QCD on the lattice - an introduction

Mike Peardon

School of Mathematics, Trinity College Dublin
Currently on sabbatical leave at JLab

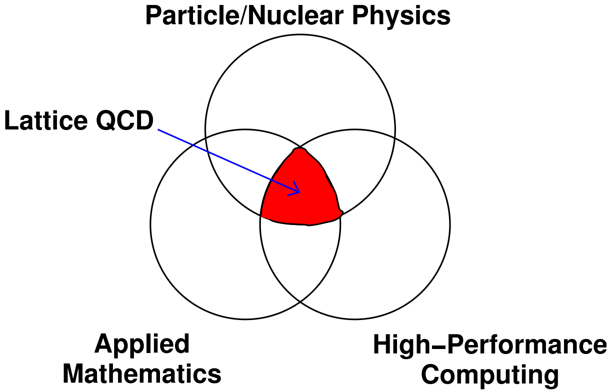
HUGS 2008 - Jefferson Lab, June 3, 2008

QCD on the lattice

Overview:

- Understanding lattice QCD
- Lattice QCD calculations : Spectroscopy
- Lattice QCD calculations : Matrix elements
- Open and unsolved problems in Lattice QCD

Why do I enjoy working in lattice QCD?



Understanding Lattice QCD

To understand the basic ingredients of lattice QCD calculations, we need to discuss:

- The path integral formulation of quantum mechanics
- Path integral quantum field theory
- Discretising fields and differential operators
- Regularisation / renormalisation
- The symmetries of QCD
- The Monte Carlo method

The path integral - reformulating quantum mechanics (1)

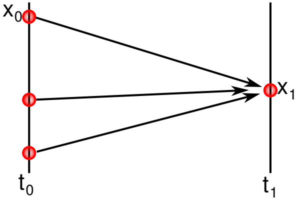
- Lattice field theory uses Feynman's path integral description of quantum mechanics.
- Quantum mechanical amplitudes can be expressed as a *"sum over histories"*.

- Consider a particle moving in one dimension. The wave-function a small time in the future can be expressed as

$$\psi(x_1, t+\epsilon) = \sqrt{\frac{im}{2\epsilon\hbar}} \int dx_0 \; \psi(x_0, t) \; e^{\frac{-i\epsilon}{\hbar}\mathcal{L}(x_0,x_1)}$$

with $\mathcal{L}$ the (classical) lagrange density.

- We have already begun to discretise; we are defining the states of the system only on time-slices.



The path integral - reformulating quantum mechanics (2)

$$\psi(x_1, t+\epsilon) = \sqrt{\frac{im}{2\epsilon\hbar}} \int dx_0\, \psi(x_0, t)\, e^{\frac{-i\epsilon}{\hbar}\mathcal{L}(x_0,x_1)}$$

- The Lagrange density is $\mathcal{L}(x,\dot{x}) = \frac{m}{2}\dot{x}^2 - V(x)$
- For time-slices closely separated, define $\dot{x} = \frac{x_1 - x_0}{\epsilon}$ then expanding for small ϵ and introducing $\eta = x_1 - x_0$ gives

$$\psi + \epsilon\frac{\partial}{\partial t}\psi = \sqrt{\frac{im}{2\epsilon\hbar}} \int d\eta\, e^{-\frac{im\eta^2}{2\hbar\epsilon}} (1 - i\frac{\epsilon}{\hbar}V)(\psi - \eta\frac{\partial}{\partial x}\psi + \frac{\eta^2}{2}\frac{\partial^2}{\partial x^2}\psi + \dots)$$

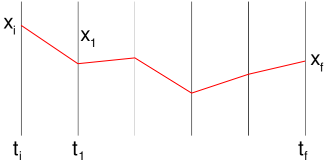
- This used a saddle-point expansion around $\epsilon = 0$. Integrating η gives

. . . ψ satisfies time-dependent Schrödinger's equation

$$\mathbf{i}\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V\psi$$

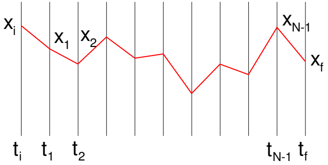
- Repeating this update many times will define an algorithm to solve the Schrödinger equation (given an initial wave-function).
- For finite updates, the corresponding integral expression relating the final wave-function $\psi(x_f, t_f)$ to the initial wave-function $\psi(x_i, t_i)$ is

$$\psi(x_f, t_f) = \int dx_i \;\; \psi(x_i, t_i) \;\; Z(x_i, x_f)$$



- with Z the *path integral*

$$Z = \int \prod_{a=1}^{N-1} \mathcal{D}x_a \quad e^{\frac{i}{\hbar} S}$$



- and S is the classical action,

$$S = \sum_{a=1}^{N-1} \frac{m}{2\epsilon^2}(x_a - x_{a-1})^2 + V(x_a)$$

Minkowski, Wick and Euclid

- Some properties of theories in Minkowski space can be related by Wick rotation to corresponding theories in Euclidean space.
- Analytic continuation: $t \to i\tau$, $\frac{-i}{\hbar} S \to \frac{1}{\hbar} S$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Correlation functions

- The path integral defines simple transition amplitudes like $\langle x_f, t_f | x_i, t_i \rangle$ but more complicated matrix elements also have representations

$$\langle x_f, t_f | x(t_2) x(t_1) | x_i, t_i \rangle = \int \prod_{a=1}^{N-1} \mathcal{D} x_a x(t_2) x(t_1) e^{\frac{i}{\hbar} S}$$

- In the Euclidean metric, the weight for field configurations is Wick rotated, so $e^{\frac{i}{\hbar} S} \longrightarrow e^{-\frac{1}{\hbar} S}$
- This gives correlation functions a useful property:

$$\begin{aligned} \langle 0 | x(\tau_2) x(\tau_1) | 0 \rangle &= \langle 0 | e^{H\tau_2} x e^{-H(\tau_2 - \tau_1)} x e^{-H\tau_1} | 0 \rangle \\ &= \sum_{n,m} \langle x | n \rangle \langle n | e^{-H(\tau_2 - \tau_1)} | m \rangle \langle m | x \rangle \\ &= \sum_{n} | \langle x | n \rangle |^2 e^{-E_n(\tau_2 - \tau_1)} \end{aligned}$$

Then

Correlation functions

- Then with $\Delta\tau = \tau_2 - \tau_1$,

Euclidean correlation functions

$$lim_{\Delta\tau \to \infty} \langle 0 | x(\tau_2) x(\tau_1) | 0 \rangle = Z e^{-E_1 \Delta\tau}$$

At large imaginary-time separations, $\Delta\tau \to \infty$ correlation functions fall exponentially with rate proportional to the energy of the lowest energy state that is excited.

- We will see another (even more) crucial property of the Euclidean metric when we look at studying quantum field theories numerically.

Path integral quantum field theory

- QCD is a relativistic quantum field theory.
- Path integral quantisation extends to quantum field theory too:

Theory	d.o.f	discretise	action	sum over all
QM	$x(t)$	t	$S[x]$	paths
QFT	$\phi(\underline{x}, t)$	$\underline{x}, t$	$S[\phi]$	field configurations

- Discretise four-dimensional space-time on a 4d lattice:
  $\underline{x}, t \longrightarrow (an_1, an_2, an_3, an_4), n_\mu \in Z, a$ is distance between sites.
- Matter fields (scalar bosons, quarks, …) are represented by integration variables at all lattice sites, ϕ_{n_1,n_2,n_3,n_4}.
- The action is a function of all these variables and the path integral is the integral over these degrees of freedom.

QFT lattice path integral

$$Z = \int \prod_{\mu, n_\mu} d\phi_{n_1,n_2,n_3,n_4} \quad e^{-S(\phi)}$$

Path integral quantum field theory

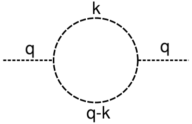
Comparing Minkowski and Euclid

	Minkowski	Euclidean
Weight of a field configuration	$e^{\frac{i}{\hbar}S}$	$e^{-\frac{1}{\hbar}S}$
Describes	QFT	Statistical mechanics
Parameter	$\hbar$	temperature, T
Numerical approach?	Not easy!	Monte Carlo

- Statistical mechanical systems often exhibit interesting phase transitions.
- "Bare" lattice couplings in the action describe "theory space"
- Correlation lengths of the lattice statistical mechanical system can diverge (in units of a) at certain points or sub-spaces of "theory space".
- Continuum field theories can live at second-order phase transition points.

Regularising path integrals

- Path integrals have continuum representation too!
- When path integrals are expanded in the continuum in powers of the coupling constant (perturbation theory), we often encounter apparent divergences.



$$\approx \int d^4k \;\; \frac{1}{k^2}\frac{1}{(q-k)^2}$$

- These can usually be regulated by a (perturbative) renormalisation procedure.
- In a finite volume, lattice path integrals are certain to be (UV) finite - the divergences are cut-off automatically.
- The lattice provides a cut-off outside perturbation theory. This is a crucial property for QCD.

First lattice field theory: scalar bosons

- The action in Euclidean space for a (real) scalar field $\phi(x)$ is

$$S[\phi] = \int d^4x \;\; \phi(-\Box + m^2)\phi$$

with $\Box = \sum_{\mu=1}^{4} \frac{\partial^2}{\partial x_\mu^2}$

- m is the (bare) boson mass.
- This action is invariant under $SO(4)$ the Euclidean rotation group in four dimensions (the equivalent of Lorentz group), so theory is one of relativistic bosons.
- A lattice version would define scalar fields that take values on lattice sites only.
- Define a to be the lattice spacing - the distance between adjacent sites.

First lattice field theory: scalar bosons (2)

- To represent the differential operator $\Box$, we can take linear combinations of the nearest neighbours of a site. A Taylor-series in a would give

$$\phi(n_1 + a, n_2, n_3, n_4) = \phi(n_1, n_2, n_3, n_4) + a\frac{\partial}{\partial x_1}\phi(n_1, n_2, n_3, n_4) + \frac{a^2}{2}\frac{\partial^2}{\partial x_1^2}\phi(n_1, n_2, n_3, n_4) + \mathcal{O}(a^3)$$

- So the linear combination of nearest neighbours

$$\frac{1}{a^2}\left(\phi(n_1 + a, n_2, n_3, n_4) + \phi(n_1 - a, n_2, n_3, n_4) - 2\phi(n_1, n_2, n_3, n_4)\right) = \frac{\partial^2}{\partial x_1^2}\phi(n_1, n_2, n_3, n_4) + \mathcal{O}(a^2)$$

gives a suitable representation of one term in $\Box$

First lattice field theory: scalar bosons (3)

- The lattice action is then

$$S[\phi] = \frac{a^4}{2} \sum_{\mathbf{x}} \left\{ (m^2+8)\phi(\mathbf{x})^2 - \sum_{\mu=1}^{4} \phi(\mathbf{x})\phi(\mathbf{x}+\hat{\mu}) \right\}$$

- The path integral is gaussian and it can be computed analytically. The propagator (in momentum space) for the theory is then

$$\langle \phi(p')\phi(p) \rangle = \delta_{p,p'} \frac{1}{\sum_{\mu=1}^{4} \frac{2}{a} sin^2(ap_\mu/2) + m^2}$$

- As $a \to 0$ the propagator for a scalar boson in the (Euclidean) continuum is recovered.

$$\langle \phi(p')\phi(p) \rangle = \delta_{p,p'} \frac{1}{\sum_{\mu=1}^{4} p_\mu^2 + m^2}$$

First lattice field theory: scalar bosons (4)

- Local interactions can be added easily, for example define

$$S[\phi] = S_0[\phi] + S_{\rm int}[\phi]$$

  with

$$S_0[\phi] + S_{\rm int}[\phi] = \lambda a^4 \sum_x \phi^4(x)$$

  and we are simulating a Higgs field.
- For finite a, the lattice theory differs from the continuum by terms proportional to a^2.
- These extra terms lead to *lattice artefacts*. Computing properties of the continuum theory will require extrapolation to $a = 0$.
- The lattice action is not unique: we could have used

$$\frac{\alpha}{4}(\phi(x+2)+\phi(x-2))+(1-\alpha)(\phi(x+1)+c_1\phi(x-1))-2(1-\frac{3}{4}\alpha)\phi(x)$$

  for any value of α
- The influence of these artefacts can be reduced by *improvement*.

Improvement

- The freedom to employ different representations of the lattice action with a common continuum limit is exploited to reduce discretisation artefacts.
- Classically, an improved version of the second-order derivative is

$$\frac{\partial^2}{\partial x^2}\phi = \frac{4}{3a^2}(\phi_1 + \phi_{-1} - 2\phi_0) - \frac{1}{12a^2}(\phi_2 + \phi_{-2} - 2\phi_0) + \mathcal{O}(a^4)$$

- The more complete procedure in a quantum field theory is the *Symanzik improvement programme*
- Symanzik improvement accounts for relative renormalisation of coefficients relating terms in the lattice action to their continuum counterparts.

Basic properties of QCD

- To discretise a theory and write a useful lattice representation, it is important to do the best possible job of respecting the symmetries of the theory.
- Symmetries define universality classes and ensure the correct approach to the continuum as we (try to) take $a \to 0$

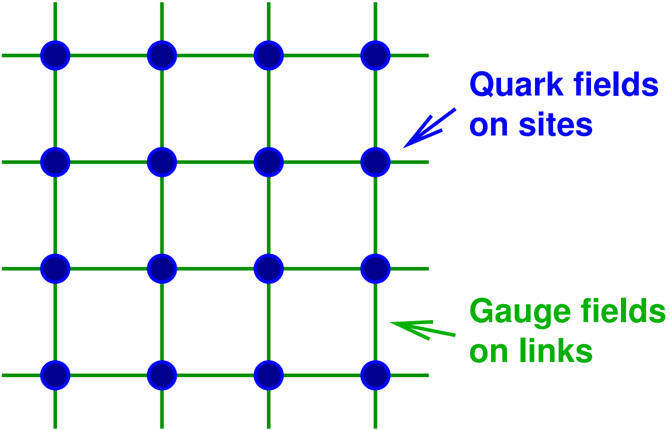
Symmetries of QCD

- Poincaré invariance (Lorentz and translation invariance)
- Gauge invariance
- Discrete symmetries: parity, time-reversal, charge conjugation
- (Near) chiral symmetry (for massless quarks).
- (Near) flavour symmetry (for mass-degenerate quarks).

- The QCD path integral is written in terms of the two fundamental fields, the quarks and the gluons.

Wilson's big idea...

- Wilson realised that ensuring *gauge invariance* means the gluon fields have to be given special treatment:



Summary

- Starting from a path integral representation, we can formulate quantum field theory on a discretised version of space-time.
- If we are interested in spectroscopy, the Euclidean metric is very useful. Correlation functions fall exponentially, with rate related to the energies of eigenstates.
- The Euclidean metric also translates QFT into a statistical mechanics problem. There are then effective numerical techniques to attack these problems.
- Symmetries define field theories - we will need to be careful what symmetries we keep or break when the field theory is put on a lattice.
- An exact gauge symmetry on the lattice will be constructed by putting *quarks on sites, gluons on links*.

Problems

1. Show

$$\frac{\partial^2}{\partial x^2}\phi = \frac{4}{3a^2}(\phi_1 + \phi_{-1} - 2\phi_0) - \frac{1}{12a^2}(\phi_2 + \phi_{-2} - 2\phi_0) + \mathcal{O}(a^4)$$

2. Compute an expression for the free scalar boson propagator:

$$\langle \tilde{\phi}(p')\tilde{\phi}(p)\rangle = \frac{1}{Z}\int \mathcal{D}\phi \;\; \tilde{\phi}(p')\tilde{\phi}(p)\; e^{-S}$$

where

$$\tilde{\phi}(p) = \sum_x \phi(x) e^{ipx}$$

and

$$S\left[\phi\right] = \frac{a^4}{2}\sum_{\mathbf{x}}\left\{(m^2+8)\phi(\mathbf{x})^2 - \sum_{\mu=1}^{4}\phi(\mathbf{x})\phi(\mathbf{x}+\hat{\mu})\right\}$$