QCD on the lattice - an introduction
Lecture 2

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The fields of QCD

- The relativistic theory of strongly-interacting quarks and gluons.
- **Quarks** are (spin $1/2$) Dirac fermions, so they have a 4-component spin index. They are also charged under the fundamental representation of the non-abelian gauge group $SU(3)$, so each spin component is in turn a three-component vector. The quark field is $\psi^a_\alpha, \alpha = 1 \ldots 4, a = 1 \ldots N_c = 3$
- **Gluons** (to preserve the gauge invariant structure) must be charged under the adjoint representation of the gauge group. They are then eight real massless vector bosons; $A^i_\mu, i = 1 \ldots 8$.
- The simplest lagrangian that preserves gauge invariance introduces interactions between quarks and gluons as well as gluon self-interactions. (cf QED).

Continuum gauge transformations

- Quark fields form a (fundamental) representation of the gauge group, $SU(3)$, that means they transform under a (space-time dependent) rotation as

$$\psi(x) \longrightarrow \psi^{(g)}(x) = \Lambda(x)\psi(x)$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}^{(g)}(x) = \bar{\psi}(x)\Lambda^{\dagger}(x)$$

  where $\Lambda(x)$ is the gauge transformation at x, and $\Lambda^{\dagger}(x)\Lambda(x) = 1$, $\det \Lambda(x) = 1$.
- To make a theory of fermion with this symmetry, another field is needed that transmits information about relative gauge transformations at nearby points.
- The derivative ∂_μ acting on the quark field must be replaced with a gauge covariant derivative D_μ with

$$D_\mu = \partial_\mu - igA_\mu$$

Continuum gauge transformations (2)

- A_μ is another field, that transforms according to

$$A_\mu \longrightarrow A_\mu^{(g)} = \frac{1}{ig}(\partial_\mu \Lambda)\Lambda^{-1} + \Lambda A_\mu \Lambda^{-1}$$

- Now under a gauge transformation, $D\psi$ transforms in the same way as ψ so the bilinear $\bar{\psi} D\psi$ is gauge invariant.
- A_μ forms an adjoint representation of the gauge transformation group.
- So A can be written in terms of an element of the Lie algebra of $SU(3)$: $A_\mu(x) = T^a A_\mu^a(x)$
- A field strength tensor can be written, which is analogous to the electromagnetic tensor (which contains electric and magnetic fields)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

- The QCD field strength tensor has a commutator that is not present for QED, which leads to gluon self-interaction.

Gauge invariant actions

- The field strength tensor has simple transformation properties

$$F_{\mu\nu} \longrightarrow F_{\mu\nu}^{(g)} = \Lambda F_{\mu\nu} \Lambda^{-1}$$

- A gauge-invariant action on the gauge fields can be defined

$$S_g = \frac{1}{4} \int d^4x \ \mathrm{Tr} \ F_{\mu\nu} F_{\mu\nu}$$

- Similarly, for a quark field, a suitable action is

$$S_q = \int d^4x \ \bar{\psi}(\gamma_\mu D_\mu + m)\psi$$

- Here, we have wick-rotated the gamma-matrices so the quark fields form a spin-1/2 representation of $SO(4)$.

$$\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu}$$

Lattice fields - the quarks

- Quark fields are discretised in the simplest way; the fields are restricted to take values only on sites of the four-dimensional space-time lattice, $\psi(\underline{x}, t) \rightarrow \psi_{n_1,n_2,n_3,n_4}$.
- Each lattice site has $4 \times N_c = 12$ degrees of freedom per quark flavour.
- Gauge transforms will be defined for sites too: $\psi_{n_1,n_2,n_3,n_4} \longrightarrow \psi^{(g)}_{n_1,n_2,n_3,n_4} = \Lambda_{n_1,n_2,n_3,n_4}\psi_{n_1,n_2,n_3,n_4}$.
- In a path integral, fermions must be represented by elements of a grassmann algebra:

$$\int d\eta = 0, \int d\eta\, \eta = 1$$

- This will make life complicated for us when it comes to simulations.
- And more problems with quarks will arise when we try to define an action...

Lattice fields - the gluons

- Wilson recognised the way to build actions with a gauge symmetry on the lattice was to put the gluon field onto the lattice in a very different way: **gluons live on links.**
- Abandon the vector potential as the fundamental degree of freedom, use instead a small path-ordered exponential connecting adjacent sites on the lattice:

$$U_\mu(x) = \mathcal{P}exp\left\{ig\int_x^{x+\hat{\mu}} ds\ \ A_\mu(s)\right\}$$

- Path-ordering is needed to give an unambiguous meaning to this expression since the gauge group is non-abelian ($A_\mu(x)$ does not commute with $A_\mu(y)$ when $x \neq y$).
- $U\mu \in SU(3)$ while $A_\mu \in \mathcal{L}(SU(3))$.
- To define a path-integral, we need to integrate over the $SU(3)$ group manifold; use an invariant Haar measure, $\mathcal{D}U$

Lattice gauge invariants

- Define the rules of gauge transformations so gauge invariants can be constructed out of lattice fields:

Gauge transformations of lattice fields

$$\begin{aligned}
\psi(x) \longrightarrow \psi^{(g)}(x) &= \Lambda(x)\psi(x) \\
\bar{\psi}(x) \longrightarrow \bar{\psi}^{(g)}(x) &= \bar{\psi}(x)\Lambda^{\dagger}(x) \\
U_\mu(x) \longrightarrow U_\mu^{(g)}(x) &= \Lambda(x)U_\mu(x)\Lambda^{\dagger}(x+\hat{\mu})
\end{aligned}$$

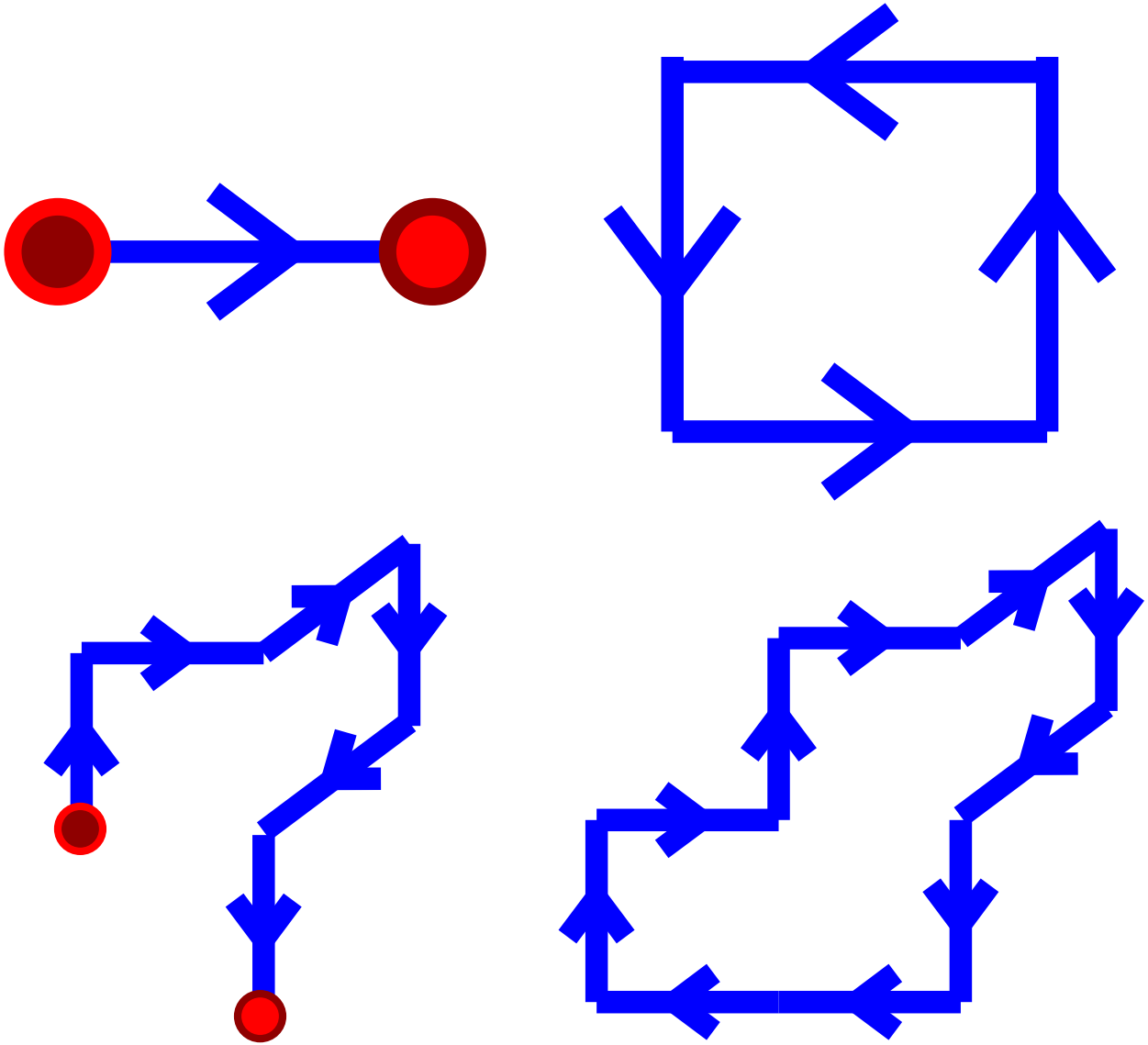
- Since $\Lambda^{\dagger}\Lambda = 1$, the following expressions are invariant under these transformations

Simple lattice gauge invariant functions

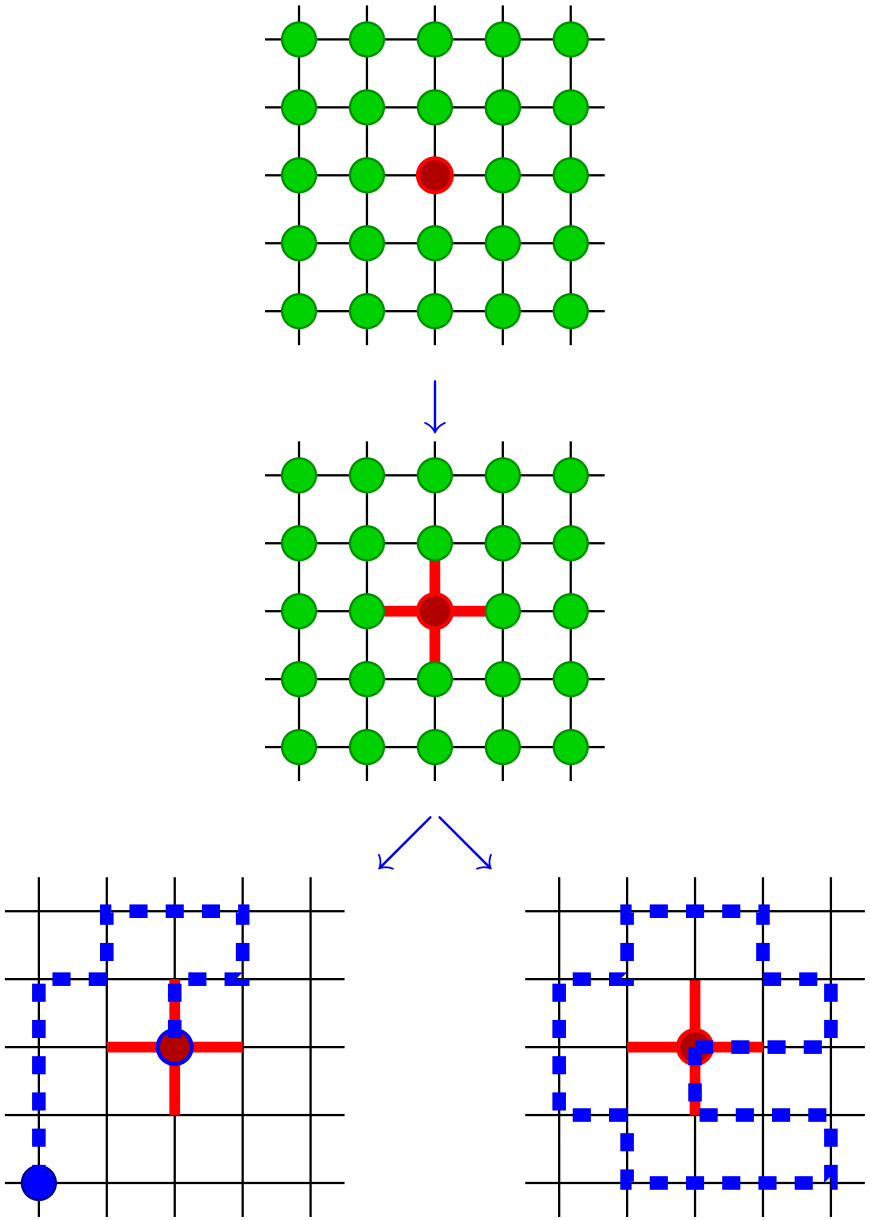
$$\bar{\psi}(x)U_\mu(x)\psi(x+\hat{\mu})$$

$$\mathrm{Tr}\ U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^{\dagger}(x+\hat{\nu})U_\nu^{\dagger}(x)$$

Lattice gauge invariants



Gauge invariance



To rotate a quark field at site x, $\psi(x) \to \psi^g(x) = g(x)\psi(x)$ ...

...we must also rotate the gauge fields that start or end at the site $U_\mu(x) \to U_\mu^g(x) = g(x)U_\mu(x)g^\dagger(x+\hat{\mu})$

The gauge invariance of the special functions is seen

Lattice action - the gluons

- To define a path integral, we also need an action
- The simplest gauge invariant function of the gauge link variables alone is the *plaquette* (the trace of a path-ordered product of links around a 1×1 square).

$$S_G[U] = \frac{\beta}{N_c} \sum_{x,\mu<\nu} \mathrm{ReTr}\ \left(1 - U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^\dagger(x+\hat{\nu}) U_\nu^\dagger(x))\right)$$

  This is the *Wilson gauge action*
- A path integral for the Yang-Mills theory of gluons would be

$$Z_{YM} = \int \prod_{\mu,x} \mathcal{D}U_\mu(x) e^{-S_G[U]}$$

- The coupling constant, g appears in $\beta = \frac{2N_c}{g^2}$
- No need to fix gauge; the gauge orbits can be trivially integrated over and the group manifold is compact.

Lattice action - the gluons

- A Taylor expansion in a shows that

$$\begin{aligned} S_G[U] &= \frac{\beta}{N_c} \sum_{x,\mu<\nu} \text{ReTr} \left(1 - U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^\dagger(x+\hat{\nu}) U_\nu^\dagger(x))\right) \\ &= \int d^4x \; -\frac{1}{4}\text{Tr}\; F_{\mu\nu}F_{\mu\nu} + \mathcal{O}(a^2) \end{aligned}$$

- As with the scalar boson lattice action, all terms proportional to odd powers in the lattice spacing vanish because the lattice action preserves a discrete parity symmetry.
- The action is also invariant under a charge-conjugation symmetry, which takes $U_\mu(x) \to U_\mu^*(x)$.
- We have kept almost all of the symmetries of the Yang-Mills sector, but broken the $SO(4)$ rotation group down to the discrete group of rotations of a hypercube.

Lattice actions - the quarks

- The continuum action is a bilinear with a first-order derivative operator inside;

$$S_Q = \int d^4x \bar{\psi}(\gamma_\mu D_\mu + m)\psi$$

- When $m = 0$, the action has an extra, chiral symmetry:

$$\psi \longrightarrow \psi^{(\chi)} = e^{i\alpha\gamma_5}\psi, \bar{\psi} \longrightarrow \bar{\psi}^{(\chi)} = \bar{\psi}e^{i\alpha\gamma_5}$$

- The simplest lattice representation of a first-order derivative that preserves reflection symmetries is the central difference:

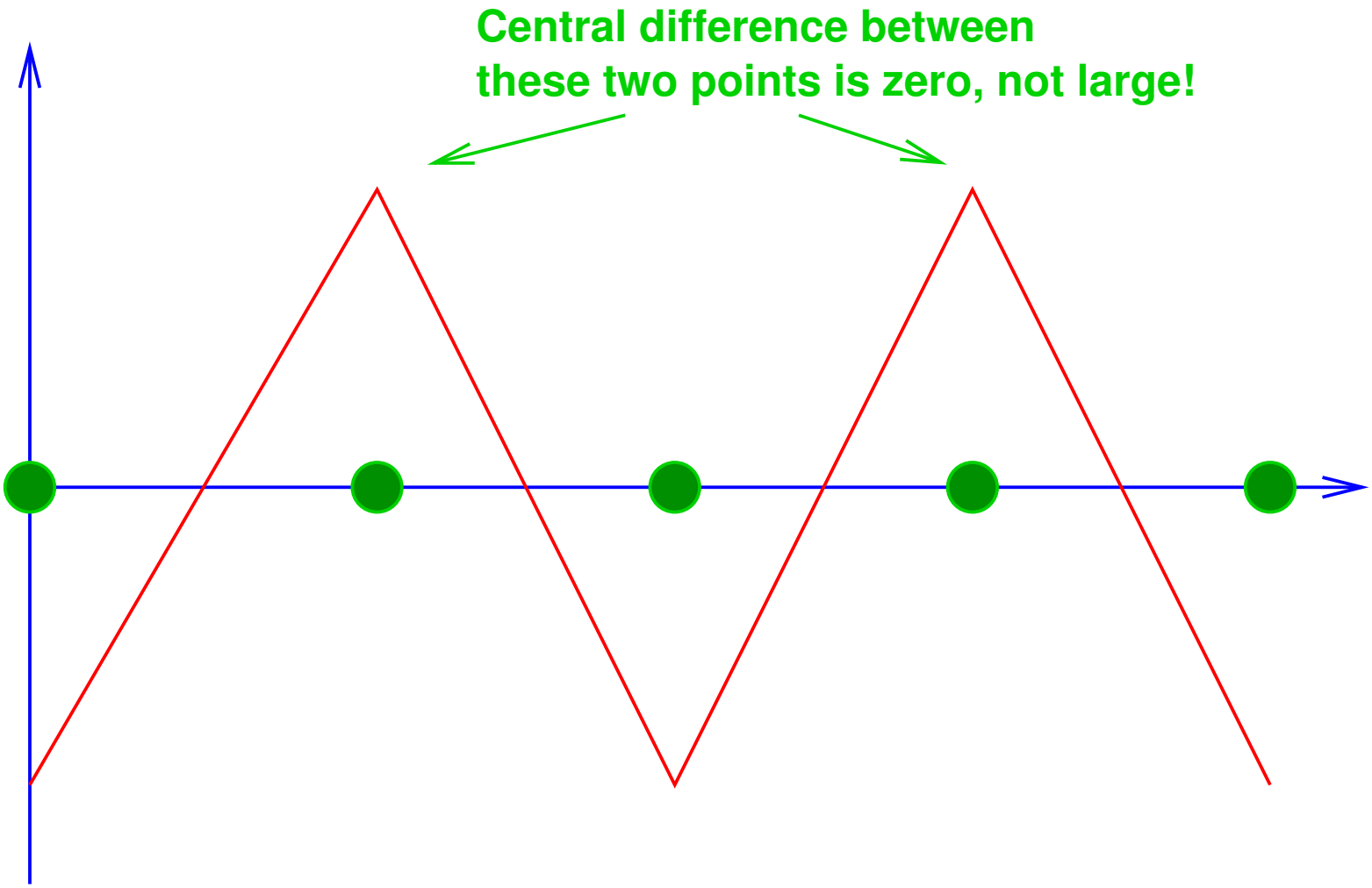
$$\partial_\mu \psi(x) = \frac{1}{2a}\left(\psi(x+\hat{\mu}) - \psi(x-\hat{\mu})\right)$$

- This can be made gauge covariant by including the gauge links:

$$D_\mu \psi(x) = \frac{1}{2a}\left(U_\mu(x)\psi(x+\hat{\mu}) - U_\mu(x-\hat{\mu})\psi(x-\hat{\mu})\right)$$

- **BUT** on closer inspection, there are more minima to this action than we want. Consider the case with no gauge fields, and when $\psi(x) = e^{ikx}$ with $k = \{\pi, 0, 0, 0\}$ or $\{\pi, \pi, 0, 0\}$ or $\{\pi, \pi, \pi, 0\}$ or $\ldots$.

Lattice doubling



Lattice actions - the quarks (3)

- This is the (in)famous **doubling problem**.

The Nielson-Ninomiya "no-go" theorem

There are **no** chirally symmetric, local, translationally invariant doubler-free fermion actions on a regular lattice.

- To put quarks on the lattice, more symmetry must be broken or else a theory with extra flavours of quarks must be simulated.
- A number of solutions are used, each with their advantages and disadvantages.
- The most commonly used are:
  - Wilson fermions
  - Kogut-Susskind (staggered) fermions
  - Ginsparg-Wilson fermions (overlap, domain wall, perfect...)
  - Twisted mass

Wilson's lattice quark action

- Wilson's original solution was to abandon chiral symmetry and add a lattice operator whose continuum limit is an irrelevant dimension-five operator. The term gives the doublers a mass $\propto 1/a$
- The extra term in the lattice action is the lattice representation of

$$a \sum_{\mu} D_{\mu}^{2} \psi \approx \sum_{\mu} U_{\mu}(x)\psi(x+\hat{\mu}) + U_{\mu}^{\dagger}(x-\mu)\psi(x-\hat{\mu})$$

- The breaking of chiral symmetry means the quark mass is not protected from additive renormalisations (short-distance gluons will now give quarks a large mass)
- Approaching the continuum limit requires fine-tuning to restore chiral symmetry and ensure quarks are light.
- Breaking chiral symmetry now introduces lattice artefacts at $\mathcal{O}(a)$.
- This action has a Symanzik-improved counterpart, the Sheikholeslami-Wohlert action, which removes all $\mathcal{O}(a)$ errors by a field redefinition and the addition of another dim-5 term, $\sigma_{\mu\nu}F_{\mu\nu}$

The Ginsparg-Wilson relation

- Actions that break chiral symmetry, but preserve a modified version can be constructed. The new chiral symmetry is

$$\{\gamma_5, \not{D}\} = 2a\,\not{D}\gamma_5\not{D} \text{ so } \{\gamma_5, \not{D}^{-1}\} = 2a\,\gamma_5$$

- In a propagator, chiral symmetry is broken by a contact term
- A number of realisations of this symmetry are in use. Neuberger's overlap uses an action

$$D = I - \frac{D_W}{\sqrt{D_W^\dagger D_W}}$$

  where D_W is the Wilson action with a large negative quark mass.
- Domain Wall quarks use a 5d lattice field (coupled to four-dimensional gluons). The boundaries in the 5th dimension are set up so left- and right-handed quarks bind to different walls in 5d. Modes are separated so chiral symmetry is (almost) maintained.
- These quarks are expensive!

Staggered quarks

- Kogut and Susskind proposed an interesting partial solution to the doubling problem.
- A field redefinition is used to scatter the sixteen components of four flavours ("tastes") of quarks across the corners of a hypercube.
- On each lattice sites there are just N_c degrees of freedom
- A remnant of chiral symmetry remains which is sufficient to ensure there is no additive mass renormalisation.
- Simulations are fast; there is no fine-tuning so the fermion matrix is well-behaved and always positive which helps the simulation algorithms
- UV gluons can change the "taste" of a quark, so flavours mix
- Practitioners simulate theories with one or two flavours by taking fractional powers of the fermion path integral. It is still a matter of debate whether this is legitimate.

Summary

- Gauge symmetry on the lattice can be maintained. This is crucial to ensure we're simulating QCD and without gauge symmetry, we would be stuck with an impossible fine-tuning problem.
- For gauge invariance, **quarks on site, gluons on links**. Now we have a set of gauge invariant functions to construct actions, observables, etc.
- A simple gluon action can be constructed from the trace of the product of links around a small square. This preserves most of the symmetries of the continuum Yang-Mills action
- Quarks suffer from **doubling**. A set of different "work-arounds" exist, but all have drawbacks (they either break a symmetry or are numerically expensive).