Lecture 1: Chiral Perturbation Theory: the basics

- Effective (field) theories: basic principles
- Introduction to Chiral Perturbation Theory
  - Chiral symmetry of QCD
  - Effective chiral Lagrangian
  - Power counting
  - Example: pion scattering
- Summary

Lecture 2: Inclusion of nucleon(s)

Lecture 3: Chiral EFT & nuclear forces

Lecture 4: Applications
Effective (Field) Theories

An effective (field) theory is an approximate theory whose scope is to describe phenomena which occur at a chosen length (or energy) range.

Examples:

1) Multipole expansion for electric potentials

\[
V \propto \int \frac{\rho(\vec{r})}{d} d^3r \\
= \int \frac{\rho(\vec{r})}{\sqrt{R^2 + 2rR \cos \theta + r^2}} d^3r \\
= \sum_{n=0}^{\infty} \frac{1}{R^{n+1}} \int r^n P_n(\cos \theta) \rho(\vec{r}) d^3r \\
= q \frac{1}{R} + P \frac{1}{R^2} + Q \frac{1}{R^3} + \ldots \quad \text{The sum converges rapidly for } a \ll R
\]
2) **Photon-photon scattering at low energy**

Euler & Kockel '35; Heisenberg & Euler '36

At low energy, $E \ll m_e$, one cannot probe details of $\gamma e$ interactions

$\Rightarrow$ $e$’s can be integrated out.

This leads to an **effective Lagrangian** which includes all possible photon interactions consistent with Lorentz & gauge symmetry.

Building blocks:

$$F^{\mu\nu} \equiv \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}, \quad \tilde{F}^{\mu\nu} = -\frac{1}{2} e^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha^2}{m_e^4} \left[ a_1 (F_{\mu\nu} F^{\mu\nu})^2 + a_2 (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right] + \ldots$$

*Each $F^{\mu\nu}$ generates photon momentum $\Rightarrow$ Amplitude $\sim E_\gamma^4$*

For 1-loop analysis based on $\mathcal{L}_{\text{eff}}$ see J.Halter, PLB 316 (1993) 155.
“if one writes down the most general possible Lagrangian, including all terms consistent with the assumed symmetry principles, and then calculates $S$-matrix elements with this Lagrangian to any order in perturbation theory, the result will simply be the most general possible $S$-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles”

S. Weinberg, Physica A96 (79) 327

- identify the symmetries of the underlying theory,
- construct the most general $\mathcal{L}_{\text{eff}}$ in terms of relevant d.o.f. and consistent with the symmetries,
- do standard quantum field theory with the effective Lagrangian.
Introduction to Chiral Perturbation Theory

What is chiral?

Lord Kelvin, 1904: “I call any geometrical figure, or group of points, chiral, and say that it has chirality, if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself.”

Chirality is an important concept in chemistry, biology, etc.

E.g. all biological polymers have definite chirality!

Some objects in real world occur with both chiralities...
Chiral symmetry of QCD

\[ \mathcal{L}_{\text{QCD}} = \bar{q}(i\not{\!D} - m)q - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \]

Left- and right-handed quark fields: \( q_{L,R} = \frac{1}{2} (1 \pm \gamma_5) q \).

Chiral group is a group of independent rotations of \( q_{L,R} \) in the flavor space.

For 2 flavors: \( G = SU(2)_L \times SU(2)_R \) and

\[
\begin{align*}
q_L &\xrightarrow{G} q'_L = g_L q_L \quad \text{with} \quad g_{L,R} \in SU(2)_{L,R} \\
q_R &\xrightarrow{G} q'_R = g_R q_R
\end{align*}
\]

Chiral \( SU(2) \) Lie algebra:

\[
\begin{align*}
[\Gamma^L_i, \Gamma^L_j] &= i \epsilon_{ijk} \Gamma^L_k \\
[\Gamma^R_i, \Gamma^R_j] &= i \epsilon_{ijk} \Gamma^R_k \\
[\Gamma^L_i, \Gamma^R_j] &= 0
\end{align*}
\]

Or, equivalently:

\[
\begin{align*}
[V_i, V_j] &= i \epsilon_{ijk} V_k \\
[A_i, A_j] &= i \epsilon_{ijk} V_k \\
[V_i, A_j] &= i \epsilon_{ijk} A_k
\end{align*}
\]

where

\[
\begin{align*}
V_i &= \Gamma_i^R + \Gamma_i^L \\
A_i &= \Gamma_i^R - \Gamma_i^L
\end{align*}
\]

Vector (isospin) generators

Axial generators
Transformation properties of $\mathcal{L}_\text{QCD}$:

$$\bar{q}(i\not{D} - m)q = \bar{q}_L i\not{D} q_L + \bar{q}_R i\not{D} q_R - m(\bar{q}_L q_R + \bar{q}_R q_L)$$

$\bar{q}_L$ and $\bar{q}_R$ are chiral invariant, but $\bar{q}_L q_R + \bar{q}_R q_L$ is not chiral invariant.

$m_{u,d} \sim 5$ MeV $\ll M_\rho \sim 770$ MeV $\implies$ $\mathcal{L}_\text{QCD}$ is approximately chiral invariant.

**Notice:** SU(2) chiral symmetry is an accurate symmetry of QCD, i.e. $M_\pi^2/M^2_\rho \sim 0.03$.

\[ H \text{ invariant under } G \]

\[ H \overset{G}{\rightarrow} H' = U^{-1}HU = H \]

**Wigner-Weyl mode**

\[ U^{-1}|0\rangle = |0\rangle \]

$\Rightarrow$ degenerate multiplets according to irred. representations of $G$

**Nambu-Goldstone mode**

\[ U^{-1}|0\rangle \neq |0\rangle \]

$\Rightarrow$ spontaneous symmetry breaking, massless Goldstone bosons
There is a strong evidence that chiral symmetry is spontaneously broken down to the isospin group:

- Only isospin but not chiral multiplets are observed in the particle spectrum (axial charges would lead to parity doublets)

- Triplet of unnaturally light pseudoscalar mesons (pions) — natural candidates for Goldstone bosons

- Scalar quark condensate:
  
  \[
  \langle 0|\bar{q}q|0 \rangle_{MS, 2 GeV} = -(262 \pm 12 \text{ MeV})^3 \quad \text{(Lattice, quenched, Hasenfratz et al. ‘02)}
  \]

- Further theoretical arguments given by:
  Vafa & Witten ‘84; ‘t Hooft ‘80; Coleman & Witten ‘80
Chiral perturbation theory

\[ e^{iZ[J]} = \int [Dq][D\bar{q}][DG] e^{\int i d^4x \mathcal{L}_{\text{QCD}}[q,\bar{q},G; J]} \Leftrightarrow \int [D\Phi] e^{\int i d^4x \mathcal{L}_{\text{eff}}[\Phi; J]}

(for details see Leutwyler, Ann. Phys. 235 (1994) 165)

Cannot derive \( \mathcal{L}_{\text{eff}} \Leftrightarrow \) write down the most general expression consistent with the chiral symmetry of QCD, i.e.:
- Include all possible \( \chi \)-invariant terms,
- Include all terms that break \( \chi \)-symmetry in the same way as \( \bar{q}mq \) in \( \mathcal{L}_{\text{QCD}} \) does.

Consider the pure Goldstone boson sector and neglect the term \( \bar{q}mq \).

How to write down most general \( \chi \)-invariant \( \mathcal{L}_{\text{eff}} \)?

How do \( \pi \)'s transform under \( G \)?
- Isospin subgroup \( H \in G \) realized linearly (\( \pi \)'s build an isospin triplet).
- Chiral group necessarily realized nonlinearly:
  \( \text{SU}(2)_L \times \text{SU}(2)_R \) is isomorphic to \( \text{SO}(4) \) \( \Leftrightarrow \) need at least 4 dimensions to construct a nontrivial linear realization.
Infinitesimal $SO(4)$ rotation of the 4-vector $(\pi_1, \pi_2, \pi_3, \sigma)$:

\[
\begin{pmatrix}
\vec{\pi}' \\
\sigma'
\end{pmatrix} = \begin{bmatrix}
1 + \vec{\theta}^V \cdot \vec{V} + \vec{\theta}^A \cdot \vec{A}
\end{bmatrix} \begin{pmatrix}
\vec{\pi} \\
\sigma
\end{pmatrix}
\]

where: \(\vec{\theta}^V \cdot \vec{V} = \begin{pmatrix}
0 & -\theta_3^V & \theta_2^V & 0 \\
\theta_3^V & 0 & -\theta_1^V & 0 \\
-\theta_2^V & \theta_1^V & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}\) and \(\vec{\theta}^A \cdot \vec{A} = \begin{pmatrix}
0 & 0 & 0 & \theta_1^A \\
0 & 0 & 0 & \theta_2^A \\
-\theta_1^A & -\theta_2^A & -\theta_3^A & 0
\end{pmatrix}\)

One reads off: \(\vec{\pi}' = \vec{\pi} + \vec{\theta}^V \times \vec{\pi} + \vec{\theta}^A \sigma\)
\(\sigma' = \sigma - \vec{\theta}^A \cdot \vec{\pi}\)

Switch to the nonlinear realization of $SO(4)$: only 3 out of 4 components of the vector \((\vec{\pi}, \sigma)\) are independent, i.e. \(\vec{\pi}^2 + \sigma^2 = F^2\)

\[
\sigma = \sqrt{F^2 - \vec{\pi}^2} \quad \Rightarrow \quad \begin{cases}
\vec{\pi} \xrightarrow{\vec{\theta}^V} \vec{\pi}' = \vec{\pi} + \vec{\theta}^V \times \vec{\pi} & \text{linear under } \vec{\theta}^V \\
\vec{\pi} \xrightarrow{\vec{\theta}^A} \vec{\pi}' = \vec{\pi} + \vec{\theta}^A \sqrt{F^2 - \vec{\pi}^2} & \text{nonlinear under } \vec{\theta}^A
\end{cases}
\]

**Notice:** it is more convenient to use a 2 x 2 matrix notation:

\[
U = \frac{1}{F} \left( \sigma I + i\vec{\pi} \cdot \vec{\tau} \right) \xrightarrow{\text{nonlinear realization}} U = \frac{1}{F} \left( I\sqrt{1 - \vec{\pi}^2} + i\vec{\pi} \cdot \vec{\tau} \right)
\]

$SO(4)$ or chiral rotations:

\[
U \rightarrow U' = LUR^\dagger \quad \text{with} \quad L = e^{-i/2(\vec{\theta}^V - \vec{\theta}^A) \cdot \vec{\tau}} \quad \text{and} \quad R = e^{-i/2(\vec{\theta}^V + \vec{\theta}^A) \cdot \vec{\tau}}
\]
The above realization of $G$ is not unique. How does this non-uniqueness affect $S$-matrix?

- All realizations of $G$ are equivalent to each other by means of nonlinear field redefinitions $\bar{\pi} \rightarrow \bar{\pi}' = \bar{\pi} F[\bar{\pi}]$, $F[0] = 1$ (Coleman, Callan, Wess & Zumino '69)
- Such field redefinitions do not affect $S$-matrix (Haag '58)

Construction of the $\chi$- and Lorentz-invariant ($\iff$ no terms with odd # of $\partial_\mu$) $\mathcal{L}_{\text{eff}}$:

- 0 derivatives: $UU^\dagger = U^\dagger U = 1$ — plays no role
- 2 derivatives: $\text{Tr}(\partial_\mu U \partial^\mu U^\dagger) \overset{g \in G}{\rightarrow} \text{Tr}(L \partial_\mu UR^\dagger R \partial^\mu U^\dagger L^\dagger) = \text{Tr}(\partial_\mu U \partial^\mu U^\dagger)$
- 4 derivatives: $[\text{Tr}(\partial_\mu U \partial^\mu U^\dagger)]^2 \; ; \; \text{Tr}(\partial_\mu U \partial_\nu U^\dagger) \text{Tr}(\partial^\mu U \partial^\nu U^\dagger)$
  ...

**Notice:** terms with $\partial_\mu \partial_\nu U$, $\partial_\mu \partial_\nu \partial_\rho U$, $\partial_\mu \partial_\nu \partial_\rho \partial_\sigma U$ can be eliminated via e.o.m. & partial integration
\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_\pi^{(2)} + \mathcal{L}_\pi^{(4)} + \ldots \quad \text{where:} \quad \mathcal{L}_\pi^{(2)} = \frac{F^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) \\
\mathcal{L}_\pi^{(4)} = l_1 [\text{Tr}(\partial_\mu U \partial^\mu U^\dagger)]^2 + l_2 \text{Tr}(\partial_\mu U \partial_\nu U^\dagger) \text{Tr}(\partial^\mu U \partial^\nu U^\dagger) \quad \ldots \\
\]

**Notice:** only derivative couplings allowed (Goldstone bosons do not interact at \( F = 0 \))

What is the meaning of \( F \)?

Axial current from \( \mathcal{L}_{\text{eff}} \):

\[ J_{A\mu}^i = i \text{Tr}[\tau^i(U^\dagger \partial_\mu U - U \partial_\mu U^\dagger)] = -F \partial_\mu \pi^i + \ldots \]

\[ \langle 0 | J_{A\mu}^i | \pi^i(p) \rangle \equiv ip_\mu F_\pi \delta^{ij} \Rightarrow F = F_\pi = 92.4 \text{ MeV} \]

What are the consequences of chiral symmetry?

\[ U = \frac{1}{F} [\sqrt{F^2 - \pi^2} + i \vec{\pi} \cdot \vec{\pi}] = 1 + \frac{i \vec{\pi} \cdot \vec{\pi}}{F} - \frac{\pi^2}{2F^2} + \ldots \quad \Rightarrow \quad \mathcal{L}_\pi^{(2)} = \frac{1}{2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} + \frac{1}{2F^2} (\partial_\mu \vec{\pi} \cdot \vec{\pi})^2 + \ldots \]

Leading (i.e. \( \sim \mathcal{O}(F^2) \)) contributions to multi-pion scattering:

\[ \mathcal{L}_\pi^{(2)} \quad \text{no free parameters} \quad \Rightarrow \quad \text{remarkable predictive power!} \]
So far only tree graphs considered. **What about quantum corrections (loops)?**

\[ p = l + p_3 + p_4 \]

\[ \Rightarrow \int \frac{d^4l}{(2\pi)^4} \frac{p_1 \cdot p_2 \ p_3 \cdot p_4}{[l^2 + i\epsilon][((l + p_3 + p_4)^2 + i\epsilon]} \sim \mathcal{O}(E^4) \]

\[ \Rightarrow \text{suppressed compared to the L.O. contribution} \]

UV divergences removed e.g. using DR, \( \int d^4l \rightarrow \mu^{d-4} \int d^d l \), and redefining the LECs \( l_i \) from \( \mathcal{L}_\pi^{(4)} \) (need only local \( \chi \)-invariant counter terms of the order \( \sim E^4 \))

**General observation:** \( n \)-loop diagrams are suppressed by the factor \( E^{2n} \) compared to the tree ones \( \sim E^2 \).
**Power counting**  (Weinberg '79)

Consider S-matrix element:  
\[ S = \delta^4(p_1 + p_2 + \ldots + p_N) M \Pi \]

The amplitude can be rewritten as:  
\[ M \equiv M(E, \mu, g^r) = E^D f \left( \frac{E}{\mu}, g^r \right) \]

Dimensional analysis:
- pion propagators:  
  \[ \frac{1}{(p^2 - M^2_\pi)} \sim \frac{1}{Q^2} \]
- momentum integrations:  
  \[ d^4l \sim Q^4 \]
- delta functions:  
  \[ \delta^4(p - p') \sim \frac{1}{Q^4} \]
- derivatives:  
  \[ \partial_\mu \sim Q \]

For a Feynman graph with \( L \) loops and \( N_d \) vertices with \( d \) derivatives:

\[ D = 2 + 2L + \sum_d N_d(d - 2) \]

**Examples:**

\[ D = 2 + 0 + 2 = 4 \quad D = 2 + 2 + 0 = 4 \quad D = 2 + 4 + 0 = 6 \]
Example: \( \pi\pi \) scattering

\[ \mathcal{O}(E^2) \]

\[ \mathcal{O}(E^4) \]

\[ \mathcal{O}(E^6) \]

Amplitude is obtained via expansion in \( \frac{E}{\Lambda_\chi} \). What is the value of \( \Lambda_\chi \)?

- Chiral expansion breaks down for \( E \sim M_\rho \implies \Lambda_\chi \sim M_\rho = 770 \text{ MeV} \)
- Consistency arguments yield: \( \Lambda_\chi \leq 4\pi F_\pi = 1.2 \text{ GeV} \) (Manohar & Georgi '84)
So far assumed exact $\chi$–symmetry. How to account for explicit $\chi$–symmetry breaking?

$$\delta \mathcal{L}_{QCD} = -\bar{q}m = -m_q \bar{q}(1 + \epsilon \tau_3)q \quad \text{where} \quad m_q = \frac{1}{2}(m_u + m_d) \quad \text{and} \quad \epsilon = \frac{m_u - m_d}{m_u + m_d}$$

4th comp. of SO(4)-vector 3rd comp. of SO(4)-vector

$$\Rightarrow \quad \text{include in $\mathcal{L}_{eff}$ all possible 3rd and 4th components of SO(4) tensors of rank $n$ multiplied by $(\epsilon m_q)^n$ and $m_q^n$ (assuming $m_q$ can be treated perturbatively).}$$

More convenient: method of external sources

$$\delta \mathcal{L}_{QCD} = -\bar{q}Mq \bigg|_{M=m}$$

The term $-\bar{q}Mq = -\bar{q}_L M_{qR} - \bar{q}_R M_{qL}$ is $\chi$–invariant if: $\mathcal{M} \xrightarrow{G} \mathcal{M}' = g_R M g_L^{-1} = g_L M g_R^{-1}$

$$\Rightarrow \quad \text{write down all possible $\chi$–invariant terms with $\mathcal{M}$ and then set $\mathcal{M} = m$}$$

The leading (i.e. no $\partial_\mu$ and $\propto \mathcal{M}$) SB term in $\mathcal{L}_{eff}$:

$$\mathcal{L}_{SB} = \frac{BF^2}{2} \text{Tr}[(U + U^\dagger)\mathcal{M}] \bigg|_{\mathcal{M}=m} = 2BF^2 m_q - B m_q \bar{\pi} \pi + O(\bar{\pi}^4) \quad \Rightarrow \quad M_\pi^2 = 2m_q B + O(m_q^2)$$

LEC $B$ is related to the scalar quark condensate via $\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = -BF^2 + O(\mathcal{M})$

Notice: the generalized scenario (Stern et al. ’91) in which $2m_q B \ll M_\pi^2$ is ruled out by recent data on $\pi\pi$ scatt. length.
CHPT = simultaneous expansion in energy and about the $\chi$-limit keeping $E \sim Q \sim M_{\pi}$

- Write down the most general $\mathcal{L}_{\text{eff}}$ consistent with the $\chi$-symmetry of QCD,
- Calculate S-matrix in perturbation theory (based on chiral expansion),
- Fix the unknown LECs from some data and make predictions.

Predictive power? \[ \mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)}_{\pi} + \mathcal{L}^{(4)}_{\pi} + \mathcal{L}^{(6)}_{\pi} + \ldots \]

\[ \begin{array}{ccc} 2 & 7 & 53 \end{array} \]

low-energy constants

Example: S-wave $\pi\pi$ scattering lengths $a_0^{I=0}$ and $a_0^{I=2}$

LO: $a_0^0 = 0.16$ (Weinberg '66)

NLO: $a_0^0 = 0.20$ (GL '83)

NNLO: $a_0^0 = 0.217$ (Bijnens et al. '95)

NNLO + dispersion relations:

$\alpha_0^0 = 0.220 \pm 0.005$

(Colangelo et al. '01)

(from Caprini et al. hep-ph/0509266)
Summary

- Effective (field) theory is a powerful method to study phenomena which occur in a certain energy range.
- The principles of Chiral Perturbation Theory, the EFT of the Standard Model, have been introduced.
- How to include nucleons ???

Some ideas will be discussed in lecture 2 …