The NJL Model for Quarks in Hadrons and Nuclei
- Part II: Diquarks and Nucleons -

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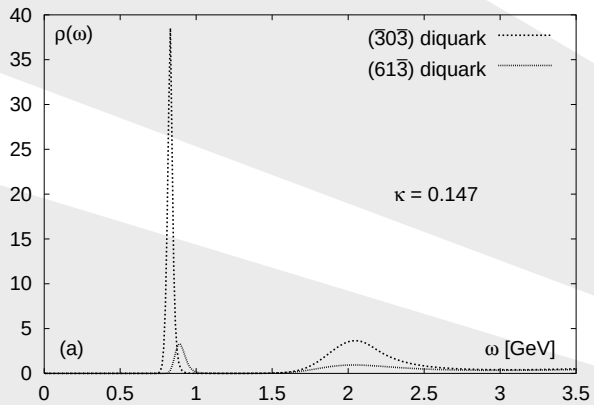
What is a diquark?

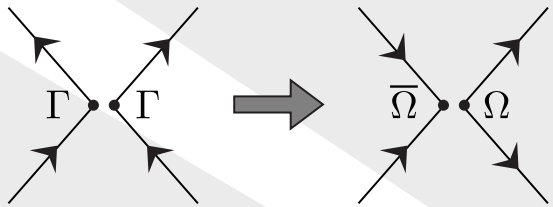
- **Diquark** is a correlated (interacting) quark-quark state. It has color $\overline{3}$ (antisymmetric) or color 6 (symmetric). Inside the nucleon, only color $\overline{3}$ is possible.
- The most important diquark is the "**scalar diquark**": $J^P = 0^+, T = 0$. (Nonrelativistic analogue: ${}^1S_0$ state.) This is a kind of "pairing" between u and d quarks, and can form a condensate in high density quark matter ($\Rightarrow$ Part III). In lattice QCD, a sharp peak in the spectral density of the correlation function in this channel is seen (**Fig.5**):



The next important diquark is the "**axial vector diquark**": $J^P = 1^+, T = 1$. (Nonrelativistic analogue: ${}^3S_1$ state.)

Diquark interaction Lagrangian (1)

To describe diquarks, it is convenient to rewrite the interaction largangian $(\overline{\psi}\Gamma\psi)^2 \to \left(\overline{\psi}\Omega\overline{\psi}^T\right)\left(\psi^T\overline{\Omega}\psi\right)$. The matrix Ω then shows the quantum numbers of the interacting qq channel.



(Time runs from left to right in this figure).

To do this rewriting, first use the identity

$$\left(\overline{\psi}_3\Gamma^1\psi_1\right)\left(\overline{\psi}_4\Gamma^2\psi_2\right) = -\left(\overline{\psi}_3\Gamma^1\psi_1\right)\left(\psi_2^T\Gamma^{2T}\overline{\psi}_4^T\right) = \left(\overline{\psi}_3\Gamma^1\psi_1\right)\left(\overline{\psi}_2'\Gamma'^2\psi_4'\right) \quad (1)$$

where

$$\begin{aligned} \psi' &= C\tau_2\overline{\psi}^T \quad (C = i\gamma_2\gamma_0) \\ \overline{\psi}' &= -\psi^T C^{-1}\tau_2\,, \quad \Gamma' = C\tau_2\Gamma^T C^{-1}\tau_2 \end{aligned}$$

and then make the usual **Fierz transformation** for (1). (See Notes!).

Diquark interaction Lagrangian (2)

For example, our interaction $\mathcal{L}_I = -G\left(\overline{\psi}\frac{\lambda_C}{2}\gamma^\mu\psi\right)^2$ can be rewritten as

$$\begin{aligned}\mathcal{L}_I &= G_s\left(\overline{\psi}\gamma_5 C\tau_2\beta^A\overline{\psi}^T\right)\left(\psi^T C^{-1}\gamma_5\tau_2\beta^A\psi\right) && (2)\\ &+ G_a\left(\overline{\psi}\gamma_\mu C(\tau_i\tau_2)\beta^A\overline{\psi}^T\right)\left(\psi^T C^{-1}\gamma^\mu(\tau_2\tau_i)\beta^A\psi\right) && (3)\\ &+ \text{other channels}\end{aligned}$$

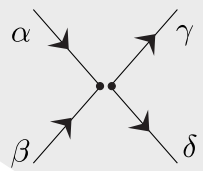
$$G_s = \frac{1}{9}G = \frac{1}{2}G_\pi, \quad G_a = \frac{1}{18}G = \frac{1}{4}G_\pi, \quad \beta^A = \sqrt{\frac{3}{2}}\lambda_C^A, \quad (A = 2, 5, 7)$$

The term (2) is the interaction in the **scalar diquark channel**: $J^P = 0^+$, $T = 0$, color $\overline{3}$. The term (3) corresponds to the **axial vector diquark channel**: $J^P = 1^+$, $T = 1$, color $\overline{3}$.

τ_2 couples the isospin of two quarks to $T = 0$, and $C^{-1}\gamma_5 \propto \Sigma_2$ couples the spins to $J = 0$. Under spinor Lorentz transformations $\psi'(x') = S(\Lambda)\psi(x)$ we have the identities $S^T\left(C^{-1}\gamma_5\right)S = \left(C^{-1}\gamma_5\right)$, and $S^T\left(C^{-1}\gamma^\mu\right)S = \Lambda^\mu{}_\nu\left(C^{-1}\gamma^\nu\right)$, etc.

BS equation for diquarks (1)

We have the **Feynman rule** for the qq interaction in the **scalar diquark** channel:

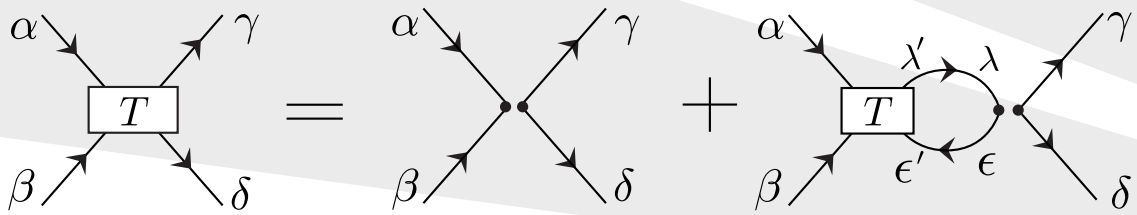


$$4iG_s\left(\gamma_5 C\tau_2\beta^A\right)_{\gamma\delta}\left(C^{-1}\gamma_5\tau_2\beta^A\right)_{\alpha\beta} \equiv K_{\gamma\delta,\alpha\beta}$$

Additional rule: For each qq intermediate state there is a symmetry factor 1/2.

Then the equation for the qq scattering matrix (**Bethe- Salpeter equation**) becomes for fixed total 4-momentum p^μ:

$$T_{\gamma\delta,\alpha\beta}(p) = K_{\gamma\delta,\alpha\beta} + \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} K_{\gamma\delta,\lambda\epsilon} S_{\epsilon'\epsilon}(-k) S_{\lambda\lambda'}(p+k) T_{\lambda'\epsilon',\alpha\beta}(p)$$



BS equation for diquarks (2)

Inserting the form $K_{\gamma\delta,\alpha\beta} = 4iG_s\,\Omega_{\gamma\delta}\,\overline{\Omega}_{\alpha\beta}$, and assuming the solution of the form

$$T_{\gamma\delta,\alpha\beta}(p) = t(p)\,\Omega_{\gamma\delta}\,\overline{\Omega}_{\alpha\beta}$$

we get for the scalar function $t(p)$ the simple equation:

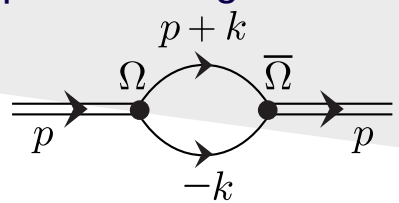
$$t(p) = 4iG_s - 2G_s\Pi(p^2)t(p) \Rightarrow t(p) = \frac{4iG_s}{1+2G_s\Pi(p^2)}$$

with the following "**bubble graph**"

$$\Pi(p^2) \equiv i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left(\overline{\Omega}S(p+k)\Omega S^T(-k)\right)$$

Using the relation $CS(-k)^TC^{-1} = S(k)$, we see that this bubble graph is the same as the previous one in the pion channel.
The pole of $t(p)$ gives the **diquark mass**: $1+2G_s\Pi(p^2 = M_D^2) = 0$.
If $G_s = G_\pi$, the scalar diquark and the pion are degenerate.

$\pi(p^2) = -i$



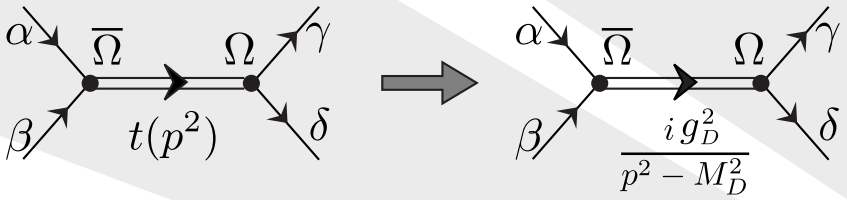
Diquark masses and vertex functions

Expanding $\Pi(p^2)$ near the pole as $\Pi(p^2) = \Pi(M_D^2) + (p^2 - M_D^2)\Pi'(M_D^2) + \ldots$, we see that near the pole

$$t(p) \to \frac{i g_D^2}{p^2 - M_D^2}$$

where $g_D^2 \equiv (-2/\Pi'(M_D^2))$.

$T_{\gamma\delta,\alpha\beta} =$

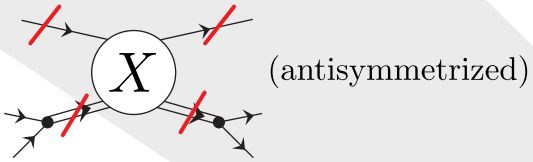


Again, this supports the interpretation of M_D as the **diquark mass** and g_D as the **quark-diquark coupling constant**.

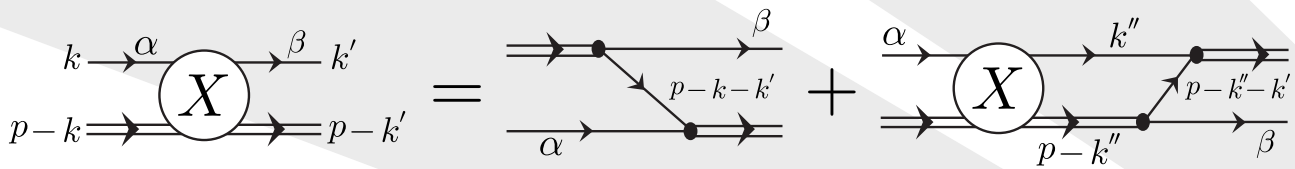
Remember the form of the vertex function for scalar diquark:
$\Omega = \gamma_5 C \tau_2 \beta^A \quad (A = 2, 5, 7)$.

Nucleon: Faddeev equation

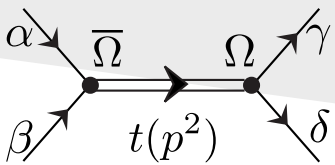
Now we describe the **nucleon** as a bound state of a quark and a diquark. The 3-quark scattering matrix can represented as follows:



Cutting the external propagators as shown, we are left with the **quark-diquark scattering matrix**, denoted as X. It satisfies



This "**Faddeev equation**" simply means the **recombination of interacting pairs**: $(12)3 \to (23)1 \to \ldots$. In the Figure, α, β refer to the quark (Dirac and isospin indices), while for the diquark we take only the scalar channel. Remember the form of our 2-body T-matrix:



Nucleon: Faddeev equation (2)

Define the "**quark exchange kernel**" as

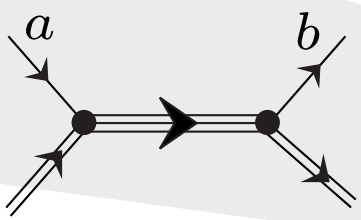
$$Z(k',k) \equiv \Omega S^T(p-k'-k)\overline{\Omega} \overset{\text{color } 0}{\Longrightarrow} (-3)\gamma_5 S(k+k'-p)\gamma_5$$

where the last form follows by coupling the quark color (3_c) and the diquark color ($\overline{3}_c$) to total color 0. Then we can write the Faddeev equation in the color singlet channel as

$$X(k',k) = Z(k',k) + \int \frac{\mathrm{d}^4 k''}{(2\pi)^4} Z(k',k'')S(k'')t(p-k'')X(k'',k)$$

where X, Z, S are Dirac matrices and t is a scalar function. This equation can be solved **numerically**. If X has a **pole** in the total momentum p^2, we can define the **nucleon mass** M_N and **vertex functions** Γ_N by the behaviour near the pole as follows:

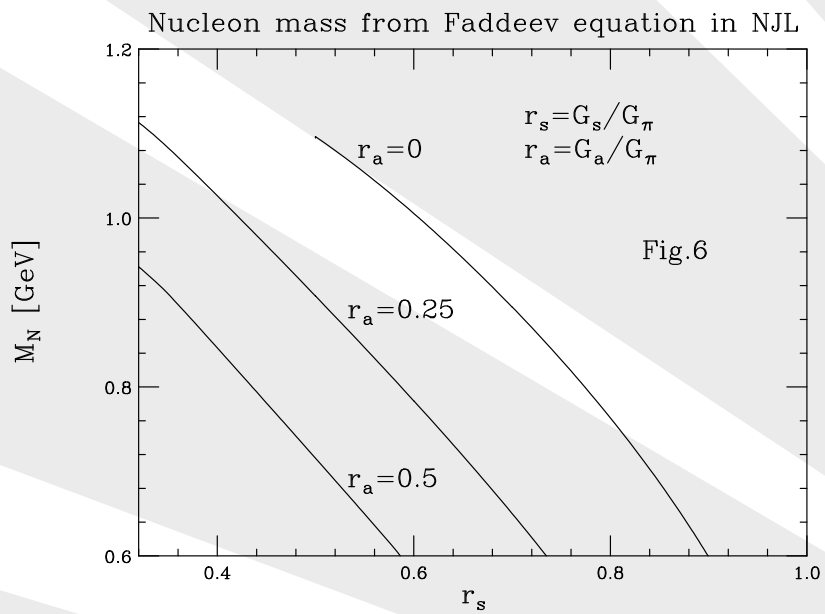
X^{ba} ⇒



$$= \frac{\Gamma_N^b(k')\overline{\Gamma}_N^a(k)}{p^2 - M_N^2}$$

Nucleon: Results from Faddeev equation

Results for the nucleon mass, including both the scalar and axial vector diquark channels:



In this calculation, G_π is fixed by the pion mass (see Notes!), and $r_s = G_s/G_\pi$, $r_a = G_a/G_\pi$ are treated as parameters.

Remember that the interaction $-G\left(\overline{\psi}\frac{\lambda^a}{2}\gamma_\mu\psi\right)^2$ gave $r_s = 0.5, r_a = 0.25 \Rightarrow$ If only the scalar diquark is included: $M_N = 1.1$ GeV; if also axial vector diquark is included: $M_N = 0.9$ GeV.

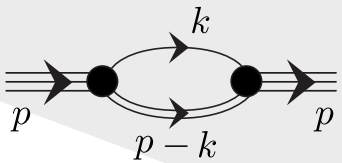
Nucleon: Static approximation

Qualitatively correct **analytic results** can be obtained with the following "**static approximation**": Neglecting the momentum dependence of the quark exchange kernel, $Z \to 3/M$. Then

$$X(p) = \frac{3}{M} - \frac{3}{M}\Pi_N(p)X(p) \Rightarrow X(p) = \frac{3}{M}\frac{1}{1+\frac{3}{M}\Pi_N(p)}$$

where the Dirac matrix $\Pi_N(p)$ is a **quark-diquark bubble graph**:

$$\Pi_N(p) \equiv -\int \frac{\mathrm{d}^4k}{(2\pi)^4} S(k)t(p-k)$$

$\pi_N(p) =$ 

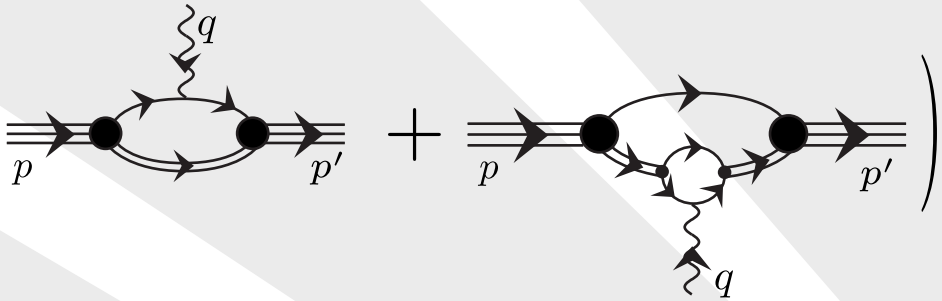
The **nucleon mass** is determined by $1 + \frac{3}{M}\Pi(\not{p} = M_N) = 0$, and the pole behaviour gives the vertex functions $\Gamma_N(p)$ defined earlier:

$$X(p) \to \frac{1}{(\not{p} - M_N)\Pi'_N(\not{p} = M_N)} = \frac{\sum_s u_N(p,s)\overline{u}_N(p,s)}{p^2 - M_N^2} \times \frac{2M_N}{\Pi'_N(\not{p} = M_N)}$$

Application 3: Nucleon electric form factors (1)

The **electromagnetic current** of the nucleon is given by the following Feynman diagrams:

$$j^\mu(q) = \frac{1}{\sqrt{4E_{p'}E_p}}$$



This is usually expressed in terms of the Dirac-Pauli form factors $F_1(Q^2), F_2(Q^2)$ as

$$j^\mu(q) = \sqrt{\frac{M_N}{E_{p'}}}\sqrt{\frac{M_N}{E_p}}\,u_N(p')\left[\gamma^\mu F_1(Q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M_N}F_2(Q^2)\right]u_N(p)$$

The **electric and magnetic form factors** are then defined by

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M_N^2}F_2(Q^2)\,, \qquad G_M(Q^2) = F_1(Q^2) + F_2(Q^2)$$

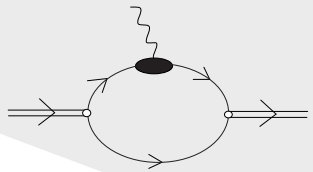
$Q^2 = -q^2 > 0$ for electron scattering. Interpretation of G_E and G_M as Fourier transforms of charge and magnetic moment densities is possible in the “Breit frame”, where $\vec{p} = -\vec{q}/2$, $\vec{p}' = \vec{q}/2$.

Nucleon electric form factors (2)

Inserting our vertex functions from the “static approximation” into the Feynman diagram, we obtain for the **current of the nucleon**

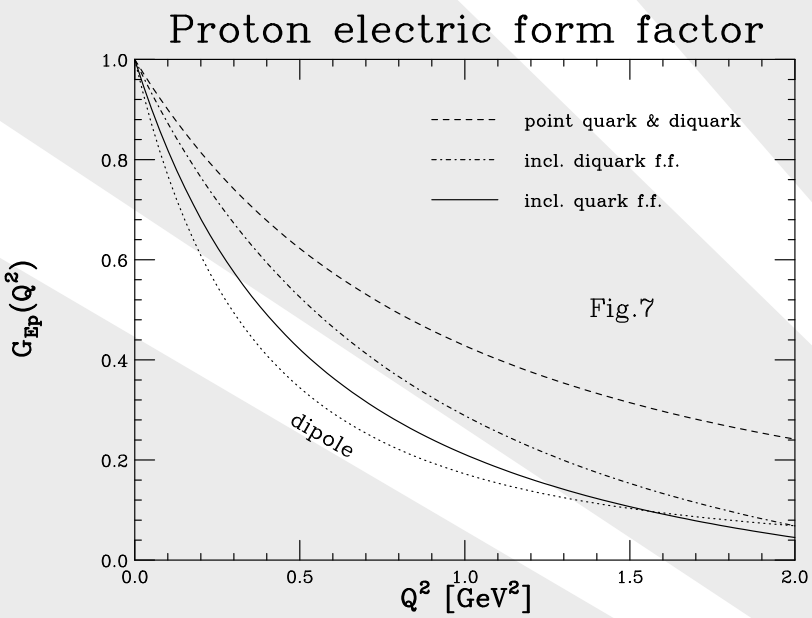
$$\begin{aligned} j^\mu(q) &= \sqrt{\frac{M_N}{E_{p'}}}\sqrt{\frac{M_N}{E_p}}\left(\frac{1}{\Pi'_N(M_N)}\right)\overline{u}_N(p')\int\frac{\mathrm{d}^4k}{(2\pi)^4} \\ &\times \left[S(p'-k)\gamma^\mu Q_q S(p-k)t(k) + i\left(t(p'-k)\Lambda_D^\mu t(p-k)\right)S(k)\right]u_N(p) \end{aligned}$$

where $Q_q = \frac{1}{6} + \frac{\tau_3}{2}$ is the quark charge, and Λ_D^μ is the diquark electromagnetic vertex:



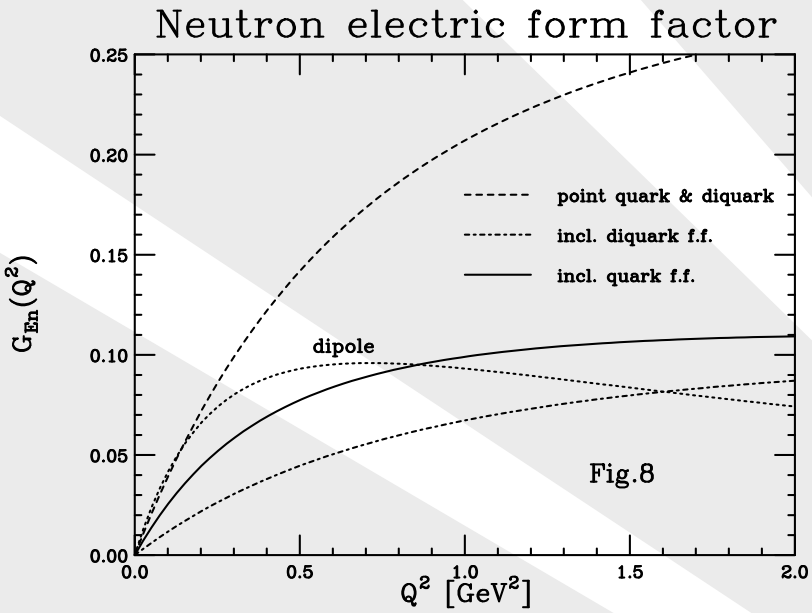
The most naive **quark-diquark model** means to approximate the diquark t-matrix by the pole term: $t(k) \to ig_D^2/(k^2 - M_D^2)$, and to make an on-shell approximation for the diquark electromagnetic vertex: $g_D^2\Lambda_D^\mu = (k+k')^\mu F_D(q^2)$, where $F_D(q^2)$ is the diquark charge form factor.

Results for proton electric form factor



- dashed line ... point diquark ($F_D(q^2) = 1$)
- dashed-dotted line ... formula on previous slide, including finite size of diquark
- solid line ... including intrinsic quark form factors from pion cloud and vector mesons.
- "dipole": Empirical form $G_{Ep} = 1/(1 + Q^2/0.71\mathrm{GeV}^2)^2$

Results for neutron electric form factor

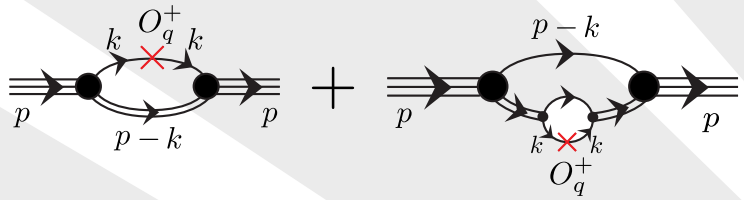


- dashed line ... point diquark ($F_D(q^2) = 1$)
- dashed-dotted line ... formula two slides before, including finite size of diquark
- solid line ... including intrinsic quark form factors from pion cloud and vector mesons.
- "dipole": Empirical form $G_{En} = \frac{Q^2}{4M_N^2} \frac{|\kappa_n|}{(1+Q^2/0.71\mathrm{GeV}^2)^2}$ ($\kappa_n = -1.91$).

Application 4: Quark distributions (1)

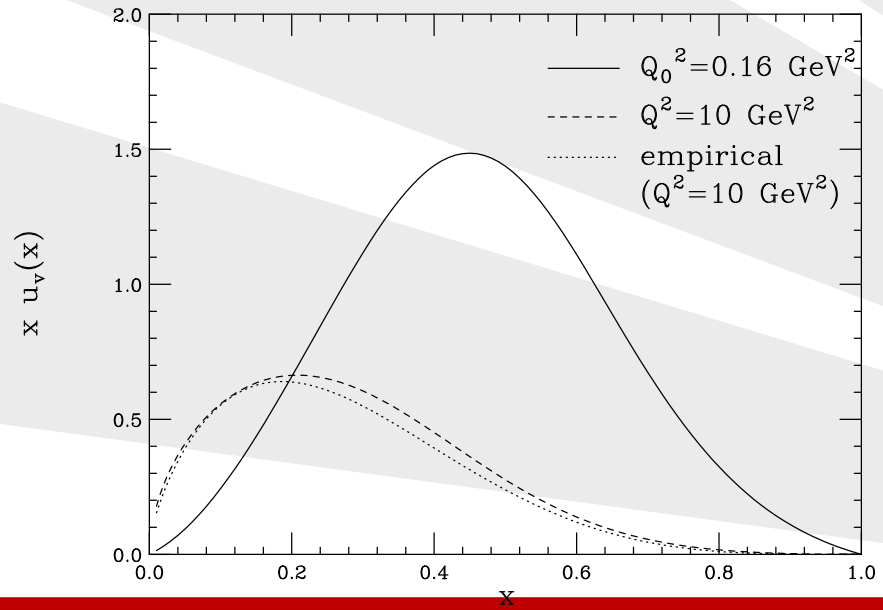
To calculate the **quark momentum distributions** in the proton, we calculate the following Feynman diagrams with operator insertion $\mathcal{O}_q^+ = (1 \pm \tau_3/2)\gamma^+\delta(x - k^+/p^+)$, where $q = u, d$:

$$f_q^N(x) = \frac{1}{2p^+}\Bigg($$



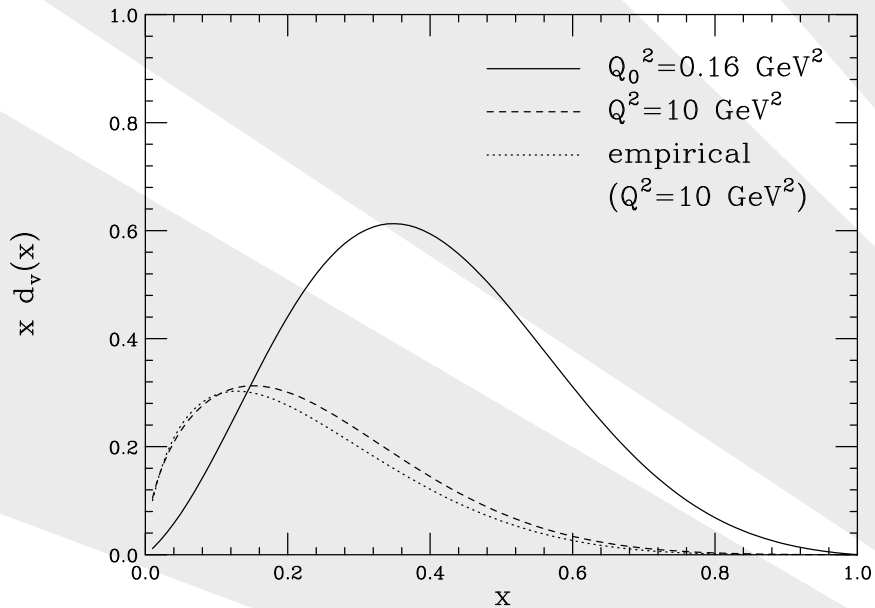
$$\Bigg)$$

Result for up-quark distribution in proton (**Fig.9**):



Quark momentum distributions in proton (2)

Result for down-quark distribution in proton (**Fig.10**):



- solid line ... NJL result
- dashed line ... Result obtained by Q^2 evolution up to $Q^2 = 10$ GeV^2, assigning a low energy scale $Q_0^2 = 0.16$ GeV^2 to the solid line
- dotted line ... empirical valence quark distribution at $Q^2 = 10$ GeV^2.

Comments on figures

- Fig. 5: See: *I. Wetzorke, F. Karsch, hep-lat/0008008, Fig.2a*. $(\overline{3}0\overline{3})$ means the scalar diquark $(\mathrm{flavor}\ \overline{3}, \mathrm{spin}0, \mathrm{color}\overline{3})$, and $(60\overline{3})$ means the axial vector diquark $(\mathrm{flavor}\ 6, \mathrm{spin}0, \mathrm{color}\overline{3})$. A delta-function like peak in the spectral density indicates a pole in the diquark propagator.

- Fig.6: See *N. Ishii et al, Nucl. Phys.* **A 587** *(1995), p. 617; Fig. 6.* Here the Euclidean cut-off is used ($\Lambda = 0.739$ GeV in the figure). The constituent quark mass is $M = 0.4$ GeV. The scalar diquark mass decreases from 0.764 GeV (for $r_s = 0.4$) to 0.14 GeV (for $r_s = 1$), while the axial vector diquark is unbound (no pole, only continuum states).

- Figs. 7, 8: See *T. Horikawa, W. Bentz, Nucl. Phys.* **A 762** *(2005) 102; Figs. 6 and 7.* The proper-time regularization is used here ($\Lambda_{\mathrm{UV}} = 0.64$ GeV, $\Lambda_{\mathrm{IR}} = 0.2$ GeV). The constituent quark mass is $M = 0.4$ GeV. The calculation of the intrinsic quark form factors from pion cloud and vector mesons is also discussed in the paper. For a general discussion of nucleon form factors, including the experimental data and the problem of extracting the neutron form factor, see: *A.W. Thomas and W. Weise, The Structure of the Nucleon*, Wiley-VCH, New York, 2001.

- Figs. 9, 10: See *H. Mineo et al, Nucl. Phys.* **A 735** *(2004), p. 482; Figs. 3 and 4.*
The proper-time regularization is used here ($\Lambda_{\mathrm{UV}} = 0.64$ GeV, $\Lambda_{\mathrm{IR}} = 0.2$ GeV). The constituent quark mass is $M = 0.4$ GeV. For the Q^2 evolution, the code of M. Miyama, S. Kumano, Comput. Phys. Commun. **94** (1996), p.185, is used. (Case of next-to-leading order with $\Lambda_{\mathrm{QCD}} = 0.25$ GeV is used in the figure.) The empirical distributions are taken from: *A.D. Martin, R.G. Roberts, W.J. Stirling and R.S. Thone, Eur. Phys. J.* **C 28** *(2003), p. 455.* They are obtained from experimental data for deep inelastic scattering of leptons off the proton, deuteron and ^{3}He.