Introduction to Quantum Chromodynamics (QCD)

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Lecture Three
Observables with ONE identified hadron

- DIS cross section is infrared divergent, and nonperturbative!

\[
\sigma_{\ell p \to \ell' X}^{\text{DIS}}(\text{everything}) \propto \sum \frac{1}{Q R}
\]

- QCD factorization (approximation!)

\[
\sigma_{\ell p \to \ell' X}^{\text{DIS}}(\text{everything}) = \sum \frac{1}{Q R}
\]
Inclusive lepton-hadron DIS – one hadron

- **Scattering amplitude:**

\[
M(\lambda, \lambda'; \sigma, q) = \bar{u}_{\lambda'}(k') \left[ -ie\gamma_\mu \right] u_\lambda(k) \]

\[
\times \left( \frac{i}{q^2} \right)(-g_{\mu\mu'})
\]

\[
\times \langle X | eJ_{\mu'}^{em}(0) | p, \sigma \rangle
\]

- **Cross section:**

\[
d\sigma_{\text{DIS}}^{\mu\nu} = \frac{1}{2s} \left( \frac{1}{2} \right)^2 \sum_{X, \lambda, \lambda', \sigma} |M(\lambda, \lambda'; \sigma, q)|^2 \left[ \prod_{i=1}^{X} \frac{d^3 l_i}{(2\pi)^3 2E_i} \right] \frac{d^3 k'}{(2\pi)^3 2E'} (2\pi)^4 \delta^4 \left( \sum_{i=1}^{X} l_i + k' - p - k \right)
\]

- **Leptonic tensor:**

- known from QED

\[
L_{\mu\nu}(k, k') = \frac{e^2}{2\pi^2} \left( k^\mu k'^\nu + k^\nu k'^\mu - k \cdot k' g^{\mu\nu} \right)
\]
**DIS structure functions**

- **Hadron tensor:**
  \[
  W_{\mu\nu}(q, p, S) = \frac{1}{4\pi} \int d^4z \ e^{iqz} \left\langle p, S \left| J^+_\mu(z)J^-_\nu(0) \right| p, S \right\rangle
  \]

- **Symmetries:**
  - Parity invariance (EM current) \( W_{\mu\nu} = W_{\nu\mu} \) symmetric for spin avg.
  - Time-reversal invariance \( W_{\mu\nu} = W^*_{\nu\mu} \) real
  - Current conservation \( q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0 \)

\[
W_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}\right)F_1(x_B, Q^2) + \frac{1}{p \cdot q} \left(p_\mu - q_\mu \frac{p \cdot q}{q^2}\right)\left(p_\nu - q_\nu \frac{p \cdot q}{q^2}\right)F_2(x_B, Q^2)
\]

\[
+iM_p \epsilon^{\mu\nu\rho\sigma} q_\rho \left[ \frac{S_\sigma}{p \cdot q} g_1(x_B, Q^2) + \frac{(p \cdot q)S_\sigma - (S \cdot q) p_\sigma}{(p \cdot q)^2} g_2(x_B, Q^2) \right]
\]

- **Structure functions – infrared sensitive:**
  \[
  F_1(x_B, Q^2), F_2(x_B, Q^2), g_1(x_B, Q^2), g_2(x_B, Q^2)
  \]

\[
Q^2 = -q^2, \quad x_B = \frac{Q^2}{2p \cdot q}
\]

No QCD parton dynamics used in above derivation!
Long-lived parton states

- Feynman diagram representation of the hadronic tensor:

\[ W_{\mu\nu} \propto \sum \text{diagrams} \]

- Perturbative pinched poles:

\[
\int d^4k \ H(Q,k) \left( \frac{1}{k^2 + i\epsilon} \right) \left( \frac{1}{k^2 - i\epsilon} \right) T(k, \frac{1}{r_0}) \Rightarrow \infty \text{ perturbatively}
\]

- Perturbative factorization:

\[
k^\mu = xp^\mu + \frac{k_T^2}{2xp \cdot n} n^\mu + k_T^\mu
\]

\[
\int \frac{dx}{x} d^2k_T \ H(Q, k^2 = 0) \int dk^2 \left( \frac{1}{k^2 + i\epsilon} \right) \left( \frac{1}{k^2 - i\epsilon} \right) T(k, \frac{1}{r_0}) + \mathcal{O} \left( \frac{\langle k^2 \rangle}{Q^2} \right)
\]

Light-cone coordinate:

\[
v^\mu = (v^+, v^-, v^\perp), \quad v^\pm = \frac{1}{\sqrt{2}}(v^0 \pm v^3)
\]

Short-distance

Nonperturbative matrix element
Collinear factorization – further approximation

- Collinear approximation, if \( Q \sim x p \cdot n \gg k_T, \sqrt{k^2} \)

- Lowest order:

\[
\int \frac{dx}{x} + O\left( \frac{k_T^2}{Q^2} \right) \times \left( \frac{\gamma \cdot n}{2 p \cdot n} \delta \left( x - \frac{k \cdot n}{p \cdot n} \right) \frac{d^4k}{(2\pi)^4} \right)
\]

Same as an elastic x-section

Parton’s transverse momentum is integrated into parton distributions, and provides a scale of power corrections

- DIS limit: \( \nu, Q^2 \to \infty, \) while \( x_B \) fixed

Feynman’s parton model and Bjorken scaling

\[
F_2(x_B, Q^2) = x_B \sum_f e_f^2 \phi_f(x_B) = 2x_B F_1(x_B, Q^2)
\]

Spin-\( \frac{1}{2} \) parton!

- Corrections: \( O(\alpha_s) + O\left( \langle k^2 \rangle / Q^2 \right) \)
PDFs as matrix elements of two parton fields:

- combine the amplitude & its complex-conjugate

\[
\phi_{q/h}(x, \mu^2) = \int \frac{p^+ dy^-}{2\pi} e^{ixp^+y^-} \langle h(p)|\bar{\psi}_q(0) \frac{\gamma^+}{2p^+} \psi_q(y^-)|h(p)\rangle \mathcal{Z}_O(\mu^2) \\
|h(p)\rangle \text{ can be a hadron, or a nucleus, or a parton state!}
\]

But, it is NOT gauge invariant! \(\psi(x) \rightarrow e^{i\alpha_a(x)t_a}\psi(x)\) \(\bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{-i\alpha_a(x)t_a}\)

- need a gauge link:

\[
\phi_{q/h}(x, \mu^2) = \int \frac{dy^-}{2\pi} e^{ixp^+y^-} \langle h(p)|\bar{\psi}_q(0) \left[ \mathcal{P} e^{-ig \int_0^{y^-} d\eta^- A^+(\eta^-)} \right] \frac{\gamma^+}{2p^+} \psi_q(y^-)|h(p)\rangle \mathcal{Z}_O(\mu^2)
\]

- corresponding diagram in momentum space:

\[
\int \frac{d^4k}{(2\pi)^4} \delta(x - k^+/p^+) \\
\text{ + UVCT}(\mu^2) \\
\mu \text{-dependence}
\]

**Universality – process independence – predictive power**
Gauge link – 1\textsuperscript{st} order in coupling “g”

- **Longitudinal gluon:**

  \[
  \int dx_1 \left[ \int \frac{p^+ dy_1^-}{2\pi} e^{ix_1 p^+(y_1^- - y^-)} n \cdot A^a(y_1^-) \right] \mathcal{M}(-igt^a) \frac{\gamma \cdot p i((x - x_1 - x_B)\gamma \cdot p + (Q^2/2x_Bp^+)\gamma \cdot n)}{p^+} (x - x_1 - x_B)Q^2/x_B + i\epsilon \]

  \[= g \int \frac{dy_1^-}{2\pi} n \cdot A^a(y_1^-) t^a \left[ \int dx_1 \frac{1}{-x_1 + i\epsilon} e^{ix_1 p^+(y_1^- - y^-)} \right] \mathcal{M} = -ig \int_{y^-}^{\infty} dy_1^- n \cdot A(y_1^-) \mathcal{M} \]

- **Left diagram:**

  \[
  \int dx_1 \left[ \int \frac{p^+ dy_1^-}{2\pi} e^{ix_1 p^+(y_1^- - y^-)} n \cdot A^a(y_1^-) \right] \mathcal{M}(igt^a) \frac{\gamma \cdot p i((x - x_1 - x_B)\gamma \cdot p + (Q^2/2x_Bp^+)\gamma \cdot n)}{p^+} (x - x_1 - x_B)Q^2/x_B - i\epsilon \]

  \[= g \int \frac{dy_1^-}{2\pi} n \cdot A^a(y_1^-) t^a \left[ \int dx_1 \frac{1}{x_1 - i\epsilon} e^{ix_1 p^+ y_1^-} \right] \mathcal{M} = ig \int_{y^-}^{\infty} dy_1^- n \cdot A(y_1^-) \mathcal{M} \]

- **Right diagram:**

  \[
  \int dx_1 \left[ \int \frac{p^+ dy_1^-}{2\pi} e^{ix_1 p^+(y_1^- - y^-)} n \cdot A^a(y_1^-) \right] \mathcal{M}(igt^a) \frac{\gamma \cdot p i((x - x_1 - x_B)\gamma \cdot p + (Q^2/2x_Bp^+)\gamma \cdot n)}{p^+} (x - x_1 - x_B)Q^2/x_B - i\epsilon \]

  \[= g \int \frac{dy_1^-}{2\pi} n \cdot A^a(y_1^-) t^a \left[ \int dx_1 \frac{1}{x_1 - i\epsilon} e^{ix_1 p^+ y_1^-} \right] \mathcal{M} = ig \int_{y^-}^{\infty} dy_1^- n \cdot A(y_1^-) \mathcal{M} \]

- **Total contribution:**

  \[-ig \left[ \int_0^{\infty} - \int_{y^-}^{\infty} \right] dy_1^- n \cdot A(y_1^-) \mathcal{M}_{\text{LO}} \]

O(g)-term of the gauge link!
NLO partonic diagram to structure functions:

\[ \propto \int_0^{\frac{-Q^2}{\mu^2}} \frac{dk_1^2}{k_1^2} \]

Dominated by

\[ \left\{ \begin{array}{l} k_1^2 \sim 0 \\ t_{AB} \to \infty \end{array} \right. \]

Diagram has both long- and short-distance physics

Factorization, separation of short- from long-distance:

\[ \int \frac{dk_1^2}{k_1^2} \]

=  

\[ \int \frac{dk_1^2}{k_1^2} \]

+  

\[ \int \frac{dk_1^2}{k_1^2} \]

\[ C^{(0)} \otimes \varphi^{(1)} \]

LO + evolution

\[ C^{(1)} \otimes \varphi^{(0)} \]

NLO

\[ k_1^2 \approx 0 \]
QCD leading power factorization

- QCD corrections: pinch singularities in $\int d^4 k_i$

- Logarithmic contributions into parton distributions:

- Factorization scale: $\mu_F^2$

To separate the collinear from non-collinear contribution

Recall: renormalization scale to separate local from non-local contribution
Time evolution:

- Long-lived parton state

Unitarity – summing over all hard jets:

\[ \sigma_{\text{tot}}^{\text{DIS}} \propto \Im \left( \left. \right|_{t \sim R} \left( t \sim \frac{1}{Q} \right) \right) \]

Interaction between the “past” and “now” are suppressed!
How to calculate the perturbative parts?

Use DIS structure function $F_2$ as an example:

$$F_{2h}(x_B, Q^2) = \sum_{q,f} C_{q/f} \left( \frac{x_B}{x}, \frac{Q^2}{\mu^2}, \alpha_s \right) \otimes \varphi_{f/h}(x, \mu^2) + O\left( \frac{\Lambda_{QCD}^2}{Q^2} \right)$$

Apply the factorized formula to parton states: $h \rightarrow q$

$$F_{2q}(x_B, Q^2) = \sum_{q,f} C_{q/f} \left( \frac{x_B}{x}, \frac{Q^2}{\mu^2}, \alpha_s \right) \otimes \varphi_{f/q}(x, \mu^2)$$

Express both SFs and PDFs in terms of powers of $\alpha_s$:

0th order:

$$F_{2q}^{(0)}(x_B, Q^2) = C_q^{(0)}(x_B / x, Q^2 / \mu^2) \otimes \varphi_{q/q}^{(0)}(x, \mu^2)$$

$$C_q^{(0)}(x) = F_{2q}^{(0)}(x)$$

$$\varphi_{q/q}^{(0)}(x) = \delta_{qq} \delta(1-x)$$

1st order:

$$F_{2q}^{(1)}(x_B, Q^2) = C_q^{(1)}(x_B / x, Q^2 / \mu^2) \otimes \varphi_{q/q}^{(0)}(x, \mu^2)$$

$$+ C_q^{(0)}(x_B / x, Q^2 / \mu^2) \otimes \varphi_{q/q}^{(1)}(x, \mu^2)$$

$$C_q^{(1)}(x, Q^2 / \mu^2) = F_{2q}^{(1)}(x, Q^2) - F_{2q}^{(0)}(x, Q^2) \otimes \varphi_{q/q}^{(1)}(x, \mu^2)$$
PDFs of a parton

- Change the state without changing the operator:

\[ \phi_{q/h}(x, \mu^2) = \int \frac{dy^-}{2\pi} e^{i p^+ y^-} \langle h(p)|\bar{\psi}_q(0) \frac{\gamma^+}{2} U^n_{[0,y^-]} \psi_2(y^-)|h(p)\rangle \]

| \( h(p) \rangle \Rightarrow |\text{parton}(p)\rangle | \phi_{f/q}(x, \mu^2) \] – given by Feynman diagrams

- Lowest order quark distribution:

  - From the operator definition:

\[
\phi^{(0)}_{q'/q}(x) = \delta_{qq'} \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \left( \frac{1}{2} \gamma \cdot p \right) \left( \frac{\gamma^+}{2p^+} \right) \right] \delta \left( x - \frac{k^+}{p^+} \right) (2\pi)^4 \delta^4(p-k)
\]

- Leading order in \( \alpha_s \) quark distribution:

  - Expand to \( (g_s)^2 \) – logarithmic divergent:

\[
\phi^{(1)}_{q/q}(x) = C_F \frac{\alpha_s}{2\pi} \int \frac{dk_T^2}{k_T^2} \left[ \frac{1 + x^2}{(1-x)^+} + \frac{3}{2} \delta(1-x) \right] + \text{UVCT}
\]

UV and CO divergence
Partonic cross sections

Projection operators for SFs:

\[
W_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}\right)F_1(x, Q^2) + \frac{1}{p \cdot q}\left(p_{\mu} - q_{\mu} \frac{p \cdot q}{q^2}\right)\left(p_{\nu} - q_{\nu} \frac{p \cdot q}{q^2}\right)F_2(x, Q^2)
\]

\[
F_1(x, Q^2) = \frac{1}{2} \left(-g_{\mu\nu} + \frac{4x^2}{Q^2} p^\mu p^\nu\right)W_{\mu\nu}(x, Q^2)
\]

\[
F_2(x, Q^2) = x \left(-g_{\mu\nu} + \frac{12x^2}{Q^2} p^\mu p^\nu\right)W_{\mu\nu}(x, Q^2)
\]

0th order:

\[
F^{(0)}_{2q}(x) = xg^{\mu\nu}W^{(0)}_{\mu\nu,q} = xg^{\mu\nu} \left(\frac{1}{4\pi} \right)
\]

\[
= \left(xg^{\mu\nu}\right) \frac{e_q^2}{4\pi} \mathrm{Tr} \left[\frac{1}{2} \gamma \cdot p \gamma \mu \gamma \cdot (p + q) \gamma \nu\right] 2\pi \delta \left( (p + q)^2 \right)
\]

\[
= e_q^2 x \delta(1 - x)
\]

\[
C^{(0)}_q(x) = e_q^2 x \delta(1 - x)
\]
NLO coefficient function – complete example

\[ C_q^{(1)}(x, Q^2 / \mu^2) = F_{2q}^{(1)}(x, Q^2) - F_{2q}^{(0)}(x, Q^2) \otimes \varphi_{q/q}^{(1)}(x, \mu^2) \]

- **Projection operators in n-dimension:**

\[
(1 - \varepsilon) F_2 = x \left( -g^{\mu\nu} + (3 - 2\varepsilon) \frac{4x^2}{Q^2} p^\mu p^\nu \right) W_{\mu\nu}
\]

- **Feynman diagrams:**

\[
W_{\mu\nu,q}^{(1)} + \text{Virtual} \quad \text{Real}
\]

- **Calculation:**

\[-g^{\mu\nu} W_{\mu\nu,q}^{(1)} \quad \text{and} \quad p^\mu p^\nu W_{\mu\nu,q}^{(1)}\]
Contribution from the trace of $W_{\mu \nu}$

- **Lowest order in n-dimension:**

  \[-g^{\mu \nu} W_{\mu \nu, q}^{(0)} = e_q^2 (1 - \varepsilon) \delta (1 - x)\]

- **NLO virtual contribution:**

  \[-g^{\mu \nu} W_{\mu \nu, q}^{(1)V} = e_q^2 (1 - \varepsilon) \delta (1 - x)\]

  \[\star \left( -\frac{\alpha_s}{\pi} \right) C_F \left[ \frac{4\pi \mu^2}{Q^2} \right]^{\varepsilon} \frac{\Gamma(1 + \varepsilon) \Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2\varepsilon)} \left[ \frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} + 4 \right]\]

- **NLO real contribution:**

  \[-g^{\mu \nu} W_{\mu \nu, q}^{(1)R} = e_q^2 (1 - \varepsilon) C_F \left( -\frac{\alpha_s}{2\pi} \right) \left[ \frac{4\pi \mu^2}{Q^2} \right]^{\varepsilon} \frac{\Gamma(1 + \varepsilon)}{\Gamma(1 - 2\varepsilon)} \]

  \[\star \left\{-\frac{1 - \varepsilon}{\varepsilon} \left[ 1 - x + \left( \frac{2x}{1 - x} \right) \left( \frac{1}{1 - 2\varepsilon} \right) \right] + \frac{1 - \varepsilon}{2(1 - 2\varepsilon)(1 - x)} + \frac{2\varepsilon}{1 - 2\varepsilon} \right\}\]
The “+” distribution:

\[
\left(\frac{1}{1-x}\right)^{1+\varepsilon} = -\frac{1}{\varepsilon} \delta(1-x) + \frac{1}{(1-x)^1_+} + \varepsilon \left(\frac{1 n(1-x)}{1-x}\right)^1_+ + O(\varepsilon^2)
\]

\[
\int dx \frac{f(x)}{(1-x)^1_+} \equiv \int dx \frac{f(x) - f(1)}{1-x} + n(1-z) f(1)
\]

One loop contribution to the trace of \( W_{\mu \nu} \):

\[
-g^{\mu \nu} W^{(1)}_{\mu \nu, q} = \epsilon^2 e_q (1-\varepsilon) \left( \frac{\alpha_s}{2\pi} \right) \left\{ -\frac{1}{\varepsilon} P_{qq} (x) + P_{qq} (x) n^{\frac{Q^2}{\mu^2 (4\pi e^{-\gamma_E})}} \right. \\
+ C_F \left[ (1+x^2) \left( \frac{1 n(1-x)}{1-x}\right)^1_+ - 3 \left( \frac{1}{1-x}\right)^1_+ - \frac{1+x^2}{1-x} n(x) \\
+ 3 - x - \left( \frac{9}{2} + \frac{\pi^2}{3} \right) \delta(1-x) \right] \right\}
\]

Splitting function:

\[
P_{qq} (x) = C_F \left[ \frac{1+x^2}{(1-x)^1_+} + \frac{3}{2} \delta(1-x) \right]
\]
One loop contribution to $p^\mu p^\nu W_{\mu \nu}$:

$$p^\mu p^\nu W^{(1)V}_{\mu \nu, q} = 0$$  
$$p^\mu p^\nu W^{(1)R}_{\mu \nu, q} = e_q^2 C_F \frac{\alpha_s Q^2}{2\pi 4x}$$

One loop contribution to $F_2$ of a quark:

$$F^{(1)}_{2q}(x, Q^2) = e_q^2 x \frac{\alpha_s}{2\pi} \left\{ \left( -\frac{1}{\epsilon} \right)_{\text{CO}} P_{qq}(x) \left( 1 + \epsilon 1 n(4\pi e^{-\gamma_E}) \right) + P_{qq}(x) 1 n \left( \frac{Q^2}{\mu^2} \right) \right\}$$
$$+ C_F \left[ \left( 1 + x^2 \right) \left( \frac{1 n(1-x)}{1-x} \right)_{+} - \frac{3}{2} \left( \frac{1}{1-x} \right)_{+} - \frac{1+x^2}{1-x} 1 n(x) + 3 + 2x - \left( \frac{9}{2} + \frac{\pi^2}{3} \right) \delta(1-x) \right] \right\}$$

$$\Rightarrow \infty \quad \text{as} \quad \epsilon \rightarrow 0$$

One loop contribution to quark PDF of a quark:

$$q_{q/q}^{(1)}(x, \mu^2) = \left( \frac{\alpha_s}{2\pi} \right) P_{qq}(x) \left\{ \left( \frac{1}{\epsilon} \right)_{\text{UV}} + \left( -\frac{1}{\epsilon} \right)_{\text{CO}} \right\} + \text{UV-CT}$$

- in the dimensional regularization

Different UV-CT = different factorization scheme!
- **Common UV-CT terms:**
  - **MS scheme:**  \[ \text{UV-CT}_{\text{MS}} = -\frac{\alpha_s}{2\pi} P_{qq}(x) \left( \frac{1}{\varepsilon} \right)_{\text{UV}} \]
  - **MS scheme:**  \[ \text{UV-CT}_{\text{MS}} = -\frac{\alpha_s}{2\pi} P_{qq}(x) \left( \frac{1}{\varepsilon} \right)_{\text{UV}} \left( 1 + \varepsilon \ln(4\pi e^{-\gamma_E}) \right) \]
  - **DIS scheme:**  choose a UV-CT, such that  \[ C^{(1)}_q(x, Q^2/\mu^2)_{\text{DIS}} = 0 \]

- **One loop coefficient function:**
  \[
  C^{(1)}_q(x, Q^2/\mu^2) = F^{(1)}_{2q}(x, Q^2) - F^{(0)}_{2q}(x, Q^2) \otimes \varphi^{(1)}_{q/q}(x, \mu^2)
  \]

\[
C^{(1)}_q(x, Q^2/\mu^2) = e_q^2 x \frac{\alpha_s}{2\pi} \left\{ P_{qq}(x) \ln \left( \frac{Q^2}{\mu^2_{\text{MS}}} \right) \right\}
\]

\[
+ C_F \left[ (1 + x^2) \left( \frac{\ln(1-x)}{1-x} \right)_+ - \frac{3}{2} \left( \frac{1}{1-x} \right)_+ - \frac{1+x^2}{1-x} \ln(x) + 3 + 2x - \left( \frac{9}{2} + \frac{\pi^2}{3} \right) \delta(1-x) \right]
\]
Physical cross sections should not depend on the factorization scale

\[ \mu_F^2 \frac{d}{d \mu_F^2} F_2(x_B, Q^2) = 0 \]

\[ F_2(x_B, Q^2) = \sum_f C_f(x_B/x, Q^2/\mu_F^2, \alpha_s) \phi_f(x, \mu_F^2) \]

\[ \text{Evolution (differential-integral) equation for PDFs} \]

\[ \sum_f \left[ \mu_F^2 \frac{d}{d \mu_F^2} C_f \left( \frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s \right) \right] \otimes \varphi_f(x, \mu_F^2) + \sum_f C_f \left( \frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s \right) \otimes \mu_F^2 \frac{d}{d \mu_F^2} \varphi_f(x, \mu_F^2) = 0 \]

PDFs and coefficient functions share the same logarithms

PDFs: \[ \log \left( \frac{\mu_F^2}{\mu_0^2} \right) \text{ or } \log \left( \frac{\mu_F^2}{\Lambda_{QCD}^2} \right) \]

Coefficient functions: \[ \log \left( \frac{Q^2}{\mu_F^2} \right) \text{ or } \log \left( \frac{Q^2}{\mu^2} \right) \]

\[ \text{DGLAP evolution equation:} \]

\[ \mu_F^2 \frac{\partial}{\partial \mu_F^2} \varphi_i(x, \mu_F^2) = \sum_j P_{i/j} \left( \frac{x}{x'}, \alpha_s \right) \otimes \varphi_j(x', \mu_F^2) \]
Calculation of evolution kernels

- Evolution kernels are process independent
  - Parton distribution functions are universal
  - Could be derived in many different ways

- Extract from calculating parton PDFs’ scale dependence

\[ Q^2 \frac{d}{dQ^2} q_i(x, Q^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dx_1}{x_1} q_i(x_1, Q^2) \gamma_{qq} \left( \frac{x}{x_1} \right) - \frac{\alpha_s}{2\pi} q_i(x, Q^2) \int_0^1 dz \gamma_{qq}(z) \]

  - Change
  - “Gain”
  - “Loss”

- Same is true for gluon evolution, and mixing flavor terms

- One can also extract the kernels from the CO divergence of partonic cross sections

Collins, Qiu, 1989
From one hadron to two hadrons

- **One hadron:**
  - $\sigma_{\text{DIS \, tot}} \sim$ 
  - Hard-part Probe

- **Two hadrons:**
  - $\sigma_{\text{DY \, tot}} \sim$
  - Predictive power: Universal Parton Distributions
Drell-Yan process – two hadrons

- **Drell-Yan mechanism:**

\[
A(P_A) + B(P_B) \rightarrow \gamma^*(q)[\rightarrow \bar{u}(q)] + X \quad \text{with} \quad q^2 \equiv Q^2 \gg \Lambda^2_{QCD} \sim 1/\text{fm}^2
\]

Lepton pair – from decay of a virtual photon, or in general, a massive boson, e.g., W, Z, H\(^0\), ... (called Drell-Yan like processes)

- **Original Drell-Yan formula:**

\[
\frac{d\sigma_{A+B\rightarrow \bar{u}+X}}{dQ^2dy} = \frac{4\pi\alpha^2_{em}}{3Q^4} \sum_{p,\bar{p}} x_A\Phi_{p/A}(x_A) x_B\Phi_{\bar{p}/B}(x_B)
\]

No color yet!

Rapidity:

\[
y = \frac{1}{2} \ln(x_A/x_B)
\]

\[
x_A = \frac{Q}{\sqrt{S}} e^y \quad x_B = \frac{Q}{\sqrt{S}} e^{-y}
\]

Right shape – But – not normalization
Drell-Yan process in QCD – factorization

- **Beyond the lowest order:**
  - Soft-gluon interaction takes place all the time
  - Long-range gluon interaction before the hard collision

- **Factorization – power suppression of soft gluon interaction:**

\[
\begin{align*}
A_{-}(x) &= \frac{e}{|x|} \\
A'_{-}(x') &= \frac{e\gamma(1+\beta)}{(x'^2_T + \gamma^2\Delta^2)^{1/2}} \\
&\implies 1 \quad \text{“not contracted!”} \\
E_3(x) &= \frac{e}{|x|^2} \\
E_3(x') &= \frac{-e\gamma\Delta}{(x'^2_T + \gamma^2\Delta^2)^{3/2}} \\
&\implies \frac{1}{\gamma^2} \quad \text{“strongly contracted!”}
\end{align*}
\]
Factorization for more than two hadrons

Factorization for high $p_T$ single hadron:

$$
\frac{d\sigma_{AB \rightarrow C+X}(p_A, p_B, p)}{dydp_T^2} = \sum_{a,b,c} \phi_{A \rightarrow a}(x, \mu_F^2) \otimes \phi_{B \rightarrow b}(x', \mu_F^2)
$$

$$
\frac{d\hat{\sigma}}{dydp_T^2} \otimes d\phi_{ab \rightarrow c+X}(x, x', z, y, p_T^2\mu_F^2) \otimes D_{c \rightarrow c}(z, \mu_F^2)
$$

Fragmentation function:

$$
D_{c \rightarrow c}(z, \mu_F^2)
$$

Choice of the scales:

$$
\mu_{Fac}^2 \approx \mu_{ren}^2 \approx p_T^2
$$

To minimize the size of logs in the coefficient functions

Nayak, Qiu, Sterman, 2006

$\gamma, W/Z, \ell(s), \text{jet}(s)$

$B, D, \Upsilon, J/\psi, \pi, ...$

$+ O(1/p_T^2)$

$p_T \gg m \gtrsim \Lambda_{QCD}$
Global QCD analysis – Testing QCD

- Factorization for observables with identified hadrons:
  - One-hadron (DIS): \[ F_2(x_B, Q^2) = \sum_f C_f (x_B/x, \mu^2/Q^2) \otimes f(x, \mu^2) \]
  - Two-hadrons (DY, Jets, W/Z, ...):
    \[ \frac{d\sigma}{dy dp_T^2} = \sum_{f,f'} f(x) \otimes \frac{d\sigma_{ff'}}{dy dp_T^2} \otimes f'(x') \]

- DGLAP Evolution:
  \[ \frac{\partial f(x, \mu^2)}{\partial \ln \mu^2} = \sum_{f'} P_{ff'}(x/x') \otimes f'(x', \mu^2) \]

- Input for QCD Global analysis/fitting:
  - World data with “Q” > 2 GeV
  - PDFs at an input scale: \[ \phi_{f/h}(x, \mu_0^2, \{\alpha_j\}) \]
PDFs from DIS

Q2-dependence is a prediction of pQCD calculation:

Physics interpretation of PDFs:

\[ f(x, Q^2) : \text{Probability density to find a parton of flavor } "f" \text{ carrying momentum fraction } "x", \text{ probed at a scale of } "Q^2" \]

\[ \star \text{ Number of partons: } \int_0^1 dx \, u_v(x, Q^2) = 2, \int_0^1 dx \, d_v(x, Q^2) = 1 \]

\[ \star \text{ Momentum fraction: } \langle x(Q^2) \rangle_f = \int_0^1 dx \, x \, f(x, Q^2) \quad \Rightarrow \quad \sum_f \langle x(Q^2) \rangle = 1 \]
Scaling and scaling violation

\( Q^2 \)-dependence is a prediction of pQCD calculation
Jet production from the LHC
Hard probes from high energy collisions

- **Lepton-lepton collisions:**
  - No hadron in the initial-state
  - Hadrons are emerged from energy
  - Not ideal for studying hadron structure

- **Hadron-hadron collisions:**
  - Hadron structure – motion of quarks, ...
  - Emergence of hadrons, ...
  - Initial hadrons broken – collision effect, ...

- **Lepton-hadron collisions:**
  - Hard collision without breaking the initial-state hadron – spatial imaging, ...
Why a lepton-hadron facility is special?

- Many complementary probes at one facility:

  - **Inclusive events**: $e+p/A \rightarrow e'+X$
    - Detect only the scattered lepton in the detector
    - (Modern Rutherford experiment!)

  - **Semi-Inclusive events**: $e+p/A \rightarrow e'+h(\pi,K,p,jet)+X$
    - Detect the scattered lepton in coincidence with identified hadrons/jets
    - (Initial hadron is broken – confined motion! – cleaner than h-h collisions)

  - **Exclusive events**: $e+p/A \rightarrow e'+p'/A'+h(\pi,K,p,jet)$
    - Detect every things including scattered proton/nucleus (or its fragments)
    - (Initial hadron is NOT broken – tomography! – almost impossible for h-h collisions)

- $Q^2 \rightarrow$ Measure of resolution
- $y \rightarrow$ Measure of inelasticity
- $x \rightarrow$ Measure of momentum fraction of the struck quark in a proton

$$Q^2 = S \times y$$
The Electron-Ion Collider (EIC) – the Future!

- A sharpest “CT” – “imagine” quark/gluon structure without breaking the hadron
  - “cat-scan” the nucleon and nuclei with a better than 1/10 fm resolution
  - “see” proton “radius” of quark/gluon density comparing with the radius of EM charge density

To discover color confining radius, hints on confining mechanism!

- A giant “Microscope” – “see” quarks and gluons by breaking the hadron

To discover/study color entanglement of the non-linear dynamics of the glue!
Backup slides
Drell-Yan process in QCD – factorization

Factorization – approximation:

- Suppression of quantum interference between short-distance \((1/Q)\) and long-distance \((\text{fm} \sim 1/\Lambda_{\text{QCD}})\) physics

Need “long-lived” active parton states linking the two

\[
\int d^4p_a \frac{1}{p_a^2 + i\varepsilon} \frac{1}{p_a^2 - i\varepsilon} \to \infty
\]

Perturbatively pinched at \(p_a^2 = 0\)

Active parton is effectively on-shell for the hard collision

- Maintain the universality of PDFs:
  Long-range soft gluon interaction has to be power suppressed

- Infrared safe of partonic parts:
  Cancelation of IR behavior
  Absorb all CO divergences into PDFs

Collins, Soper, Sterman, 1988
Drell-Yan process in QCD – factorization

- Leading singular integration regions (pinch surface):
  - **Hard:** all lines off-shell by $Q$
  - **Collinear:**
    - lines collinear to $A$ and $B$
    - One “physical parton” per hadron
  - **Soft:** all components are soft

- **Collinear gluons:**
  - Collinear gluons have the polarization vector: $\epsilon^{\mu} \sim k^{\mu}$
  - The sum of the effect can be represented by the eikonal lines,

*which are needed to make the PDFs gauge invariant!*
Drell-Yan process in QCD – factorization

- Trouble with soft gluons:

- Soft gluon exchanged between a spectator quark of hadron B and the active quark of hadron A could rotate the quark’s color and keep it from annihilating with the antiquark of hadron B.

- The soft gluon approximations (with the eikonal lines) need $k^\pm$ not too small. But, $k^\pm$ could be trapped in “too small” region due to the pinch from spectator interaction:
  \[ k^\pm \sim M^2/Q \ll k_\perp \sim M \]

Need to show that soft-gluon interactions are power suppressed.
Drell-Yan process in QCD – factorization

- Most difficult part of factorization:

  - Sum over all final states to remove all poles in one-half plane
    - no more pinch poles
  - Deform the $k^\pm$ integration out of the trapped soft region
  - Eikonal approximation → soft gluons to eikonal lines
    - gauge links
  - Collinear factorization: Unitarity → soft factor = 1

All identified leading integration regions are factorizable!
Factorized Drell-Yan cross section

- **TMD factorization** ($q_\perp \ll Q$):
  \[
  \frac{d\sigma_{AB}}{d^4q} = \sigma_0 \int d^2k_a d^2k_b d^2k_s \delta^2(q_\perp - k_a - k_b - k_s) F_{a/A}(x_A, k_A) F_{b/B}(x_B, k_b) S(k_s) + O(q_\perp/Q) \\
  x_A = \frac{Q}{\sqrt{s}} e^y \quad x_B = \frac{Q}{\sqrt{s}} e^{-y}
  \]

  The soft factor, $S$, is universal, could be absorbed into the definition of TMD parton distribution.

- **Collinear factorization** ($q_\perp \sim Q$):
  \[
  \frac{d\sigma_{AB}}{d^4q} = \int dx_a f_{a/A}(x_a, \mu) \int dx_b f_{b/B}(x_b, \mu) \frac{d\hat{\sigma}_{ab}}{d^4q}(x_a, x_b, \alpha_s(\mu), \mu) + O(1/Q)
  \]

- **Spin dependence**:

  The factorization arguments are independent of the spin states of the colliding hadrons.

  same formula with polarized PDFs for $\gamma^*, W/Z, H^0$...
Global QCD analysis – Testing QCD

**Input DPFs at $Q_0$**

$$\varphi_{f/h}(x, \{a_j\})$$

**DGLAP**

**$\varphi_{f/h}(x)$ at $Q > Q_0$**

**QCD calculation**

**Minimize $\chi^2$**

**Vary $\{a_j\}$**

**Comparison with Data at various $x$ and $Q$**

Procedure: Iterate to find the best set of $\{a_j\}$ for the input DPFs
Uncertainties of PDFs

“non-singlet” sector

“singlet” sector
Partonic luminosities

\[ q - q\bar{q} \]

NNLO \( \Sigma_q(q\bar{q}) \) luminosity at LHC (\( \sqrt{s} = 8 \text{ TeV} \))

\[ g - g \]

NNLO \( gg \) luminosity at LHC (\( \sqrt{s} = 8 \text{ TeV} \))