A COLLECTION OF FORMULAS FOR SPIN DEPENDENT DEEP INELASTIC SCATTERING

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ABSTRACT

The analysis of the longitudinal spin structure functions of the proton, neutron and deuteron requires the use of a large number of formulas and numerical inputs taken from various unpolarised experiments. The aim of this report is to collect this information which is usually scattered in many textbooks, Ph.D thesis and publications.

To observ the internal structure of elementary particles, the only solution is to use a probe which has a smallest structure, which means a wavelength smaller than the object size that it has to study. Therefore, the smaller the wavelength is , the better the space resolution will be. Now, as the particle's wavelength is proportionnal to the inverse of its momentum, the probe must be very energetic to have a small wavelength. To understand the structures of some compouned particles like the hadrons, and more particulary those of the nucleon, the Nature offers us two probes : the electromagnetic one (the photon) and the weak one (the Z_0). These probes are generated by electrons beams about $30\,GeV$ and muons beams about $200\,GeV$. With a maximum transfert momentum of order of $100\,GeV^2$, we can neglect the weak interaction since the ratio Q^2/M_z^2 is of order of 10^{-2}. Then, we will assume that the electromagnetic interaction is widely dominant and we will approximate the deep inelastic scattering like the exchange of one virtual photon. Interacting with the nucleon, it explodes it to generate hadronical fragments.

Deep inelastic scattering experiments measure the double differential cross sections $\frac{d^2\sigma}{dE'\,d\Omega}$ for lepton-nucleons interactions on an unpolarised or a polarised target as a function of the outgoing lepton energy (E') and the solid angle ($d\Omega = d\phi\, dcos\theta$).

In the first part, we define kinematical variables. Then, we show how these cross sections can be expressed in terms some functions describing the internal structure of the nucleon. We present also the relation between cross sections and structure functions for unpolarised and polarised charged lepton-nucleon scattering in the most general case and in the scaling limit of large quadri-momentum transfer (Q^2) and large energy transfer (ν). In the next section, we define the spin asymmetry in terms of polarised structure functions g_1, g_2 or of virtual photon-nucleon asymmetries.
In the following section, we determine the first moment of the spin dependent structure functions and their interpretations in the parton model. Then, we define the differents sum rules. The last section describes the effects on the observables of a spin 1 target like the deuteron.

1. The kinematical variables

The figure 1 shows the deep inelastic scattering of a muon off a nucleon at the lowest order of the Feynman graphes. The four-momenta of incoming and outgoing muons are defined in the laboratory frame by the relations:

$$k = (E, k_x, k_y, k_z) \qquad\qquad k' = (E', k'_x, k'_y, k'_z)$$

whereas those of the nucleon and the virtual photon are given by :

$$p = (M, 0, 0, 0) \qquad\qquad q = (\nu, q_x, q_y, q_z)$$

where M is the target nucleon mass which is supposed to be at rest. From the energy-momentum conservation laws at the vertex, the four-momentum of the virtual photon

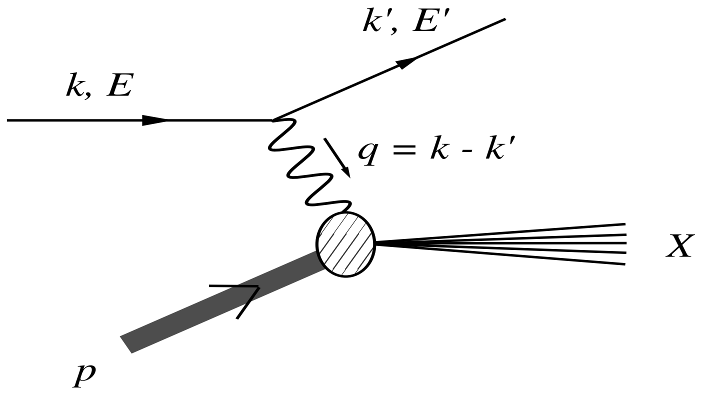


Fig. 1. *The basic diagram for deep inelastic lepton hadron scattering. The virtual photon momentum is q. The final hadronic state is not measured,and is denoted by X.*

is given by :

$$q = k - k' \qquad Q^2 = -q^2 = 2EE'(1-\cos\theta) = 4EE'\sin^2\theta/2$$

where θ is the muon scattering angle in the laboratory frame versus the incoming direction. The energy of the virtual photon is defined by :

$$\nu = E - E' = \frac{p\cdot q}{M}$$

Here again from the energy-momentum conservation, the mass of the hadronical residual is known :

$$W^2 = \left(p^2 + q^2\right) = M^2 + 2p\cdot q + q^2$$

now, as a fonction of the virtual photon kinematical variables, we get :

$$W^2 = M^2 + 2M\nu - Q^2 \tag{1}$$

we define then the Bjorken dimensionless variables by the relations :

$$y = \frac{p\cdot q}{p\cdot k} = \frac{\nu}{E}$$

$$x = \frac{Q^2}{2M\nu} = \frac{Q^2}{2p\cdot q} = \frac{Q^2}{2MEy}$$

we can now treat the deep inelastic scattering problematic of a muon off a nucleon in two separate ways :

- The leptonic part which consists in the emission of a virtual photon of energy ν and momentum q by a muon of momentum k.
- The hadronic part which consists in the absorption of this same virtual photon by a nucleon of momentum p.

2. The differential cross-section

Usually, the cross section for a scattering process in which we have :

- Initial state : 2 incoming particles (a,b)
- Final state : N outgoing particles (1,2,..,N)

is defined by :

$$d\sigma\Big(\underbrace{a+b}_{i};\underbrace{1;2,\ldots,N}_{f}\Big) = \Phi|\langle f \mid \mathcal{M} \mid i\rangle|^2(2\pi)^4\delta^4\Big(p_a+p_b-\sum_{j=1}^{N}p_j\Big)\prod_{j=1}^{N}\frac{d^3\vec{p}_j}{(2\pi)^3\,2p_j^0}$$

with the flux factor defined by

$$\Phi = \frac{1}{4[(p_a.p_b)^2 - m_a^2m_b^2]^{1/2}}$$

In deep inelastic scattering of polarised muons on a fixed nucleon target (see fig 1),we get for this factor :

$$\Phi = \frac{1}{\Big(2E\Big)\Big(2M\Big)}$$

where M is the mass of the target nucleon and E the energy of the incident muon in the laboratory frame. So that :

$$d\sigma = \frac{\Big(2\pi\Big)^4}{\Big(2E\Big)\Big(2M\Big)}|\langle f \mid M \mid i\rangle|^2\delta^4\Big(p+q-\sum_{n=1}^{X}p_n\Big)\frac{d^3k'}{\Big(2\pi\Big)^3\Big(2k'^0\Big)}\prod_{n=1}^{X}\frac{d^3p_n}{\Big(2\pi\Big)^3\Big(2p_n^0\Big)}$$

The scattering amplitude $\mathcal{M}$ for a given X state is defined by :

$$i\mathcal{M} \;=\; (-ie)^2\left(\frac{-ig_{\mu\nu}}{q^2}\right)\langle k's_l'|\; j_l^{\mu}(0)\;|k,s_l\rangle\langle Xs_X'|\; j_h^{\nu}(0)\;|p,s_n\rangle$$

where

- $s_l(s_l')$ is the polarisation of the incoming (outgoing) muon
- $s_n(s_X')$ is the polarisation of the initial (final) nucleon (hadrons) and can be choosen to be the value of the spin along an arbitrary quantisation axis, usually the muon axis, so that $s_n \;=\; \pm\frac{1}{2}$.
- j_l^{μ}, j_h^{ν} are the leptonic and hadronic electromagnetic currents.

The differential cross section is obtained from $\mathcal{M}$ by squaring and multipling by the phase space factor. The polarisation of the final lepton is not determined. Final state hadrons are not measured and we have to sum over all possible states. This gives the following differential cross section :

$$d\sigma = \frac{\left(2\pi\right)^4}{\left(2E\right)\left(2M\right)} \cdot \frac{d^3k'}{\left(2\pi\right)^3\left(2E'\right)} \cdot \sum_X \sum_{s',\lambda'} \int \prod_{n=1}^{X} \frac{d^3p_n}{\left(2\pi\right)^3\left(2E_n\right)} \cdot \frac{e^4}{Q^4} \cdot \delta^4\left(p+q-\sum_{n=1}^{X} p_n\right)$$

$$\times \langle k, s_l|\, j_l^\mu(0)\,|k's_l'\rangle\langle k's_l'|\, j_l^\nu(0)\,|k, s_l\rangle\ \langle p, s_n|\, j_{\mu h}(0)\,|X, s_X'\rangle\langle X, s_X'|\, j_{\nu h}(0)\,|p, s_n\rangle$$

From the equation 2, we can define the leptonic tensor (see appendix A) :

$$L^{\mu\nu} = \sum_{s_l'} \langle k's_l'|\, j_e^\nu(0)\,|k, s_l\rangle\langle k, s_l|\, j_e^\mu(0)\,|k's_l'\rangle$$

As the muon is a pointlike fermion, we can deduce that :

$$\langle k, s_l|\, j_e^\nu(0)\,|k, s_l\rangle = \bar{u}(k, s_l)\,\gamma^\nu\, u(k, s_l)$$

Using the several properties of the γ matrices, the leptonic tensor can be written as :

$$L^{\mu\nu} = 2\left[\left(k'^\mu k^\nu + k'^\nu k^\mu\right) - \left(k \cdot k'\right)g^{\mu\nu} - i\epsilon^{\mu\nu\rho\sigma} q_\rho s^l_\sigma\right] \tag{2}$$

where $g^{\mu\nu}$ is the metric tensor, $\epsilon^{\mu\nu\rho\sigma}$ Levi-Civita's antisymetric tensor and s^l_σ is the incoming muon spin four-vector. We can immediatly deduce that this tensor can be separated in symetric and antisymetric parts. This antisymetric tensor is the only one to contain an information about the incoming muon spin.

The hadronic tensor can be defined from the hadronic current j_ν^h by the following relation (see appendix B) :

$$W_{\mu\nu}(p, q) = \frac{1}{4\pi} \sum_{X, s_X'} \left[(2\pi)^4 \delta^4\left(p + q - p_X\right)\langle p, s_n|\, j_\mu^h(0)\,|X, s_X'\rangle\langle X, s_X'|\, j_\nu^h(0)\,|p, s_n\rangle\right]$$

where the sum contains all hadronic states X of spin s_X' that we do not measure. The hadronic tensor has to be defined like a function of all available Lorentz vectors which are present in the absorption process of a virtual photon off a polarised nucleon being p_μ, q_μ, s_n^σ. The general form of this tensor is a linear combination of all the tensors built from these Lorentz vectors, from the metric tensor $g_{\mu\nu}$ and from Levi-Civita $\epsilon_{\mu\nu\alpha\beta}$. The coefficients of this combination will be Lorentz invariants, *i.e.*functions of ν et Q^2. Several constraints like the parity conservation, the time reversal invariance, the hermiticity, translation invariance and the conservation of the hadronic current allow to write in a conventionnal way the tensor in the form[1] :

$$\begin{aligned} W_{\mu\nu} &= W_1\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)+\frac{W_2}{M^2}\left(p_\mu-\frac{p\cdot q\; q_\mu}{q^2}\right)\left(p_\nu-\frac{p\cdot q\; q_\nu}{q^2}\right) \\ &+ iMG_1\,\epsilon_{\mu\nu\lambda\sigma}q^\lambda s_n^\sigma+\frac{i}{M}G_2\epsilon_{\mu\nu\lambda\sigma}q^\lambda\left(p\cdot q\, s_n^\sigma - s\cdot q\, p^\sigma\right) \end{aligned} \tag{3}$$

where the coefficients W_1, W_2, G_1 et G_2 are the so-called structure functions. They depend on ν and Q^2 and summarize all our ignorance in the nucleon structure. Again, this tensor can be divided in a symetric part and an antisymetric part where appear the unpolarised structure functions W_1 and W_2 and an antisymetric part where occur the spin-dependent structure functions G_1 et G_2.

Substituting for $L^{\mu\nu}$and $W_{\mu\nu}$in Eq (2) gives :

$$\frac{d^2\sigma}{dE'\,d\Omega}=\frac{e^4}{16\pi^2Q^4}\Big(\frac{E'}{E}\Big)\,L^{\mu\nu}\,W_{\mu\nu} \tag{4}$$

The contraction of both tensors allows to compute the cross section introducing the nucleon structure functions. Knowing that the contraction of a symetric tensor with and antisymetric tensor is null and that the information on the nucleon spin is contained in the antisymetric part of the hadronic tensor, we will need to polarise both the muon beam and the nucleon target (see appendix C).

The contraction of the symmetric parts of $L^{\mu\nu}$ and $W_{\mu\nu}$ gives the spin independent cross section which in its most general form is written as :

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{4\alpha^2E'^2}{Q^4}\left[\cos^2\frac{\theta}{2}W_2+2\sin^2\frac{\theta}{2}W_1\right] \tag{5}$$

The contraction of the antisymmetric parts and the difference of both cross sections where leptons and hadrons are polarised parallel ($\sigma^{\uparrow\uparrow}$) or antiparallel ($\sigma^{\uparrow\downarrow}$), give the spin dependent cross section :

$$\begin{aligned} \frac{d^2\Delta\sigma}{d\Omega dE'} &= \frac{d^2\sigma^{\uparrow\downarrow}}{d\Omega dE'}-\frac{d^2\sigma^{\uparrow\uparrow}}{d\Omega dE'} \\ \frac{d^2\Delta\sigma}{d\Omega dE'} &= \frac{4\alpha^2}{Q^2}\frac{E'}{E}\left[MG_1\Big(E+E'\cos\theta\Big)-Q^2G_2\right] \end{aligned} \tag{6}$$

W_1, W_2, G_1, G_2 are four structure functions depending on Q^2 and ν that we can expressed in terms of the dimensionless variable x by the relations :

$$F_1(x,Q^2) = MW_1(\nu,Q^2) \tag{7a}$$
$$F_2(x,Q^2) = \nu W_2(\nu,Q^2) \tag{7b}$$
$$g_1(x,Q^2) = M^2\nu G_1(\nu,Q^2) \tag{7c}$$
$$g_2(x,Q^2) = M\nu^2 G_2(\nu,Q^2) \tag{7d}$$

Now, we can define the hadronic tensor versus both kinematical variables x and Q^2 characterising the process in forms of :

$$\begin{aligned} W_{\mu\nu} &= F_1\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)+\frac{F_2}{p\cdot q}\left(p_\mu-\frac{p\cdot q\; q_\mu}{q^2}\right)\left(p_\nu-\frac{p\cdot q\; q_\nu}{q^2}\right) \\ &+ g_1\frac{i}{p\cdot q}\,\epsilon_{\mu\nu\lambda\sigma}q^\lambda s_n^\sigma+g_2\frac{i}{(p\cdot q)^2}\epsilon_{\mu\nu\lambda\sigma}q^\lambda\left(p\cdot q\, s_n^\sigma - s\cdot q\, p^\sigma\right) \end{aligned}$$

and the spin independent and spin dependent deep inelastic cross sections from the equations 5 and 6 :

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4x}\left[\left(1-\frac{Q^2}{2MEx}-\frac{Q^2}{4E^2}\right)F_2(x,Q^2)\;+\frac{Q^4}{4M^2E^2x}F_1(x,Q^2)\right] \tag{8}$$

$$\frac{d^2\Delta\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{MExQ^2}\left[\left(2-\frac{Q^2}{2MEx}-\frac{Q^2}{2E^2}\right)g_1(x,Q^2)\;-\frac{2Mx}{E}g_2(x,Q^2)\right] \tag{9}$$

From time to time, we can find these cross sections defined from the so-called Bjorken variables x and y. Using :
$Q^2 \;= -q^2 = 2EE'(1-\cos\theta) = 4EE'\sin^2\frac{\theta}{2} = 2MxyE,$
one get :

$$\sin^2\frac{\theta}{2} = \frac{Mxy}{2E\big(1-y\big)} = \frac{Q^2}{4EE'} \tag{10}$$

$$\frac{1}{tg^2\frac{\theta}{2}} = \frac{2E\big(1-y\big)-Mxy}{Mxy} = 4\frac{EE'}{Q^2}-1$$

and

$$\cos\theta = \frac{E\big(1-y\big)-Mxy}{E\big(1-y\big)} = 1-\frac{Q^2}{2EE'}$$

Thus, the differential cross-sections are defined by the relations :

$$\frac{d^2\ldots}{dxdy} = \frac{2\pi M\nu}{E'}\frac{d^2\ldots}{d\Omega dE'} \tag{11}$$

Thus, we can rewrite the spin independent and dependent differential cross sections as :

$$Eq\ (5)\ \Rightarrow\ \frac{d^2\sigma}{dxdy} = \frac{2\pi\alpha^2}{MEx^2y^2}\left[\left(1-y-\frac{Mxy}{2E}\right)F_2(x,Q^2)\;+xy^2F_1(x,Q^2)\right] \tag{12}$$

$$Eq\ (6)\ \Rightarrow\ \frac{d^2\Delta\sigma}{dxdy} = \frac{4\pi\alpha^2}{MExy}\left[\left(2 - y - \frac{Mxy}{E}\right)g_1(x,Q^2)\ - \frac{2Mx}{E}g_2(x,Q^2)\right] \quad (13)$$

From the equations 8 and 9, we deduce expressions what link structure functions to deep inelastic cross sections. In a imaginary world, we can create an experiment to do an absolute measurement of these cross sections. Unfortunatly, from an experimental point of view where the systematic and statistic errors are not negligeable, this mission seems to be impossible. So, we have to measure an other quantity.

3. The asymmetry and the structure function g_1

The experimentalists are not able to measure very accuratly the polarised deep inelastic scattering cross section and therefore are not able to perform a significant spin structure functions measurement. For this reason, we are measuring an asymmetry of lepton-nucleon cross sections defined by :

$$\mathcal{A}(x,Q^2)\ =\ \frac{\frac{d^2\sigma^{\uparrow\Downarrow}}{dxdQ^2} - \frac{d^2\sigma^{\uparrow\Uparrow}}{dxdQ^2}}{\frac{d^2\sigma^{\uparrow\Downarrow}}{dxdQ^2} + \frac{d^2\sigma^{\uparrow\Uparrow}}{dxdQ^2}}$$

We get from the equations 8 et 9.

$$\mathcal{A}(x,Q^2)\ =\ \frac{\Delta\sigma}{2\sigma}\ =\ xy\frac{\left(2 - y - \frac{Mxy}{E}\right)g_1(x,Q^2)\ - \frac{2Mx}{E}g_2(x,Q^2)}{\left(1 - y - \frac{Mxy}{2E}\right)F_2(x,Q^2)\ + xy^2F_1(x,Q^2)} \quad (14)$$

Assuming we can measure from others experiments the unpolarised structure functions, we obtain one equation with two unknowns.

The probe, we use, is a virtual photon. The incoming lepton presence is only required for the creation of this probe. If we could build a virtual photons factory, the muon beams and leptons from a more general point of view will be completly obsolete. Therefore, the spin dependent structure functions has to be defined from the virtual photon-nucleon cross sections.

3.1. Virtual photon-nucleon asymmetry

As previously mentionned, we consider deep inelastic scattering as a process composed of two parts (see also appendix A and B). First, the upper arm of Fig.1, completely described by Q.E.D, consists in the emission of a virtual photon by the incident muon. The second arm consists in the absorption of this same virtual photon by the nucleon. This decomposition separates the deep inelastic scattering cross section into hadronic and leptonic parts. Using the optical theorem, we express the absorption cross section of the photon by the hadronic target as the imaginary ('absorptive') part of the forward virtual photon-nucleon Compton scattering amplitude.

These amplitudes $\mathcal{M}_{ab,cd}$, where a, b, c and d represent the helicities of the incident virtual photon, incident nucleon, scattered virtual photon and scattered nucleon respectivly, are derived from the hadronic tensor $W_{\mu\nu}$ through :

$$\mathcal{M}_{ab,cd} \;=\; \epsilon^{\mu\dagger}\Big(\lambda_a\Big)\epsilon^{\nu}\Big(\lambda_c\Big)W_{\mu\nu} \tag{15}$$

where $\epsilon^{\mu}\Big(\lambda\Big)$ is the polarisation vector of the virtual photon (see ref[2]).

As we demonstrate in the appendix D, for all the scattering amplitudes between photons with three polarisation states and nucleons with two polarisation states, the differents conservation and invariance laws imply that only four independent helicity amplitudes contribute to the forward Compton scattering. We have chosen :

$$\mathcal{M}_{1\frac{1}{2},1\frac{1}{2}}, \qquad \mathcal{M}_{1-\frac{1}{2},1-\frac{1}{2}}, \qquad \mathcal{M}_{0\frac{1}{2},0\frac{1}{2}}, \qquad \mathcal{M}_{0\frac{1}{2},0-\frac{1}{2}}$$

Substituting the expression of the hadronic tensor given by the equation 3 and aplying the optical theorem, we get for the virtual photon-nucleon cross sections the following relations :

$$\sigma^T_{\frac{1}{2}} \;=\; \tfrac{4\pi^2\alpha}{K}\mathcal{M}_{1\frac{1}{2},1\frac{1}{2}} \;=\; \frac{4\pi^2\alpha}{MK}\Big(F_1 + g_1 - \frac{2Mx}{\nu}g_2\Big) \tag{16a}$$

$$\sigma^T_{\frac{3}{2}} \;=\; \tfrac{4\pi^2\alpha}{K}\mathcal{M}_{1-\frac{1}{2},1-\frac{1}{2}} \;=\; \frac{4\pi^2\alpha}{MK}\Big(F_1 - g_1 + \frac{2Mx}{\nu}g_2\Big) \tag{16b}$$

$$\sigma^L_{\frac{1}{2}} \;=\; \tfrac{4\pi^2\alpha}{K}\mathcal{M}_{0\frac{1}{2},0\frac{1}{2}} \;=\; \frac{4\pi^2\alpha}{K}\Big[\frac{F_2}{\nu}\Big(1 + \frac{\nu^2}{Q^2}\Big) - \frac{1}{M}F_1\Big] \tag{16c}$$

$$\sigma^{TL}_{\frac{1}{2}} \;=\; \tfrac{4\pi^2\alpha}{K}\mathcal{M}_{0\frac{1}{2},0-\frac{1}{2}} \;=\; \frac{4\pi^2\alpha}{K}\frac{\sqrt{Q^2}}{M\nu}\Big(g_1 + g_2\Big) \tag{16d}$$

where $1/K$ is the arbitrary factor representing the incoming photon flux. For real photons beam ($Q^2 \to 0$), the flux is equal to $1/\nu$. If now, we consider the invariant mass of the final state X as we have written it in the equation 1, we have for a real photon where $K = \nu$ and $Q^2 = 0$:

$$W^2 \;=\; M^2 + 2MK$$

The Hand's convention for K requires that this relation has to be always verified for a virtual photon, being :

$$K \;=\; \frac{W^2 - M^2}{2M} \;=\; \nu - \frac{Q^2}{2M}$$

The quantity $\sigma^T_{1/2}(\sigma^T_{3/2})$ is the virtual photoabsorption cross section when the projection of the photon-nucleon system spin along the virtual photon axis is 1/2 (3/2). σ^T is the total transverse cross section defined by

$$\sigma^T \;=\; \frac{1}{2}\Big(\sigma^T_{1/2} + \sigma^T_{3/2}\Big) \;=\; \frac{4\pi^2\alpha}{MK}F_1(x,Q^2) \tag{17}$$

σ^L is the corresponding cross section for the longitudinally polarised photon. σ^{TL} is a term arising from the interference between transverse and longitudinal amplitudes. Now, we can introduce the virtual photon-nucleon asymmetries. The first one is the transverse asymmetry :

$$\mathcal{A}_1(x, Q^2) \; = \; \frac{\sigma^T_{1/2} - \sigma^T_{3/2}}{\sigma^T_{1/2} + \sigma^T_{3/2}}$$

Substituting the cross sections $\sigma^T_{1/2}$ and $\sigma^T_{3/2}$ defined by the equations 16a and 16b, the asymmetry becomes :

$$\mathcal{A}_1(x, Q^2) \; = \; \frac{g_1(x, Q^2) \; - \gamma^2 \, g_2(x, Q^2)}{F_1(x, Q^2)} \tag{18}$$

où γ est un facteur cinématique défini par :

$$\gamma \; = \; \sqrt{\frac{2Mx}{\nu}} \; = \; \sqrt{\frac{Q^2}{\nu^2}}$$

which goes to 0 in the Bjorken limit where ν goes to the infinity at a fixed x.
The so-called second asymmetry is defined by :

$$\mathcal{A}_2(x, Q^2) \; = \; \frac{\sigma_{TL}}{\sigma_T}$$

Here again, with the same substitutions, we obtain :

$$\mathcal{A}_2(x, Q^2) \; = \; \frac{\gamma \Big[g_1(x, Q^2) \; + \; g_2(x, Q^2) \Big]}{F_1(x, Q^2)} \tag{19}$$

The ratio R of the longitudinal to transversal cross sections that we can defined from the equations 16c and 17, is given by :

$$R(x, Q^2) \; = \; \frac{\sigma_L}{\sigma_T} \; = \; \frac{\Big(1 + \gamma^2\Big)}{2xF_1(x, Q^2)} F_2(x, Q^2) \; - 1 \tag{20}$$

This equation allows to deduce a relation between the two unpolarised structure functions $F_1(x, Q^2)$ et $F_2(x, Q^2)$:

$$F_1(x, Q^2) \; = \; \frac{\Big(1 + \gamma^2\Big)}{2x\Big(1 + R\Big)} F_2(x, Q^2) \tag{21}$$

3.2. The spin dependent structure function $g_1(x, Q^2)$

The firs step consists to give the expression of the both structure functions g_1 et g_2 versus the asymmetries $\mathcal{A}_1$ et $\mathcal{A}_2$ from equations 18 and 19 :

$$g_1(x,Q^2) = \frac{F_1(x,Q^2)}{1+\gamma^2}\Big(\mathcal{A}_1 + \gamma\mathcal{A}_2\Big) = \frac{F_2(x,Q^2)}{2x(1+R(x,Q^2))}\Big(\mathcal{A}_1 + \gamma\mathcal{A}_2\Big) \tag{22}$$

$$g_2(x,Q^2) = \frac{F_1(x,Q^2)}{1+\gamma^2}\Big(\frac{\mathcal{A}_2}{\gamma} - \mathcal{A}_1\Big) = \frac{F_2(x,Q^2)}{2x(1+R(x,Q^2))}\Big(\frac{\mathcal{A}_2}{\gamma} - \mathcal{A}_1\Big) \tag{23}$$

Subtituting for the previous equations in equation 14, we get :

$$\mathcal{A} = \frac{1}{\Big(1+\gamma^2\Big)}\frac{y\Big(2-y-\frac{\gamma^2y^2}{2}\Big)}{2\,\Big(1-y-\frac{\gamma^2y^2}{4}\Big)\Big(\frac{1+R}{1+\gamma^2}\Big) + y^2} \times$$
$$\Big[\mathcal{A}_1\Big(1 + \frac{\gamma^2 y}{\Big(2-y-\frac{\gamma^2y^2}{2}\Big)}\Big) + \gamma\mathcal{A}_2\Big(1 - \frac{y}{\Big(2-y-\frac{\gamma^2y^2}{2}\Big)}\Big)\Big]$$

From the above equation, we can express the asymmetry in the form :

$$\mathcal{A} = \mathcal{D}\,\Big(\mathcal{A}_1 + \eta\,\mathcal{A}_2\Big) \tag{24}$$

where $\mathcal{D}$ is the depolarisation factor. In polarised deep inelastic scattering, we align the spins of the incoming muons and at rest nucleons. The virtual photon spin is parallel or perpendicular to the propagation axis which is not necessary parallel to the incoming muon propagation axis. The depolarisation factor takes this particularity into account. The full expression of the kinematical factors $\mathcal{D}$ et η are given by the relations :

$$\mathcal{D} = \frac{y\Big(2-y\Big)\Big(1+\frac{1}{2}\gamma^2y\Big)}{\Big(1+\gamma^2\Big)\Big[2\,\Big(1-y-\frac{\gamma^2y^2}{4}\Big)\Big(\frac{1+R}{1+\gamma^2}\Big) + y^2\Big]} \tag{25}$$

$$\eta = \frac{2\gamma\Big(1-y-\frac{1}{4}\gamma^2y^2\Big)}{\Big(2-y\Big)\Big(1+\frac{1}{2}\gamma^2y\Big)} \tag{26}$$

As previously mentionned, the kinematical factor γ goes to 0 in the Bjorken limit. Thus, we get the approximate expression of the depolarisation factor $\mathcal{D}$:

$$\mathcal{D} \simeq \frac{y\Big(2-y\Big)}{2\Big(1-y\Big)\Big(1+R\Big)+y^2} \tag{27}$$

Aswell the kinematical factor η becomes equal to 0. Therefore, the asymmetry lepton-nucleon of the equation 24 becomes :

$$\mathcal{A} \simeq \mathcal{D}\mathcal{A}_1 \simeq \mathcal{D}\,\frac{g_1(x,Q^2)}{F_1(x,Q^2)}$$

so

$$g_1(x,Q^2) \simeq \frac{F_2(x,Q^2)}{2x(1+R(x,Q^2))}\frac{\mathcal{A}}{\mathcal{D}}$$

3.3. The external parameters

As shown by the previous equation, in order to compute the spin dependent structure function g_1, we have to measure $\mathcal{A}$. In addition, we need to know some external parameters like the unpolarised structure functions $F_2(x,Q^2)$ and $R(x,Q^2)$ and the errors generated by the neglect of $A_2(x,Q^2)$

3.3.1. The structure function $F_2(x,Q^2)$

The unpolarised structure function $F_2(x,Q^2)$ has been the first one to be measured in the eighties mainly because the polarisation technics for the beam or the target were not under control. The measurement of this structure function gives the quarks momentum distributions inside the nucleon. The first experiments were performed at SLAC[4] then at CERN by the EMC,[5],[6] BCDMS,[78] and NMC,[910] collaborations. The following parametrisation takes into account all available data except those of the EMC collaboration and is given by :

$$F_2(x,Q^2) = A(x)\cdot\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{B(x)}\cdot\left[1+\frac{C(x)}{Q^2}\right] \tag{28}$$

$$Q_0^2 = 20GeV^2, \Lambda = 250MeV \tag{29}$$

$$A(x) = x^{a_1}\left(1-x\right)^{a_2}\left[a_3 + a_4\left(1-x\right) + a_5\left(1-x\right)^2 + a_6\left(1-x\right)^3 + a_7\left(1-x\right)^4\right] \tag{30}$$

$$B(x) = b_1 + b_2 x + \frac{b_3}{x+b_4} \tag{31}$$

$$C(x) = c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \tag{32}$$

The kinematical range covered by the whole data set is given by (see table 1) :

$$0.006 \leq x \leq 0.6 \qquad \text{and} \qquad 0.5 \leq Q^2 \leq 55\,GeV^2 \tag{33}$$

New results coming from the HERA experiments on the unpolarised structure function at very low x are now published and compatible with the previous results. These data are very usefull for the SMC collaboration where no data was available in its low x range ($x < 0.005$). New parametrisations taken into account these measurements will be soon published.

$F_2(x,Q^2)$ fit		
$Parameters$	$proton$	$deuton$
a_1	-0.1011	-0.0996
a_2	2.562	2.489
a_3	0.4121	0.4684
a_4	-0.518	-1.924
a_5	5.967	8.159
a_6	-10.197	-10.893
a_7	4.685	4.535
b_1	0.364	0.252
b_2	-2.764	-2.713
b_3	0.0150	0.0254
b_4	0.0186	0.0299
c_1	-1.179	-1.221
c_2	8.24	7.50
c_3	-36.36	-30.49
c_4	47.76	40.23

Table 1. *The fit parameters of the SLAC, BCDMS et NMC data of the unpolarised structure function $F_2(x,Q^2)$ published by the NMC collaboration.*

3.3.2. The ratio $R(x,Q^2)$

We can deduce from the equations 5, 7a and 7b, the unpolarised deep inelastic scattering cross section versus the unpolarised structure functions F_1 et F_2 defined by the relation :

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{4\alpha^2 E'^2}{Q^4}\left[\frac{1}{\nu}F_2(x,Q^2) + \frac{2}{M}F_1(x,Q^2)tan^2(\frac{\theta}{2})\right] \tag{34}$$

We have also defined in the section 3.1, the ratio of the longitudinal over transversal cross sections of nucleon photoabsorption by the relation :

$$R(x,Q^2) \;=\; \frac{\sigma_L}{\sigma_T} \;=\; \frac{\left(1+\gamma^2\right)}{2xF_1(x,Q^2)}F_2(x,Q^2) \;-1 \tag{35}$$

Thus, we can rewrite the unpolarised cross section as :

$$\frac{d^2\sigma}{d\Omega dE'} = \Phi_{gamma}\,\sigma_T(x,Q^2)\left[1+\epsilon R(x,Q^2)\right] \tag{36}$$

where the virtual photons flux Φ_{gamma} is given by the relation :

$$\Phi_{gamma} \;=\; \frac{\alpha E'}{4\pi^2 ME}\left(\frac{1}{x}\right)\frac{1}{1-\epsilon} \tag{37}$$

and :

$$\epsilon = \left[1 + \left(1 + \frac{\nu^2}{Q^2}\right) tan^2(\frac{\theta}{2})\right]^{-1} \tag{38}$$

Starting from the results of eight deep inelastic scattering experiments performed at SLAC between 1970 and 1985, L.W. Whitlow et al. have determined a parametrisation of $R(x, Q^2)$ for the proton and the deuton[3] which is valid for the Q^2 greater than $0.3\,GeV^2$ (This is not important for this purpose since the Bjorken experimental lower limit is fixed at $1\,GeV^2$).

The parametrisation used is an arithmetic mean of three differents fits of these data and is therefore defined by the relation (see fig 2) :

$$R^{1990}(x, Q^2) = \frac{1}{3}\Big(R_a(x, Q^2) + R_b(x, Q^2) + R_c(x, Q^2)\Big) \tag{39}$$

où

$$R_a(x, Q^2) = \frac{a_1}{\ln\left(Q^2/0.04\right)}\,\Theta(x, Q^2) + \frac{a_2}{\sqrt[4]{Q^8 + a_3^4}} \tag{40a}$$

$$R_b(x, Q^2) = \frac{b_1}{\ln\left(Q^2/0.04\right)}\,\Theta(x, Q^2) + \frac{b_2}{Q^2} + \frac{b_3}{Q^4 + .3^2} \tag{40b}$$

$$R_c(x, Q^2) = \frac{c_1}{\ln\left(Q^2/0.04\right)}\,\Theta(x, Q^2) + c_2\left[\left(Q^2 - Q^2_{thr}\right)^2 + c_3^2\right]^{-1/2} \tag{40c}$$

with a theoretical transfer energy fitted by $Q^2_{thr} = 5\left(1-x\right)^5$ and the $\Theta(x, Q^2)$ function given by :

$$\Theta(x, Q^2) = 1 + 12\frac{Q^2}{Q^2+1}.\frac{0.125^2}{0.125^2 + x^2} \tag{41}$$

The coefficients used in the equations 40a, 40b and 40c being given by :

$$\begin{array}{llllll} a_1 & = .0672 & a_2 & = .4671 & a_3 & = 1.8979 \\ b_1 & = .0635 & b_2 & = .5747 & b_3 & = -.3534 \\ c_1 & = .0599 & c_2 & = .5088 & c_3 & = 2.1081 \end{array} \tag{42}$$

3.3.3. The upper limit over A_2

The asymmetry $A_2(x, Q^2)$ is given by the ratio of the cross section coming from the interference term between the absorption for virtual photons of longitudinal and transverse polarisation and absorption cross section for virtual photons of transversal polarisation (see §3.1) :

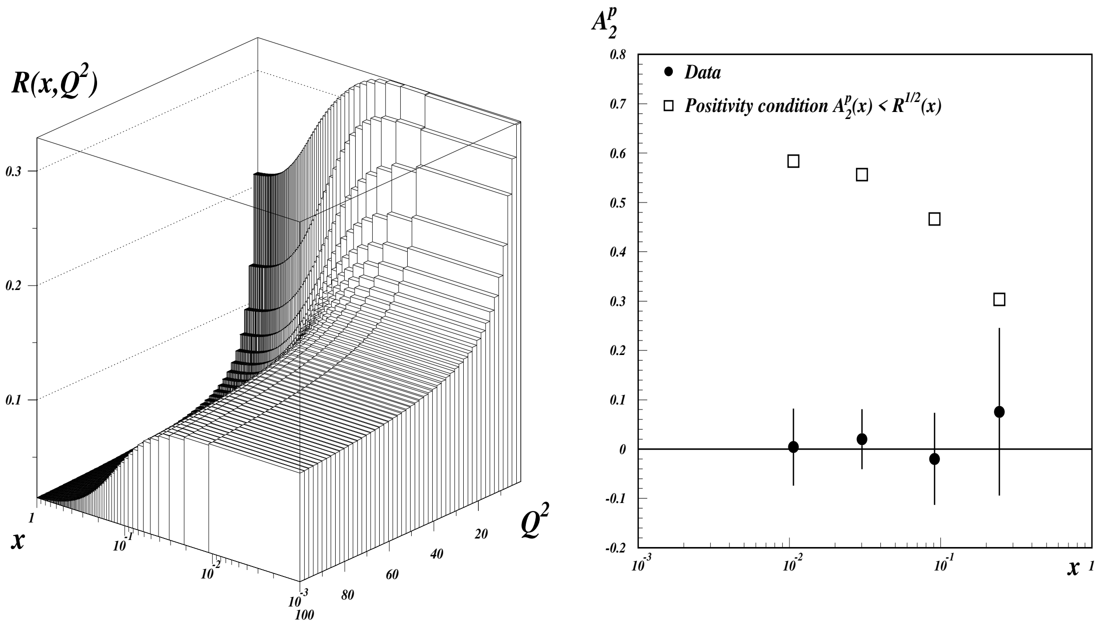


Fig. 2. *The structure function $R(x,Q^2)$ parametrised from the measurements of deep inelastic scattering cross section of electrons off protons and deutons at SLAC and the spin asymmetry in muon-proton deep inelastic scattering on a transversely-polarised target A_2^p*

$$\mathcal{A}_2(x,Q^2) \;=\; \frac{\sigma_{TL}}{\sigma_T} \tag{43}$$

The asymmetry virtual photon-nucleon is defined from the equation 24 by :

$$\mathcal{A}_1 \;=\; \frac{\mathcal{A}}{\mathcal{D}} - \;\eta\,\mathcal{A}_2 \tag{44}$$

with

$$\eta \;=\; \frac{2\gamma\Big(1-y-\frac{1}{4}\gamma^2y^2\Big)}{\Big(2-y\Big)\Big(1+\frac{1}{2}\gamma^2 y\Big)} \qquad \text{et} \qquad \gamma \;=\; \sqrt{\frac{2Mx}{\nu}} \tag{45}$$

Typically, the kinematical term η is small in the high energy experiments. For the SMC collaboration, the lower transfer energy is $\nu\;=\;20\,GeV$ and the upper value of the Bjorken variable x is 0.7. Therefore, the upper value of this kinematical term γ is of order of 3.10^{-2}. The upper limit of η is smaller than this number. Then, we get two possibilities.

- We are not able to measure this asymmetry and we assume the positivity condition is right which implies that $|A_2(x)| \leq \sqrt{R(x)}$.[11] Thus, we can assume that this contribution is so small that we include it in the systematic error.

- We are able to measure this quantity and we take it into account in the $g_1(x, Q^2)$ determination (see fig 2). Therefore, we can reduce the systematic error on the asymmetry $g_1(x, Q^2)$.

The contribution of the $\mathcal{A}_2$ asymmetry is extracted from the relation 22 where we use the previous definition of $\mathcal{A}_1$ This equation becomes :

$$g_1(x, Q^2) \;=\; \frac{F_1(x, Q^2)}{1+\gamma^2}\left[\frac{\mathcal{A}}{\mathcal{D}} + \Big(\gamma - \eta\Big)\mathcal{A}_2\right] \tag{46}$$

We can apply there the same assumption than for the asymmetry. If we do not measure $\mathcal{A}_2$, we neglect the $\Big(\gamma - \eta\Big)\mathcal{A}_2$ term that we take into account in the total systematic error.

Now, knowing the kinematical factor $\mathcal{D}$, the both unpolarised structure functions F_2 and R which have been measured by several experiments, we are able to determine the spin dependent structure function. But before doing it, with a simple model, we are going to give an intrepretation of the spin dependent structure function which allows to better understand the internal dynamic of the nucleon.

4. The first moment of the spin dependent structure function

The theory is not predictive for the spin dependent structure function itself but only for its first moment. We are going to developp in this section the necessary formalism for the sum rules concerning the first moments of the spin dependent structure function of the nucleon.

4.1. Interpretation of the structure functions in the parton model

First of all, the simplest picture of the nucleon is given by the partons model developped by Feynman at the end of sixties. The nucleon is discribed as a particle made of pointlike objects : the partons. At this time, this particles did not identify to the quarks and gluons. Neglecting the interactions between these partons, the deep inelastic scattering of virtual photons off nucleons is equivalent to an incoherent sum of elastics scattering of a virtual photon off a parton(see fig 3). A very pedagogical approach is given in the "Halzen and Martin" Bible.[12]

In this model, the Bjorken variable x is defined like the momentum fraction carried by the quarks and the unpolarised structure functions are given by :

$$F_1(x) \;=\; \tfrac{1}{2}\textstyle\sum_i e_i^2 f_i(x) \tag{47}$$

$$F_2(x) \;=\; \textstyle\sum_i e_i^2 x f_i(x) \tag{48}$$

where e_i is the charge of the parton i and $f_i(x)$ is the density of probability that the parton i carries a fraction x of the proton momentum. We can immediatly remark

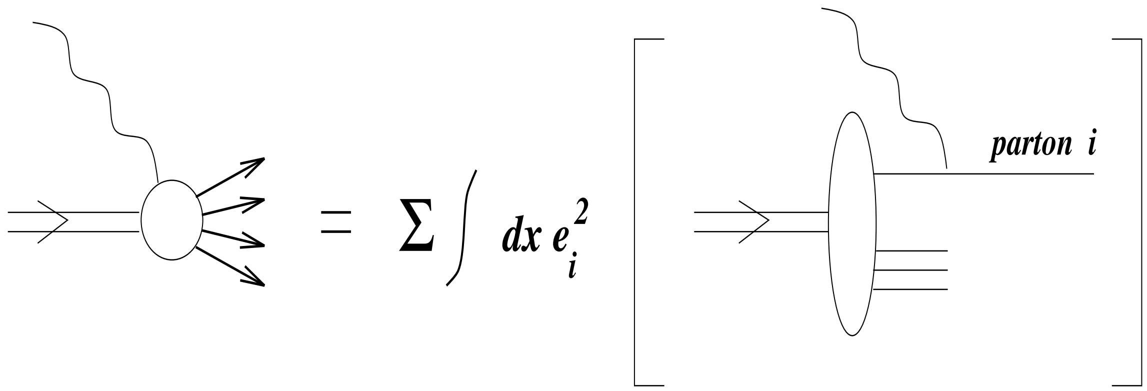


Fig. 3. *In the infinite momentum frame, the deep inelastic scattering of virtual photons off nucleons is equivalent to an incoherent sum of elastic scatterings of virtual photons off free partons.*

that these structure functions are independent of Q^2 (scaling invariance) which is easily understandable since we assume the partons poinlike. With the three lightest quark flavours, we get :

$$F_2^p(x) = x\left[\frac{4}{9}\left(u(x)+\bar{u}(x)\right)+\frac{1}{9}\left(d(x)+\bar{d}(x)\right)+\frac{1}{9}\left(s(x)+\bar{s}(x)\right)\right]$$

where $u(x)$ (resp. $\bar{u}(x)$) the density of probability that the quark (resp. antiquark) of flavour u carrying a fraction x of the proton momentum. For the spin dependent structure function, we obtain :

$$g_1(x) = \frac{1}{2}\sum_i e_i^2\left(f_i^+(x)-f_i^-(x)\right) \tag{49}$$

$$g_2(x) = 0 \tag{50}$$

where $f_i^+(x)\,(f_i^-(x))$ is the density of probability that the parton i carries a fraction x of the proton momentum and an helicity equal (opposite) to the proton's one. Thus, we get for the proton :

$$\begin{aligned} g_1^p(x) = \frac{1}{2}\Big[& \frac{4}{9}\left(u^+(x)-u^-(x)+\bar{u}^+(x)-\bar{u}^-(x)\right) \\ & +\frac{1}{9}\left(d^+(x)-d^-(x)+\bar{d}^+(x)-\bar{d}^-(x)\right) \\ & +\frac{1}{9}\left(s^+(x)-s^-(x)+\bar{s}^+(x)-\bar{s}^-(x)\right)\Big] \end{aligned} \tag{51}$$

and applying the isospin symmetry $\mathcal{SU}(2)$, $p \to n$, $u = u^p \to d^n$, $d = d^p \to u^n$, we obtain the neutron spin dependent structure function :

$$\begin{aligned} g_1^n(x) = \frac{1}{2}\Big[& \frac{1}{9}\left(u^+(x)-u^-(x)+\bar{u}^+(x)-\bar{u}^-(x)\right) \\ & +\frac{4}{9}\left(d^+(x)-d^-(x)+\bar{d}^+(x)-\bar{d}^-(x)\right) \\ & +\frac{1}{9}\left(s^+(x)-s^-(x)+\bar{s}^+(x)-\bar{s}^-(x)\right)\Big] \end{aligned} \tag{52}$$

4.2. The first moment of the spin dependent structure function

The first moment of the spin dependent structure function is defined like the integral over the whole x range of $g_1(x)$. We define the spin fraction carried by a quark i of flavour q by the relation :

$$\Delta q_i = \int_0^1 dx \left[f_i^+(x) - f_i^-(x) + \bar{f}_i^+(x) - f_i^-(x)\right]$$

Now, if we compute the integral of g_1^p starting from its partonic expresion (51), we get :

$$\Gamma_1^p = \int_0^1 g_1^p(x)dx = \frac{1}{2}\Big(\frac{4}{9}\Delta u + \frac{1}{9}\Delta d + \frac{1}{9}\Delta s\Big)$$

and for the neutron :

$$\Gamma_1^n = \int_0^1 g_1^n(x)dx = \frac{1}{2}\Big(\frac{1}{9}\Delta u + \frac{4}{9}\Delta d + \frac{1}{9}\Delta s\Big)$$

4.3. The decomposition in flavour $\mathcal{SU}(3)$ matrix elements of the first moment

If we assume that the three quark fields (u, d, s) forms a flavour triplet, we can group some baryons $(p, n, \Sigma^+, \Sigma^-, ...)$ and mesons ($\pi^+, \pi^-, K^+, ...$) in an octet[12] :

$$\vec{\Psi} = \begin{pmatrix} \psi_u \\ \psi_d \\ \psi_s \end{pmatrix} \tag{53}$$

This triplet under a transformation $\mathcal{U}$ de $\mathcal{SU}(3)$ becomes :

$$\vec{\Psi}' = \mathcal{U}\vec{\Psi}$$

$$U = e^{ia_k\frac{\lambda^k}{2}}$$

where the coefficients $\frac{\lambda^k}{2}$ are the group generators of $\mathcal{SU}(3)$. With the Wigner-Eckart theorem, we can define the matrix elements of the axial-vector currents $j_0^{A\mu}$, $j_3^{A\mu}$ et $j_8^{A\mu}$, where :

$$j_a^{A\mu} = \bar{\vec{\Psi}}\gamma^\mu\gamma_5\frac{\lambda_a}{2}\vec{\Psi} \tag{54}$$

that we are writting explicitly (i, j flavour indexes)

$$j_a^{A\mu} = \sum_{i,j=1}^{3} \psi_i\gamma^\mu\gamma_5\left(\frac{\lambda_a}{2}\right)_{ij}\psi_j$$

versus two independents matrix elements : the so-called $\mathcal{F}$ and $\mathcal{D}$ constants. We will use the three currents $j_a^{A\mu}$, $a = 0, 3, 8$ given by :

$$\begin{array}{lcl} j_0^{A\mu} & = & \frac{1}{2}\left[\bar{\psi}_u\gamma^\mu\gamma_5\psi_u + \bar{\psi}_d\gamma^\mu\gamma_5\psi_d + \bar{\psi}_s\gamma^\mu\gamma_5\psi_s\right] \\ j_3^{A\mu} & = & \frac{1}{2}\left[\bar{\psi}_u\gamma^\mu\gamma_5\psi_u - \bar{\psi}_d\gamma^\mu\gamma_5\psi_d\right] \\ j_8^{A\mu} & = & \frac{1}{2}\left[\bar{\psi}_u\gamma^\mu\gamma_5\psi_u + \bar{\psi}_d\gamma^\mu\gamma_5\psi_d - 2\bar{\psi}_s\gamma^\mu\gamma_5\psi_s\right] \end{array} \tag{55}$$

The eight axial-vector currents of the equation 54 $a = 1, 8$ form an octet. They are the Gamow-Teller transition vectors (weak decay) in the baryons octet. The current $j_0^{A\mu}$ is an $\mathcal{SU}(3)$ singlet. To obtain relations with the $\mathcal{F}$ and $\mathcal{D}$ constants, we introduce the A_a parameters :

$$s^\mu A_a \;=\; \langle ps|\, j_a^{A\mu}\, |ps\rangle$$

and the notation :

$$\Delta\psi_q s^\mu \;=\; \Delta q s^\mu \;=\; \langle ps|\, \bar{\psi}_q\gamma_\mu\gamma_5\psi_q\, |ps\rangle$$

where $|ps\rangle$ is the state vector of a a proton with a spin s. We obtain :

$$2s^\mu A_0 \;=\; s^\mu\left[\Delta u + \Delta d + \Delta s\right] \tag{56a}$$

$$2s^\mu A_3 \;=\; s^\mu\left[\Delta u - \Delta d\right] \tag{56b}$$

$$2s^\mu A_8 \;=\; s^\mu\left[\Delta u + \Delta d - 2\Delta s\right] \tag{56c}$$

The A_3 constant is determined by the β decay measurement :

$$2s^\mu A_3 \;=\; s^\mu\left[\Delta u - \Delta d\right] \;=\; g_a s^\mu \;=\; (F + D)\, s^\mu \tag{57}$$

where g_a is the nucleon axial-vector charge. Its parametrisation in terms of $\mathcal{F}$ et $\mathcal{D}$ is only valid if we assume that $\mathcal{SU}(3)$ is a good symmetry to explain the baryon decays. Assuming that, the experiments on hyperon decays gives a measurement of A_8,[1314] :

$$2s^\mu A_8 \;=\; s^\mu\left[\Delta u + \Delta d - 2\Delta s\right] \;=\; s^\mu\,(3F - D) \;=\; s^\mu g_a \left(\frac{3\frac{F}{D} - 1}{\frac{F}{D} + 1}\right) \tag{58}$$

The nucleon axial charge Σ is given by :

$$\Sigma \;=\; A_0 \;=\; \Delta u + \Delta d + \Delta s$$

which defines in a more partonic language the spin fraction carried by the quarks u, d and s. Now, we obtain three equations with three unknowns $(\Delta u, \Delta d, \Delta s)$:

$$g_a \;=\; \Delta u - \Delta d$$

$$3F - D \;=\; \Delta u + \Delta d - 2\Delta s$$

$$\Gamma_1^p = \int_0^1 g_1^p(x)dx = \frac{1}{2}\Big(\frac{4}{9}\Delta u + \frac{1}{9}\Delta d + \frac{1}{9}\Delta s\Big)$$

Several experiments has performed measurements of the neutron β decay constant g_a[14] and of both antisymmetric and symmetric $\mathcal{SU}(3)$ couplings $\mathcal{F}$ and $\mathcal{D}$.[15] Slightly different values of $\mathcal{F}$ and $\mathcal{D}$ have been derived in different fits[16]:

$$\begin{array}{ll} g_A = 1.254 \pm 0.006 & \\ \frac{\mathcal{F}}{\mathcal{D}} = 0.631 \pm 0.018 & \text{M. Bourquin } et\ al. \\ \mathcal{F} = 0.485 \pm 0.010 & \text{Z.Phys C21 (1983) 27} \\ \mathcal{D} = 0.769 \pm 0.016 & \\ & \\ g_A = 1.259 \pm 0.004 & \\ \frac{\mathcal{F}}{\mathcal{D}} = 0.575 \pm 0.024 & \text{Z. Dziembowski and J. Franklin} \\ \mathcal{F} = 0.461 \pm 0.014 & \text{J.Phys.G 17 (1991) 213} \\ \mathcal{D} = 0.798 \pm 0.013 & \end{array} \tag{59}$$

With these results, we can also expressed the proton spin dependent structure function as a function of Δs :

$$\Gamma_1^p = \int_0^1 g_1^p(x)dx = \frac{g_a}{12}\Big(\frac{5}{3}\frac{3\frac{F}{D}-1}{\frac{F}{D}+1}+1\Big)+\frac{1}{3}\Delta s \tag{60}$$

and for the neutron :

$$\Gamma_1^n = \int_0^1 g_1^n(x)dx = \frac{g_a}{12}\Big(\frac{5}{3}\frac{3\frac{F}{D}-1}{\frac{F}{D}+1}-1\Big)+\frac{1}{3}\Delta s \tag{61}$$

As a function of the axial charge Σ :

$$\Gamma_1^p = \int_0^1 g_1^p(x)dx = \frac{g_a}{12}\Big(\frac{1}{3}\frac{3\frac{F}{D}-1}{\frac{F}{D}+1}+1\Big)+\frac{1}{9}\Sigma \tag{62}$$

$$\Gamma_1^n = \int_0^1 g_1^n(x)dx = \frac{g_a}{12}\Big(\frac{1}{3}\frac{3\frac{F}{D}-1}{\frac{F}{D}+1}-1\Big)+\frac{1}{9}\Sigma \tag{63}$$

These equations are the so-called "Proton and Neutron spin sum rules".

5. The sum rules

5.1. The Bjorken sum rule

In the parton model, we can find a relation between the first moments of the proton and the neutron spin structure functions subtracting the equations 62 et 63 :

$$\Gamma_1^p - \Gamma_1^n = \int_0^1 \Big[g_1(x)^p - g_1(x)^n\Big]dx = \frac{1}{6}g_a \tag{64}$$

This relation is the so-called Bjorken sum rule,[17].[18] This sum rule is a bridge between spin dependent high energies phenomena and low energy processes, the neutron β decay. Using $g_a = 1.254 \pm 0.006$, the theoretical prediction of this sum rule is :

$$\Gamma_1^p - \Gamma_1^n \;=\; 0.209 \pm 0.007 \tag{65}$$

5.2. The Ellis-Jaffe sum rule

To define the Bjorken sum rule, we only assume the $\mathcal{SU}(2)$ symmetry. Now, let's assume that $\mathcal{SU}(3)$ is a good symmetry to explain the hyperon β decay which allows us to use the equations 58 et 57 and that the strange quark sea is unpolarised $\Delta s \;=\; 0$.[19] Starting from the equations 62 et 63, we get the Ellis-Jaffe sum rules for the proton and the neutron :

$$\Gamma_1^p \;=\; \int_0^1 g_1^p(x)dx \;=\; \frac{g_a}{12}\left[1 + \frac{5}{3}\frac{3\mathcal{F}/\mathcal{D} - 1}{\mathcal{F}/\mathcal{D} + 1}\right] \tag{66}$$

$$\Gamma_1^n \;=\; \int_0^1 g_1^n(x)dx \;=\; \frac{g_a}{12}\left[-1 + \frac{5}{3}\frac{3\mathcal{F}/\mathcal{D} - 1}{\mathcal{F}/\mathcal{D} + 1}\right] \tag{67}$$

Using here again $g_a \;=\; 1.254 \pm 0.006$ and $F/D \;=\; 0.631 \pm 0.018$,[13] the theoretical predictions of these sum rules are :

$$\Gamma_1^p \;=\; +0.199 \pm 0.005 \qquad\qquad \Gamma_1^n \;=\; -0.009 \pm 0.005$$

Until now, we have assumed that the Bjorken limit in which we can consider a scattering of virtual photons off free partons was valid. Experimentally, the Bjorken limit corresponds to a $Q^2 > 1\,GeV^2$. At this transfer momentum, we cannot neglect the interactions between quarks and gluons. We have to switch on the theory taking this effects into account : QCD.

5.3. The sum rules with QCD corrections

At the α_s order and only including the leading twist terms (twist 2),[20],[21],[22] we get :

$$\int_0^1 g_1^p(x)dx \;=\; C_N\frac{g_a}{12}\Big(\frac{1}{3}\frac{3\frac{F}{D} - 1}{\frac{F}{D} + 1} + 1\Big) + C_0\frac{1}{9}\Sigma \tag{68}$$

$$\int_0^1 g_1^n(x)dx \;=\; C_N\frac{g_a}{12}\Big(\frac{1}{3}\frac{3\frac{F}{D} - 1}{\frac{F}{D} + 1} - 1\Big) + C_0\frac{1}{9}\Sigma \tag{69}$$

with

$$C_N = 1 - \frac{\alpha_s(Q^2)}{\pi} \tag{70}$$

$$C_0 = 1 - \frac{33-8f}{33-2f}\frac{\alpha_s(Q^2)}{\pi} \tag{71}$$

The number of active flavors (f) is taken equal to 4 (this is strictly valid for $1.5 \leq Q^2 \leq 5. GeV^2$, *i.e.* for a large fraction of the data).
The study of the variation of α_s with Q^2 is the subject of Q.C.D analysis. At N.L.O, the relation between $\alpha_S(Q^2)$ and $\alpha_S(M_Z^2)$ is :

$$\begin{aligned}\frac{1}{\alpha_S(Q^2)} \;+\; b_1\, ln\left(\frac{b_1\,\alpha_S(Q^2)}{1\;+\;b_1\,\alpha_S(Q^2)}\right) \;+\; b_0\, ln\,\frac{M_Z^2}{Q^2} \;= \\ \frac{1}{\alpha_S(M_Z^2)} \;+\; b_1\, ln\left(\frac{b_1\,\alpha_S(M_Z^2)}{1\;+\;b_1\,\alpha_S(M_Z^2)}\right)\end{aligned} \tag{72}$$

with

$$b_0 \;=\; \frac{11-\frac{2}{3}f}{4\pi} \tag{73}$$

and

$$b_1 \;=\; \frac{102-\frac{38}{3}f}{4\pi\left(11-\frac{2}{3}f\right)} \tag{74}$$

This equation is of the type $x = F(x)$ and has to be solved iteratively, taking into account possible changes in the number of active flavors (f).
A recent N.L.O fit using S.L.A.C and B.C.D.M.S data in ref,[23] gives

$$\alpha_s(M_Z^2) = 0.113 \pm 0.005 \tag{75}$$

The following tables gives the values of α_s corresponding to some Q^2 values in the range of the S.M.C experiment :

Q^2	α_s
1.00	0.384
1.46	0.339
2.09	0.308
2.41	0.298
3.45	0.275

Q^2	α_s
4.50	0.261
5.05	0.255
7.00	0.240
9.85	0.226
14.42	0.212

As a function of the strange quark sea polarisation Δs, we get :

$$\Gamma_1^p \;=\; \frac{g_a}{12}\left[\frac{1}{3}\frac{3\frac{F}{D}-1}{\frac{F}{D}+1}\left(C_N + 4C_0\right) + C_N\right] + \frac{C_0}{3}\Delta s \tag{76}$$

$$\Gamma_1^n \;=\; \frac{g_a}{12}\Big[\frac{1}{3}\frac{3\frac{F}{D}-1}{\frac{F}{D}+1}\Big(C_N+4C_0\Big)-C_N\Big]-\frac{C_0}{3}\Delta s \tag{77}$$

Normally, we have to take into account in these definitions, the terms coming from the upper twists, but they are from a theoretical point of view very difficult to estimate,[24].[25] This terms being a $1/Q^2$ serie, experimentaly we are measuring the structure function at a Q^2 as higher as possible to be able to neglect their effects.

Substracting both equatins 68 and 69, the Bjorken sum rule becomes :

$$\int_0^1 g_1^p(x)dx \;-\; \int_0^1 g_1^n(x) \;=\; C_N\frac{g_a}{6} \tag{78}$$

For the Ellis-Jaffe sum rule, we assume that the strange quark sea is unpolarised : $\Delta s \;=\; 0$.From the equations 56a and 56c, we deduce :

$$\Sigma \;=\; A_0 \;=\; \Delta u+\Delta d \;=\; A_8 \;=\; g_a\left(\frac{3\frac{F}{D}-1}{\frac{F}{D}+1}\right)$$

For the proton sum rule, we get :

$$\Gamma_1^p(\text{E-J}) \;=\; \frac{g_a}{12}\Big[\frac{1}{3}\frac{3\frac{F}{D}-1}{\frac{F}{D}+1}\Big(C_N+4C_0\Big)+C_N\Big] \tag{79}$$

and for the neutron :

$$\Gamma_1^n(\text{E-J}) \;=\; \frac{g_a}{12}\Big[\frac{1}{3}\frac{3\frac{F}{D}-1}{\frac{F}{D}+1}\Big(C_N+4C_0\Big)-C_N\Big] \tag{80}$$

Another way to get a determination of the proton or deuton spin structure function, if you have already measured one of both, is to measure the deuteron spin structure function.

6. Effects due to the spin 1 of the deuteron

The deuteron is a particle with a total orbital moment equal to 1 and of parity $+$: $J^P \;=\; 1^+$. The wave function associated to this nucleus is a S state or a D state. Therefore, in the D state, there exists one configuration in which the proton and neutron spins are not aligned with the deuteron's one. We are going to develop in this section the effects of this particularity on the asymmetries and spin dependent structure functions. But first of all, let's define the experimental way which allow to determine the cross sections asymmetry.

6.1. Measured asymmetry for a spin $\frac{1}{2}$ target

As mentionned in the previous sections, the spin asymmetry is defined as :

$$\mathcal{A} = \frac{\sigma_{\uparrow\downarrow}-\sigma_{\uparrow\uparrow}}{\sigma_{\uparrow\downarrow}+\sigma_{\uparrow\uparrow}} \tag{81}$$

where the two arrows refer to the spin orientations of the beam particle and target nucleon. Therefore, for a particle population, we need to define the polarisation. This is done fixing a quantification axis on which all particles spins will be projected. The spin 1/2 particles polarisation is defined by the substraction of populations when the spin is aligned or anti-aligned with the chosen direction of this quantification axis :

$$P = \frac{n_{+1} - n_{-1}}{n_{+1} + n_{-1}} = \frac{n_{+1} - n_{-1}}{n_{TOT}} \tag{82}$$

Now, the experimental asymmetry is given by the number of scattered muons (N_1, N_2) for two opposite polarisations ($+P, -P$).

$$N_1 = \Phi \frac{N_{Tot}}{2} \left[(1 + \mathcal{P}) \sigma^{\uparrow\Downarrow} + (1 - \mathcal{P}) \sigma^{\uparrow\Uparrow} \right] \tag{83}$$

$$N_2 = \Phi \frac{N_{Tot}}{2} \left[(1 - \mathcal{P}) \sigma^{\uparrow\Downarrow} + (1 + \mathcal{P}) \sigma^{\uparrow\Uparrow} \right] \tag{84}$$

where Φ is the incident flux and N_{Tot} the number of target nucleons, so that

$$\mathcal{A}_{meas} = \frac{N_1 - N_2}{N_1 + N_2} = \mathcal{P} \frac{\sigma^{\uparrow\Downarrow} - \sigma^{\uparrow\Uparrow}}{\sigma^{\uparrow\Downarrow} + \sigma^{\uparrow\Uparrow}} \tag{85}$$

6.2. The measured asymmetry for a spin 1 target

For a spin 1 system, the situation is a bit more complicated because some particles have a spin whom the projection on the quantification axis is null. Thus, in a deuteron target made of n_{TOT} particles, these exists three kinds of populations :

- n_{+1} for the deuteron spin aligned with the polarisation
- n_{-1} for the deuteron spin anti-aligned with the polarisation
- n_0 for the deuterons which have a null spin projection.

and so that $n_{TOT} = n_{+1} + n_{-1} + n_0$. The polarisation of such a target is by analogy with the spin 1/2 target given by :

$$P = \frac{n_{+1} - n_{-1}}{n_{+1} + n_{-1} + n_0} \tag{86}$$

Now, the number of scattered muons is given by :

$$N_u = \Phi \left(n_{+1} \sigma^{\uparrow\Uparrow} + n_{-1} \sigma^{\uparrow\Downarrow} + n_0 \sigma^{\uparrow 0} \right) \tag{87}$$

$$N_d = \Phi \, , \left(n_{-1} \sigma^{\uparrow\Uparrow} + n_{+1} \sigma^{\uparrow\Downarrow} + n_0 \sigma^{\uparrow 0} \right) \tag{88}$$

where as previously, Φ is the incident flux, $\sigma^{\uparrow\Uparrow}$, $\sigma^{\uparrow\Downarrow}$ et $\sigma^{\uparrow 0}$ are the deep inelastic cross sections of polarised muons $\uparrow$ off polarised or unpolarised deuteron. The counting rates asymmetry is given by :

$$\mathcal{A}_{mes} = \frac{N_u - N_d}{N_u + N_d} = \frac{\Big(n_1 - n_{-1}\Big)\Big(\sigma^{\uparrow\Downarrow} - \sigma^{\uparrow\Uparrow}\Big)}{\Big(n_1 + n_{-1}\Big)\Big(\sigma^{\uparrow\Downarrow} + \sigma^{\uparrow\Uparrow}\Big) + 2n_0\sigma^{\uparrow 0}} \tag{89}$$

As a function of the P polarisation defined by the equation 86, we get :

$$\mathcal{A}_{mes} = \mathcal{P}\frac{\Big(n_1 + n_{-1} + n_0\Big)\Big(\sigma^{\uparrow\Downarrow} - \sigma^{\uparrow\Uparrow}\Big)}{\Big(n_1 - n_{-1}\Big)\Big(\sigma^{\uparrow\Downarrow} + \sigma^{\uparrow\Uparrow}\Big) + 2n_0\sigma^{\uparrow 0}} \tag{90}$$

Let's define the $\mathcal{A}_{||}^0$ asymmetry by :

$$\mathcal{A}_{||}^0 = \frac{\sigma^{\uparrow\Downarrow} + \sigma^{\uparrow\Uparrow} - 2\sigma^{\uparrow 0}}{\sigma^{\uparrow\Downarrow} + \sigma^{\uparrow\Uparrow} + 2\sigma^{\uparrow 0}} \tag{91}$$

The unpolarised cross section becomes :

$$2\sigma^{\uparrow 0} = \frac{\Big(1 - \mathcal{A}_{||}^0\Big)}{\Big(1 + \mathcal{A}_{||}^0\Big)}\Big(\sigma^{\uparrow\Downarrow} + \sigma^{\uparrow\Uparrow}\Big) \tag{92}$$

Inserting this equation in the equation 90, we obtain :

$$\mathcal{A}_{mes} = \mathcal{P}\frac{\Big(n_1 + n_{-1} + n_0\Big)\Big(\sigma^{\uparrow\Downarrow} - \sigma^{\uparrow\Uparrow}\Big)}{\Big[n_1 + n_{-1} + n_0\Big(\frac{\big(1-\mathcal{A}_{||}^0\big)}{\big(1+\mathcal{A}_{||}^0\big)}\Big)\Big]\Big(\sigma^{\uparrow\Downarrow} + \sigma^{\uparrow\Uparrow}\Big)} \tag{93}$$

The asymmetry $\mathcal{A}_{||}^0$ is directly linked to the deuteron structure function $b_1(x)$ given by :

$$b_1(x) = \frac{1}{2}\sum_i e_i^2\Big(q^0(x) - q^1(x)\Big) \tag{94}$$

where $q^0(x)$ and $q^1(x)$ are the spin-averaged quark densities for deuterons with helicity 0 and 1 respectivly. Assuming this structure function being negligeable,[27] we consider that the quarks distribution is independent of the deuteron helicity which allow to rewrite the equation 93 as :

$$\mathcal{A}_{mes} = \mathcal{P}\ \frac{\sigma^{\uparrow\Downarrow}\ -\ \sigma^{\uparrow\Uparrow}}{\sigma^{\uparrow\Downarrow}\ +\ \sigma^{\uparrow\Uparrow}} \tag{95}$$

6.3. The relations between the proton, neutron and deuteron structure functions

As we have shown that the cross sections asymmetry has the same form for a spin 1/2 target than for a spin 1 target, we are going to demonstrate that there exists a relation between the proton, neutron and deuteron asymmetries.

The deuteron is a system made of a neutron and a proton and can be in a S or D state since its total orbital moment and its parity are $J^P = 1^+$. The deuteron S state has a null orbital moment L and its spin S is equal to 1 :

$$|S>= |J=1; J_z=1>= |L=0, L_z=0> |S=1,\ S_Z=1> \qquad (96)$$

For the D state, the spin is always equal to 1 but the total orbital moment is now equal to 2, therefore :

$$\begin{aligned} |D>= |J=1; J_z=1>= & \sqrt{\frac{3}{5}}|L=2, L_z=2> |S=1, S_z=-1> - \\ & \sqrt{\frac{3}{10}}|L=2, L_z=1> |S=1, S_z=0> + \\ & \sqrt{\frac{1}{10}}|L=2, L_z=0> |S=1, S_z=1> \end{aligned} \qquad (97)$$

Let's consider now the deuteron as a system made off one proton and one neutron.

For a S state, the deuteron spin is equal to 1 as the proton-neutron system spin. So, the proton and neutron spins are parallel to the deuteron's one :

$$|L=0, L_z=0> |S=1,\ S_Z=1> = \ |\Downarrow>_p |\Downarrow>_n \qquad (98)$$

For a D state, since the deuteron spin projected on the quantification axis can takes three differents values $(-1,\ 0,\ 1)$, we have to take into account the differents proton and neutron spins combinations inside the deuteron what we can writte as :

- The both nucleons have the same spin projection than the deuteron.

$$|S=1\,,\ S_z=1>= |\Downarrow>_p\ |\Downarrow>_n \qquad (99)$$

- The both nucleons have the opposite spin projection to the deuteron's one.

$$|S=1\,,\ S_z=-1>= |\Downarrow>_p\ |\Downarrow>_n \qquad (100)$$

- One of both nucleons has the same spin projection than the deuteron's one.

$$|S=1, S_z=0> = \ \sqrt{\frac{1}{2}}\,|\Uparrow>_p\ |\Downarrow>_n\ + \sqrt{\frac{1}{2}}\,|\Downarrow>_p\ |\Uparrow>_n \qquad (101)$$

From the equation 97, we deduce the probability for the D state to have at least one nucleon whom its spin is not parallel to the deuteron's one :

$$P(s_{n,p} \neq s_d) \;=\; \frac{3}{5}(1) + \frac{3}{10}(\frac{1}{2}) = \frac{3}{4} \tag{102}$$

In addition, the probability ω_d to find deuterons in the D state is estimated in the Paris potential model and is defined by $\omega_d \;=\; 0.058$.[26] We are now able to compute the polarised deep inelastic cross section of muons off deuterons versus this same cross section but off nucleons :

$$\sigma_d^{\uparrow\Downarrow} \;=\; \Big(1 - \frac{3}{4}\omega_d\Big)\sigma_N^{\uparrow\Downarrow} \;+\; \frac{3}{4}\omega_d\sigma_N^{\uparrow\Uparrow} \tag{103a}$$

$$\sigma_d^{\uparrow\Uparrow} \;=\; \Big(1 - \frac{3}{4}\omega_d\Big)\sigma_N^{\uparrow\Uparrow} \;+\; \frac{3}{4}\omega_d\sigma_N^{\uparrow\Downarrow} \tag{103b}$$

where we have taken the convention :

$$\sigma_N \;=\; \frac{1}{2}\Big(\sigma_p + \sigma_n\Big) \tag{104}$$

Injecting these equations in the lepton-deuteron polarised cross sections asymmetry, we get :

$$\mathcal{A}^d_{mes} = \mathcal{P}\;\Big(1 - \frac{3}{2}\omega_d\Big)\frac{\sigma_N^{\uparrow\downarrow} \;-\; \sigma_N^{\uparrow\uparrow}}{\sigma_N^{\uparrow\downarrow} \;+\; \sigma_N^{\uparrow\uparrow}} \tag{105}$$

being :

$$\mathcal{A}^d_{mes} = \mathcal{P}\;\Big(1 - \frac{3}{2}\omega_d\Big)\frac{\Delta\sigma_p \;+\; \Delta\sigma_n}{2\sigma_p \;+\; 2\sigma_n} \tag{106}$$

Ce qui en fonction des asymétries mesurées nous donne :

$$\mathcal{A}^d_{mes} = \;\Big(1 - \frac{3}{2}\omega_d\Big)\Big(\frac{\sigma_p}{\sigma_d}\mathcal{A}^p_{mes} \;+\; \frac{\sigma_n}{\sigma_d}\mathcal{A}^n_{mes}\Big) \tag{107}$$

We get also versus virtual photons nucleons asymmetries and unpolarised structure functions the relation :

$$\mathcal{A}^d_1 \;=\; \frac{\Big(1 - \frac{3}{2}\omega_d\Big)}{1 + F_2^n/F_2^p}\Big(\mathcal{A}^p_1 \;+\; \frac{F_2^n}{F_2^p}\mathcal{A}^n_1\Big) \tag{108}$$

For the spin dependent structure functions, we get :

$$2g_1^d(x,Q^2) \;=\; \Big(1 - \frac{3}{2}\omega_d\Big)\Big[g_1^p(x,Q^2) + g_1^n(x,Q^2)\Big] \tag{109}$$

If ω_d is independent of x, we obtain the following relation for the first moment of g_1^d :

$$2\Gamma_1^d \;=\; \Big(1 - \frac{3}{2}\omega_d\Big)\Big(\Gamma_1^p + \Gamma_1^n\Big) \tag{110}$$

A. The leptonic tensor $L^{\mu\nu}$

To determine the contribution of the leptonic arm (Upper part of the Feynmann's graph), we use Q.E.D since the muons are pointlike fermions.

1.1. Normalisation

We adopt the following spinors normalisation :

$$\bar{u}^{\alpha}(k).u^{\beta}(k) = 2m_l\delta^{\alpha\beta}$$

$$\bar{u}^{\alpha}(k).v^{\beta}(k) = 0$$

1.2. Dirac's particle polarisation

A Dirac's particle with a four-momentum k and a polarisation defined along a $\vec{n}$ axis. This axis is associated to a four-vector n such as $n.k = 0$. We define the projection operator on polarisation state n, $P(n)$, by :

$$P(n) = \frac{1}{2}(1 + \gamma^5.\frac{\not{n}}{m_l})$$

where

$$\not{n} \;=\; \gamma^{\mu}\,n_{\mu}$$

This projector $P(n)$ satisfies the relation :

$$P(n)u(k,n) = u(k,n)$$

The initial state spinor product can be written in terms of the standard spin projection operator :

$$u(k,n)\otimes\bar{u}(k,n) = (\not{k} + m_l)P(n)$$

which, when we sum over the polarisation states, become :

$$\sum_{s=1,2} u(k,s)\otimes\bar{u}(k,s) = \not{k} + m_l \tag{111}$$

1.3. Helicity operator

It is a particular choice of the four-vector n

$$n(k) = \left(\mid\vec{k}\mid; k^0\frac{\vec{k}}{\mid\vec{k}\mid}\right) \tag{112}$$

We take the polarisation axis along the lepton momentum with $n.k = 0$ and $n^2 = -m_l^2$.

1.4. *Tensor calculation*

From the differential cross section given by Eq (2), we have defined the leptonic tensor $L^{\mu\nu}$ (See Eq (2)) :

$$L^{\mu\nu} = \sum_{s'} \langle k's' |\ j_l^{\nu}(0)\ | k, s_l\rangle\langle k, s_l |\ j_l^{\mu}(0)\ | k's'\rangle$$

Since the muons are pointlike fermions,

$$L^{\mu\nu} = \sum_{s'} \bar{u}(k', s')\gamma^{\mu} u(k, s)\bar{u}(k, s)\gamma^{\nu} u(k', s')$$

Writting explicitly the spinor indices, we get :

$$L^{\mu\nu} = \sum_{s'} \bar{u}(k', s')_{\alpha}\gamma^{\mu}_{\alpha\beta} u(k, s)_{\beta}\bar{u}(k, s)_{\gamma}\gamma^{\nu}_{\gamma\delta} u(k', s')_{\delta}$$

Thus,

$$L^{\mu\nu} = \left(\sum_{s'} \bar{u}(k', s')_{\alpha} u(k', s')_{\delta}\right)\left(u(k, s)_{\beta}\bar{u}(k, s)_{\gamma}\right)\gamma^{\mu}_{\alpha\beta}\gamma^{\nu}_{\gamma\delta}$$

Using relations Eq (111) and Eq (112), we have :

$$L^{\mu\nu} = \left(\not{k}' + m_l\right)_{\delta\alpha}\left((\not{k} + m_l)\frac{1 + \gamma^5 \frac{\not{s}}{m_l}}{2}\right)_{\beta\gamma}\gamma^{\mu}_{\alpha\beta}\gamma^{\nu}_{\gamma\delta}$$

Finaly :

$$L^{\mu\nu} = Tr\left((\not{k}' + m_l)\gamma^{\mu}(\not{k} + m_l)\frac{1 + \gamma^5 \frac{\not{s}}{m_l}}{2}\gamma^{\nu}\right)$$

if we isolate the terms in γ^5 :

$$\begin{aligned} 2L^{\mu\nu} = \quad & Tr\left((\not{k}' + m_l)\gamma^{\mu}(\not{k} + m_l)\ \gamma^{\nu}\right) + \\ & Tr\left(0 + (\not{k}' + m_l)\gamma^{\mu}(\not{k} + m_l)\gamma^5 \tfrac{\not{s}}{m_l}\gamma^{\nu} + 0\right) \end{aligned}$$

Knowing that :

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{-i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}$$

and that the trace of an odd number of γ matrices is null:

$$\begin{aligned} 2L^{\mu\nu} = \quad & Tr\left(\not{k}'\gamma^{\mu}\not{k}\ \gamma^{\nu} + 0 + 0 + m_l^2\gamma^{\mu}\gamma^{\nu}\right) + \\ & Tr\left(m_l\gamma^{\mu}\ \not{k}\gamma^5 \tfrac{\not{s}}{m_l}\gamma^{\nu} + \not{k}'\gamma^{\mu} m_l\gamma^5 \tfrac{\not{s}}{m_l}\gamma^{\nu}\right) \end{aligned}$$

With the explicit formulas for $\not{k}$, $\not{k}' and$ $\not{s}$, we get :

$$2L^{\mu\nu} = Tr\left(k'_\rho k_\sigma \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu + m_l^2\gamma^\mu\gamma^\nu\right) + Tr\left(k_\rho s_\sigma \gamma^\mu\gamma^\rho\gamma^5\gamma^\sigma\gamma^\nu + k'_\rho s_\sigma \gamma^\rho\gamma^\mu\gamma^5\gamma^\sigma\gamma^\nu\right)$$

with

$$Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

$$Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$$

$$Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\{\gamma^5, \gamma^\mu\} = 0$$

so finaly

$$2L^{\mu\nu} = 4k'_\rho k_\sigma(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}) + 4m_l^2 g^{\mu\nu} + Tr\left(k_\rho s_\sigma \gamma^5\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu + k'_\rho s_\sigma \gamma^5\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\right)$$

$$2L^{\mu\nu} = 4(k'^\mu k^\nu + k'^\nu k^\mu - \left(k\cdot k'\right)g^{\mu\nu}) + 4m_l^2 g^{\mu\nu} + \left(k_\rho s_\sigma(-4i\epsilon^{\mu\rho\sigma\nu}) + k'_\rho s_\sigma(-4i\epsilon^{\rho\mu\sigma\nu})\right)$$

$$L^{\mu\nu} = 2(k'^\mu k^\nu + k'^\nu k^\mu) + 2(m_l^2 - \left(k\cdot k'\right))g^{\mu\nu} + 2\left(k_\rho s_\sigma(-i\epsilon^{\mu\nu\rho\sigma}) + k'_\rho s_\sigma(+i\epsilon^{\mu\nu\rho\sigma})\right)$$

$$L^{\mu\nu} = 2(k'^\mu k^\nu + k'^\nu k^\mu) + 2(m_l^2 - \left(k\cdot k'\right))g^{\mu\nu} - 2i\epsilon^{\mu\nu\rho\sigma}(k-k')_\rho s_\sigma$$

$$L^{\mu\nu} = 2\left((k'^\mu k^\nu + k'^\nu k^\mu) + (m_l^2 - \left(k\cdot k'\right))g^{\mu\nu} - i\epsilon^{\mu\nu\rho\sigma}q_\rho s_\sigma)\right)$$

The Livi-Civita tensor is antisymetric, so

$$L^{\mu\nu} = L_S^{\mu\nu} + L_A^{\mu\nu}$$

with

$$L_S^{\mu\nu} = 2\left[k'^\mu k^\nu + k'^\nu k^\mu + \left[m_l^2 - \left(k\cdot k'\right)\right]g^{\mu\nu}\right] \tag{113}$$

and

$$L_A^{\mu\nu} = -2i\epsilon^{\mu\nu\rho\sigma}q_\rho s_\sigma \tag{114}$$

B. The hadronic tensor $W_{\mu\nu}$

2.1. The hadronic tensor $W_{\mu\nu}$ for spin 1/2 target

The structure of the hadron relevant for deep inelastic scattering can be completely characterised by the hadronic tensor $W_{\mu\nu}$. The tensor is also defined by Eq (3) :

$$W^{\mu\nu}(p,q) = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\, \langle p,\lambda|\, j^\mu(x) j^\nu(0)\, |p,\lambda\rangle \tag{115}$$

In its most general form for a spin $\frac{1}{2}$ target, this tensor must be expressed in terms of p, q, s_h and the invariant tensors $g_{\mu\nu}$and $\epsilon_{\mu\nu\alpha\beta}$. The additionnal constraints provided by :

- parity conservation
- time reversal invariance
- hermiticity
- translation invariance
- current conservation

will be discuss below.
It is conventional to write $W_{\mu\nu}$ in the form

$$\begin{aligned} W_{\mu\nu} &= \frac{1}{M}F_1\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)+\frac{1}{M}\frac{F_2}{p\cdot q}\left(p_\mu-\frac{p\cdot q\; q_\mu}{q^2}\right)\left(p_\nu-\frac{p\cdot q\; q_\nu}{q^2}\right) \\ &+ \frac{ig_1}{p\cdot q}\,\epsilon_{\mu\nu\lambda\sigma}q^\lambda s^\sigma+\frac{ig_2}{(p\cdot q)^2}\,\epsilon_{\mu\nu\lambda\sigma}q^\lambda\,(p\cdot q\, s^\sigma - s\cdot q\, p^\sigma)\end{aligned}$$

where the coefficients of the different tensor structures are called structure functions. The leptonic current is also conserved, so that $q^\mu L_{\mu\nu} = q^\nu L_{\mu\nu} = 0$. Thus it is allowed to simplify the expression for $W_{\mu\nu}$ by omitting all q_μ and q_ν terms before contracting with $L_{\mu\nu}$. In this way, we obtain for symmetric and antisymmetric parts,

$$W^S_{\mu\nu} = -\frac{1}{M}F_1\, g_{\mu\nu} + \frac{F_2}{p\cdot q}\, p_\mu p_\nu \tag{116}$$

$$W^A_{\mu\nu} = \frac{ig_1}{\left(p\cdot q\right)}\,\epsilon_{\mu\nu\lambda\sigma}q^\lambda s_h^\sigma \;+\; \frac{ig_2}{\left(p\cdot q\right)^2}\,\epsilon_{\mu\nu\lambda\sigma}q^\lambda\,(p\cdot q\, s_h^\sigma - s_h\cdot q\, p^\sigma) \tag{117}$$

and

$$W_{\mu\nu} \;=\; W^S_{\mu\nu} \;+\; W^A_{\mu\nu}$$

2.2. The invariance properties of $W^{\mu\nu}$.

This part is largely inspirated from the excellent reference.[1]

2.2.1. Parity conservation

The strong interaction is invariant under parity. The parity transform of the electromagnetic current is

$$\mathcal{P} j^{\mu}(x) \mathcal{P}^{-1} = j^{\mu_P}(x_P)$$

with
$\mu_P = \mu$ for $\mu = 0$
$\mu_P = -\mu$ for $\mu = 1, 2, 3, ...$
We will use the convention $j^{-1} = -j^1$, *etc...*, for the transformation of the various components of a vector under parity. The parity transform of the point x is $x_P = (t, -\vec{x})$, and the parity transform of the polarisation density matrix is

$$\mathcal{P}\rho\mathcal{P}^{-1} = \rho_P, \qquad Tr\,\rho_P = 1, \qquad MTr\rho_P\vec{\sigma} = \vec{s}_h,$$

since s_h is an axial vector. This implies

$$\begin{aligned} W^{\mu\nu}(p,q,s_h) &= \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} Tr\Big(\rho\, j^{\mu}(x) j^{\nu}(0)\Big) \\ &= \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} Tr\Big(\mathcal{P}\rho\mathcal{P}^{-1}\ \mathcal{P} j^{\mu}(x) j^{\nu}(0)\mathcal{P}^{-1}\Big) \\ &= \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} Tr\Big(\rho_P\, j^{\mu_P}(x_P) j^{\nu_P}(0)\Big) \end{aligned} \tag{118}$$

Defining $q_P = (q^0, -\vec{q})$, $(s_h)_P = (-s_h^0, \vec{s}_h)$, and using $q_P \cdot x_P = q \cdot x$, one can rewrite the last equality in Eq (118) as

$$W^{\mu\nu}(p,q,s_h) = W^{\mu_P\nu_P}(p_P, q_P, (s_h)_P)$$

which is the condition that $W_{\mu\nu}$ is parity invariant.

2.2.2. time-reversal invariance

The time reversal is an antiunitary transformation. The time-reversal of momentum is defined by $p_T = (p^0, -\vec{p})$, spin by $(s_h)_T = (s_h^0, -\vec{s}_h)$, and coordinates by $x_T = (-t, \vec{x})$. Note that time reversal takes x^0 to $-x^0$, but still takes p^0 to $+p^0$, *i.e.*time reversal changes the sign of time, but not of energy. Time reversal invariance then implies that

$$\begin{aligned} W^{\mu\nu}(p,q,s_h) &= \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} Tr\Big(\rho\, j^{\mu}(x) j^{\nu}\Big)(0) \\ &= \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\Big(Tr\Big(\mathcal{T}\rho\mathcal{T}^{-1}\ \mathcal{T} j^{\mu}(x) j^{\nu}(0)\mathcal{T}^{-1}\Big)\Big)^* \\ &= \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\Big(Tr\Big(\rho_T\, j^{\mu_T}(x_T) j^{\nu_T}(0)\Big)\Big)^* \end{aligned}$$

where the complex conjugation in the second line is present because time reversal is antiunitary. Using the identity $q \cdot x = -q_T \cdot x_T$, we get

$$e^{iq\cdot x} = \left(e^{iq_T\cdot x_T}\right)^*$$

so that time reversal invariance implies that

$$W^{\mu\nu}(p,q,s_h) = W^{\mu_T\nu_T}(p_T,q_T,(s_h)_T)^*. \tag{119}$$

This invariance has been confirmed experimentaly. The measurements of asymmetries in resonance region inelastic scattering of unpolarised electrons by protons polarised normal to the electron scattering plane yield null asymmetries.[28]

2.2.3. Hermiticity

The electromagnetic currents and density matrix are hermitian, which implies

$$\begin{aligned} W^{\mu\nu}(p,q,s_h)^* &= \frac{1}{4\pi}\int d^4x\, e^{-iq\cdot x} Tr\Big(\rho^\dagger\, j^\mu(x) j^\nu(0)^\dagger\Big) \\ &= \frac{1}{4\pi}\int d^4x\, e^{-iq\cdot x} Tr\Big(\rho\, j^\nu(0) j^\mu(x)\Big) \\ &= \frac{1}{4\pi}\int d^4x\, e^{-iq\cdot x} Tr\Big(\rho\, j^\nu(-x) j^\mu(0)\Big) \\ &= W^{\nu\mu}(p,q,s_h) \end{aligned} \tag{120}$$

where the first line follows from the identity $(Tr\,AB)^* = Tr(AB)^\dagger$ and the third line follows from translation invariance. Note that all three identities, Eq (118), Eq (119), and Eq (120) relate $W^{\mu\nu}$ for q^0 to a transformed $W^{\mu\nu}$ at the same value of q^0.

2.2.4. Translation invariance

There is a final identity that relates $W_{\mu\nu}$ for $q^0 > 0$ to $W_{\mu\nu}$ for $q^0 < 0$ which follows from the antisymmetry of the commutator and translation invariance:

$$\begin{aligned} W^{\mu\nu}(p,q,s_h) &= -\frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} Tr\Big(\rho\, j^\nu(0) j^\mu(x)\Big) \\ &= -\frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} Tr\Big(\rho\, j^\nu(-x) j^\mu(0)\Big) \\ &= -\frac{1}{4\pi}\int d^4x\, e^{-iq\cdot(-x)} Tr\Big(\rho\, j^\nu(-x) j^\mu(0)\Big) \\ &= -W^{\nu\mu}(p,-q,s_h). \end{aligned} \tag{121}$$

This identity is the photon crossing symmetry identity for $W_{\mu\nu}$, since exchanging $\Big(\mu,\nu\Big)$ and changing $q \to -q$ is equivalent to exchanging the outgoing and incoming photons.

2.2.5. current conservation

Finally current conservation, $\partial_\mu j^\mu(x) = 0$, implies that

$$q_\mu W^{\mu\nu}(p,q,s_h) = q_\nu W^{\mu\nu}(p,q,s_h) = 0 \qquad (122)$$

C. Tensor contraction

The contraction of symmetric and antisymmetric tensor is null. So, we only consider the contraction of both symmetric and antisymmetric parts of the leptonic and hadronic tensors.

3.1. Contraction of symmetric parts

$$L_S^{\mu\nu} W_{\mu\nu}^S = \frac{2}{M}\Big(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu} k\cdot k'\Big)\Big(-F_1(x,Q^2)\ g_{\mu\nu} + \frac{F_2(x,Q^2)}{\big(p\cdot q\big)} p_\mu p_\nu\Big)$$

One obtains :

$$\begin{aligned} L_S^{\mu\nu} W_{\mu\nu}^S = \quad & 2\frac{F_1(x,Q^2)}{M}\Big[\Big(k\cdot k'\Big) + \Big(k\cdot k'\Big) - 4\Big(k\cdot k'\Big)\Big] + \\ & 2\frac{F_2(x,Q^2)}{M\big(p\cdot q\big)}\Big[\Big(k\cdot p\Big)\Big(k'\cdot p\Big) + \Big(k\cdot p\Big)\Big(k'\cdot p\Big) - \Big(p\cdot p\Big)\Big(k\cdot k'\Big)\Big] \end{aligned}$$

$$L_S^{\mu\nu} W_{\mu\nu}^S = -4F_1(x,Q^2)\ \frac{\big(k\cdot k'\big)}{M} + 2\frac{F_2(x,Q^2)}{M\big(p\cdot q\big)}\Big[2\Big(k\cdot p\Big)\Big(k'\cdot p\Big) - M^2\Big(k\cdot k'\Big)\Big]$$

The definition of the Bjorken variable is given by the following relations :

$$x = \frac{Q^2}{2p\cdot q} = \frac{Q^2}{2MEy} \qquad y = \frac{\nu}{E} = \frac{\big(p\cdot q\big)}{\big(p\cdot k\big)} \qquad q^2 = -2\Big(k\cdot k'\Big) \tag{123}$$

One obtains :

$$-4\Big(k\cdot k'\Big) \ = \ 2q^2 \ = \ 4M\nu x \ = \ 4xyME$$

$$\begin{aligned} 4\frac{\big(k\cdot p\big)\big(k'\cdot p\big)}{\big(p\cdot q\big)} &= 4\tfrac{MEME'}{M\nu} &&= 4\frac{MEE'}{\nu} \\ &= 4\tfrac{ME'}{y} &&= 4\frac{ME}{y}\Big(1-y\Big) \end{aligned}$$

$$-2M^2\frac{\big(k\cdot k'\big)}{\big(p\cdot q\big)} \ = \ \frac{M^2q^2}{M\nu} \ = \ \frac{Mq^2}{\nu} \ = \ -2M^2x \ = \ \frac{4ME}{y}\Big(\frac{Mxy}{2E}\Big)$$

We get :

$$L_S^{\mu\nu} W_{\mu\nu}^S = \frac{4E}{y}\Big[xy^2 F_1(x,Q^2) \ + \Big(1-y-\frac{Mxy}{2E}\Big) F_2(x,Q^2) \ \Big]$$

Using Eq (10), we can deduce that :

$$cos^2\frac{\theta}{2} \ = \ \frac{1}{1-y}\Big(1-y-\frac{Mxy}{2E}\Big)$$

So, using Eq (7b) :

$$\frac{4E}{y}\Big(1-y-\frac{Mxy}{2E}\Big)F_2(x,Q^2) \ \ = \ \frac{4E}{y}\Big(1-y\Big)cos^2\frac{\theta}{2}\nu W_2 \ = \ 4EE'cos^2\frac{\theta}{2}W_2$$

and using Eq (10) and Eq (7a), we can get :

$$\begin{aligned} \frac{4E}{y}xy^2 MW_1 \ = \ & \quad 4EMxyW_1 \ = \quad 4E2E\Big(1-y\Big)sin^2\frac{\theta}{2}W_1 \\ = \ & \ 4EE'\Big(2W_1 sin^2\tfrac{\theta}{2}\Big) \end{aligned}$$

Finally, we obtain :

$$L_S^{\mu\nu} W_{\mu\nu}^S = 4EE'\Big(\cos^2\frac{\theta}{2}W_2 + 2\sin^2\frac{\theta}{2}W_1\Big)$$

3.2. Contraction of antisymmetric parts

$$\begin{aligned} L_A^{\mu\nu} W_{\mu\nu}^A = & \quad \Big(-2i\epsilon^{\mu\nu\alpha\beta} q_\alpha s_\beta^l\Big) \\ & \Big[\frac{ig_1(x,Q^2)}{M\nu} \ \epsilon_{\mu\nu\rho\sigma} q^\rho s_n^\sigma + \frac{ig_2(x,Q^2)}{M^2\nu^2}\epsilon_{\mu\nu\rho\sigma} q^\rho\Big(\Big(p\cdot q\Big)s_n^\sigma - \Big(s_h\cdot q\Big)p^\sigma\Big)\Big] \\ = & \quad 2\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\frac{g_1(x,Q^2)}{\nu} q_\alpha s_\beta^l q^\rho s_n^\sigma \\ & + 2\frac{g_2(x,Q^2)}{\nu^2}\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\Big[q_\alpha s_\beta^l q^\rho s_n^\sigma\Big(p\cdot q\Big) - q_\alpha s_\beta^l q^\rho p^\sigma\Big(s_h\cdot q\Big)\Big] \end{aligned}$$

Using the following identity

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\rho\sigma} \ = -2\Big(g_\rho^\alpha g_\sigma^\beta - g_\sigma^\alpha g_\rho^\beta\Big)$$

One obtains :

$$\begin{aligned} L_A^{\mu\nu} W_{\mu\nu}^A = & \quad \frac{-4g_1(x,Q^2)}{M\nu}\Big(g_\rho^\alpha g_\sigma^\beta - g_\sigma^\alpha g_\rho^\beta\Big) q_\alpha s_\beta^l q^\rho s_n^\sigma \\ & - \frac{4g_2(x,Q^2)}{M^2\nu^2}\Big(g_\rho^\alpha g_\sigma^\beta - g_\sigma^\alpha g_\rho^\beta\Big)\Big[q_\alpha s_\beta^l q^\rho s_n^\sigma\Big(p\cdot q\Big)\Big] \\ & + \frac{4g_2(x,Q^2)}{M^2\nu^2}\Big(g_\rho^\alpha g_\sigma^\beta - g_\sigma^\alpha g_\rho^\beta\Big)\Big[q_\alpha s_\beta^l q^\rho p^\sigma\Big(s_h\cdot q\Big)\Big] \end{aligned}$$

$$
\begin{aligned}
L_A^{\mu\nu} W_{\mu\nu}^A = \quad & \frac{-4 g_1(x,Q^2)}{M\nu} \Big(q_\rho s_\sigma^l q^\rho s_n^\sigma - q_\sigma s_\rho^l q^\rho s_n^\sigma \Big) \\
& -4 \frac{g_2(x,Q^2)}{M^2\nu^2} \Big(p \cdot q \Big) \Big(q_\rho s_\sigma^l q^\rho s_n^\sigma - q_\sigma s_\rho^l q^\rho s_n^\sigma \Big) \\
& +4 \frac{g_2(x,Q^2)}{M^2\nu^2} \Big(s_h \cdot q \Big) \Big(q_\rho s_\sigma^l q^\rho p^\sigma - q_\sigma s_\rho^l q^\rho p^\sigma \Big)
\end{aligned}
$$

$$
\begin{aligned}
L_A^{\mu\nu} W_{\mu\nu}^A = \quad & \frac{-4 g_1(x,Q^2)}{M\nu} \Big[q^2 \Big(s_l \cdot s_h \Big) - \Big(s_h \cdot q \Big) \Big(s_h \cdot q \Big) \Big] \\
& - \frac{4 g_2(x,Q^2)}{M^2\nu^2} \Big(p \cdot q \Big) \Big[q^2 \Big(s_l \cdot s_h \Big) - \Big(s_h \cdot q \Big) \Big(s_h \cdot q \Big) \Big] \\
& + \frac{4 g_2(x,Q^2)}{M^2\nu^2} \Big(s_h \cdot q \Big) \Big[q^2 \Big(s_l \cdot p \Big) - \Big(s_h \cdot q \Big) \Big(p \cdot q \Big) \Big]
\end{aligned}
$$

$$
\begin{aligned}
L_A^{\mu\nu} W_{\mu\nu}^A = \quad & -4 g_1(x,Q^2) \ \tfrac{q^2}{M\nu} \Big(s_l \cdot s_h \Big) \ + \ 4 \frac{g_1(x,Q^2)}{M\nu} \Big(s_h \cdot q \Big) \Big(s_h \cdot q \Big) \\
& -4 \frac{g_2(x,Q^2)}{M^2\nu^2} q^2 \Big(s_l \cdot s_h \Big) \Big(p \cdot q \Big) \ + \ 4 \frac{g_2(x,Q^2)}{M^2\nu^2} \Big(p \cdot q \Big) \Big(s_h \cdot q \Big) \Big(s_h \cdot q \Big) \\
& +4 \frac{g_2(x,Q^2)}{M^2\nu^2} q^2 \Big(s_h \cdot q \Big) \Big(s_l \cdot p \Big) \ - \ 4 \frac{g_2(x,Q^2)}{M^2\nu^2} \Big(s_h \cdot q \Big) \Big(s_h \cdot q \Big) \Big(p \cdot q \Big)
\end{aligned}
$$

$$
\begin{aligned}
L_A^{\mu\nu} W_{\mu\nu}^A = \quad & -4 g_1(x,Q^2) \ \tfrac{q^2}{M\nu} \Big(s_l \cdot s_h \Big) \ + \ 4 \frac{g_1(x,Q^2)}{M\nu} \Big(s_h \cdot q \Big) \Big(s_h \cdot q \Big) \\
& -4 g_2(x,Q^2) \ \tfrac{q^2}{M^2\nu^2} \Big(s_l \cdot s_h \Big) \Big(p \cdot q \Big) \ + \ 4 g_2(x,Q^2) \ \tfrac{q^2}{M^2\nu^2} \Big(s_h \cdot q \Big) \Big(s_l \cdot p \Big)
\end{aligned}
$$

Using Eq (123), we can develop each previous factors in terms of Bjorken variable.

$$
\begin{aligned}
-4 g_1(x,Q^2) \ \frac{q^2}{M\nu} \Big(s_l \cdot s_h \Big) &= \ -4 g_1(x,Q^2) \ \tfrac{q^2}{M\nu} \Big(\tfrac{2ME}{y} \Big) \Big(\tfrac{y}{2ME} \Big) \Big(s_l \cdot s_h \Big) \\
&= \quad \Big(\tfrac{2ME}{y} \Big) \Big(4 g_1(x,Q^2) \ \Big) \Big(s_l \cdot s_h \Big) \tfrac{xy}{ME} \\
&= \quad \Big(\tfrac{2ME}{y} \Big) \Big(4 g_1(x,Q^2) \ \Big) \frac{\Big(s_l \cdot s_h \Big)}{\Big(p \cdot q \Big)} \tfrac{xyM\nu}{ME} \\
&= \quad \Big(\tfrac{2ME}{y} \Big) \frac{\Big(s_l \cdot s_h \Big)}{\Big(p \cdot q \Big)} 4 g_1(x,Q^2) \ xy^2
\end{aligned}
$$

$$
\begin{aligned}
4 \frac{g_1(x,Q^2)}{M\nu} \Big(s_h \cdot q \Big) \Big(s_h \cdot q \Big) &= \ \Big(\tfrac{2ME}{y} \Big) \frac{\Big(s_h \cdot q \Big)}{\Big(p \cdot q \Big)} \frac{\Big(s_h \cdot q \Big)}{\Big(p \cdot q \Big)} \Big(4 g_1(x,Q^2) \ \Big) \frac{\Big(p \cdot q \Big)^2}{M\nu} \Big(\tfrac{y}{2ME} \Big) \\
&= \quad \Big(\tfrac{2ME}{y} \Big) \frac{\Big(s_h \cdot q \Big)}{\Big(p \cdot q \Big)} \frac{\Big(s_h \cdot q \Big)}{\Big(p \cdot q \Big)} \Big(4 g_1(x,Q^2) \ \Big) \tfrac{M^2\nu^2}{M\nu} \Big(\tfrac{y}{2ME} \Big)
\end{aligned}
$$

$$= \quad \left(\tfrac{2ME}{y}\right)\frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}\frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}\left(2yg_1(x,Q^2)\ \right)\tfrac{\nu}{E}$$

$$= \quad \left(\tfrac{2ME}{y}\right)\frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}\frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}2y^2g_1(x,Q^2)$$

$$-4g_2(x,Q^2)\ \frac{q^2}{M^2\nu^2}\left(s_l \cdot s_h\right)\left(p \cdot q\right) = \left(\tfrac{2ME}{y}\right)\frac{\left(s_l \cdot s_h\right)}{\left(p \cdot q\right)}\left(-4g_2(x,Q^2)\ \right)\tfrac{q^2}{M^2\nu^2}\left(p \cdot q\right)^2\tfrac{y}{2ME}$$

$$= \quad \left(\tfrac{2ME}{y}\right)\frac{\left(s_l \cdot s_h\right)}{\left(p \cdot q\right)}4\left(\tfrac{-q^2}{2M}\right)\tfrac{M^2\nu^2}{M^2\nu^2}\tfrac{y}{E}g_2(x,Q^2)$$

$$= \quad \left(\tfrac{2ME}{y}\right)\frac{\left(s_l \cdot s_h\right)}{\left(p \cdot q\right)}4x\nu\tfrac{y}{E}g_2(x,Q^2)$$

$$= \quad \left(\tfrac{2ME}{y}\right)\frac{\left(s_l \cdot s_h\right)}{\left(p \cdot q\right)}4xy^2g_2(x,Q^2)$$

$$4g_2(x,Q^2)\ \frac{q^2}{M^2\nu^2}\left(s_h \cdot q\right)\left(s_l \cdot p\right) = \left(\tfrac{2ME}{y}\right)\frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}\frac{\left(s_l \cdot p\right)}{\left(p \cdot q\right)}\left(4g_2(x,Q^2)\ \right)\tfrac{q^2}{M^2\nu^2}\left(p \cdot q\right)^2\tfrac{y}{2ME}$$

$$= \quad \left(\tfrac{2ME}{y}\right)\frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}\frac{\left(s_l \cdot p\right)}{\left(p \cdot q\right)}\left(4g_2(x,Q^2)\ \right)\tfrac{M^2\nu^2}{M^2\nu^2}\tfrac{q^2}{2M}\tfrac{y}{E}$$

$$= \quad \left(\tfrac{2ME}{y}\right)\frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}\frac{\left(s_l \cdot p\right)}{\left(p \cdot q\right)}\left(-4g_2(x,Q^2)\ \right)xy\tfrac{\nu}{E}$$

$$= \quad -\left(\tfrac{2ME}{y}\right)\frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}\frac{\left(s_l \cdot p\right)}{\left(p \cdot q\right)}4xy^2g_2(x,Q^2)$$

So, we can rewrite Eq (124) as follows :

$$\begin{aligned} L_A^{\mu\nu}W_{\mu\nu}^A = \quad & \left(\tfrac{2ME}{y}\right)\frac{\left(s_l \cdot s_h\right)}{\left(p \cdot q\right)}4g_1(x,Q^2)\ xy^2 + \left(\tfrac{2ME}{y}\right)\frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}\frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}2y^2g_1(x,Q^2) \\ & + \left(\tfrac{2ME}{y}\right)\frac{\left(s_l \cdot s_h\right)}{\left(p \cdot q\right)}4xy^2g_2(x,Q^2)\ - \left(\tfrac{2ME}{y}\right)\frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}\frac{\left(s_l \cdot p\right)}{\left(p \cdot q\right)}4xy^2g_2(x,Q^2) \end{aligned}$$

Finally, we obtain

$$L_A^{\mu\nu}W_{\mu\nu}^A \ = \ 4\frac{ME}{y} \quad \Big[y^2g_1(x,Q^2)\ \Big(2x\frac{\left(s_l \cdot s_h\right)}{\left(p \cdot q\right)} + \frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}\frac{\left(s_h \cdot q\right)}{\left(p \cdot q\right)}\Big)$$

$$+2xy^2 g_2(x,Q^2)\ \Big(\frac{\big(s_l\cdot s_h\big)}{\big(p\cdot q\big)}-\frac{\big(s_l\cdot p\big)}{\big(p\cdot q\big)}\frac{\big(s_h\cdot q\big)}{\big(p\cdot q\big)}\Big)\Big]$$

3.3. Differential cross section explicit expression

From the equation 4, substituting for the tensors contraction in this equation and using equation 11, we deduce the differential cross section versus x and y :

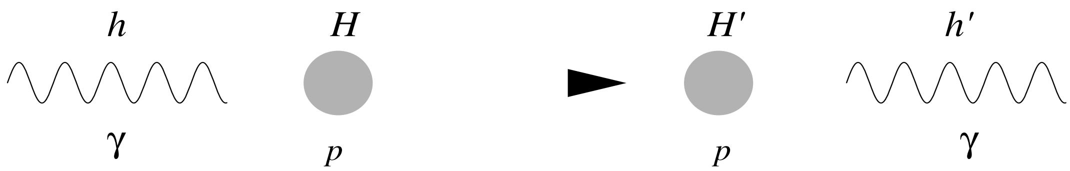
$$\begin{aligned}\frac{d^2\sigma}{dxdy} \;=\;& \tfrac{2\pi\alpha^2}{MEx^2y^2}\Big[xy^2F_1(x,Q^2)\ +(1-y-\tfrac{Mxy}{2E}\Big)F_2(x,Q^2)\ + \\ & My^2g_1(x,Q^2)\ \Big(2x\frac{\big(s_l\cdot s_h\big)}{\big(p\cdot q\big)}+\frac{\big(s_h\cdot q\big)}{\big(p\cdot q\big)}\frac{\big(s_h\cdot q\big)}{\big(p\cdot q\big)}\Big)+ \\ & 2xy^2M^2g_2(x,Q^2)\ \Big(\frac{\big(s_l\cdot s_h\big)}{\big(p\cdot q\big)}-\frac{\big(s_l\cdot p\big)}{\big(p\cdot q\big)}\frac{\big(s_h\cdot q\big)}{\big(p\cdot q\big)}\Big)\Big]\end{aligned} \tag{124}$$

which is the same expression than the equations 12 and 13.

D. Why four structure functions ?

In the following $\Rightarrow$ indicates the particle momentum direction, whereas $\rightarrow$ stands for the particle spin direction. The z axis will be the direction of the forward scattering. In the Compton helicity amplitudes, we define initial (rep. final) photon helicity $h_\gamma = 0, \pm 1$ ($h'_\gamma = 0, \pm 1$) and nucleon helicty $h_N = \pm\frac{1}{2}$ ($h'_N = \pm\frac{1}{2}$), so that the total angular momentun for the γN system is

$$J_z = h_\gamma - h_N = h'_\gamma - h'_N$$



The following table list all the possible γN helicity combinaisons.

Initial Photon Helicity	Initial Nucleon Helicity	Total Angular Momentum J_z $\rightarrow_z$	Final Photon Helicity	Final Nucleon Helicity	Forward Compton helicity Amplitude
1 $\rightarrow$ $\Rightarrow$	$\frac{1}{2}$ $\leftarrow$ $\Leftarrow$	$\frac{1}{2}$	0 $\uparrow$ $\Rightarrow$	$-\frac{1}{2}$ $\rightarrow$ $\Leftarrow$	$M_{1,\frac{1}{2},0,-\frac{1}{2}}$ I
1 $\rightarrow$ $\Rightarrow$	$\frac{1}{2}$ $\leftarrow$ $\Leftarrow$	$\frac{1}{2}$	1 $\rightarrow$ $\Rightarrow$	$\frac{1}{2}$ $\leftarrow$ $\Leftarrow$	$M_{1,\frac{1}{2},1,\frac{1}{2}}$ II
1 $\rightarrow$ $\Rightarrow$	-$\frac{1}{2}$ $\rightarrow$ $\Leftarrow$	$\frac{3}{2}$	1 $\rightarrow$ $\Rightarrow$	$-\frac{1}{2}$ $\rightarrow$ $\Leftarrow$	$M_{1,-\frac{1}{2},1,-\frac{1}{2}}$ III
0 $\uparrow$ $\Rightarrow$	$\frac{1}{2}$ $\leftarrow$ $\Leftarrow$	$-\frac{1}{2}$	-1 $\leftarrow$ $\Rightarrow$	$-\frac{1}{2}$ $\rightarrow$ $\Leftarrow$	$M_{0,\frac{1}{2},-1,-\frac{1}{2}}$ IV
0 $\uparrow$ $\Rightarrow$	$\frac{1}{2}$ $\leftarrow$ $\Leftarrow$	$-\frac{1}{2}$	0 $\uparrow$ $\Rightarrow$	$\frac{1}{2}$ $\leftarrow$ $\Leftarrow$	$M_{0,\frac{1}{2},0,\frac{1}{2}}$ V

Initial Photon Helicity	Initial Nucleon Helicity	Total Angular Momentum J_z $\rightarrow_z$	Final Photon Helicity	Final Nucleon Helicity	Forward Compton helicity Amplitude
-1 $\leftarrow$ $\Rightarrow$	$-\frac{1}{2}$ $\rightarrow$ $\Leftarrow$	$-\frac{1}{2}$	0 $\uparrow$ $\Rightarrow$	$\frac{1}{2}$ $\leftarrow$ $\Leftarrow$	$M_{-1,-\frac{1}{2},0,\frac{1}{2}}$ Ib
-1 $\leftarrow$ $\Rightarrow$	$-\frac{1}{2}$ $\rightarrow$ $\Leftarrow$	$-\frac{1}{2}$	-1 $\leftarrow$ $\Rightarrow$	$-\frac{1}{2}$ $\rightarrow$ $\Leftarrow$	$M_{-1,-\frac{1}{2},-1,-\frac{1}{2}}$ IIb
-1 $\leftarrow$ $\Rightarrow$	$\frac{1}{2}$ $\leftarrow$ $\Leftarrow$	$-\frac{3}{2}$	-1 $\leftarrow$ $\Rightarrow$	$\frac{1}{2}$ $\leftarrow$ $\Leftarrow$	$M_{-1,\frac{1}{2},-1,\frac{1}{2}}$ $IIIb$
0 $\uparrow$ $\Rightarrow$	-$\frac{1}{2}$ $\rightarrow$ $\Leftarrow$	$\frac{1}{2}$	1 $\rightarrow$ $\Rightarrow$	$\frac{1}{2}$ $\leftarrow$ $\Leftarrow$	$M_{0,-\frac{1}{2},1,\frac{1}{2}}$ IVb
0 $\uparrow$ $\Rightarrow$	-$\frac{1}{2}$ $\rightarrow$ $\Leftarrow$	$\frac{1}{2}$	0 $\uparrow$ $\Rightarrow$	$-\frac{1}{2}$ $\rightarrow$ $\Leftarrow$	$M_{0,-\frac{1}{2},0,-\frac{1}{2}}$ Vb

By invariance under reflexion with respect to any plane containing the nucleon and the γ momenta :

$$M_{\alpha,\beta,\gamma,\delta} = M_{-\alpha,-\beta,-\gamma,-\delta}$$

We are left with 5 helicity amplitudes. But by time reversal

$$M_{0,-\frac{1}{2},1,\frac{1}{2}} = M_{1,\frac{1}{2},0,-\frac{1}{2}}$$

Finally there are only 4 independent helicity amplitudes :

$$\mathcal{M}_{1\frac{1}{2},1\frac{1}{2}},\ \mathcal{M}_{1-\frac{1}{2},1-\frac{1}{2}},\ \mathcal{M}_{0\frac{1}{2},0\frac{1}{2}},\ \mathcal{M}_{0\frac{1}{2},0-\frac{1}{2}}$$

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