In order to prove that, we first show that $f(-x) = -\bar{f}(x)$

⇒ Although $f(x)$ does not vanish for $x<0$, it is determined by $f(x)$ for $x>0$.

We first go back to the definition Eq.(2.40) for $f(x)$, and Eq.(2.41) for $\bar{f}(x)$

$$f(x) = \int dz^- e^{-ixp^+z^-} \langle p| \psi_+^\dagger(z^-)\, \psi(0) |p\rangle$$

$$\bar{f}(x) = \int dz^- e^{ixp^+z^-} \langle p| \psi_+(0)\, \psi_+^\dagger(z^-) |p\rangle$$

$$= -\int dz^- e^{ixp^+z^-} \langle p| \psi_+^\dagger(z^-)\, \psi_+(0) |p\rangle$$

$$= -f(-x) \quad \ldots$$

Now, we perform the x integral over the all possible x range:

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$$= -f(-x) \quad \ldots$$

Now, we perform the x integral over the all possible x range:

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We define the parton distribution through Eq. (2.40):

$$f(x) = \sqrt{2} \sum_n \delta\left(p^+(1-x) - p_n^+\right) \left| \langle n | \psi_+(0) | p \rangle \right|^2 , \quad (2.42)$$

while Eq. (2.41) defines the q_n tiquark distribution:

$$\bar{f}(x) = -H(-x) =$$

$$= \sqrt{2} \sum_n \delta\left(p^+(1-x) - p_n^+\right) \left| \langle n | \psi_+^\dagger(0) | p \rangle \right|^2 . \quad (2.43)$$

It is now easy to verify that

$$\int_\alpha^{+\infty} dx\, H(x) = \text{number of quarks} - \text{number of antiquarks} = \text{valence quarks}$$

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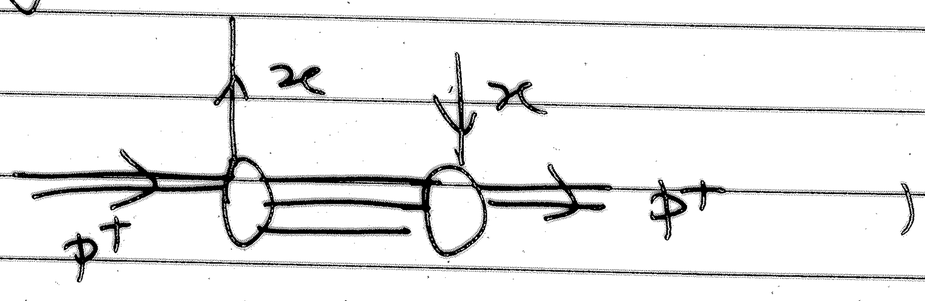
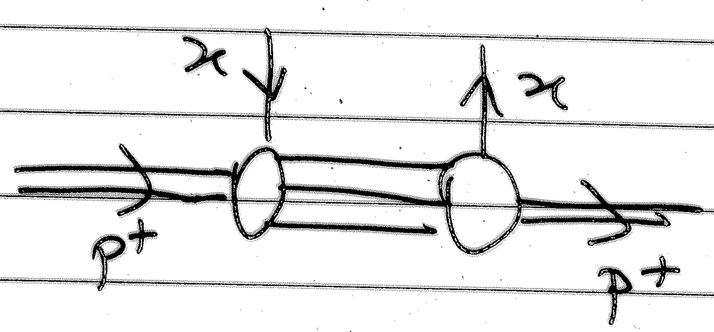
Hence, the two representations of $H(x)$ say that

$$-1 \leq x \leq 1$$

Moreover, for $0 \leq x \leq 1$, $H(x)$ is given by:



while for $-1 \leq x \leq 0$, $H(x)$ is given by:



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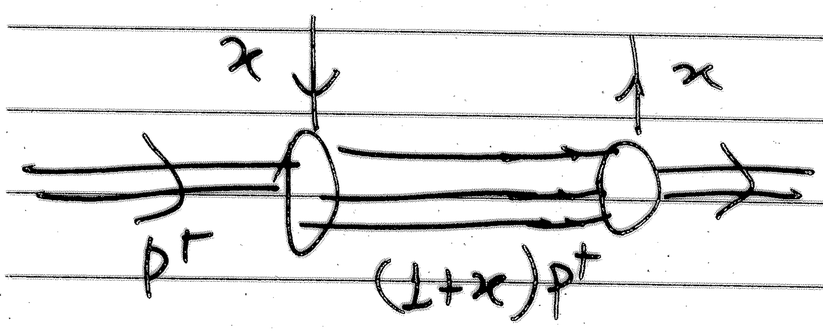
However, the negative representation of Eq. (2.38) gives:

$$H_1(x) = -\sqrt{2}\,p^+ \sum_n \delta\left(p^+ + x p^- - p_n^+\right) \left|\langle n | \psi_+^+(0) | p\rangle\right|^2 \tag{2.41}$$

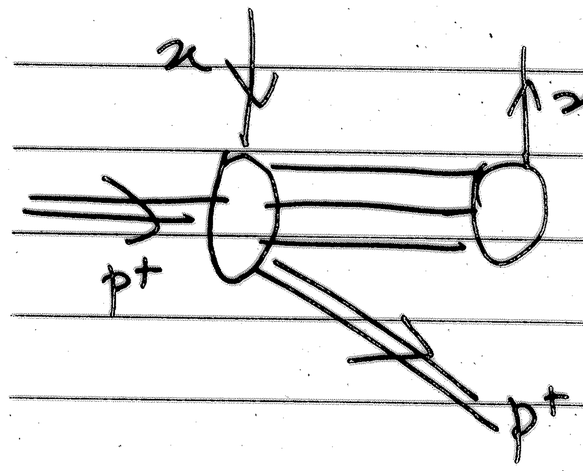
We now have:

(a)



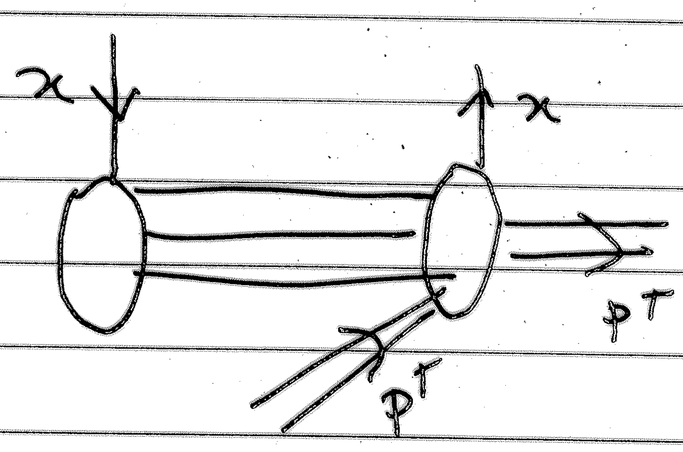
$x \geq -1$!

(b)



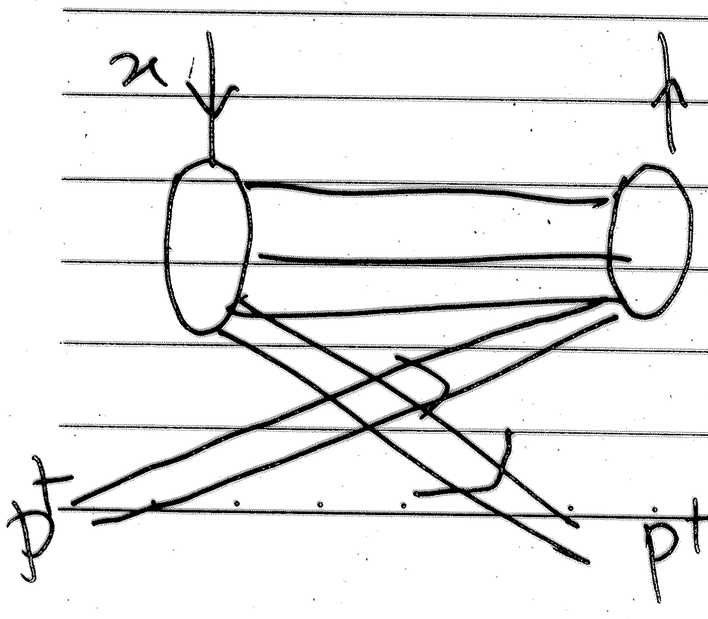
$x \geq 0$

(c)



$x \geq 0$

(d)



$x \geq 1$

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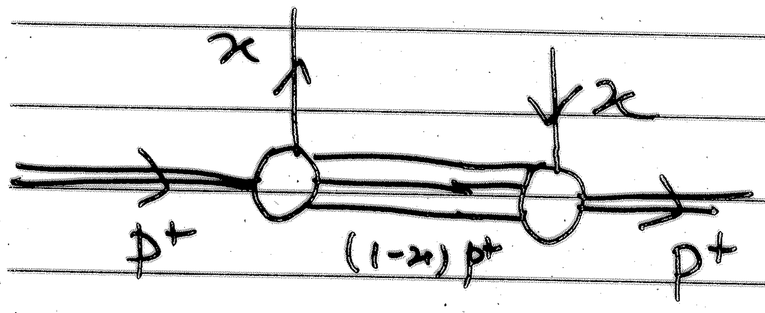
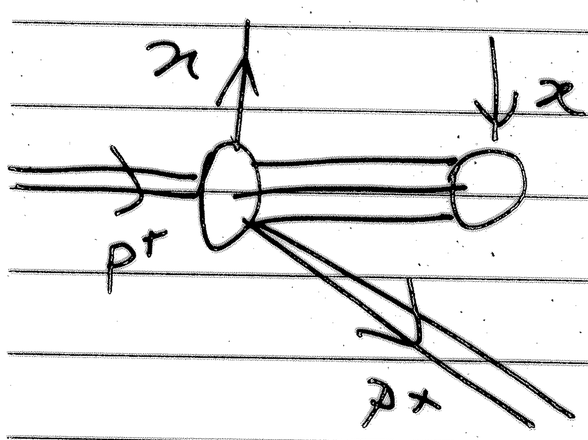
(a)



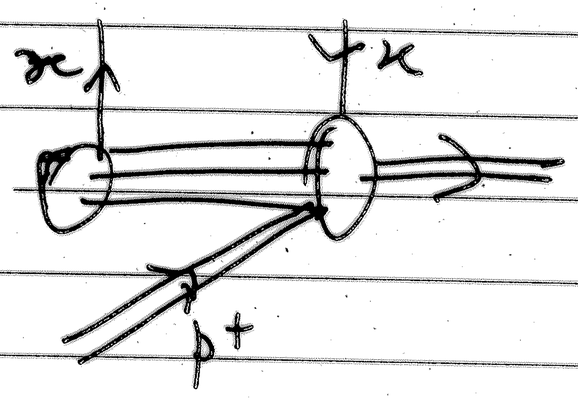
if $p_n^+ \geq 0 \Rightarrow x \leq 1$!

(b)



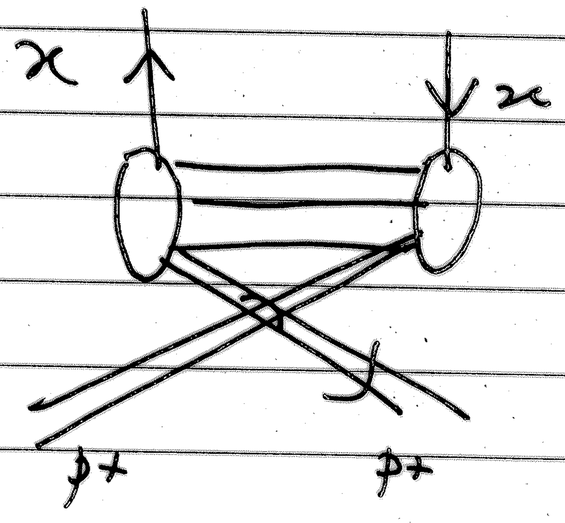
$\Rightarrow x \leq 0$

(c)



$\Rightarrow x \leq 0$

(d)



$\Rightarrow x \leq -1$

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So,

$$H_1(x) = \frac{\sqrt{2}\,p^+}{2\pi}\int_{-\infty}^{+\infty} dz^-\, e^{-ixp^+z^-}\,\langle p|\psi_+^\dagger(z^-)\,\psi_+(0)|p\rangle$$

$$= \frac{\sqrt{2}\,p^+}{2\pi}\int_{-\infty}^{+\infty} dz^-\,\sum_n\, e^{-ixp^+z^-}\cdot \langle p|e^{i\hat{P}_+z^-}\psi_+(0)e^{-i\hat{P}_+z^-}|n\rangle\langle n|\psi_+(0)|p\rangle$$

$$= \frac{\sqrt{2}\,p^+}{2\pi}\sum_n \int_{-\infty}^{+\infty} e^{-i\left(p^+x - p^+ + p_n^+\right)z^-}\cdot \langle p|\psi_+^\dagger(0)|n\rangle\langle n|\psi_+(0)|p\rangle$$

$$\boxed{H_1(x) = \sqrt{2}\,p^+\sum_n \delta\left(p^+(1-x) - p_n^+\right)\left|\langle n|\psi_+(0)|p\rangle\right|^2} \qquad (2.40)$$

what does it mean?

Hence:

$$\langle p|\, T\left[\bar{\psi}(z^-)\,\gamma_+\,\psi_{10}\right]|p\rangle =$$

$$= \langle p|\,\bar{\psi}(z^-)\,\gamma_+\,\psi_{10}\,|p\rangle =$$

$$= -\langle p|\,\psi_{10}\,\bar{\psi}(z^-)\,\gamma_+\,|p\rangle \qquad (2.38)$$

let's take the positive part:

$$\bar{\psi}\gamma_+\psi = \psi^\dagger\gamma_0\,\frac{(\gamma_0+\gamma_3)}{\sqrt{2}}\,\psi =$$

$$= \frac{1}{\sqrt{2}}\,\psi^\dagger(1+\gamma_0\gamma_3)\,\psi = \sqrt{2}\,\psi^\dagger P_+\,\psi,$$

with $P_+ = \frac{1}{2}(1+\gamma_0\gamma_3)$, $P_+^2 = P_+ \rightarrow$ projector

$$\boxed{\bar{\psi}\gamma_+\psi = \sqrt{2}\;\psi_+^\dagger\,\psi_+} \qquad (2.39)$$

with $\psi_+ = P_+\psi$

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So, we have:

$$\int_{-\alpha}^{+\alpha} H_i(x)\, x^{n-1}\, dx = A_i^n \qquad (2.36)$$

$A_i^n = ?$ We can work out a simpler expression.

Let's look at Jaffe, Nucl. Phys. B229 (1983) 205.

then, he proves that:

$$\langle p | \{ \bar{\psi}(z^-)\gamma_+, \psi(0) \} | p \rangle_c = 0 \qquad (2.37)$$

Hence, the $\top$ product in the definition of the H distribution is illusory!

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Hence:

$$A_n^i = \frac{1}{(p^+)^n} \langle P | \bar{\psi}(0) \gamma^+ (i\partial^+)^{n-1} \frac{\lambda^i}{2} \psi(0) | P \rangle \tag{2.34}$$

Let:

$$H_i(x) = \frac{p^+}{2\pi} \int_{-\infty}^{+\infty} dz^- e^{-ixp^+z^-} \langle p | T\left[\bar{\psi}(z^-) \gamma^+ \frac{\lambda_i}{2} \psi(0)\right] | p \rangle_c$$

Or:

$$\int_{-\infty}^{+\infty} H_i(x) x^{n-1} dx = \frac{p^+}{2\pi} \int_{-\infty}^{+\infty} dz^- \int_{-\infty}^{+\infty} dx \left[\left\{ \frac{1}{-ip^+} \frac{\partial}{\partial z^-} \right\}^{n-1} e^{-ixp^+z^-} \right] \langle p | \; | p \rangle$$

$$= \frac{p^+}{2\pi} \int_{-\infty}^{+\infty} \frac{dz^- dx}{(-ip^+)^{n-1}} e^{-ixp^+z^-} (-1)^{n-1} \left(\frac{\partial}{\partial z^-}\right)^{n-1} \langle p | \; | p \rangle$$

integration by parts is done $(n-1)$ times

It is assumed that $e^{-ixp^+z^-} \langle p | \; | p \rangle \to 0$ at the infinity

The integration in x gives a δ function:

$$\int_{-\infty}^{+\infty} H_i(x)\, x^{n-1}\, dx = \frac{p^+}{2\pi} \int_{-\infty}^{+\infty} \frac{dz^-}{(p^+)^{n-1}}\, 2\pi (-i)^{n-1}\, \delta(p^+ z^-)$$

$$\cdot \langle p| T\left[\frac{\partial^{n-1} \psi^+(z^-)}{\partial z^-}\, \gamma_+\, \frac{\lambda_i}{2}\, \psi(0) \right] |p\rangle$$

$$= \frac{1}{(p^+)^{n-1}} \int_{-\infty}^{+\infty} dy\, (-i)^{n-1}\, \delta(y)$$

$$\cdot \langle p| T\left[(p^+)^{n-1} \frac{\partial^{n-1}}{\partial y^{n-1}}\, \psi^+\left(\frac{y}{p^+}\right) \gamma_+\, \frac{\lambda_i}{2}\, \psi(0) \right] |p\rangle$$

$$= \frac{1}{(p^+)^{n-1}} \langle p| T\left[(-ip^+)^{n-1}\, \psi^+(0)\, \gamma_+\, \frac{\lambda_i}{2}\, \psi(0) \right] |p\rangle, \qquad (2.35)$$

where

$$\psi^+_{(0)} \equiv \frac{\partial^{n-1}}{\partial x^{n-1}}\, \psi^+\left(\frac{x}{p^+}\right)\Bigg|_{y=0}$$

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We now shaw that the A_n terms can be represented as parton distributions

As we saw:

$$\langle P | O^i_{\mu_1 \dots \mu_n} | P \rangle = A^i_n \; P_{\mu_1} \dots P_{\mu_n} \qquad (2.32)$$

let the Operator O be:

$$O^i_{\mu_1 \dots \mu_n} = \bar{\psi}(0) \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_n} \frac{\lambda^i}{2} \psi(0) \qquad (2.33)$$

Two important steps:

- We work on the light front coordinate system, with $\mu_1 \dots \mu_n = + \dots +$
- We work in the light cone gauge:

$$A_+ = 0$$

✓ gluon field

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We integrate Eq. (2.29) in w,

Use that $\frac{1}{2\pi i}\oint_C \frac{\tau_{\mu\nu}}{w^{n+1}}\,dw = 2\int_0^1 dx\, x^{n-1} W_{\mu\nu}$,

and use Eq. (2.30) for $W_{\mu\nu}$, to get:

$$\int_0^1 dx\, x^{n-2} F_L(x,Q^2) = \sum_j A_n^{j,L}\, C_n^{j,L}(Q^2), \quad n=0,2,4\ldots$$

$$\int_0^1 dx\, x^{n-2} F_2(x,Q^2) = \sum_j A_n^{j,2}\, C_n^{j,2}(Q^2), \quad n=0,2,4$$

$A_n^{j,2}$: nonperturbative; $C_n^{j,2}$: perturbative

(2.31)

Can we get the parton model from here?

Yes!

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As we saw before,

$$W_{\mu\nu} = \frac{1}{\pi} \text{Abs}\, T_{\mu\nu}$$

But:

$$\frac{1}{2\pi i} \oint_C \frac{T_{\mu\nu}}{\omega^n} = \frac{2}{\pi} \int_1^\infty \frac{d\omega}{\omega^n} \text{Abs}\, T_{\mu\nu} = 2 \int_0^1 dx\, x^{n-2} W_{\mu\nu}$$

↓

can be proven. Who is interested, look for me!

Also. $\frac{1}{2\pi i} \oint_C d\omega\, \omega^{m-n} = \delta_{m,n-1}$

Finally, using Eq. (1.15) with $M W_1 = F_1$, $\nu W_2 = F_2$, $F_L = F_2 - 2x F_1$:

$$\boxed{2x M W_{\mu\nu} = q_\mu q_\nu F_L(x) + d_{\mu\nu} F_2(x)} \quad (2.30)$$

Finally, we define:

$$C_L^{n,j}(-q^2) \equiv -\frac{(-q^2)^{n+1}}{2}\left(\frac{\partial}{\partial q^2}\right)^n \int d^4z\, C_L^{n,j}(z^2-i\varepsilon)\, e^{-iq\cdot z}$$

$$C_2^{n,j}(-q^2) \equiv -\frac{(-q^2)^{n+2}}{8}\left(\frac{\partial}{\partial q^2}\right)^n \int d^4z\, C_2^{n,j}(z^2-i\varepsilon)\, e^{-iq\cdot z}$$

Hence:

$$T_{\mu\nu} = 2\sum_{n,j}\left[q_\mu q_\nu A_n^{j,1} C_1^{n,j}(Q^2) + g_{\mu\nu} A_n^{j,2} C_2^{n,j}(Q^2)\right]\omega^n \qquad (2.29)$$

Beatiful result! But

how useful is it?

PS: n is even. Because $T_{\mu\nu}(\omega) = T_{\mu\nu}(-\omega)$

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We now include Eqs. (2.23) and (2.24) into Eq. (2.22):

$$T_{\mu\nu} = \sum_{n,j} i^2 \Big\{ (q_\mu q_\nu - q^2 g_{\mu\nu})(2P\cdot q)^n \cdot A_n^{j,1} \left(\frac{\partial}{\partial q^2}\right)^n \int d^4z\, C_1^{n,j}(z^2 - i\varepsilon) e^{iq\cdot z}$$

$$+ \left(P\cdot q\, P_\mu q_\nu + P\cdot q\, q_\mu P_\nu - q^2 P_\mu P_\nu - g_{\mu\nu}(P\cdot q)^2 \right) \cdot (2P\cdot q)^n A_{n+2}^{j,2} \left(\frac{\partial}{\partial q^2}\right)^n \int d^4z\, C_2^{n,j}(z^2 - i\varepsilon) e^{iq\cdot z}$$

$$= \sum_{n,j} \Big\{ e_{\mu\nu}\, w^n (-q^2)^{n+1} A_n^{j,1} \left(\frac{\partial}{\partial q^2}\right)^n \int d^4z\, C_1^{n,j}(z^2 - i\varepsilon) e^{iq\cdot z}$$

$$+ d_{\mu\nu}\, w^{n+2} \frac{(-q^2)^{n+2}}{4} A_{n+2}^{j,2} \left(\frac{\partial}{\partial q^2}\right)^n \int d^4z\, C_2^{n,j}(z^2 - i\varepsilon) e^{iq\cdot z} \Big\} \tag{2.25}$$

where: $w^2 = \dfrac{2P\cdot q}{-q^2}$ (2.26)

$$e_{\mu\nu} = -\frac{q_\mu q_\nu}{q^2} + g_{\mu\nu} \tag{2.27}$$

$$d_{\mu\nu} = (P_\mu q_\nu + P_\nu q_\mu)\frac{1}{P\cdot q} - \frac{q^2}{(P\cdot q)^2} P_\mu P_\nu - g_{\mu\nu} \tag{2.28}$$

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The matrix elements of the operators $\mathcal{O}$ have to be a function of the proton momenta p:

$$\langle p | \mathcal{O}^{\hat{a},L}_{\mu_1 \dots \mu_n} | p \rangle = A^{\hat{a},L}_n p_{\mu_1} \dots p_{\mu_2} + B^{\hat{a},L}_n p^2 g_{\mu_1\mu_2} p_{\mu_3} \dots p_{\mu_n} + \dots \quad (2.23)$$

$$\langle p | \mathcal{O}^{\hat{a},\eta p,2}_{\mu_1 \dots \mu_n} | p \rangle = A^{\hat{a},2}_{n+2} p_{\mu_1} \dots p_{\mu_n} p^\lambda p^\rho + \dots \quad (2.24)$$

Exercise: Show that in the Bjorken limit, the dominant terms are the ones proportional to $A^{\hat{a},L}_n$ and $A^{\hat{a},2}_n$ only!

If possible, generalize all the following conclusions by including the $B^{\hat{a},L}_n$, $B^{\hat{a},2}_{n+2}$ terms!

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Hence:

$$T_{\mu\nu} = \int d^4z \sum_{n,j} \Big\{ i^2 C_L^{n,j}(z^2)\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right) \langle p | O^{j,L}_{\mu_1\ldots\mu_n} | p \rangle (2)^n q_{\mu_1}\ldots q_{\mu_n}\left(\frac{\partial}{\partial q^2}\right)^n$$

$$+ i^2 C_2^{n,j}\left(g_{\mu\lambda} q_\rho q_\nu + g_{\rho\lambda} q_\mu q_\nu - g_{\mu\lambda} g_{\rho\nu} q^2 - g_{\mu\nu} q_\lambda q_\rho\right) \cdot \langle p | O^{j,\lambda\rho,2}_{\mu_1\ldots\mu_n} | p \rangle (2)^n q_{\mu_1}\ldots q_{\mu_n}\left(\frac{\partial}{\partial q^2}\right)^n \Big\} e^{iq\cdot x} \quad (2.22)$$

where the q_μ, etc, come from

$$\partial_\mu e^{iq\cdot x} = i q_\mu e^{iq\cdot x}$$

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The singularities are buried in the O_L and O_2 operators. We perform a Wilson expansion on them:

$$O_L(z,0) = \sum_{n,j} i^{n-1} C_L^{n,j}(z^2 - i\varepsilon)\, O^{j,L}_{\mu_1 \dots \mu_n} z_{\mu_1} \dots z_{\mu_n} \tag{2.19}$$

$$O_2(z,0) = \sum_{n,j} i^{n-1} C_2^{n,j}(z^2 - i\varepsilon)\, O^{j,2}_{\mu_1 \dots \mu_n} z_{\mu_1} \dots z_{\mu_n} \tag{2.20}$$

We remember that we have to calculate

$$T_{\mu\nu} = \int d^4 z \langle p | T[j_\mu(x) j_\nu(0)] | p \rangle e^{iq\cdot z}$$

Hence, the $z_{\mu_1} \dots z_{\mu_n}$ can be replaced by derivatives:

$$z_{\mu_1} \dots z_{\mu_n} \longrightarrow (-i)^n \frac{\partial}{\partial q^{\mu_1}} \dots \frac{\partial}{\partial q^{\mu_n}} = (-2i)^n q_{\mu_1} \dots q_{\mu_n} \left(\frac{\partial}{\partial q^2}\right)^n \tag{2.21}$$

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Let's now look at the case of two electromagnetic currents.

The general form for the product is.

$$T[J_\mu(z) J_\nu(0)] =$$

$$= -(g_{\mu\nu}\Box - \partial_\mu\partial_\nu) O_1 +$$

$$+ (g_{\mu\lambda}\partial_\rho\partial_\nu + g_{\rho\nu}\partial_\mu\partial_\lambda - g_{\mu\lambda}g_{\rho\nu}\Box$$

$$- g_{\mu\nu}\partial_\rho\partial_\lambda) O_2^{\lambda\rho} \qquad (2.13)$$

See Hey and Mandula

Phys. Rev. D5 (1972) 2610

This representation for the product of currents is completely similar to the representation of $W_{\mu\nu}$ (in terms of $g_{\mu\nu}, p, q$) in momentum space. See eq. (1.15).

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In QCD, $\psi = \begin{pmatrix} u \\ d \\ s \\ \vdots \end{pmatrix}$

The operator (2.16) is then a singlet (in flavour) operator, as it does not distinguish quark flavour.

Similarly, we can have:

$$O^{a,n}_{\mu_1 \cdots \mu_n} = i^{n-1} S \bar{\psi} \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_n} \frac{\lambda^a}{2} \psi \quad , (2.17)$$

(λ^a: Gell-Mann Matrix)

and that is a non-singlet (in flavour) operator because of the λ^a matrix.

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So, the smaller the twist is, more singular is the operator $\mathcal{O}^n$

The simplest operator.

$\bar{\psi}\gamma_\mu\psi$: $d_\psi = \frac{3}{2}$; $n=1$

$\Rightarrow \boxed{\tau = 2}$ which is the lowest possible twist.

We can add as many derivatives into this operator as we want (each derivative will have a corresponding ∂^μ factor, producing no extra dimension to the operator):

$$\mathcal{O}^n_{\mu_1\mu_2\cdots\mu_n} = i^{n-1} S\, \bar{\psi}\gamma_{\mu_1} D_{\mu_2} D_{\mu_3} \cdots D_{\mu_n}\psi \qquad (2.16)$$

S → Permutation over the Lorentz indexes

D → Covariant derivative

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$C_j^n(z^2)$ is called the Wilson coefficient of the expansion.

What is the dimension of C_j^n?

. Let $\underline{d_j}$ be the dimension of the current. (units of mass) ; $[M]^{d_j}$

. Let $\underline{d_0}$ be the mass dimension of the Operator O ; $[M]^{d_0}$

. Let $\underline{n}$ be the mass dimension of the $z^{\mu_1} \dots z^{\mu_n}$ product. ; $[M]^{-n}$

So, the mass dimension of C^n is:

$$[C^n] = 2d_j + n - d_0 \equiv 2d_j - \tau_j \qquad (2.15)$$

defined as the "twist" of the operator: $\tau = d_0 - n$

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Main idea: The product of any two operators can be expanded in a series of well defined operators.

look at: W. Zimmermann
Ann. Phys. 77 (1973) 570
Brand and Preparata
Nucl. Phys. B27 (71) 541

From now on, i will call z the position four vector.

$$T[j_\mu(z) j_\nu(0)] \longrightarrow \sum_{n=0}^{\infty} i^{n-1} C_j^n(z^2) z^{\mu_1} z^{\mu_2} \dots z^{\mu_n} O^{j,n}_{\mu_1 \mu_2 \dots \mu_n}(0) \qquad (2.14)$$

j: labels different Operators O

They have the same quantum numbers of $j_\mu j_\nu$ for each n

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Similarly, using the anticommutation relation:

$$\{\psi(x), \bar{\psi}(0)\} = \frac{1}{2\pi} \not{p}_x \, \varepsilon(x^0)\, \delta(x^2) \quad , \quad (2.13)$$

$$\varepsilon(x^0) = \frac{x^0}{|x^0|}$$

(See page 666 of Itzykson/Zuber),

We have:

$$[J_\mu(x), J_\nu(0)] \propto \not{\partial} \frac{\varepsilon(x^0)}{2\pi} \delta(x^2) \quad \text{as}$$

$$x^2 \to 0.$$

which also shows that the current commutator is singular at $x^2 \sim 0$, or on the light cone.

Remedy: Expand the product of the operators in decreasing singularity

Wilson, Phys. Rev. 179 (1969) 1499

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S_F can be rewritten as

$$S_F(x) = (i\not{\partial}_x + m)\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x)}}{m^2 - p^2 - i\varepsilon}$$

$$\equiv (i\not{\partial}_x + m)\,\Delta(x) \qquad (2.11)$$

The important point here is that $\Delta(x)$ can be expanded in its singularities:

$$\Delta(x) = \frac{1}{4\pi^2}\frac{1}{x^2 - i\varepsilon} + \text{less singular terms} \qquad (2.12)$$

$\downarrow$

has a singularity at $x^2 \sim 0$

Exercise: Calculate $\Delta(x) = \frac{1}{4\pi^2}\frac{1}{x^2 - i\varepsilon}$

Look at Cheng and Li, page 219, for directions.

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On the other hand, we can use the Wick theorem to express the time ordering of the electromagnetic current in terms of propagators:

$$T[j_\mu(x) j_\nu(0)] =$$

$$=: \bar{\psi}(x)\gamma_\mu \psi(x)\bar{\psi}(0)\gamma_\nu \psi(0):$$

$$+ : \bar{\psi}(x)\gamma_\mu \langle 0|T\psi(x)\bar{\psi}(0)|0\rangle \gamma_\nu \psi(0):$$

$$+ \ldots$$

where $::$ is the normal ordering.

Now, $\langle 0|T\psi(x)\bar{\psi}(0)|0\rangle$ is the propagator for the fermion field:

$$i\langle 0|T\psi(x)\bar{\psi}(0)|0\rangle = S_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{m - \not{p} - i\varepsilon} e^{-ip\cdot x} \qquad (2.12)$$

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So,

$$T_{\mu\nu}(q_0 + i\varepsilon) - T_{\mu\nu}(q_0 - i\varepsilon) =$$

$$= (2\pi)^4 i\, \delta^4(p_x - q - p) \sum_x \langle p | j_\mu(0) | x \rangle \langle x | j_\nu(0) | p \rangle$$

$$= 2\pi i\, W_{\mu\nu} \,,$$

as we can see from Eq. (2.1)

$$W_{\mu\nu} = \frac{1}{2\pi i} \left[T_{\mu\nu}(q_0 + i\varepsilon) - T_{\mu\nu}(q_0 - i\varepsilon) \right] \qquad (2.9)$$

$$= \frac{1}{\pi} \text{Abs}\, T_{\mu\nu}$$

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The second term of $T_{\mu\nu}$ does not have a pole:

$P_{x0} + q_0 - P_0$ is never zero!

As long $q_0 > 0$

Thus, when calculating the Abs $T_{\mu\nu}$, we need only the first term.

$$\Rightarrow T_{\mu\nu}(q_0 + i\varepsilon) - T_{\mu\nu}(q_0 - i\varepsilon) =$$

$$= (2\pi)^3 \delta(\vec{P}_x - \vec{q} - \vec{P}) \langle p | \cdots | p \rangle \left\{ \frac{1}{P_{x0} - q_0 - P_0 - i\varepsilon} - \frac{1}{P_{x0} - q_0 - P_0 + i\varepsilon} \right\}$$

$$= (2\pi)^3 \delta(\ ) \langle p | \cdots | p \rangle \left\{ \frac{2i\varepsilon}{(P_{x0} - q_0 - P_0)^2 + \varepsilon^2} \right\}$$

However,

$$\delta(P_{x0} - q_0 - P_0) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{(P_{x0} - q_0 - P_0)^2 + \varepsilon^2}$$

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$$T_{\mu\nu} = i \sum_X \int dx_0 \, e^{i(q+p-p_x)^0 x_0} \, \vartheta(x_0) \cdot$$

$$\cdot \int d^3x \, e^{-i(\vec{q}+\vec{p}-\vec{p}_x)\cdot\vec{x}} \langle p | j_\mu(0) | x \rangle \langle x | j_\nu(0) | p \rangle$$

$$+ \ldots$$

Using that $\vartheta(x_0) = \begin{cases} 0 & \text{for } x_0 < 0 \\ 1 & \text{for } x_0 > 0 \end{cases}$

and: $\int d^3x \, e^{-i(\vec{q}+\vec{p}-\vec{p}_x)\cdot\vec{x}} = (2\pi)^3 \delta(\vec{p}_x - \vec{q} - \vec{p})$

$$T_{\mu\nu} = \sum_X \frac{(2\pi)^3 \delta(\vec{p}_x - \vec{q} - \vec{p})}{p_{x0} - q_0 - p_0 - i\varepsilon} \langle p | j_\mu(0) | x \rangle \langle x | j_\nu(0) | p \rangle$$

$$+ \sum_X \frac{(2\pi)^3 \delta(\vec{p}_x + \vec{q} - \vec{p})}{p_{x0} + q_0 - p_0 - i\varepsilon} \langle p | j_\nu(0) | x \rangle \langle x | j_\mu(0) | p \rangle \qquad (2.7)$$

We can also see a symmetry:

$$T_{\mu\nu}(q) = T_{\mu\nu}(-q) \qquad (2.8)$$

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We first use the fact that the current $j_\mu(x)$ is translationally invariant: $j_\mu(x) = e^{i\hat{P}\cdot x} j_\mu(0) e^{-i\hat{P}\cdot x}$.

So,

$$T_{\mu\nu} = i\sum_x \int d^4x\, e^{iq\cdot x} \langle p|\, T(x_0)\, e^{i\hat{P}\cdot x} j_\mu(0)\, e^{-i\hat{P}\cdot x} |X\rangle\langle X| j_\nu(0)|p\rangle$$

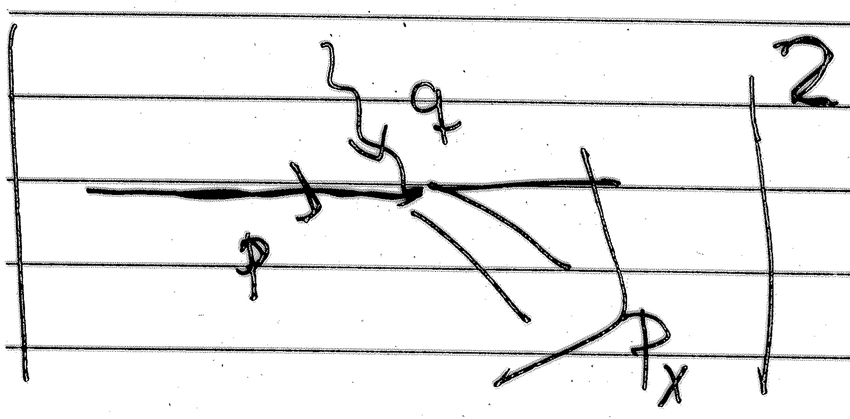
$$+ i\sum_x \int d^4x\, e^{iq\cdot x} \langle p|\, T(-x_0)\, j_\nu(0) |X\rangle\langle X|\, e^{i\hat{P}\cdot x} j_\mu(0) e^{-i\hat{P}\cdot x} |p\rangle$$

where a set of complete intermediate states was introduced.

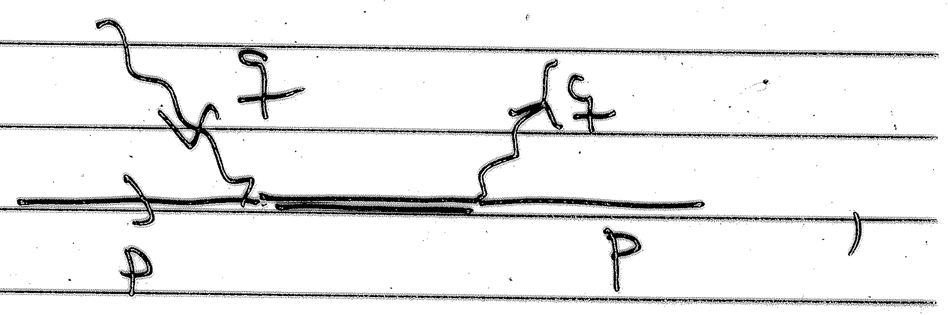
$$T_{\mu\nu} = i\sum_x \int d^4x\, e^{i(q+p-P_x)\cdot x}\, T(x_0)\langle p| j_\mu(0)|X\rangle\langle X| j_\nu(0)|p\rangle$$

$+ \ldots$

In other words,



Can be related to



through the following way:

$$\boxed{W_{\mu\nu} = \frac{1}{\pi} \text{Abs}\, T_{\mu\nu}}, \tag{2.5}$$

with $$\text{Abs}\, T_{\mu\nu} = \frac{1}{2i}\left[T_{\mu\nu}(q_0 + i\varepsilon) - T_{\mu\nu}(q_0 - i\varepsilon)\right] \tag{2.6}$$

We will prove this relationship between these two quantities.

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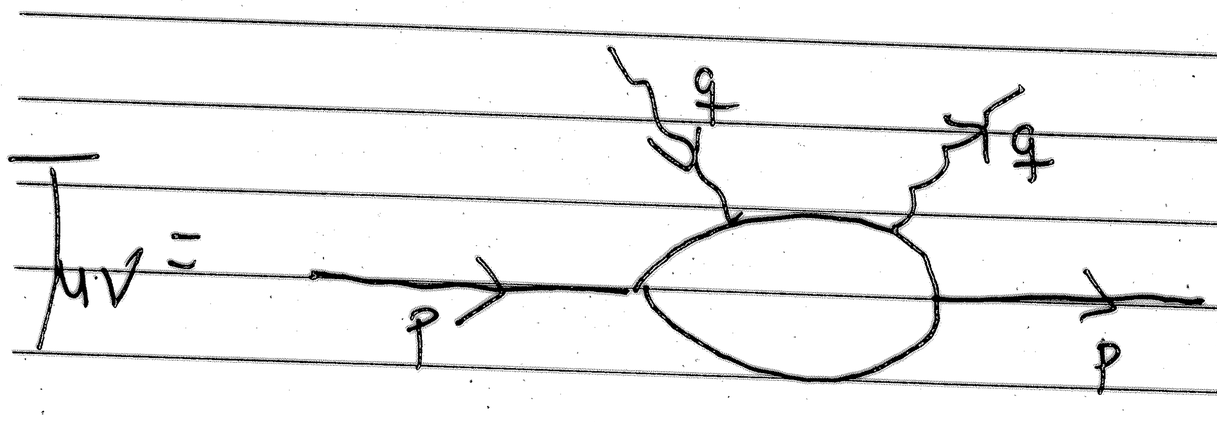
In quantum field theory we usally work with time ordered products:

$$T(A(x)B(y)) = \theta(x_0 - y_0)\, A(x)\, B(y) + \theta(y_0 - x_0)\, B(y)\, A(x)$$

We can actually relate the hadronic tensor to this kind of product, through the use of the forward Compton amplitude (virtual):

$$T_{\mu\nu} = i \int d^4x\, e^{iq\cdot x} \langle p | T[j_\mu(x) j_\nu(0)] | p \rangle \quad (2.4)$$



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It means:

Deep inelastic scattering ($W_{\mu\nu}$) is light cone dominated

Not short distance ($x \to 0$), but $x^2 \to 0$.

Hence, Bjorken limit does not mean that we are deep inside the proton. the scattering can occur at the centre of the proton, as it can also occur at 2 fm from the centre

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We also have:

$$q \cdot x = q^+ x^- + q^- x^+ - \vec{q}_\perp \cdot \vec{x}_\perp$$

$$= q^+ x^- + q^- x^+ \quad \text{because } \vec{q}_\perp = \vec{0}$$

Hence, $q \cdot x$ is finite only if $x^- \sim \frac{1}{x}$, $x^+ \sim \frac{1}{\nu}$

Causality also ensures that

$$x^2 = 2x^+x^- - \vec{x}_\perp \cdot \vec{x}_\perp \geq 0$$

$$\sim \frac{2}{x\nu} - \vec{x}_\perp \cdot \vec{x}_\perp \to -\vec{x}_\perp \cdot \vec{x}_\perp$$

in the Bjorken limit

$$\Rightarrow \vec{x}_\perp \cdot \vec{x}_\perp \sim 0$$

thus $x^2 \sim 0$ in the Bjorken limit !!

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Where in x is this contribution?

Consider the scattering in the target rest frame, where the photon momentum transfer is:

$$q = (\nu, 0, 0, -\sqrt{\nu^2 + Q^2})$$

Such that $q^2 = -Q^2$

We now take the Bjorken limit ($\nu, Q^2 \to \infty$ with Q^2/ν fixed) in the light cone coordinates

$$q^+ = \frac{1}{\sqrt{2}}(q^0 + q^3) = \nu\left(1 - \sqrt{1 + \frac{Q^2}{\nu^2}}\right)$$

$$\simeq -\frac{Q^2}{\sqrt{2}\,2\nu} = -\frac{M x_B}{\sqrt{2}}$$

$$q^- = \frac{1}{\sqrt{2}}(q^0 - q^3) \simeq \sqrt{2}\,\nu$$

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. First, by causality:

$$[j_\mu(x), j_\nu(0)] = 0 \quad \text{for } x^2 < 0$$

See page 27 of Peskin Schroeder

. Second, the Riemann-Lebesgue lemma says that if a function $f(q)$ is an ordinary function absolutely integrable, and $g(x)$ is its Fourier transform, then

$$g(x) \to 0 \quad \text{as} \quad |x| \to \infty$$

Hence, the main contribution in $W_{\mu\nu}$ comes from the region where $q.x$ is finite

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On the other hand we have:

$$\int d^4x\, e^{iq\cdot x} \langle p|J_\mu(0) J_\nu(x)|p\rangle =$$

$$= \sum_X (2\pi)^4 \delta^4(q-p+p_x) \langle p|J_\nu(0)|X\rangle\langle X|J_\mu(0)|p\rangle$$

which implies that

$p_x = p-q$, which is not physical!

Hence, $W_{\mu\nu}$ can be written as:

$$W_{\mu\nu} = \frac{1}{2\pi} \int d^4x\, e^{iq\cdot x} \frac{1}{2} \sum_S \langle p,S|[J_\mu(x), J_\nu(0)]|p,S\rangle \quad (2.3)$$

Why is it interesting to express the hadronic tensor as an expectation value of the commutator of currents??

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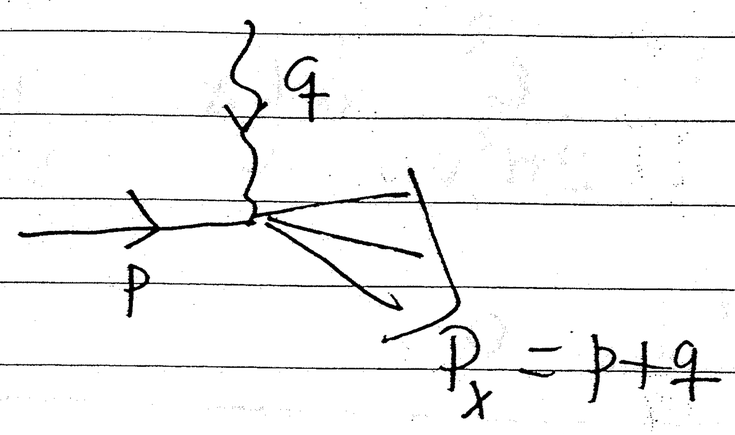
The OPE and partons

We just saw that the hadronic tensor is given by:

$$W_{\mu\nu} = \frac{1}{2\pi} \sum_X (2\pi)^4 \delta^4(P_X - p - q) \frac{1}{2} \sum_S \langle P,S|J_\mu(0)|X\rangle\langle X|J_\nu(0)|P,S\rangle$$

(2.1)



$$= \frac{1}{2\pi} \int d^4x\, e^{-i(P_X - p - q)\cdot x} \frac{1}{2} \sum_S \langle P,S|e^{iP\cdot x} J_\mu(x) e^{-iP\cdot x}|X\rangle\langle X|J_\nu(0)|P,S\rangle$$

$$= \frac{1}{2\pi} \int d^4x\, e^{iq\cdot x} \frac{1}{2} \sum_S \langle P,S|J_\mu(x) J_\nu(0)|P,S\rangle \quad (2.2)$$

PS: x here is not the Bjorken x. It is a four vector.