QCD on the lattice - an introduction
Lecture 2

Mike Peardon

School of Mathematics, Trinity College Dublin
Currently on sabbatical leave at JLab

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The fields of QCD

- The relativistic theory of strongly-interacting quarks and gluons.

- **Quarks** are (spin 1/2) Dirac fermions, so they have a 4-component spin index. They are also charged under the fundamental representation of the non-abelian gauge group $SU(3)$, so each spin component is in turn a three-component vector. The quark field is $\psi^a_\alpha, \alpha = 1 \ldots 4, a = 1 \ldots N_c = 3$

- **Gluons** (to preserve the gauge invariant structure) must be charged under the adjoint representation of the gauge group. They are then eight real massless vector bosons; $A^i_\mu, i = 1 \ldots 8$.

- The simplest lagrangian that preserves gauge invariance introduces interactions between quarks and gluons as well as gluon self-interactions. (cf QED).
Continuum gauge transformations

- Quark fields form a (fundamental) representation of the gauge group, $SU(3)$, that means they transform under a (space-time dependent) rotation as

$$\psi(x) \rightarrow \psi^g(x) = \Lambda(x)\psi(x)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}^g(x) = \bar{\psi}(x)\Lambda^\dagger(x)$$

where $\Lambda(x)$ is the gauge transformation at $x$, and $\Lambda^\dagger(x)\Lambda(x) = 1$, $\det \Lambda(x) = 1$.

- To make a theory of fermion with this symmetry, another field is needed that transmits information about relative gauge transformations at nearby points.

- The derivative $\partial_\mu$ acting on the quark field must be replaced with a gauge covariant derivative $D_\mu$ with

$$D_\mu = \partial_\mu - igA_\mu$$
Continuum gauge transformations (2)

- $A_\mu$ is another field, that transforms according to

$$A_\mu \rightarrow A_\mu^{(g)} = \frac{1}{ig} (\partial_\mu \Lambda) \Lambda^{-1} + \Lambda A_\mu \Lambda^{-1}$$

- Now under a gauge transformation, $D\psi$ transforms in the same way as $\psi$ so the bilinear $\bar{\psi}D\psi$ is gauge invariant.
- $A_\mu$ forms an adjoint representation of the gauge transformation group.
- So $A$ can be written in terms of an element of the Lie algebra of $SU(3)$: $A_\mu(x) = T^a A^a_\mu(x)$
- A field strength tensor can be written, which is analogous to the electromagnetic tensor (which contains electric and magnetic fields)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

- The QCD field strength tensor has a commutator that is not present for QED, which leads to gluon self-interaction.
Gauge invariant actions

- The field strength tensor has simple transformation properties
  \[ F_{\mu\nu} \rightarrow F^{(g)}_{\mu\nu} = \Lambda F_{\mu\nu} \Lambda^{-1} \]

- A gauge-invariant action on the gauge fields can be defined
  \[ S_g = \frac{1}{4} \int d^4x \; \text{Tr} \; F_{\mu\nu} F_{\mu\nu} \]

- Similarly, for a quark field, a suitable action is
  \[ S_q = \int d^4x \; \bar{\psi}(\gamma_\mu D_\mu + m)\psi \]

- Here, we have wick-rotated the gamma-matrices so the quark fields form a spin-1/2 representation of $SO(4)$.
  \[ \{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu} \]
Lattice fields - the quarks

- Quark fields are discretised in the simplest way; the fields are restricted to take values only on sites of the four-dimensional space-time lattice, $\psi(x, t) \rightarrow \psi_{n_1, n_2, n_3, n_4}$.

- Each lattice site has $4 \times N_c = 12$ degrees of freedom per quark flavour.

- Gauge transforms will be defined for sites too:
  $\psi_{n_1, n_2, n_3, n_4} \rightarrow \psi^{(g)}_{n_1, n_2, n_3, n_4} = \Lambda_{n_1, n_2, n_3, n_4} \psi_{n_1, n_2, n_3, n_4}$.

- In a path integral, fermions must be represented by elements of a grassmann algebra:
  $\int d\eta = 0, \int d\eta \eta = 1$

- This will make life complicated for us when it comes to simulations.

- And more problems with quarks will arise when we try to define an action...
Wilson recognised the way to build actions with a gauge symmetry on the lattice was to put the gluon field onto the lattice in a very different way: **gluons live on links**.

Abandon the vector potential as the fundamental degree of freedom, use instead a small path-ordered exponential connecting adjacent sites on the lattice:

\[ U_\mu(x) = \mathcal{P} \exp \left\{ ig \int_x^{x+\hat{\mu}} ds \, A_\mu(s) \right\} \]

Path-ordering is needed to give an unambiguous meaning to this expression since the gauge group is non-abelian \((A_\mu(x)\) does not commute with \(A_\mu(y)\) when \(x \neq y)\).

\(U_\mu \in SU(3)\) while \(A_\mu \in \mathcal{L}(SU(3))\).

To define a path-integral, we need to integrate over the \(SU(3)\) group manifold; use an invariant Haar measure, \(\mathcal{D}U\).
Lattice gauge invariants

- Define the rules of gauge transformations so gauge invariants can be constructed out of lattice fields:

\[
\begin{align*}
\psi(x) &\longrightarrow \psi^{(g)}(x) = \Lambda(x)\psi(x) \\
\bar{\psi}(x) &\longrightarrow \bar{\psi}^{(g)}(x) = \bar{\psi}(x)\Lambda^\dagger(x) \\
U_\mu(x) &\longrightarrow U_\mu^{(g)}(x) = \Lambda(x)U_\mu(x)\Lambda^\dagger(x + \hat{\mu})
\end{align*}
\]

- Since $\Lambda^\dagger\Lambda = 1$, the following expressions are invariant under these transformations

\[
\begin{align*}
\bar{\psi}(x)U_\mu(x)&\psi(x + \hat{\mu}) \\
\text{Tr }U_\mu(x)U_\nu(x + \hat{\mu})U_\mu^\dagger(x + \hat{\nu})U_\nu^\dagger(x)
\end{align*}
\]
Lattice gauge invariants
Gauge invariance

To rotate a quark field at site $x$, $\psi(x) \rightarrow \psi^g(x) = g(x)\psi(x)$ ...

...we must also rotate the gauge fields that start or end at the site $U_\mu(x) \rightarrow U^g_\mu(x) = g(x)U_\mu(x)g^\dagger(x + \hat{\mu})$

The gauge invariance of the special functions is seen
Lattice action - the gluons

- To define a path integral, we also need an action
- The simplest gauge invariant function of the gauge link variables alone is the *plaquette* (the trace of a path-ordered product of links around a $1 \times 1$ square).

\[
S_G[U] = \frac{\beta}{N_c} \sum_{x, \mu < \nu} \text{ReTr} \left( 1 - U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\nu}) U_\nu^\dagger(x) \right)
\]

This is the *Wilson gauge action*

- A path integral for the Yang-Mills theory of gluons would be

\[
Z_{YM} = \int \prod_{\mu, x} D U_\mu(x) e^{-S_G[U]}
\]

- The coupling constant, $g$ appears in $\beta = \frac{2N_c}{g^2}$
- No need to fix gauge; the gauge orbits can be trivially integrated over and the group manifold is compact.
Lattice action - the gluons

- A Taylor expansion in $a$ shows that

$$S_G[U] = \frac{\beta}{N_c} \sum_{x, \mu < \nu} \text{ReTr} \left( 1 - U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\nu}) U_\nu^\dagger(x) \right)$$

$$= \int d^4x - \frac{1}{4} \text{Tr} \ F_{\mu\nu} F_{\mu\nu} + \mathcal{O}(a^2)$$

- As with the scalar boson lattice action, all terms proportional to odd powers in the lattice spacing vanish because the lattice action preserves a discrete parity symmetry.

- The action is also invariant under a charge-conjugation symmetry, which takes $U_\mu(x) \rightarrow U_\mu^*(x)$.

- We have kept almost all of the symmetries of the Yang-Mills sector, but broken the $SO(4)$ rotation group down to the discrete group of rotations of a hypercube.
Lattice actions - the quarks

- The continuum action is a bilinear with a first-order derivative operator inside:

\[ S_Q = \int d^4x \bar{\psi}(\gamma_\mu D_\mu + m)\psi \]

- When \( m = 0 \), the action has an extra, chiral symmetry:

\[ \psi \rightarrow \psi(x) = e^{i \alpha \gamma_5} \psi, \quad \bar{\psi} \rightarrow \bar{\psi}(x) = \bar{\psi} e^{i \alpha \gamma_5} \]

- The simplest lattice representation of a first-order derivative that preserves reflection symmetries is the central difference:

\[ \partial_\mu \psi(x) = \frac{1}{2a} (\psi(x + \hat{\mu}) - \psi(x - \hat{\mu})) \]

- This can be made gauge covariant by including the gauge links:

\[ D_\mu \psi(x) = \frac{1}{2a} (U_\mu(x)\psi(x + \hat{\mu}) - U_\mu(x - \hat{\mu})\psi(x - \hat{\mu})) \]

- **BUT** on closer inspection, there are more minima to this action than we want. Consider the case with no gauge fields, and when \( \psi(x) = e^{ikx} \) with \( k = \{\pi, 0, 0, 0\} \) or \( \{\pi, \pi, 0, 0\} \) or \( \{\pi, \pi, \pi, 0\} \) or \ldots \.
Lattice doubling

Central difference between these two points is zero, not large!
This is the (in)famous **doubling problem**.

**The Nielson-Ninomiya “no-go” theorem**

There are **no** chirally symmetric, local, translationally invariant doubler-free fermion actions on a regular lattice.

- To put quarks on the lattice, more symmetry must be broken or else a theory with extra flavours of quarks must be simulated.
- A number of solutions are used, each with their advantages and disadvantages.
- The most commonly used are:
  - Wilson fermions
  - Kogut-Susskind (staggered) fermions
  - Ginsparg-Wilson fermions (overlap, domain wall, perfect…)
  - Twisted mass
Wilson’s lattice quark action

- Wilson's original solution was to abandon chiral symmetry and add a lattice operator whose continuum limit is an irrelevant dimension-five operator. The term gives the doublers a mass $\propto 1/a$
- The extra term in the lattice action is the lattice representation of
  \[ a \sum_\mu D^2_\mu \psi \approx \sum_\mu U_\mu(x)\psi(x + \hat{\mu}) + U^\dagger_\mu(x - \mu)\psi(x - \hat{\mu}) \]
- The breaking of chiral symmetry means the quark mass is not protected from additive renormalisations (short-distance gluons will now give quarks a large mass)
- Approaching the continuum limit requires fine-tuning to restore chiral symmetry and ensure quarks are light.
- Breaking chiral symmetry now introduces lattice artefacts at $O(a)$.
- This action has a Symanzik-improved counterpart, the Sheikholeslami-Wohlert action, which removes all $O(a)$ errors by a field redefinition and the addition of another dim-5 term, $\sigma_{\mu\nu}F_{\mu\nu}$
The Ginsparg-Wilson relation

- Actions that break chiral symmetry, but preserve a modified version can be constructed. The new chiral symmetry is
  \[ \{ \gamma_5, D \} = 2a \frac{D \gamma_5 D}{D} \text{ so } \{ \gamma_5, D^{-1} \} = 2a \gamma_5 \]

- In a propagator, chiral symmetry is broken by a contact term

- A number of realisations of this symmetry are in use. Neuberger’s overlap uses an action
  \[ D = I - \frac{D_W}{\sqrt{D_W^\dagger D_W}} \]
  where \( D_W \) is the Wilson action with a large negative quark mass.

- Domain Wall quarks use a 5d lattice field (coupled to four-dimensional gluons). The boundaries in the 5th dimension are set up so left- and right-handed quarks bind to different walls in 5d. Modes are separated so chiral symmetry is (almost) maintained.

- These quarks are expensive!
Staggered quarks

- Kogut and Susskind proposed an interesting partial solution to the doubling problem.
- A field redefinition is used to scatter the sixteen components of four flavours ("tastes") of quarks across the corners of a hypercube.
- On each lattice sites there are just $N_c$ degrees of freedom.
- A remnant of chiral symmetry remains which is sufficient to ensure there is no additive mass renormalisation.
- Simulations are fast; there is no fine-tuning so the fermion matrix is well-behaved and always positive which helps the simulation algorithms.
- UV gluons can change the "taste" of a quark, so flavours mix.
- Practitioners simulate theories with one or two flavours by taking fractional powers of the fermion path integral. It is still a matter of debate whether this is legitimate.
Summary

- Gauge symmetry on the lattice can be maintained. This is crucial to ensure we’re simulating QCD and without gauge symmetry, we would be stuck with an impossible fine-tuning problem.

- For gauge invariance, **quarks on site, gluons on links**. Now we have a set of gauge invariant functions to construct actions, observables, etc.

- A simple gluon action can be constructed from the trace of the product of links around a small square. This preserves most of the symmetries of the continuum Yang-Mills action.

- Quarks suffer from **doubling**. A set of different “work-arounds” exist, but all have drawbacks (they either break a symmetry or are numerically expensive).