into factors for the jet and target

\[ W^{\mu\nu}(P, q) = \text{Tr} \left[ H^\mu(Q^2) \left( \int \frac{dl^+}{2\pi} J(l^+) \right) H^\nu(Q^2) \left( \int \frac{dk^-d^2k_T}{(2\pi)^3} L(\tilde{k}, P) \right) \right] + O\left(\frac{m^2}{Q^2}\right) W^{\mu\nu} \]
Collinear Factorization

- To complete the factorization
  - Express the Jet and Lower factors in a basis of Dirac matrices.

\[
J^{(l^+)} = \gamma_\mu \Delta_\mu^{(l^+)} + \Delta_S^{(l^+)} + \gamma_5 \Delta_P^{(l^+)} + \gamma_5 \gamma_\mu \Delta_A^{(l^+)} + \sigma_{\mu\nu} \Delta_T^{(l^+)} ,
\]
\[
L(\vec{k}, P) = \gamma_\mu \Phi_\mu (\vec{k}, P) + \Phi_S (\vec{k}, P) + \gamma_5 \Phi_P (\vec{k}, P) + \gamma_5 \gamma_\mu \Phi_A (\vec{k}, P) + \sigma_{\mu\nu} \Phi_T^{(\mu\nu)} (\vec{k}, P)
\]

- Focusing on spin and azimuthally independent cross sections, only the vector part of those expressions contributes

- Remembering that \( Q \gg m \)

\[
J^{(l^+)} = \gamma^+ \Delta^- (l^+)^{\mu} + \mathcal{O} \left( \frac{m^2}{Q^2} \right) J + \text{(spin dep.)}
\]
\[
= \frac{\vec{k}'}{4q} \text{Tr} \left[ \gamma^- J^{(l^+)} \right] + \mathcal{O} \left( \frac{m^2}{Q^2} \right) J + \text{(spin dep.)},
\]
\[
L(\vec{k}, P) = \gamma^- \Phi^+ (\vec{k}, P) + \mathcal{O} \left( \frac{m^2}{Q^2} \right) L + \text{(spin dep.)}
\]
\[
= \frac{\vec{k}}{4s_{\vec{P}}^+} \text{Tr} \left[ \gamma^+ L(\vec{k}, P) \right] + \mathcal{O} \left( \frac{m^2}{Q^2} \right) L + \text{(spin dep.)},
\]
Collinear Factorization

Now we have

$$W^{\mu\nu}(P, q) = \frac{1}{2Q^2} \text{Tr} \left[ H^\mu(Q^2) \vec{k}' H^{\nu}(Q^2) \vec{k} \right] \left( \int \frac{d^4l^+}{4\pi} \text{Tr} \left[ \frac{\gamma^-}{2} J(l^+) \right] \right)$$

$$\times \left( \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ \frac{\gamma^+}{2} \Lambda(k, P) \right] \right) + O \left( \frac{m^2}{Q^2} \right) W^{\mu\nu}.$$

Perform the $l^+$ integration to obtain the desired factorized structure

$$W^{\mu\nu}(P, q) = \frac{1}{2Q^2} \text{Tr} \left[ H^\mu(Q^2) \vec{k}' H^{\nu}(Q^2) \vec{k} \right] \left( \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ \frac{\gamma^+}{2} \Lambda(k, P) \right] \right)$$

$$+ O \left( \frac{m^2}{Q^2} \right) W^{\mu\nu}.$$
Collinear Factorization

- For a specific structure function

\[ F_i(x_{bj}, Q^2) = \mathcal{H}_i(Q^2) f(x_{bj}) + O \left( \frac{m^2}{Q^2} \right), \quad i = 1, 2, \]

- Where

\[ \mathcal{H}_i(Q^2) \equiv P_i^{\mu\nu} \frac{1}{2Q^2} \text{Tr} \left[ H_\mu(Q^2) \not{k} \not{k}^\nu H_\nu(Q^2) \not{k} \right] \]
Collinear Factorization

- Factorization of the simple QFT
  - At the large $Q$ limit, Figures B and C are suppressed by powers of $m/Q$
  - Only need to factorize Figure A
  - The hard functions are
    \[ H(Q^2)^\mu = \gamma^\mu, \quad H^\dagger(Q^2)^\nu = \gamma^\nu \]
  - The projected hard functions are
    \[ \mathcal{H}_1(Q^2) = 1, \]
    \[ \mathcal{H}_2(Q^2) = \frac{2Q^2 x_{bj} (Q^2 - M^2 x_{bj}^2)}{(Q^2 + M^2 x_{bj}^2)^2} \]
    \[ = 2 x_{bj} \left( 1 + O \left( \frac{M^2 x_{bj}^2}{Q^2} \right) \right) \]
The lower part is given by:

\[ f(x_{bij}) = \int \frac{dk^{-} dk_T}{(2\pi)^3} \left( \frac{1}{k^2 - m_q^2} \right)^2 \text{Tr} \left[ \frac{\gamma^+}{2} (\tilde{k} + m_q)(P + M)(\tilde{k} + m_q) \right] \times (2\pi) \delta_+ \left( (P - \tilde{k})^2 - m_s^2 \right). \]

Integrating over \( k^{-} \) yields:

\[ k^{-} = -\frac{x_{bj} [k_T^2 + m_s^2 + (x_{bj} - 1)M^2]}{\sqrt{2Q(1 - x_{bj})}} \]

The parton virtuality is:

\[ \tilde{k}^2 = -\frac{k_T^2 + x_{bj} [m_s^2 + (x_{bj} - 1)M^2]}{1 - x_{bj}} \]

The \( k_T \)-unintegrated functions \( \mathcal{F}_{1,2} \) (equivalent to what was defined in the exact case are:

\[ \mathcal{F}_1(x_{bj}, Q^2, k_T^2) = \mathcal{F}_2(x_{bj}, Q^2, k_T^2) = \frac{1}{(2\pi)^2} \frac{(1 - x_{bj}) [k_T^2 + (m_q + x_{bj}M)^2]}{[k_T^2 + x_{bj}m_s^2 + (1 - x_{bj}) m_q^2 + x_{bj}(x_{bj} - 1)M^2]^2} \]
Collinear Factorization

- Expanding exact solutions in powers of $1/Q$

$$\xi = x_{bj} \left[ 1 + \frac{k_T^2 + m_\perp^2 - x_{bj}^2 M^2}{Q^2} - \frac{x_{bj} M^2 (k_T^2 + m_\perp^2) + x_{bj} (k_T^2 + m_\perp^2) (k_T^2 + m_\perp^2 - M^2) - 2M^4 x_{bj}^4 (x_{bj} - 1)}{Q^4 (x_{bj} - 1)} \right] + O\left(\frac{m^6}{Q^6}\right),$$

$$k^- = - \frac{x_n}{Q \sqrt{2}} \left[ \frac{k_T^2 + m_\perp^2 + (x_n - 1) M^2}{1 - x_n} - \frac{x_n (k_T^2 + m_\perp^2) (k_T^2 + m_\perp^2)}{Q^2 (x_n - 1)^2} \right] + O\left(\frac{m \cdot m_\perp}{Q^6}\right),$$

$$k^2 = - \frac{k_T^2 + x_n [m_\perp^2 + (x_n - 1) M^2]}{1 - x_n} - \frac{x_n (k_T^2 + m_\perp^2) (k_T^2 + m_\perp^2) [m_\perp + (x_n - 1) M] [m_\perp - (x_n - 1) M]}{Q^2 (x_n - 1)^2} + O\left(\frac{m^2 \cdot m_\perp^4}{Q^4}\right).$$
Comparison Between the Exact Calculation and the Standard Approximation

- Want to choose a set of masses that mimics QCD
  - For $M$, use the proton mass (0.938 GeV)
  - Choose values of $m_q$ and $m_s$ such that $|k|$ is on the order of a few MeV and the $k_T$ distribution peaks at not more than 300 MeV
    - $m_q$ should be on the order of a few MeV
    - $m_s$ is chosen on a case by case basis:
      - In QCD, the remnant mass would grow with $Q$. The mass used here should behave similarly.
      - The mass in the quark-diquark rest frame is constrained
        \[ M - m_q < m_s \leq W(x_{bj}, Q) - m_q \]
      - Solve $v \equiv \sqrt{-k^2}$ at $k_T = 0$ for $m_s$. 
Comparison Between the Exact Calculation and the Standard Approximation

- Plots of exact and approximate $k_T F_1$ for $x_{bj} = 0.6$
Comparison Between the Exact Calculation and the Standard Approximation

- Plot $v \equiv \sqrt{-k^2}$ vs. $k_T$ ($x_{bj} = 0.6$, $m_q = 0.3$ GeV, and $m_s$ corresponding to $v(k_T = 0) = 0.5$ GeV)
Comparison Between the Exact Calculation and the Standard Approximation

- Integrated Structure Functions
  - Exact:
    \[ I(x_{bj}, Q) \equiv \int_0^{k_{T_{\text{max}}}} \, dk_T \, k_T \, F_1^{\text{exact}}(x_{bj}, Q, k_T) \]
  - Approximate:
    \[ \hat{I}(x_{bj}, Q, k_{\text{cut}}) \equiv \int_0^{k_{\text{cut}}} \, dk_T \, k_T \, F_1^{\text{approx}}(x_{bj}, Q, k_T) \]

<table>
<thead>
<tr>
<th></th>
<th>(Q = 2) GeV</th>
<th>(Q = 20) GeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_q) (GeV)</td>
<td>0.3 0.5 0.3 0.5</td>
<td>0.3 0.5 0.3 0.5</td>
</tr>
<tr>
<td>(m_s) (GeV)</td>
<td>0.67 0.65 0.75 0.73</td>
<td>0.64 0.64 0.72 0.72</td>
</tr>
<tr>
<td>(I/I(k_{T_{\text{max}}}))</td>
<td>0.88 0.64 0.76 0.57</td>
<td>1.00 1.00 1.00 1.00</td>
</tr>
<tr>
<td>(I/\hat{I}(Q))</td>
<td>0.67 0.45 0.49 0.35</td>
<td>0.90 0.88 0.86 0.85</td>
</tr>
</tbody>
</table>
Our analysis provides a means of clearly defining purely kinematic TMCs.

Expand exact solutions in powers of $m/Q$, but keep only powers of $M/Q$ (assume powers of $k_T/Q$, $m_q/Q$, and $m_s/Q$ are still negligible):

\[ \xi \rightarrow \xi_{\text{TMC}} \equiv x_{bj} \left[ 1 - \frac{x_{bj}^2 M^2}{Q^2} + \frac{2M^4 x_{bj}^4}{Q^4} + \cdots \right] = x_n \]

\[ k^- \rightarrow k^-_{\text{TMC}} = -\frac{x_n \left[ k_T^2 + m_s^2 + (x_n - 1)M^2 \right]}{\sqrt{2Q(1-x_n)}} \]

\[ k^2 \rightarrow k^2_{\text{TMC}} \equiv -\frac{k_T^2 + x_n \left[ m_s^2 + (x_n - 1)M^2 \right]}{1-x_n} \]

This is equivalent to inserting $x_n$ in place of $x_{bj}$ in the collinear factorized equations for these quantities.

Define purely kinematic TMCs as those corrections obtained from this substitution.
Purely Kinematic TMCs

- Plots of $k_T F_1$ (exact, approximate, and approximate with $x_{bj} \rightarrow x_n$) ($x_{bj} = 0.6$, $m_q = 0.3$ GeV, and $m_s$ corresponding to $v(k_T = 0) = 0.5$ GeV)

![Plot Diagram]

$Q = 3$ GeV

$M \rightarrow 2M$
Summary of Findings

- Analysis using the simple QFT demonstrates that the most accurate QCD factorization theorem for low-$Q$ and large-$x_{bj}$ would need to account for corrections due to parton mass, parton transverse momentum, and parton virtuality as well as the target mass.
- This type of analysis using a simple QFT can be used as a testing ground for any factorization theorem.
- From this analysis, we can define purely kinematical TMCs as corrections that result from substituting $x_n$ in place of $x_{bj}$ in the collinear factorized formula.