Transverse (Spin) Structure of Hadrons

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electromagnetic form factor ⇒ charge distribution in position space

depth-inelastic lepton-nucleon scattering
(DIS) ⇒ momentum distribution of quarks in nucleon

deply virtual Compton scattering
(DVCS) ⇒ generalized parton distributions (GPDs)
  ⇒ simultaneous determination of (longitudinal) momentum and
  (transverse) position of quarks in the nucleon
  ⇒ 3-d images of the nucleon where $x - y$ plane in position space
  and $z$ axis in momentum space

what is orbital angular momentum?

single-spin asymmetries (SSA)

transverse force on quarks in polarized DIS
Physical Interpretation of Form Factors

- Nonrelativistic quantum mechanics (NRQM)
- Relativistic effects
- Breit frame interpretation
- Infinite momentum frame (IMF)
Form Factors in nonrel. QM

- Potential scattering

\[
\frac{d\sigma}{d\Omega} = |f(\vec{q})|^2
\]

- Born approx. ("1-photon exchange" for Coulomb interaction)

\[
f(\vec{q}) = \int d^3r e^{-i\vec{q} \cdot \vec{r}} V(\vec{r})
\]

- Electro-magnetic interactions: \( V(\vec{r}) = e A^0(\vec{r}) \) with

\[
\nabla^2 A^0(\vec{r}) = -4\pi \rho(\vec{r}), \text{ i.e. } f(\vec{q}) = \frac{4\pi e}{q^2} \int d^3r e^{-i\vec{q} \cdot \vec{r}} \rho(\vec{r})
\]

\[
\Rightarrow \quad \frac{d\sigma}{d\Omega} = |f(\vec{q})|^2 = \frac{d\sigma}{d\Omega}\bigg|_{\text{point}} \times |F(\vec{q})|^2 \quad \text{with} \quad F(\vec{q}) = \int d^3r e^{-i\vec{q} \cdot \vec{r}} \rho(\vec{r})
\]

\[
\frac{d\sigma}{d\Omega} \rightarrow F(\vec{q}) \rightarrow \rho(\vec{r})
\]
Form Factor (Interpretation)

- small $q^2$ expansion

$$F(q) = 1 - \frac{1}{6} \langle r^2 \rangle q^2 + O(q^4)$$

- Hofstadter (NP 1961): first measurement of $F(q)$ using electron nucleon scattering

$\sqrt{\langle r_p^2 \rangle} \approx 0.86 \text{ fm}$
Issues

- Form factor of a “moving target”
- Relativistic effects
- Breit frame
- Infinite momentum frame (IMF)
Moving Target (Meson, Nucleon,..)

- Fixed target: $F(q) \overset{FT}{\leftrightarrow} \rho(r)$ trivial
Fixed target: \( F(\vec{q}) \overset{FT}{\leftrightarrow} \rho(\vec{r}) \) trivial

Moving target (nonrel.) \( H_{int} = e \int d^3 \vec{r} A^0(\vec{r}) j^0(\vec{r}) \)

\[
\frac{d\sigma}{d\Omega} \propto |\langle \vec{p}' \vert H_{int} \vert \vec{p} \rangle|^2
\]

\[
\rightarrow \quad \frac{d\sigma}{d\Omega} = \left. \frac{d\sigma}{d\Omega} \right|_{point} \times |F(\vec{q})|^2 \quad \text{with} \quad F(\vec{q}) \equiv \langle \vec{p}' \vert j^0(\vec{0}) \vert \vec{p} \rangle
\]

\((\vec{q} = \vec{p} - \vec{p}')\)
Fixed target: $F(q) \overset{FT}{\leftrightarrow} \rho(r)$ trivial

Moving target (nonrel.) $H_{int} = e \int d^3 r A^0(r) j^0(r)$

$$\frac{d\sigma}{d\Omega} \propto |\langle \vec{p}' | H_{int} | \vec{p} \rangle|^2$$

$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}_{point} \times |F(q)|^2 \quad \text{with} \quad F(q) \equiv \langle \vec{p}' | j^0(\vec{0}) | \vec{p} \rangle$$

$(q = \vec{p} - \vec{p}')$

Relation to $\rho(r)$?
Moving Target (Meson, Nucleon,..)

- Fixed target: $F(\vec{q}) \overset{FT}{\leftrightarrow} \rho(\vec{r})$ trivial
- Moving target (nonrel.) $H_{int} = e \int d^3\vec{r} A^0(\vec{r}) j^0(\vec{r})$

$$\frac{d\sigma}{d\Omega} \propto |\langle \vec{p}' | H_{int} | \vec{p} \rangle|^2$$

$$\leftrightarrow \quad \frac{d\sigma}{d\Omega} = \left. \frac{d\sigma}{d\Omega} \right|_{\text{point}} \times |F(\vec{q})|^2 \quad \text{with} \quad F(\vec{q}) \equiv \left\langle \vec{p}' | j^0(\vec{0}) | \vec{p} \right\rangle$$

$(\vec{q} = \vec{p} - \vec{p}')$

- Relation to $\rho(\vec{r})$?
- What is $\rho(\vec{r})$?
plane wave states have uniform charge distribution

meaningful definition of $\rho(\vec{r})$ requires that state is localized in position space!

define localized state (center of mass frame)

$$| \vec{R} = \vec{0} \rangle \equiv \mathcal{N} \int d^3 \vec{p} | \vec{p} \rangle$$

define charge distribution (for this localized state)

$$\rho(\vec{r}) \equiv \langle \vec{R} = \vec{0} | j^0(\vec{r}) | \vec{R} = \vec{0} \rangle$$
use translational invariance to relate to same matrix element that appears in def. of form factor

\[ \rho(\vec{r}) \equiv \langle \vec{R} = 0 | j^0(\vec{r}) | \vec{R} = 0 \rangle \]

\[ = |\mathcal{N}|^2 \int d^3 \vec{p} \int d^3 \vec{p}' \langle \vec{p}' | j^0(\vec{r}) | \vec{p} \rangle \]

\[ = |\mathcal{N}|^2 \int d^3 \vec{p} \int d^3 \vec{p}' \langle \vec{p}' | j^0(\vec{0}) | \vec{p} \rangle e^{i\vec{r} \cdot (\vec{p}' - \vec{p})} , \]

\[ = |\mathcal{N}|^2 \int d^3 \vec{p} \int d^3 \vec{p}' F \left( - (\vec{p}' - \vec{p})^2 \right) e^{i\vec{r} \cdot (\vec{p}' - \vec{p})} \]

\[ \rho(\vec{r}) = \int \frac{d^3 \vec{q}}{(2\pi)^3} F(-q^2) e^{i\vec{q} \cdot \vec{r}} \]
Form Factors (relativistic)

- Lorentz invariance, parity, current conservation \( \Rightarrow \)

\[
\langle p' \mid j^\mu(0) \mid p \rangle = \begin{cases} 
(p^\mu + p'^\mu') \ F(q^2) & \text{(spin 0)} \\
\bar{u}(p') \left[ \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q^\nu}{2M} F_2(q^2) \right] u(p) & \text{(spin } \frac{1}{2} \text{)}
\end{cases}
\]

with \( q^\mu = p^\mu - p'^\mu' \).

- issues:
  - “energy factors” spoil simple interpretation of form factors as FT of charge distributions
  - \( q^2 \) depends on \( \vec{p} + \vec{p}' \)
wave packet

\[ |\Psi\rangle = \int \frac{d^3p}{\sqrt{2E_p(2\pi)^3}} \psi(\vec{p}) |\vec{p}\rangle, \]

\[ E_{\vec{p}} = \sqrt{M^2 + \vec{p}^2} \] and covariant normalization \( \langle \vec{p}' | \vec{p} \rangle = 2E_{\vec{p}} \delta(\vec{p}' - \vec{p}) \)

Fourier transform of charge distribution in the wave packet

\[ \tilde{\rho}(\vec{q}) \equiv \int d^3x e^{-i\vec{q} \cdot \vec{x}} \langle \Psi | j^0(\vec{x}) | \Psi \rangle \]

\[ = \int \frac{d^3p}{\sqrt{2E_p2E_{\vec{p}'}}} \Psi^*(\vec{p} + \vec{q}) \Psi(\vec{p}) \langle \vec{p}' | j^0(\vec{0}) | \vec{p} \rangle \]

\[ = \frac{1}{2} \int d^3p \frac{E_{\vec{p}} + E_{\vec{p}'}}{\sqrt{E_{\vec{p}}E_{\vec{p}'}}} \Psi^*(\vec{p} + \vec{q}) \Psi(\vec{p}) F(q^2). \]
Nonrelativistic case:

\[
\frac{E_{\vec{p}} + E_{\vec{p}'}'}{2 \sqrt{E_{\vec{p}} E_{\vec{p}'}}} = 1 \quad \text{and} \quad q^2 = -\bar{q}^2
\]

\[\rightarrow\] Fourier transform of charge distribution in the wave packet

\[
\tilde{\rho}(\vec{q}) = \int d^3 p \Psi^*(\vec{p} + \vec{q}) \Psi(\vec{p}) F(\bar{q}^2)
\]

choose \( \Psi(\vec{p}) \) very localized in position space

\[\Psi^*(\vec{p} + \vec{q}) \approx \Psi^*(\vec{p})\]

\[\rightarrow\] \( \tilde{\rho}(\vec{q}) = F(\bar{q}^2) \)
Relativistic corrections (example rms radius):

\[ \tilde{\rho}(q^2) = 1 - \frac{R^2}{6} q^2 - \frac{R^2}{6} \int d^3 p |\Psi(\vec{p})|^2 \frac{(|\vec{q} \cdot \vec{p}|)^2}{E^2_\vec{p}} \]

\[ + \int d^3 p \left| \vec{q} \cdot \vec{\nabla} \Psi(\vec{p}) \right|^2 - \frac{1}{8} \int d^3 p |\Psi(\vec{p})|^2 \frac{(|\vec{q} \cdot \vec{p}|)^2}{E^4_\vec{p}} , \]

\[ R^2 \text{ defined as usual: } F(q^2) = 1 + \frac{R^2}{6} q^2 + \mathcal{O}(q^4) \]

If one completely localizes the wave packet, i.e. \( \int d^3 p \left| \vec{q} \cdot \vec{\nabla} \Psi(\vec{p}) \right|^2 \to 0 \), then relativistic corrections diverge \( (\Delta x \Delta p \sim 1) \)

\[ \frac{R^2}{6} \int d^3 p |\Psi(\vec{p})|^2 \frac{(|\vec{q} \cdot \vec{p}|)^2}{E^2_\vec{p}} \to \infty , \quad \frac{1}{8} \int d^3 p |\Psi(\vec{p})|^2 \frac{(|\vec{q} \cdot \vec{p}|)^2}{E^4_\vec{p}} \to \infty \]
in rest frame, rel. corrections contribute $\Delta R^2 \sim \chi_C^2 = \frac{1}{M^2}$

identification of charge distribution in rest frame with Fourier transformed form factor only unique down to scale $\lambda_C$

standard remedy: interpret $F(q)$ as Fourier transform of charge distribution in “Breit frame” $\vec{p}' = -\vec{p}$

$G_E(q^2) \equiv F_1(q^2) + \frac{q^2}{4M^2} F_2(q^2)$  $G_M(q^2) \equiv F_1(q^2) + F_2(q^2)$

Physics (Breit “frame”):

$\langle \vec{q}/2 | j^0(0) | - \vec{q}/2 \rangle = 2MG_E(q^2)\delta_{s's}$

$\langle \vec{q}/2 | \vec{j}(0) | - \vec{q}/2 \rangle = G_M(q^2)\chi^\dagger_{s'}i\vec{\sigma} \times \vec{q}\chi_s$

$\leftrightarrow G_E \xrightarrow{FT} \text{charge density}; \quad G_M \xrightarrow{FT} \text{magnetization density};$

flaw of this interpretation: Breit “frame” is not one single frame but a different frame for each momentum transfer
Infinite Momentum Frame Interpretation

- rel. corrections governed by $\frac{\vec{p} \cdot \vec{q}}{E^2_p}$ and $\frac{\vec{q}^2}{E^2_p}$

consider wave packet $\Psi(\vec{p}_\perp)$ in transverse direction, with

- sharp longitudinal momentum $P_z \rightarrow \infty$
- transverse size of wave packet $r_\perp$, with $R \gg r_\perp \gg \frac{1}{P_z}$
- take purely transverse momentum transfer

$\rightarrow \tilde{\rho}(\vec{q}_\perp) = F(\vec{q}_\perp^2)$

$\rightarrow$ form factor can be interpreted as Fourier transform of charge distribution w.r.t. impact parameter in $\infty$ momentum frame (without $\lambda_C$ uncertainties!)

- impact parameter measured w.r.t. $\perp$ center of momentum

$R_\perp = \sum_{i \in q,g} x_i r^i_\perp$
Derivation in LF-Coordinates

- light-front (LF) coordinates

\[ p^+ = \frac{1}{\sqrt{2}} (p^0 + p^3) \quad p^- = \frac{1}{\sqrt{2}} (p^0 - p^3) \]

- form factor for spin \( \frac{1}{2} \) target (Lorentz invariance, parity, charge conservation)

\[ \langle p' | j^\mu (0) | p \rangle = \bar{u}(p') \left[ \gamma^\mu F_1 (q^2) + \frac{i \sigma^{\mu \nu} q^\nu}{2M} F_2 (q^2) \right] u(p) \]

with \( q^\mu = p^\mu - p'^\mu \).

- If \( q^+ = 0 \) (Drell-Yan-West frame) then

\[ \langle p', \uparrow | j^+(0) | p, \uparrow \rangle = 2p^+ F_1 (-q^2) \]
$F(q^2_{\perp}) \rightarrow \rho(r_{\perp})$ in LF-Coordinates

- define state that is localized in $\perp$ position:

$$|p^+, R_{\perp} = 0_{\perp}, \lambda\rangle \equiv \mathcal{N} \int d^2p_{\perp} |p^+, p_{\perp}, \lambda\rangle$$

Note: $\perp$ boosts in IMF form Galilean subgroup $\Rightarrow$ this state has $R_{\perp} \equiv \frac{1}{p^+} \int dx^- d^2x_{\perp} x_{\perp} T^{++}(x) = 0_{\perp}$

(cf.: working in CM frame in nonrel. physics)

- define charge distribution in impact parameter space

$$\rho(b_{\perp}) \equiv \frac{1}{2p^+} \langle p^+, R_{\perp} = 0_{\perp} | j^+(0^-, b_{\perp}) | p^+, R_{\perp} = 0_{\perp}\rangle$$
\[ F(q_{\perp}^2) \rightarrow \rho(r_{\perp}) \text{ in LF-Coordinates} \]

- use translational invariance to relate to same matrix element that appears in def. of form factor

\[
\rho(b_{\perp}) \equiv \frac{1}{2p^+} \left< p^+, R_{\perp} = 0_{\perp} \mid j^+(0^-, b_{\perp}) \mid p^+, R_{\perp} = 0_{\perp} \right>
\]

\[
= \frac{|N|^2}{2p^+} \int d^2p_{\perp} \int d^2p'_{\perp} \left< p^+, p'_{\perp} \mid j^+(0^-, b_{\perp}) \mid p^+, p_{\perp} \right>
\]

\[
= \frac{|N|^2}{2p^+} \int d^2p_{\perp} \int d^2p'_{\perp} \left< p^+, p'_{\perp} \mid j^+(0^-, 0_{\perp}) \mid p^+, p_{\perp} \right> e^{i q_{\perp} \cdot b_{\perp}}
\]

\[
= |N|^2 \int d^2p_{\perp} \int d^2p'_{\perp} F_1(-q_{\perp}^2) e^{i q_{\perp} \cdot b_{\perp}}
\]

\[
\leftrightarrow \quad \rho(b_{\perp}) = \int \frac{d^2q_{\perp}}{(2\pi)^2} F_1(-q_{\perp}^2) e^{i q_{\perp} \cdot b_{\perp}}
\]
Boots in NRQM

\[ \vec{x}' = \vec{x} + \vec{v}t \quad t' = t \]

purely kinematical (quantization surface \( t = 0 \) inv.)

1. boosting wavefunctions very simple

\[ \Psi_{\vec{v}}(\vec{p}_1, \vec{p}_2) = \Psi_{\vec{0}}(\vec{p}_1 - m_1 \vec{v}, \vec{p}_2 - m_2 \vec{v}). \]

2. dynamics of center of mass

\[ \vec{R} = \sum_i x_i \vec{r}_i \quad \text{with} \quad x_i = \frac{m_i}{M} \]

decouples from the internal dynamics
Relativistic Boosts

\[ t' = \gamma \left( t + \frac{v}{c^2} z \right), \quad z' = \gamma (z + vt), \quad x'_\perp = x_\perp \]

- generators satisfy Poincaré algebra:

\[
\begin{align*}
[P^\mu, P^\nu] &= 0 \\
[M^{\mu\nu}, P^\rho] &= i \left( g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu \right) \\
[M^{\mu\nu}, M^{\rho\lambda}] &= i \left( g^{\mu\lambda} M^{\nu\rho} + g^{\nu\rho} M^{\mu\lambda} - g^{\mu\rho} M^{\nu\lambda} - g^{\nu\lambda} M^{\mu\rho} \right)
\end{align*}
\]

- rotations: \( M_{ij} = \varepsilon_{ijk} J_k \), boosts: \( M_{i0} = K_i \).

- \( [K_i, P_j] = i \delta_{ij} P^0 \) 

\( \leftrightarrow \) boost operator contains interactions!

\( \leftrightarrow \) in general, no useful generalization of concept of center of mass to a relativistic theory
introduce generator of \( \perp \) ‘boosts’:

\[
B_x \equiv M^{+x} = \frac{K_x + J_y}{\sqrt{2}} \quad B_y \equiv M^{+y} = \frac{K_y - J_x}{\sqrt{2}}
\]

Poincaré algebra \( \implies \) commutation relations:

\[
[J_3, B_k] = i \varepsilon_{kl} B_l \quad [P_k, B_l] = -i \delta_{kl} P^+
\]

\[
[P^-, B_k] = -i P_k \quad [P^+, B_k] = 0
\]

with \( k, l \in \{x, y\} \), \( \varepsilon_{xy} = -\varepsilon_{yx} = 1 \), and \( \varepsilon_{xx} = \varepsilon_{yy} = 0 \).
Galilean subgroup of $\perp$ boosts

Together with $[J_z, P_k] = i\varepsilon_{kl} P_l$, as well as

$$
\begin{align*}
[P^-, P_k] &= [P^-, P^+] = [P^-, J_z] = 0 \\
[P^+, P_k] &= [P^+, B_k] = [P^+, J_z] = 0.
\end{align*}
$$

Same as commutation relations among generators of nonrel. boosts, translations, and rotations in x-y plane, provided one identifies

$$
\begin{align*}
P^- &\quad\rightarrow\quad \text{Hamiltonian} \\
P_{\perp} &\quad\rightarrow\quad \text{momentum in the plane} \\
P^+ &\quad\rightarrow\quad \text{mass} \\
L_z &\quad\rightarrow\quad \text{rotations around } z\text{-axis} \\
B_{\perp} &\quad\rightarrow\quad \text{generator of boosts in the plane},
\end{align*}
$$
many results from NRQM carry over to $\perp$ boosts in IMF, e.g.

$\perp$ boosts kinematically

$$
\Psi_{\Delta_\perp}(x, k_\perp) = \Psi_{0_\perp}(x, k_\perp - x\Delta_\perp)
$$

$$
\Psi_{\Delta_\perp}(x, k_\perp, y, l_\perp) = \Psi_{0_\perp}(x, k_\perp - x\Delta_\perp, y, l_\perp - y\Delta_\perp)
$$

Transverse center of momentum $R_\perp \equiv \sum_i x_i r_\perp,i$ plays role similar to NR center of mass, e.g. $\int d^2p_\perp |p^+, p_\perp\rangle$ corresponds to state with $R_\perp = 0_\perp$. 
Summary: Form Factor vs. Charge Distribution

- fixed target: FT of form factor = charge distribution in position space
- “moving” target:
  - nonrelativistically: FT of form factor = charge distribution in position space, where position is measured relative to center of mass
  - relativistic corrections usually make identification
    \[ F(q^2) \leftrightarrow F_T(\vec{r}) \] ambiguous at scale \( \Delta R \sim \lambda_C = \frac{1}{M} \)
  - Sachs form factors have interpretation as charge and magnetization density in Breit “frame”
  - Infinite momentum frame: form factors can be interpreted as transverse charge distribution in fast moving proton (without rel. corrections)
Deep Inelastic Scattering (DIS)

- high-energy lepton \((e^\pm, \mu^\pm, \nu, )\) nucleon scattering usually inelastic

- for \(Q^2 = -q^2 = -(p - p')^2 \gg M_p^2 \sim 1\text{GeV}^2\), probe can resolve distance scales much smaller than the proton size

\[ \rightarrow \text{deep inelastic scattering} \]

- because of high \(Q^2\), inclusive (i.e. sum over final states) cross section obtained from incoherent superposition of charged constituents

\[ \rightarrow \text{DIS provided first direct evidence for the existence of quarks inside nucleons (Nobel Prize 1990: Freedman, Kendall, Taylor)} \]
Deep Inelastic Scattering (DIS)

\[ \nu = E - E' \]
\[ Q^2 \equiv -q^2 = 4EE' \sin^2 \frac{\theta}{2} \]

\[
\frac{d^2\sigma}{d\Omega dE'} = \frac{4\pi\alpha^2}{MQ^4} \left\{ W_2(Q^2, \nu) \cos^2 \frac{\theta}{2} + 2W_1(Q^2, \nu) \sin^2 \frac{\theta}{2} \right\}
\]

exp. result: Bjorken scaling

\[
Q^2 \to \infty, \quad \nu \to \infty \quad \text{fixed} \quad \begin{cases} 2M W_1(Q^2, \nu) = F_1(x_{Bj}) \\ \nu W_2(Q^2, \nu) = F_2(x_{Bj}) \end{cases}
\]
Physical Meaning of $x_{Bj} = \frac{Q^2}{2p\cdot q}$

- Go to frame where $q_{\perp} = 0$, i.e.
  \[ Q^2 = -q^2 = -2q^+ q^- \quad 2p \cdot q = 2q^- p^+ + 2q^+ p^- \]

- Bjorken limit: $q^- \to \infty$, $q^+$ fixed

\[ x_{Bj} = -\frac{q^+ q^-}{q^- p^+ + q^+ p^-} \to -\frac{q^+}{p^+} \]
Physical Meaning of $x_{Bj} = \frac{Q^2}{2p\cdot q}$

- $x_{Bj} = -\frac{q^+}{p^+}$

- LC energy-momentum dispersion relation

$$k^- = \frac{m^2 + k^2_{\perp}}{2k^+}$$

→ struck quark with $k^{-'} = k^- + q^- \rightarrow \infty$ can only be on mass shell if $k^{+'} = k^+ + q^+ \approx 0$

→

$$k^+ = -q^+ \quad \Rightarrow \quad x \equiv \frac{k^+}{p^+} = x_{Bj}$$

→ $x_{Bj}$ has physical meaning of light-cone momentum fraction carried by struck quark before it is hit by photon
opt. theorem:
inclusive cross-section ⇔ virtual, forward Compton amplitude

\[ \sum_X \sigma(ep \rightarrow e' X) = \mathcal{F} \]
\[ = \mathcal{F} + \mathcal{F} \]

struck quark carries large momentum: \( Q^2 \gg \Lambda_{\text{QCD}}^2 \)

- crossed diagram suppressed (wavefunction!)
- asymptotic freedom \( \Rightarrow \) neglect interactions of struck quark
- struck quark propagates along light-cone \( x^2 = 0 \)
suppression of crossed diagrams

Flow of the large momentum $q$ through typical diagrams contributing to the forward Compton amplitude. a) ‘handbag’ diagrams; b) ‘cat’s ears’ diagram. Diagram b) is suppressed at large $q$ due to the presence of additional propagators.
opt. theorem:
inclusive cross-section ⇔ virtual, forward Compton amplitude

\[ \sum_X \sigma (e p \rightarrow e' X) = \mathcal{F} \]

\[ = \mathcal{F} + \mathcal{F} \]

struck quark carries large momentum: \( Q^2 \gg \Lambda_{QCD}^2 \)

- crossed diagram suppressed (wavefunction!)
- asymptotic freedom ⇒ neglect interactions of struck quark
- struck quark propagates along light-cone \( x^2 = 0 \)
DIS → light-cone correlations

light-cone coordinates:

\[ x^+ = \left( x^0 + x^3 \right) / \sqrt{2} \]

\[ x^- = \left( x^0 - x^3 \right) / \sqrt{2} \]

DIS related to correlations along light–cone

\[ q(x_{Bj}) = \int \frac{dx^-}{2\pi} \langle P|\bar{q}(0^-, 0_\perp)^+ q(x^-, 0_\perp)|P\rangle e^{ix^-x_{Bj}P^+} \]

**Probability interpretation!**

No information about transverse position of partons!
\[ q(x_{Bj}) = \int \frac{dx^-}{2\pi} \langle P|\overline{q}(0^-, 0_\perp)\gamma^+ q(x^-, 0_\perp)|P\rangle e^{ix^-x_{Bj}P^+} \]

- Fourier transform along \( x^- \) filters out quarks with light-cone momentum \( k^+ = x_{Bj}P^+ \)
- momentum distribution = FT of equal time correlation function
- boost to IMF tilts equal time plane arbitrarily close to \( x^+ = \text{const.} \) plane
- \( q(x) = \text{light-cone momentum distribution or momentum distribution in IMF} \)
unpolarized $ep$ DIS

$\sigma_{elast}^{eq} \propto Q_q^2$

$e + p \rightarrow e' + X$ sensitive to:

\[
\frac{4}{9} \left[ u_p^\uparrow(x) + \bar{u}_p^\uparrow(x) + u_p^\downarrow(x) + \bar{u}_p^\downarrow(x) \right] + \\
\frac{1}{9} \left[ d_p^\uparrow(x) + d_p^\downarrow(x) + d_p(x) + d_p(x) \right] + \\
\frac{1}{9} \left[ s_p^\uparrow(x) + s_p^\downarrow(x) + s_p(x) + s_p(x) \right] + \ldots
\]

where e.g. $u_p^\uparrow(x), u_p^\downarrow(x), \bar{u}_p^\uparrow(x), \bar{u}_p^\downarrow(x), \ldots$ are the distribution of $u, \bar{u}, \ldots$ in the proton with spin parallel/antiparallel to the proton’s spin.

neutron target (charge symmetry)

\[ u_n(x) = d_p(x), \quad d_n(x) = u_p(x), \quad s_n(x) = s_p(x) \]

sensitive to $\frac{4}{9} d_p(x) + \frac{1}{9} u_p(x) + \frac{1}{9} s_p(x)$

different linear combination of the same distribution functions!
many important results from DIS:

- discovery of elementary, charged, spin $\frac{1}{2}$ constituents in the nucleon $\rightarrow$ quarks
- failure of momentum sum rule, i.e. quarks carry only about 50% of the nucleon’s momentum $\rightarrow$ gluons

some recent puzzles:

- nuclear binding effect on structure functions (EMC collaboration): large and systematic modification of the nucleon’s parton distribution in a bound nucleus
- failure of the "Ellis-Jaffe sum rule" for the spin dependent structure function $g_1(x)$ (EMC collaboration, also SMC, E142): spin fraction carried by quarks $\equiv$

$$\sum_{q=u,d,s} \int_0^1 dx \left[ q^\uparrow(x) + \bar{q}^\uparrow(x) - q^\downarrow(x) - \bar{q}^\downarrow(x) \right] \ll 1$$

(nonrel quark model: 1) $\Rightarrow$ "spin crisis"
Longitudinally Polarized DIS

- $e^-/q$ helicity conserved in high-energy interactions
- Longitudinally polarized $e^-$ preferentially scatter off $q$ with spin opposite to that of the $e^-$
- Scattering long. pol. $e^-$ off long. pol. nucleons ⇒ quark/nucleon spin correlation

$$d\sigma^{\uparrow\uparrow} - d\sigma^{\uparrow\downarrow} \propto g_1(x) \quad \text{with} \quad g_1(x) = \sum_q e_q^2 \left[ q^\uparrow(x) - q^\downarrow(x) \right]$$

- $q^\uparrow(x)/q^\downarrow(x)$ = probability that $q$ has same/opposite spin as $N$
- $q(x) = q^\uparrow(x) + q^\downarrow(x)$
- $g_1(x)$ has been measured at CERN, SLAC, DESY, JLab, ...
- Future, more precise, measurements from JLab@12GeV, EIC
Longitudinally Polarized DIS

\[ \frac{q_\uparrow(x)}{q_\downarrow(x)} = \text{probability that } q \text{ has same/opposite spin as } N \]

\[ \text{spin sum rule (→ R.Jaffe)} \]

\[ \frac{1}{2} = \frac{1}{2} \Delta \Sigma + \Delta G + L_{\text{parton}} \]

\[ \Delta \Sigma = \sum_q \Delta q \equiv \sum_q \int_0^1 dx \left[ q_\uparrow(x) - q_\downarrow(x) \right] = \text{fraction of the nucleon spin due to quark spins} \]

\[ \Delta G = \text{fraction of the nucleon spin due to gluon spins} \]

\[ L_{\text{parton}} = \text{angular momentum due to quarks & gluon} \]

EMC collaboration (1987): only small fraction of the proton spin due to quark spins

incl. more recent data (CERN,SLAC,DESY): \( \sim 30\% \)

was called ‘spin crisis’, because \( \Delta \Sigma \) much smaller than the quark model result \( \Delta \Sigma = 1 \)

quest for the remaining 70\%
• Gluons, like photons, described by a massless vector field and carry intrinsic angular momentum (spin) \( \pm 1 \)

• Gluon contribution to nucleon momentum known to be large

\[
\langle x_g \rangle \equiv \int_0^1 dx \, x g(x) = 1 - \sum_q \int_0^1 dx \, x q(x) \approx 0.5
\]

(Physics of \( \langle x_g \rangle \): think of \( \mathbf{E} \times \mathbf{B} \))

• Conceivable that \( \Delta G \) is of the same order of magnitude as \( \langle x_g \rangle \) (or larger)

• Several ‘explanations’ of the ‘spin crisis’ even suggested \( \Delta G \sim 4 - 6 \)

• \( \Delta G \) accessible e.g. through
  • ‘QCD-evolution’ of \( \Delta q(x) \)
  • \( A_{LL} \) in \( \overrightarrow{p} \overrightarrow{p} \rightarrow \gamma + \text{jet} \)
Sometimes a gluon is not just a gluon (quantum fluctuations):
- $g \rightarrow \bar{q}q$
- $g \rightarrow gg, ggg$

Similar for quarks: $q \rightarrow qg$

Quantum fluctuations short distance effects become more ‘visible’ as $Q^2$ of the probe increases.

Resulting $Q^2$ dependence of PDFs described by QCD evolution equations (DGLAP evolution): coupled integro-differential equations with perturbatively calculable kernel.

$\bar{q}q$ pair ‘inherits’ gluon spin in $g \rightarrow \bar{q}q$

Infer $\Delta G$ from $Q^2$ dependence of $\Delta q$.

Need coverage down to small $x$ ($g(x)$ concentrated at very small $x$) and wide $Q^2$ range (‘QCD evolution’ slow).

Planned EIC (electron ion collider).
ΔG from $\vec{p} + \vec{p}$ (RHIC-spin)

- Double longitudinal-spin asymmetry: $A_{LL}$

- Study helicity dependent structure functions (Gluon polarization)

- $gg$ and $gq$ scattering sensitive to (relative) helicity

- Use double-spin asymmetry $A_{LL} = \frac{\sigma^{++} - \sigma^{+-}}{\sigma^{++} + \sigma^{+-}}$ in inelastic $pp$-scattering at RHIC to infer $\Delta G$ directly
\( \Delta \! \! \! G \) from \( \vec{p} + \vec{p} \) (RHIC-spin)

- **‘global analysis’** (DIS & RHIC data):
  - RHIC-spin substantially reduced error band (yellow) between \( x = .05 \) and \( x = .2 \)
  - despite remaining uncertainties, now evident that \(|\Delta \! \! \! G|\) significantly less than 1

- **big deal**: rules out a significant role of gluonic corrections to the quark spin as explanation for spin puzzle

- 500GeV at RHIC run with improved forward acceptance will reduce error band down to \( x \sim .002 \)
perturbative effects in general well understood, e.g. $Q^2$ dependence (=“evolution”) calculable in QCD (Altarelli, Parisi, Gribov, Lipatov eqs.): given $q(x, Q_0^2)$ one can calculate $q(x, Q_1^2)$ for $Q_2^2 > Q_1^2 > Q_0^2 > \text{a few } GeV^2$

important applications:

- compare two experiments at two different $Q^2$
- compare low $Q^2$ models or sum rules with experiments at high $Q^2$

nonperturbative effects difficult!

- power law behavior for $x \to 0$ from Regge phenomenology
- typically, $q(x, Q_0^2)$ from some QCD-inspired models
- lowest moments from lattice QCD
Calculating PDFs in lattice QCD

- **direct evaluation**: NO:
  On a Euclidean lattice, all distances are spacelike \((x^0 \rightarrow ix^0_E)\). Therefore, a direct calculation of lightlike correlation functions on a Euclidean lattice is not possible!

- **indirect evaluation**: yes! Using analyticity, one can show that moments of parton distributions for a hadron \(h\) are related to expectation values of certain local operators in that hadron state:

\[
\int_0^1 dx f(x)x^n \leftrightarrow \langle h|\bar{\psi}D^n\psi|h\rangle
\]

\(\rightarrow\) r.h.s. of this equation can be calculated in Euclidean space (and then one could reconstruct \(f(x)\) from its moments)! In practice: replace n-th derivative on the r.h.s. by appropriate finite differences. Problem: statistical noise (Euclidean lattice calculations are done using Monte Carlo techniques) makes it very hard to calculate any moment \(n \gg 1\).
DIS $\xrightarrow{Bj}$ PDF $q(x)$

$q(x)$ is probability to find quark carrying fraction $x$ of light-cone momentum (total momentum in IMF)

no information about position of partons

major results:

- 50% of nucleon momentum carried by glue
- significant $[O(10\%)]$ modification of quark distributions in nuclei (“EMC effect”)
- only 30% of nucleon spin carried by quark spin (“spin crisis”)
- more $\bar{d}$ than $\bar{u}$ in proton (“violation of Gottfried sum rule”)
- $\Delta G$ not very large

theory:

- perturbative “$Q^2$ evolution”
- lattice calculations: lowest moments of PDFs
- and many “QCD-inspired” models ...
3D imaging of the nucleon
Motivation (GPDs)

X. Ji, PRL 78, 610 (1997):

\[ \text{DVCS} \iff \text{GPDs} \iff \vec{J}_q \]

→ GPDs are interesting physical observable!

But:
- do GPDs have a simple physical interpretation?
- what more can we learn from GPDs about the structure of the nucleon?
Probabilistic interpretation of GPDs as Fourier transforms of impact parameter dependent PDFs

- $H(x, 0, -\Delta^2_{\perp}) \rightarrow q(x, b_{\perp})$
- $\tilde{H}(x, 0, -\Delta^2_{\perp}) \rightarrow \Delta q(x, b_{\perp})$
- $E(x, 0, -\Delta^2_{\perp}) \rightarrow \perp$ distortion of PDFs when the target is $\perp$ polarized

Chromodynamik lensing and $\perp$ SSAs

transverse distortion of PDFs + final state interactions $\Rightarrow \perp$ SSA in $\gamma N \rightarrow \pi + X$

Summary
Generalized Parton Distributions (GPDs)

GPDs: decomposition of form factors at a given value of $t$, w.r.t. the average momentum fraction $x = \frac{1}{2}(x_i + x_f)$ of the active quark

\[
\begin{align*}
\int dx H_q(x, t) &= F_1^q(t) & \int dx \tilde{H}_q(x, t) &= G_A^q(t) \\
\int dx E_q(x, t) &= F_2^q(t) & \int dx \tilde{E}_q(x, t) &= G_P^q(t),
\end{align*}
\]

$x_i$ and $x_f$ are the momentum fractions of the quark before and after the momentum transfer

$F_1^q(t), F_2^q(t), G_A^q(t)$, and $G_P^q(t)$ are the Dirac, Pauli, axial, and pseudoscalar formfactors, respectively ($t \equiv q^2 = (P' - P)^2$)

\[
\langle P', S' | j^\mu(0) | P, S \rangle = \bar{u}(P', S') \left[ \gamma^\mu F_1(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2M} F_2(q^2) \right] u(P, S)
\]

GPDs can be probed in Deeply Virtual Compton Scattering (DVCS).
Deeply Virtual Compton Scattering (DVCS)

- virtual Compton scattering: $\gamma^* p \rightarrow \gamma p$ (actually: $e^- p \rightarrow e^- \gamma p$)

- ‘deeply’: $-q^2_{\gamma} \gg M^2_p, |t|$ $\rightarrow$ Compton amplitude dominated by (coherent superposition of) Compton scattering off single quarks

- only difference between form factor (a) and DVCS amplitude (b) is replacement of photon vertex by two photon vertices connected by quark propagator (depends on quark momentum fraction $x$)

- DVCS amplitude provides access to momentum-decomposition of form factor (GPDs).
Deeply Virtual Compton Scattering (DVCS)

- need $\gamma^*$ with several GeV$^2$ for Bjorken scaling
- DVCS X-section factor $\alpha = \frac{1}{137}$ smaller than elastic X-section
- need high luminosity $e^-$ beam with $> 10$ GeV
- facilities suitable for detailed GPD studies:
  - 12 GeV upgrade at Jefferson Lab (higher $x$)
  - $e^-$ Ion Collider (EIC): lower $x$, higher $Q^2$
Deeply Virtual Compton Scattering (DVCS)

$$T^\mu\nu = i \int d^4 z \ e^{i \vec{q} \cdot \vec{z}} \langle p' \bigg| T J^\mu \left( -\frac{\vec{z}}{2} \right) J^\nu \left( \frac{\vec{z}}{2} \right) \bigg| p \rangle$$

$$Bj \mapsto g_{\perp}^{\mu\nu} \int_{-1}^{1} dx \left( \frac{1}{x - \xi + i\epsilon} + \frac{1}{x + \xi - i\epsilon} \right) H(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ u(p) + \ldots$$

$$\bar{q} = (q + q')/2 \quad \Delta = p' - p \quad x_{Bj} = -q^2 / 2 p \cdot q = 2\xi(1 + \xi)$$
\[
\int \frac{dx^–}{2\pi} e^{ix^-\bar{p}^+x} \left\langle p' \left| \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \right| p \right\rangle = H(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ u(p) \\
+ E(x, \xi, \Delta^2) \bar{u}(p') \frac{i\sigma^{+\nu} \Delta_{\nu}}{2M} u(p)
\]

\[
\int \frac{dx^–}{2\pi} e^{ix^-\bar{p}^+x} \left\langle p' \left| \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ \gamma_5 q \left( \frac{x^-}{2} \right) \right| p \right\rangle = \tilde{H}(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ \gamma_5 u(p) \\
+ \tilde{E}(x, \xi, \Delta^2) \bar{u}(p') \frac{\gamma_5 \Delta^+}{2M} u(p)
\]

where \( \Delta = p' - p \) is the momentum transfer and \( \xi \) measures the longitudinal momentum transfer on the target \( \Delta^+ = \xi(p^+ + p'^+) \).
What is Physics of GPDs?

\[ \langle p' | \hat{O} | p \rangle = H(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ u(p) + E(x, \xi, \Delta^2) \bar{u}(p') \frac{i\sigma^{+\nu} \Delta^\nu}{2M} u(p) \]

with \( \hat{O} = \int \frac{dx^-}{2\pi} x^- \bar{n}^+ x^- \Gamma \left( x^- \right) + \Gamma \left( x^- \right) \)

\[ \leftarrow \text{relation between PDFs and GPDs similar to relation between a charge and a form factor} \]

\[ \leftarrow \text{If form factors can be interpreted as Fourier transforms of charge distributions in position space, what is the analogous physical interpretation for GPDs?} \]
Form Factors vs. GPDs

<table>
<thead>
<tr>
<th>operator</th>
<th>forward matrix elem.</th>
<th>off-forward matrix elem.</th>
<th>position space</th>
</tr>
</thead>
<tbody>
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<td>$\bar{q} \gamma^+ q$</td>
<td>$Q$</td>
<td>$F(t)$</td>
<td>$\rho(\vec{r}^*)$</td>
</tr>
<tr>
<td>$\int \frac{dx^- e^{ix^+ x^-}}{4\pi} \bar{q}\left(-\frac{x^-}{2}\right) \gamma^+ q\left(\frac{x^-}{2}\right)$</td>
<td>$q(x)$</td>
<td>$H(x, \xi, t)$</td>
<td>?</td>
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### Form Factors vs. GPDs

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<td>( \int \frac{d\vec{r}^-}{4\pi} e^{ix^+\vec{p}^-} \bar{q}(-\frac{x^-}{2}) \gamma^+ q(\frac{x^-}{2}) )</td>
<td>( q(x) )</td>
<td>( H(x, 0, t) )</td>
<td>( q(x, \vec{b}_\perp) )</td>
</tr>
</tbody>
</table>

\( q(x, \vec{b}_\perp) \) = impact parameter dependent PDF
Impact parameter dependent PDFs

- define state that is localized in \( \perp \) position:

\[
|p^+, \mathbf{R}_\perp = 0_\perp, \lambda\rangle \equiv \mathcal{N} \int d^2 \mathbf{p}_\perp |p^+, \mathbf{p}_\perp, \lambda\rangle
\]

Note: \( \perp \) boosts in IMF form Galilean subgroup \( \Rightarrow \) this state has

\[
\mathbf{R}_\perp \equiv \frac{1}{p^+} \int dx^- d^2 \mathbf{x}_\perp \mathbf{x}_\perp T^{++}(x) = 0_\perp
\]

(cf.: working in CM frame in nonrel. physics)

- define impact parameter dependent PDF

\[
q(x, \mathbf{b}_\perp) \equiv \int \frac{dx^-}{4\pi} \langle p^+, \mathbf{R}_\perp = 0_\perp | \bar{\psi} \left(-\frac{x^-}{2}, \mathbf{b}_\perp\right) \gamma^+ \psi \left(\frac{x^-}{2}, \mathbf{b}_\perp\right) | p^+, \mathbf{R}_\perp = 0_\perp\rangle e^{ixp^+x^-}
\]
use translational invariance to relate to same matrix element that appears in def. of GPDs

\[ q(x, b_\perp) \equiv \int dx^- \langle p^+, R_\perp = 0_\perp | \bar{\psi}(-\frac{x^-}{2}, b_\perp) \gamma^+ \psi(\frac{x^-}{2}, b_\perp) | p^+, R_\perp = 0_\perp \rangle e^{ixp^+x^-} = |N|^2 \int d^2p_\perp \int d^2p'_\perp \int dx^- \langle p^+, p'_\perp | \bar{\psi}(\frac{x^-}{2}, b_\perp) \gamma^+ \psi(\frac{x^-}{2}, b_\perp) | p^+, p_\perp \rangle e^{ixp^+x^-} \]
Impact parameter dependent PDFs

- use translational invariance to relate to same matrix element that appears in def. of GPDs

\[ q(x, b_{\perp}) \equiv \int dx^- \left\langle p^+, R_{\perp} = 0_{\perp} \right\vert \bar{\psi}(-\frac{x^-}{2}, b_{\perp}) \gamma^+ \psi(\frac{x^-}{2}, b_{\perp}) \left\vert p^+, R_{\perp} = 0_{\perp} \right\rangle e^{ixp^+x^-} \]

\[ = |N|^2 \int d^2 p_{\perp} \int d^2 p'_{\perp} \int dx^- \left\langle p^+, p'_{\perp} \right\vert \bar{\psi}(-\frac{x^-}{2}, b_{\perp}) \gamma^+ \psi(\frac{x^-}{2}, b_{\perp}) \left\vert p^+, p_{\perp} \right\rangle e^{ixp^+x^-} \]

\[ = |N|^2 \int d^2 p_{\perp} \int d^2 p'_{\perp} \int dx^- \left\langle p^+, p'_{\perp} \right\vert \bar{\psi}(-\frac{x^-}{2}, 0_{\perp}) \gamma^+ \psi(\frac{x^-}{2}, 0_{\perp}) \left\vert p^+, p_{\perp} \right\rangle e^{ixp^+x^-} \times e^{ib_{\perp} \cdot (p_{\perp} - p'_{\perp})} \]
Impact parameter dependent PDFs

- use translational invariance to relate to same matrix element that appears in def. of GPDs

\[
q(x, b_\perp) \equiv \int dx^- \langle p^+, R_\perp = 0_\perp | \bar{\psi}(-\frac{x^-}{2}, b_\perp) \gamma^+ \psi(\frac{x^-}{2}, b_\perp) | p^+, R_\perp = 0_\perp \rangle e^{ixp^+x^-} \\
= |N|^2 \int d^2 p_\perp \int d^2 p'_\perp \int dx^- \langle p^+, p'_\perp | \bar{\psi}(-\frac{x^-}{2}, b_\perp) \gamma^+ \psi(\frac{x^-}{2}, b_\perp) | p^+, p_\perp \rangle e^{ixp^+x^-} \\
= |N|^2 \int d^2 p_\perp \int d^2 p'_\perp \int dx^- \langle p^+, p'_\perp | \bar{\psi}(-\frac{x^-}{2}, 0_\perp) \gamma^+ \psi(\frac{x^-}{2}, 0_\perp) | p^+, p_\perp \rangle e^{ixp^+x^-} \\
\times e^{ib_\perp \cdot (p_\perp - p'_\perp)} \\
= |N|^2 \int d^2 p_\perp \int d^2 p'_\perp H \left(x, 0, -(p'_\perp - p_\perp)^2 \right) e^{ib_\perp \cdot (p_\perp - p'_\perp)}
\]

\[\rightarrow \quad q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} H(x, 0, -\Delta_\perp^2) e^{-ib_\perp \cdot \Delta_\perp}\]
Impact parameter dependent PDFs

GPDs allow simultaneous determination of longitudinal momentum and transverse position of partons

\[ q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} H(x, 0, -\Delta_\perp^2) e^{-i b_\perp \cdot \Delta_\perp} \]

\( q(x, b_\perp) \) has interpretation as density (positivity constraints!)

\[ q(x, b_\perp) \sim \langle p^+, 0_\perp | b^\dagger(xp^+, b_\perp) b(xp^+, b_\perp) | p^+, 0_\perp \rangle \]

\[ = | b(xp^+, b_\perp) |p^+, 0_\perp \rangle |^2 \geq 0 \]

\[ q(x, b_\perp) \geq 0 \quad \text{for} \quad x > 0 \]

\[ q(x, b_\perp) \leq 0 \quad \text{for} \quad x < 0 \]
Impact parameter dependent PDFs

- No relativistic corrections (Galilean subgroup!)
- corollary: interpretation of 2d-FT of $F_1(Q^2)$ as charge density in transverse plane also free from relativistic corrections
- $q(x, b_\perp)$ has probabilistic interpretation as number density ($\Delta q(x, b_\perp)$ as difference of number densities)
- Reference point for IPDs is transverse center of (longitudinal) momentum $R_\perp \equiv \sum_i x_i r_{i,\perp}$
- for $x \to 1$, active quark ‘becomes’ COM, and $q(x, b_\perp)$ must become very narrow ($\delta$-function like)
- $H(x, -\Delta^2_\perp)$ must become $\Delta_\perp$ indep. as $x \to 1$ (MB, 2000)
- consistent with lattice results for first few moments (→J.Negele)
- Note that this does not necessarily imply that ‘hadron size’ goes to zero as $x \to 1$, as separation $r_\perp$ between active quark and COM of spectators is related to impact parameter $b_\perp$ via $r_\perp = \frac{1}{1-x} b_\perp$. 
$q(x, b_\perp)$ for unpol. $p$

$x = \text{momentum fraction of the quark}$

$\vec{b} = \perp \text{ position of the quark}$
Summary

- form factor $\mathcal{F}_T \leftrightarrow \rho(\vec{r})$
- relativistic corrections! Can be avoided in ‘infinite momentum frame’ interpretation of 2D FT of form factors
- DIS $\rightarrow q(x)$ light-cone momentum distribution of quarks in nucleon
- DVCS $\rightarrow$ GPDs $\mathcal{F}_T \leftrightarrow q(x, b_\perp)$ ‘impact parameter dependent PDFs
Transversely Deformed Distributions and $E(x, -\Delta_{\perp}^2)$


- Distribution of unpol. quarks in unpol (or long. pol.) nucleon:

$$q(x, b_{\perp}) = \int d^2\Delta_{\perp} \frac{1}{(2\pi)^2} H(x, -\Delta_{\perp}^2) e^{-ib_{\perp} \cdot \Delta_{\perp}} \equiv \mathcal{H}(x, b_{\perp})$$

- Unpol. quark distribution for nucleon polarized in $x$ direction:

$$q(x, b_{\perp}) = \mathcal{H}(x, b_{\perp}) - \frac{1}{2M} \frac{\partial}{\partial b_y} \int d^2\Delta_{\perp} \frac{1}{(2\pi)^2} E(x, -\Delta_{\perp}^2) e^{-ib_{\perp} \cdot \Delta_{\perp}}$$

- Physics: $j^+ = j^0 + j^3$, and left-right asymmetry from $j^3$
Intuitive connection with $\vec{J}_q$

- DIS probes quark momentum density in the infinite momentum frame (IMF). Quark density in IMF corresponds to $j^+ = j^0 + j^3$ component in rest frame ($\vec{p}_{\gamma^*}$ in $-\hat{z}$ direction)

- $j^+$ larger than $j^0$ when quark current towards the $\gamma^*$; suppressed when away from $\gamma^*$

- For quarks with positive orbital angular momentum in $\hat{x}$-direction, $j^z$ is positive on the $+\hat{y}$ side, and negative on the $-\hat{y}$ side

- Details of $\perp$ deformation described by $E_q(x, -\Delta_{\perp}^2)$

- not surprising that $E_q(x, -\Delta_{\perp}^2)$ enters Ji relation!

$$\langle J_q^i \rangle = S^i \int dx \left[ H_q(x, 0) + E_q(x, 0) \right] x.$$
Transversely Deformed PDFs and $E(x, 0, -\Delta^2_\perp)$

- $q(x, b_\perp)$ in $\perp$ polarized nucleon is deformed compared to longitudinally polarized nucleons!

- Mean $\perp$ deformation of flavor $q$ ($\perp$ flavor dipole moment)

$$d^q_y \equiv \int dx \int d^2b_\perp q(x, b_\perp)b_y = \frac{1}{2M} \int dx E_q(x, 0) = \frac{\kappa_{q/p}}{2M}$$

with $\kappa_{q/p} \equiv F_2^{u/d}(0)$ contribution from quark flavor $q$ to the proton anomalous magnetic moment

- $\kappa_p = 1.793 = \frac{2}{3}\kappa_{u/p} - \frac{1}{3}\kappa_{d/p}$
  $\kappa_n = -2.033 = \frac{2}{3}\kappa_{d/p} - \frac{1}{3}\kappa_{u/p}$

  $\Rightarrow \kappa_{u/p} = 2\kappa_p + \kappa_n = 1.673$
  $\kappa_{d/p} = 2\kappa_n + \kappa_p = -2.033.$

  $d^q_y = O(0.2 \text{ fm})$
p polarized in $+\hat{x}$ direction

\[ u(x, b_\perp) \quad d(x, b_\perp) \]

\[ b_x \quad b_y \]

\[ x = 0.1 \quad x = 0.3 \quad x = 0.5 \]

\[ \vec{p}_\gamma \]

$\hat{y}$

$\hat{z}$

$\hat{x}$

$j^z > 0$

$j^z < 0$
SSAs in SIDIS ($\gamma + p \uparrow \rightarrow \pi^+ + X$)

- **SIDIS** = semi-inclusive DIS
- **Single-Spin-Asymmetry (SSA)** = left-right asymmetry in the X-section when only one spin is measured (e.g. target spin)

  example: nucleon transversely (relative to $e^-$ beam) polarized $\rightarrow$ left-right asymmetry of produced $\pi$-mesons relative to target pol.

- infer transverse momentum distribution $q(x, k_\perp)$ of quarks in target from transverse momentum distribution of produced $\pi$
  (note: left-right asymmetry can also arise in ‘fragmentation’ process (Collins effect), but resulting asymmetry has different angular dependence...)
Sivers: distribution of unpol. quarks in ⊥ pol. proton

\[ f_{q/p}^{↑}(x, k_{\perp}) = f_{1}^{q}(x, k_{\perp}^2) - f_{1T}^{\perp q}(x, k_{\perp}^2) \frac{\hat{P} \times k_{\perp}}{M} \cdot S \]

- without FSI, \( f(x, k_{\perp}) = f(x, -k_{\perp}) \Rightarrow f_{1T}^{\perp q}(x, k_{\perp}^2) = 0 \)
- with FSI, \( f_{1T}^{\perp q}(x, k_{\perp}^2) \neq 0 \) (Brodsky, Hwang, Schmidt)

Why interesting?
- (like \( \kappa \)), Sivers requires matrix elements between wave function components that differ by one unit of OAM (Brodsky, Diehl, ..)
- probe for orbital angular momentum
- Sivers requires nontrivial final state interaction phases
- learn about FSI
GPD $\leftrightarrow$ SSA (Sivers)

- example: $\gamma p \rightarrow \pi X$

- $u, d$ distributions in $\perp$ polarized proton have left-right asymmetry in $\perp$ position space (T-even!); sign “determined” by $\kappa_u$ & $\kappa_d$

- attractive FSI deflects active quark towards the center of momentum

- FSI translates position space distortion (before the quark is knocked out) in $+\hat{y}$-direction into momentum asymmetry that favors $-\hat{y}$ direction

- correlation between sign of $\kappa_q^p$ and sign of SSA: $f_{1T}^\perp q \sim -\kappa_q^p$

- $f_{1T}^\perp q \sim -\kappa_q^p$ confirmed by HERMES data (also consistent with COMPASS deuteron data $f_{1T}^\perp u + f_{1T}^\perp d \approx 0$)
Consider quark in ground state hadron polarized out of the plane

\[ \rightarrow \text{expect counterclockwise net current } \vec{j} \text{ associated with the magnetization density in this state} \]

\[ \rightarrow \text{virtual photon ‘sees’ enhancement of quarks (polarized out of plane) at the top, i.e.} \]

\[ \rightarrow \text{virtual photon ‘sees’ enhancement of quarks with polarization up (down) on the left (right) side of the hadron} \]
lowest moment of distribution $q(x, b_\perp)$ for unpol. quarks in $\perp$ pol. proton (left) and of $\perp$ pol. quarks in unpol. proton (right):
Boer-Mulders Function

- SIDIS: attractive FSI expected to convert position space asymmetry into momentum space asymmetry

  \( \rightarrow \) e.g. quarks at negative \( b_x \) with spin in \( +\hat{y} \) get deflected (due to FSI) into \( +\hat{x} \) direction

  \( \rightarrow \) (qualitative) connection between Boer-Mulders function \( h_{1\perp}^+(x, k_\perp) \) and the chirally odd GPD \( \bar{E}_T \) that is similar to (qualitative) connection between Sivers function \( f_{1T}^+(x, k_\perp) \) and the GPD \( E \).

- Boer-Mulders: distribution of \( \perp \) pol. quarks in \( \text{unpol.} \) proton

  \[
  f_{q^\perp/p}(x, k_\perp) = \frac{1}{2} \left[ f_{1}^q(x, k_\perp^2) - h_{1\perp}^q(x, k_\perp^2) \left( \hat{P} \times k_\perp \right) \cdot S_q \right]
  \]

- \( h_{1\perp}^q(x, k_\perp^2) \) can be probed in Drell-Yan (RHIC, J-PARC, GSI) and tagged SIDIS (JLab, eRHIC), using Collins-fragmentation
how do you measure the transversity distribution of quarks without measuring the transversity of a quark?

consider semi-inclusive pion production off unpolarized target

spin-orbit correlations in target wave function provide correlation between (primordial) quark transversity and impact parameter

(attractive) FSI provides correlation between quark spin and \( \perp \) quark momentum \( \Rightarrow \) BM function

Collins effect: left-right asymmetry of \( \pi \) distribution in fragmentation of \( \perp \) polarized quark \( \Rightarrow \) ‘tag’ quark spin

\( \cos(2\phi) \) modulation of \( \pi \) distribution relative to lepton scattering plane

\( \cos(2\phi) \) asymmetry proportional to: Collins \( \times \) BM
probing BM function in tagged SIDIS

Primordial Quark Transversity Distribution

→⊥ quark pol.
\(\perp\) polarization and \(\gamma^*\) absorption

QED: when the \(\gamma^*\) scatters off \(\perp\) polarized quark, the \(\perp\) polarization gets modified:
- gets reduced in size
- gets tilted symmetrically w.r.t. normal of the scattering plane

quark pol. before \(\gamma^*\) absorption

\[\text{lepton scattering plane}\]

quark pol. after \(\gamma^*\) absorption
probing BM function in tagged SIDIS

Primordial Quark Transversity Distribution

→ \perp \text{quark pol.}
probing BM function in tagged SIDIS

Quark Transversity Distribution after $\gamma^*$ absorption

$\rightarrow \perp$ quark pol.

lepton scattering plane

quark transversity component in lepton scattering plane flips
probing BM function in tagged SIDIS

⊥ momentum due to FSI

→ ⊥ quark pol.

↓ $k_q^\perp$ due to FSI

lepton scattering plane

on average, FSI deflects quarks towards the center
Collins effect

- When a $\perp$ polarized struck quark fragments, the structure of jet is sensitive to polarization of quark.
- Distribution of hadrons relative to $\perp$ polarization direction may be left-right asymmetric.
- Asymmetry parameterized by Collins fragmentation function.
- Artru model:
  - Struck quark forms pion with $\bar{q}$ from $q\bar{q}$ pair with $^{3}P_{0}$ ‘vacuum’ quantum numbers.
  - Pion ‘inherits’ OAM in direction of $\perp$ spin of struck quark.
  - Produced pion preferentially moves to left when looking into direction of motion of fragmenting quark with spin up.
- Artru model confirmed by HERMES experiment.
- More precise determination of Collins function under way (KEK).
probing BM function in tagged SIDIS

$\mathbf{k}_\perp$ due to Collins

$\mathbf{k}^q_\perp$ due to FSI

lepton scattering plane

$\perp$ momentum due to Collins

SSA of $\pi$ in jet emanating from $\perp$ pol. $q$
probing BM function in tagged SIDIS

net $\perp$ momentum (FSI+Collins)

- $k^\perp$ due to Collins
- $k^q$ due to FSI
- net $k^q$

lepton scattering plane

$\rightarrow$ in this example, enhancement of pions with $\perp$ momenta $\perp$ to lepton plane
probing BM function in tagged SIDIS

\[ \text{net } k_{\perp}^\pi \ (\text{FSI + Collins}) \]

\[ \downarrow \text{net } k_{\perp}^q \]

lepton scattering plane

\[ \leftrightarrow \text{expect enhancement of pions with } \perp \text{ momenta } \perp \text{ to lepton plane} \]
What is Orbital Angular Momentum
polarized DIS: only $\sim 30\%$ of the proton spin due to quark spins

→ ‘spin crisis’ $\rightarrow$ ‘spin puzzle’, because $\Delta \Sigma$ much smaller than the quark model result $\Delta \Sigma = 1$

→ quest for the remaining 70%
  - quark orbital angular momentum (OAM)
  - gluon spin
  - gluon OAM

→ How are the above quantities defined?
→ How can the above quantities be measured
example: angular momentum in QED

consider, for simplicity, QED without electrons:

\[ \vec{J} = \int d^3r \vec{x} \times (\vec{E} \times \vec{B}) = \int d^3r \vec{x} \times [\vec{E} \times (\vec{\nabla} \times \vec{A})] \]

integrate by parts

\[ \vec{J} = \int d^3r \left[ E^j (\vec{x} \times \vec{\nabla}) A^j + (\vec{x} \times \vec{A}) \vec{\nabla} \cdot \vec{E} + \vec{E} \times \vec{A} \right] \]

drop 2\textsuperscript{nd} term (eq. of motion \( \vec{\nabla} \cdot \vec{E} = 0 \)), yielding \( \vec{J} = \vec{L} + \vec{S} \) with

\[ \vec{L} = \int d^3r E^j (\vec{x} \times \vec{\nabla}) A^j \quad \vec{S} = \int d^3r \vec{E} \times \vec{A} \]

note: \( \vec{L} \) and \( \vec{S} \) not separately gauge invariant
total angular momentum of isolated system uniquely defined

ambiguities arise when decomposing $\mathbf{J}$ into contributions from different constituents

gauge theories: changing gauge may also shift angular momentum between various degrees of freedom

decomposition of angular momentum in general depends on ‘scheme’ (gauge & quantization scheme)

does not mean that angular momentum decomposition is meaningless, but

one needs to be aware of this ‘scheme’-dependence in the physical interpretation of exp/lattice/model results in terms of spin vs. OAM

and, for example, not mix ‘schemes’, e.t.c.
What is Orbital Angular Momentum?

- Ji decomposition
- Jaffe decomposition
- recent lattice results (Ji decomposition)
- model/QED illustrations for Ji v. Jaffe
The nucleon spin pizza(s)

Ji

\[ J_g \]

\[ \frac{1}{2} \Delta \Sigma \]

\[ L_q \]

Jaffe & Manohar

\[ \mathcal{L}_q \]

\[ \frac{1}{2} \Delta \Sigma \]

\[ \Delta G \]

\[ \mathcal{L}_g \]

‘pizza tre stagioni’

‘pizza quattro stagioni’

\[ \text{only } \frac{1}{2} \Delta \Sigma \equiv \frac{1}{2} \sum_q \Delta q \text{ common to both decompositions!} \]
energy momentum tensor $T^{\mu\nu} = T^{\nu\mu}$; \( \partial_\mu T^{\mu\nu} = 0 \)

$T^{00}$ energy density; \( T^{0i} \) momentum density

\( \tilde{P}_\mu \equiv \int d^3x T^{\mu0} \) conserved

\[
\frac{d}{dt} \tilde{P}_\mu = \int d^3x \frac{\partial}{\partial x^0} T^{\mu0} \partial_\mu T^{\mu\nu} = 0 \int d^3x \frac{\partial}{\partial x^i} T^{\mu i} = 0
\]

$T^{\mu\nu}$ contains interactions, e.g. \( T_{q\mu}^{\mu\nu} = \frac{i}{2} \bar{\psi} (\gamma^\mu D^\nu + \gamma^\nu D^\mu) \psi \)

$T^{\mu0}$ contains time derivative (don’t want a Hamiltonian/momentum operator that contains time derivative!)

\( \rightarrow \) replace by space derivative, using equation of motion, e.g. \( (i\gamma^\mu D_\mu - m) \psi = 0 \) to replace \( iD^0 \psi \rightarrow \gamma^0 \left( i\gamma^k D_k - m \right) \psi \)

some of the resulting space derivatives add up to total derivative terms which do not contribute to volume integral

\[
P^\mu \equiv \int d^3x \left[ T^{\mu0} + \text{‘eq. of motion terms’} + \text{‘surface terms’} \right]
\]
Angular Momentum Operator

- Angular momentum tensor $M^{\mu \nu \rho} = x^\mu T^{\nu \rho} - x^\nu T^{\mu \rho}$
- $\partial_\rho M^{\mu \nu \rho} = 0$

$\tilde{J}^i = \frac{1}{2} \varepsilon^{ijk} \int d^3 r M^{jk0}$ is conserved

$$\frac{d}{dt} \tilde{J}^i = \frac{1}{2} \varepsilon^{ijk} \int d^3 x \partial_0 M^{jk0} = \frac{1}{2} \varepsilon^{ijk} \int d^3 x \partial_l M^{jkl} = 0$$

- $M^{\mu \nu \rho}$ contains time derivatives (since $T^{\mu \nu}$ does)
  - Use eq. of motion to get rid of these (as in $T^{0i}$)
  - Integrate total derivatives appearing in $T^{0i}$ by parts
  - Yields terms where derivative acts on $x^i$ which then ‘disappears’

$J^i$ usually contains both
  - ‘Extrinsic’ terms, which have the structure ‘$\vec{x} \times$ Operator’, and can be identified with ‘OAM’
  - ‘Intrinsic’ terms, where the factor $\vec{x} \times$ does not appear, and can be identified with ‘spin’
following this general procedure, one finds in QCD

\[
\vec{J} = \int d^3 x \left[ \psi^\dagger \Sigma \psi + \psi^\dagger \vec{x} \times \left( i \vec{\partial} - g \vec{A} \right) \psi + \vec{x} \times \left( \vec{E} \times \vec{B} \right) \right]
\]

with \( \Sigma^i = \frac{i}{2} \varepsilon^{ijk} \gamma^j \gamma^k \)

Ji does not integrate gluon term by parts, nor identify gluon spin/OAM separately

Ji-decomposition valid for all three components of \( \vec{J} \), but usually only applied to \( \hat{z} \) component, where the quark spin term has a partonic interpretation

(+) all three terms manifestly gauge invariant

(+) DVCS can be used to probe \( \vec{J}_q = \vec{S}_q + \vec{L}_q \)

(-) quark OAM contains interactions

(-) only quark spin has partonic interpretation as a single particle density
Ji-decomposition

\[ \frac{1}{2} = \sum_q J_q + J_g = \sum_q \left( \frac{1}{2} \Delta q + L_q \right) + J_g \]

with \( P^\mu = (M, 0, 0, 1), S^\mu = (0, 0, 0, 1) \)

\[ \frac{1}{2} \Delta q = \frac{1}{2} \int d^3x \langle P, S | q^\dagger(\vec{x}) \Sigma^3 q(\vec{x}) | P, S \rangle \]

\[ L_q = \int d^3x \langle P, S | q^\dagger(\vec{x}) (\vec{x} \times i\vec{D})^3 q(\vec{x}) | P, S \rangle \]

\[ J_g = \int d^3x \langle P, S | \left[ \vec{x} \times (\vec{E} \times \vec{B}) \right]^3 | P, S \rangle \]

\[ i\vec{D} = i\vec{\partial} - g\vec{A} \]
The Ji-relation (poor man’s derivation)

What distinguishes the Ji-decomposition from other decompositions is the fact that $L_q$ can be constrained by experiment:

$$\langle \vec{J}_q \rangle = \vec{S} \int_{-1}^{1} dx \ x \ [H_q(x, \xi, 0) + E_q(x, \xi, 0)]$$

(nucleon at rest; $\vec{S}$ is nucleon spin)

$\rightarrow \quad L^z_q = J^z_q - \frac{1}{2} \Delta q$

derivation (MB-version):

- consider nucleon state that is an eigenstate under rotation about the $\hat{x}$-axis (e.g. nucleon polarized in $\hat{x}$ direction with $\vec{p} = 0$ (wave packet if necessary)

- for such a state, $\langle T^{00}_q y \rangle = 0 = \langle T^{zz}_q y \rangle$ and $\langle T^{0y}_q z \rangle = -\langle T^{0z}_q y \rangle$

$\rightarrow \quad \langle T^{++}_q y \rangle = \langle T^{0y}_q z - T^{0z}_q y \rangle = \langle J^x_q \rangle$

$\rightarrow \quad$ relate 2nd moment of $\perp$ flavor dipole moment to $J^x_q$
The Ji-relation (poor man’s derivation)

- derivation (MB-version):
  - consider nucleon state that is an eigenstate under rotation about the \( \hat{x} \)-axis (e.g. nucleon polarized in \( \hat{x} \) direction with \( \vec{p} = 0 \) (wave packet if necessary)
  - for such a state, \( \langle T_{q}^{00} y \rangle = 0 = \langle T_{q}^{zz} y \rangle \) and \( \langle T_{q}^{0y} z \rangle = -\langle T_{q}^{0z} y \rangle \)
  - \( \langle T_{q}^{++} y \rangle = \langle T_{q}^{0y} z - T_{q}^{0z} y \rangle = \langle J_{q}^{x} \rangle \)
  - relate 2\( \text{nd} \) moment of \( \perp \) flavor dipole moment to \( J_{q}^{x} \)

- effect sum of two effects:
  - \( \langle T_{q}^{++} y \rangle \) for a point-like transversely polarized nucleon
  - \( \langle T_{q}^{++} y \rangle \) for a quark relative to the center of momentum of a transversely polarized nucleon

- 2\( \text{nd} \) moment of \( \perp \) flavor dipole moment for point-like nucleon

\[
\psi = \left( \frac{f(r)}{\vec{\sigma} \cdot \vec{p}} \right) \chi \quad \text{with} \quad \chi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]
derivation (MB-version):

\[ T^0_{qz} = i\bar{q} \left( \gamma^0 \partial^z + \gamma^z \partial^0 \right) q \]

since \( \psi \dagger \partial_y \psi \) is even under \( y \rightarrow -y \), \( i\bar{q} \gamma^0 \partial^z q \) does not contribute to \( \langle T^{0z} y \rangle \)

\[ \langle T^{0z} b_y \rangle = E \int d^3r \bar{\psi} \gamma^0 \gamma^z \psi y = E \int d^3r \bar{\psi} \left( \begin{array}{cc} 0 & \sigma^z \\ \sigma^z & 0 \end{array} \right) \psi y \]

\[ = \frac{2E}{E+M} \int d^3r \bar{\chi} \sigma^z \sigma^y \chi f(r)(-i)\partial_y f(r)y = \frac{E}{E+M} \int d^3r f^2(r)y \]

consider nucleon state with \( \vec{p} = 0 \), i.e. \( E = m \) & \( \int d^3r f^2(r) = 1 \)

\[ 2^{nd} \text{ moment of } \perp \text{ flavor dipole moment is } \frac{1}{2M} \]

\[ \langle T^{++} y \rangle \]

\[ \frac{1}{2} \int dx \, x H_q(x,0,0) \] to \( \langle T^{++} y \rangle \)
derivation (MB-version):

- intrinsic distortion adds \( \frac{1}{2} \int dxx E_q(x, 0, 0) \) to that

\[ J^x_q = \frac{1}{2} \int dxx \left[ H_q(x, 0, 0) + E_q(x, 0, 0) \right] \]

- rotational invariance: should apply to each vector component

\[ \text{Ji relation} \]
Ji-decomposition

\[ \vec{J} = \sum_q \frac{1}{2} q^\dagger \vec{\Sigma} q + q^\dagger \left( \vec{r} \times i \vec{D} \right) q + \vec{r} \times \left( \vec{E} \times \vec{B} \right) \]

applies to each vector component of nucleon angular momentum, but Ji-decomposition usually applied only to \( \hat{z} \) component where at least quark spin has parton interpretation as difference between number densities

- \( \Delta q \) from polarized DIS
- \( J_q = \frac{1}{2} \Delta q + L_q \) from exp/lattice (GPDs)
- \( L_q \) in principle independently defined as matrix elements of \( q^\dagger \left( \vec{r} \times i \vec{D} \right) q \), but in practice easier by subtraction \( L_q = J_q - \frac{1}{2} \Delta q \)
- \( J_g \) in principle accessible through gluon GPDs, but in practice easier by subtraction \( J_g = \frac{1}{2} - J_q \)
- further decomposition of \( J_g \) into intrinsic (spin) and extrinsic (OAM) that is local and manifestly gauge invariant has not been found
$L_q$ for proton from Ji-relation (lattice)

- lattice QCD $\Rightarrow$ moments of GPDs (LHPC; QCDSF)
- insert in Ji-relation

$$\langle J^i_q \rangle = S^i \int dx \left[ H_q(x,0) + E_q(x,0) \right] x.$$ 

- $L^z_q = J^z_q - \frac{1}{2} \Delta q$
- $L_u, L_d$ both large!
- present calcs. show $L_u + L_d \approx 0$, but
- disconnected diagrams ..?
- $m^2_\pi$ extrapolation
- parton interpret. of $L_q$...
Angular Momentum in QCD (Jaffe & Manohar)

- define OAM on a light-like hypersurface rather than a space-like hypersurface

\[ \tilde{J}^3 = \int d^2 x_\perp \int dx^- M^{12+} \]

where \( x^- = \frac{1}{\sqrt{2}} (x^0 - x^-) \) and \( M^{12+} = \frac{1}{\sqrt{2}} (M^{120} + M^{123}) \)

- Since \( \partial_\mu M^{12\mu} = 0 \)

\[ \int d^2 \mathbf{x}_\perp \int dx^- M^{12+} = \int d^2 \mathbf{x}_\perp \int dx^3 M^{120} \]

(compare electrodynamics: \( \tilde{\nabla} \cdot \mathbf{B} = 0 \) \( \Rightarrow \) flux in = flux out)

- use eqs. of motion to get rid of ‘time’ (\( \partial_+ \) derivatives) & integrate by parts whenever a total derivative appears in the \( T^{i+} \) part of \( M^{12+} \)
in light-cone framework & light-cone gauge

\(A^+ = 0\) one finds for \(J^z = \int dx^- d^2r_\perp M^{+xy}\)

\[
\frac{1}{2} = \frac{1}{2} \Delta \Sigma + \sum_q \mathcal{L}_q + \Delta G + \mathcal{L}_g
\]

where \((\gamma^+ = \gamma^0 + \gamma^z)\)

\[
\mathcal{L}_q = \int d^3r \langle P, S | \bar{q}(\vec{r}) \gamma^+ (\vec{r} \times i \vec{\partial})^\dagger q(\vec{r}) | P, S \rangle
\]

\[
\Delta G = \varepsilon^{+-ij} \int d^3r \langle P, S | \text{Tr} F^{+i} A^j | P, S \rangle
\]

\[
\mathcal{L}_g = 2 \int d^3r \langle P, S | \text{Tr} F^{+j} (\vec{x} \times i \vec{\partial})^\dagger A^j | P, S \rangle
\]
Jaffe/Manohar decomposition

\[ \frac{1}{2} = \frac{1}{2} \Delta \Sigma + \sum_q L_q + \Delta G + L_g \]

- $\Delta \Sigma = \sum_q \Delta q$ from polarized DIS (or lattice)
- $\Delta G$ from $p \rightarrow p$ or polarized DIS (evolution)
- $\Delta G$ gauge invariant, but local operator only in light-cone gauge
- $\int dxx^n \Delta G(x)$ for $n \geq 1$ can be described by manifestly gauge inv. local op. (\(\rightarrow\) lattice)
- $L_q, L_g$ independently defined, but
  - no exp. identified to access them
  - not accessible on lattice, since nonlocal except when $A^+ = 0$
- parton net OAM $L = L_g + \sum_q L_q$ by subtr. $L = \frac{1}{2} - \frac{1}{2} \Delta \Sigma - \Delta G$
- in general, $L_q \neq L_q$   $L_g + \Delta G \neq J_g$
- makes no sense to ‘mix’ Ji and JM decompositions, e.g. $J_g - \Delta G$

has no fundamental connection to OAM
\[ L_q \neq \mathcal{L}_q \]

- \( L_q \) matrix element of

\[
q^\dagger \left[ \vec{r} \times \left( i \vec{\partial} - g \vec{A} \right) \right] \vec{z} \ q = \bar{q} \gamma^0 \left[ \vec{r} \times \left( i \vec{\partial} - g \vec{A} \right) \right] \vec{z} \ q
\]

- \( \mathcal{L}^z_q \) matrix element of \((\gamma^+ = \gamma^0 + \gamma^z)\)

\[
\bar{q} \gamma^+ \left[ \vec{r} \times i \vec{\partial} \right] \vec{z} \ q \bigg|_{A^+ = 0}
\]

- For nucleon at rest, matrix element of \( L_q \) same as that of

\[
\bar{q} \gamma^+ \left[ \vec{r} \times \left( i \vec{\partial} - g \vec{A} \right) \right] \vec{z} \ q
\]

\[ \leftrightarrow \] even in light-cone gauge, \( L^z_q \) and \( \mathcal{L}^z_q \) still differ by matrix element

\[
q^\dagger \left( \vec{r} \times g \vec{A} \right) \vec{z} \ q \bigg|_{A^+ = 0} = q^\dagger \left( xgA^y - ygA^x \right) \ q \bigg|_{A^+ = 0}
\]
Summary part 1:

- Ji: \( J^z = \frac{1}{2} \Delta \Sigma + \sum_q L_q + J_g \)

- Jaffe: \( J^z = \frac{1}{2} \Delta \Sigma + \sum_q \mathcal{L}_q + \Delta G + \mathcal{L}_g \)

\( \Delta G \) can be defined without reference to gauge (and hence gauge invariantly) as the quantity that enters the evolution equations and/or represented by simple (i.e. local) operator only in LC gauge and corresponds to the operator that one would naturally identify with ‘spin’ only in that gauge.

- in general \( L_q \neq \mathcal{L}_q \) or \( J_g \neq \Delta G + \mathcal{L}_g \), but

- how significant is the difference between \( L_q \) and \( \mathcal{L}_q \), etc.?
OAM in scalar diquark model

[M.B. + Hikmat Budathoki Chhetri (BC), PRD 79, 071501 (2009)]

- Toy model for nucleon where nucleon (mass $M$) splits into quark (mass $m$) and scalar ‘diquark’ (mass $\lambda$)

$\rightarrow$ Light-cone wave function for quark-diquark Fock component

$$
\psi_{1/2}^\uparrow (x, k_\perp) = \left( M + \frac{m}{x} \right) \phi \\
\psi_{-1/2}^\uparrow = -\frac{k^1 + ik^2}{x} \phi
$$

with $\phi = \frac{c/\sqrt{1-x}}{M^2 - \frac{k^2 + m^2}{x} - \frac{k^2 + \lambda^2}{1-x}}$.

- Quark OAM according to JM: $L_q = \int_0^1 dx \int \frac{d^2k_\perp}{16\pi^3} (1 - x) \left| \psi_{-1/2}^\uparrow \right|^2$

- Quark OAM according to Ji: $L_q = \frac{1}{2} \int_0^1 dx \left[ q(x) + E(x, 0, 0) \right] - \frac{1}{2} \Delta q$

$\rightarrow$ (using Lorentz inv. regularization, such as Pauli Villars subtraction) both give identical result, i.e. $L_q = \mathcal{L}_q$

- Not surprising since scalar diquark model is not a gauge theory
But, even though $L_q = \mathcal{L}_q$ in this non-gauge theory,

$$\mathcal{L}_q(x) \equiv \int \frac{d^2k_\perp}{16\pi^3} (1-x) \left| \psi_{\frac{1}{2}} \right|^2 \neq \frac{1}{2} \left\{ x \left[ q(x) + E(x, 0, 0) \right] - \Delta q(x) \right\} \equiv L_q(x)$$

‘unintegrated Ji-relation’ does not yield $x$-distribution of OAM
**OAM in QED**

- Light-cone wave function in $e\gamma$ Fock component

\[
\Psi_{+\frac{1}{2}+1}(x, k_{\perp}) = \sqrt{2} \frac{k^1 - ik^2}{x(1-x)} \phi \\
\Psi_{+\frac{1}{2}-1}(x, k_{\perp}) = -\sqrt{2} \frac{k^1 + ik^2}{1-x} \phi \\
\Psi_{-\frac{1}{2}+1}(x, k_{\perp}) = \sqrt{2} \left( \frac{m}{x} - m \right) \phi \\
\Psi_{-\frac{1}{2}-1}(x, k_{\perp}) = 0
\]

- OAM of $e^-$ according to Jaffe/Manohar

\[
L_e = \int_0^1 dx \int d^2k_{\perp} \left[ (1-x) \left| \Psi_{+\frac{1}{2}-1}(x, k_{\perp}) \right|^2 - \left| \Psi_{+\frac{1}{2}+1}(x, k_{\perp}) \right|^2 \right]
\]

- $e^-$ OAM according to Ji

\[
L_e = \frac{1}{2} \int_0^1 dx \ x \ [q(x) + E(x, 0, 0)] - \frac{1}{2} \Delta q
\]

\[
\leadsto L_e = L_e + \frac{\alpha}{4\pi} \neq L_e
\]

- Likewise, computing $J_\gamma$ from photon GPD, and $\Delta_\gamma$ and $L_\gamma$ from light-cone wave functions and defining $\hat{L}_\gamma \equiv J_\gamma - \Delta_\gamma$ yields

\[
\hat{L}_\gamma = L_\gamma + \frac{\alpha}{4\pi} \neq L_\gamma
\]

- $\frac{\alpha}{4\pi}$ appears to be small, but here $L_e, L_e$ are all of $O(\frac{\alpha}{\pi})$
1-loop QCD: \( L_q - L_q = \frac{\alpha_s}{3\pi} \)

- recall (lattice QCD): \( L_u \approx -0.15; L_d \approx +0.15 \)

- QCD evolution yields negative correction to \( L_u \) and positive correction to \( L_d \)

- evolution suggested (A.W. Thomas) to explain apparent discrepancy between quark models (low \( Q^2 \)) and lattice results (\( Q^2 \sim 4 GeV^2 \))

- above result suggests that \( L_u > L_u \) and \( L_d > L_d \)

- additional contribution (with same sign) from vector potential due to spectators (MB, to be published)

- possible that lattice result consistent with \( L_u > L_d \)
inclusive $\vec{e} \vec{p} / \vec{p} \vec{p}$ provide access to
- quark spin $\frac{1}{2} \Delta q$
- gluon spin $\Delta G$
- parton grand total OAM $\mathcal{L} \equiv \mathcal{L}_g + \sum_q \mathcal{L}_q = \frac{1}{2} - \Delta G - \sum_q \Delta q$

DVCS & polarized DIS and/or lattice provide access to
- quark spin $\frac{1}{2} \Delta q$
- $J_q$ & $L_q = J_q - \frac{1}{2} \Delta q$
- $J_g = \frac{1}{2} - \sum_q J_q$

$J_g - \Delta G$ does not yield gluon OAM $\mathcal{L}_g$

$L_q - \mathcal{L}_q = \mathcal{O}(0.1 \ast \alpha_s)$ for $\mathcal{O}(\alpha_s)$ dressed quark
(longitudinally) polarized DIS at leading twist →
‘polarized quark distribution’ \( g_1^q(x) = q^\uparrow(x) + \bar{q}^\uparrow(x) - q^\downarrow(x) - \bar{q}^\downarrow(x) \)

\( \frac{1}{Q^2} \)-corrections to X-section involve ‘higher-twist’ distribution functions, such as \( g_2(x) \)

\[
\sigma_{TT} \propto g_1 - \frac{2Mx}{\nu} g_2
\]

\( g_2(x) \) involves quark-gluon correlations and does not have a parton interpretation as difference between number densities

for \( \perp \) polarized target, \( g_1 \) and \( g_2 \) contribute equally to \( \sigma_{LT} \)

\[
\sigma_{LT} \propto g_T \equiv g_1 + g_2
\]

→ ‘clean’ separation between higher order corrections to leading twist \( (g_1) \) and higher twist effects \( (g_2) \)

what can one learn from \( g_2 \)?
Quark-Gluon Correlations (QCD analysis)

- \( g_2(x) = g_{2WW}(x) + \bar{g}_2(x) \), with \( g_{2WW}(x) \equiv -g_1(x) + \int_x^1 \frac{dy}{y} g_1(y) \)

- \( \bar{g}_2(x) \) involves quark-gluon correlations, e.g.

\[
\int dxx^2 \bar{g}_2(x) = \frac{1}{3} d_2 = \frac{1}{6MP^2} \langle P, S | \bar{q}(0) g^{+y}(0) \gamma^+ q(0) | P, S \rangle
\]

- \( \sqrt{2}G^{+y} \equiv G^{0y} + G^{zy} = -E^y + B^x \)

- matrix elements of \( \bar{q}B^x \gamma^+ q \) and \( \bar{q}E^y \gamma^+ q \) are sometimes called color-electric and magnetic polarizabilities

\[
2M^2 \tilde{S}_{\chi_E} = \langle P, S | \vec{j}_a \times \vec{E}_a | P, S \rangle \quad \text{and} \quad 2M^2 \tilde{S}_{\chi_B} = \langle P, S | j_0^a \vec{B}_a | P, S \rangle
\]

with \( d_2 = \frac{1}{4} (\chi_E + 2\chi_M) \) — but these names are misleading!
\( \bar{g}_2(x) \) involves quark-gluon correlations, e.g.

\[
\int dx x^2 \bar{g}_2(x) = \frac{1}{3} d_2 = \frac{1}{6 M P^2 S^x} \langle P, S | \bar{q}(0) g G^+ y(0) \gamma^+ q(0) | P, S \rangle
\]

QED: \( \bar{q}(0) e F^+ y(0) \gamma^+ q(0) \) correlator between quark density \( \bar{q} \gamma^+ q \) and (\( \hat{y} \)-component of the) Lorentz-force

\[
F^y = e \left[ \vec{E} + \vec{v} \times \vec{B} \right]_y = e (E^y - B^x) = -e \left( F^0 y + F^z y \right) = -e \sqrt{2} F^+ y.
\]

for charged paricle moving with \( \vec{v} = (0, 0, -1) \) in the \( -\hat{z} \) direction

\(-\) matrix element of \( \bar{q}(0) e F^+ y(0) \gamma^+ q(0) \) yields \( \gamma^+ \) density (density relevant for DIS in Bj limit!) weighted with the Lorentz force that a charged particle with \( \vec{v} = (0, 0, -1) \) would experience at that point

\(-\) \( d_2 \) a measure for the color Lorentz force acting on the struck quark in SIDIS in the instant after being hit by the virtual photon

\[
\langle F^y(0) \rangle = -M^2 d_2 \quad \text{(rest frame; } S^x = 1)\]
Interpretation of $d_2$ with the transverse FSI force in DIS also consistent with $\langle k_y^\perp \rangle \equiv \int_0^1 dx \int d^2 k_\perp k_\perp^2 f_{1T}^T(x, k_\perp^2)$ in SIDIS (Qiu, Sterman)

$$\langle k_y^\perp \rangle = -\frac{1}{2p^+} \left\langle P, S \left| \bar{q}(0) \int_0^\infty dx^- gG^{+y}(x^-) \gamma^+ q(0) \right| P, S \right\rangle$$

semi-classical interpretation: average $k_\perp$ in SIDIS obtained by correlating the quark density with the transverse impulse acquired from (color) Lorentz force acting on struck quark along its trajectory to (light-cone) infinity

matrix element defining $d_2$ same as the integrand (for $x^- = 0$) in the QS-integral:

$$\langle k_y^\perp \rangle = \int_0^\infty dt F^y(t) \quad \text{(use } dx^- = \sqrt{2} dt)$$

$\leftrightarrow$ first integration point $\rightarrow F^y(0)$

$\leftrightarrow$ (transverse) force at the begin of the trajectory, i.e. at the moment after absorbing the virtual photon
Quark-Gluon Correlations (Interpretation)

- $x^2$-moment of twist-4 polarized PDF $g_3(x)$
  \[
  \int dxx^2g_3(x) \sim \left\langle P, S \left| \bar{q}(0)g\tilde{G}^{\mu\nu}(0)\gamma_\nu q(0) \right| P, S \right\rangle \sim f_2
  \]

  → different linear combination $f_2 = \chi_E - \chi_B$ of $\chi_E$ and $\chi_M$

  → combine with $d_2 \Rightarrow$ disentangle electric and magnetic force

- What should one expect (sign)?
  - $\kappa_q^p \rightarrow$ signs of deformation ($u/d$ quarks in $\pm \hat{y}$ direction for proton polarized in $+\hat{x}$ direction $\rightarrow$ expect force in $\mp \hat{y}$

  → $d_2$ positive/negative for $u/d$ quarks in proton

  - large $N_C$: $d_2^u/p = -d_2^d/p$

  - consistent with $f_{1T}^{\perp u} + f_{1T}^{\perp d} \approx 0$

  → lattice (Göckeler et al.): $d_2^u \approx 0.010$ and $d_2^d \approx -0.0056$

- $(M^2 \approx 5 \frac{\text{GeV}}{\text{fm}} \quad \langle F_u^y(0) \rangle \approx -50 \frac{\text{MeV}}{\text{fm}} \quad \langle F_d^y(0) \rangle \approx 28 \frac{\text{MeV}}{\text{fm}}$

- $x^2$-moment of chirally odd twist-3 PDF $e(x) \rightarrow$ transverse force on transversely polarized quark in unpolarized target ($\leftrightarrow$ Boer-Mulders $h_1^\perp$)
GPDs provide decomposition of form factors w.r.t. the momentum of the active quark

\[ \int \frac{dx^-}{2\pi} e^{ixp^+} x^- \left\langle p' \left| \bar{q} \left(-\frac{x^-}{2}\right) \gamma^+ q \left(\frac{x^-}{2}\right) \right| p \right\rangle \]

GPDs resemble both PDFs and form factors: defined through matrix elements of light-cone correlator, but \( \Delta \equiv p' - p \neq 0 \).

t-dependence of GPDs at \( \xi = 0 \) (purely \( \perp \) momentum transfer) \( \Rightarrow \) Fourier transform of impact parameter dependent PDFs \( q(x, b_{\perp}) \)

\[ q(x, b_{\perp}) = \int \frac{d^2\Delta_{\perp}}{(2\pi)^2} H(x, 0, -\Delta^2_{\perp}) e^{ib_{\perp} \cdot \Delta_{\perp}} \]

\[ \Delta q(x, b_{\perp}) = \int \frac{d^2\Delta_{\perp}}{(2\pi)^2} \tilde{H}(x, 0, -\Delta^2_{\perp}) e^{ib_{\perp} \cdot \Delta_{\perp}} \]

\( q(x, b_{\perp}) \) has probabilistic interpretation, e.g. \( q(x, b_{\perp}) > 0 \) for \( x > 0 \)
\[ \frac{\Delta_{\perp}}{2M} E(x, 0, -\Delta^2_{\perp}) \] describes how the momentum distribution of unpolarized partons in the \( \perp \) plane gets transversely distorted when is nucleon polarized in \( \perp \) direction.

(attractive) final state interaction in semi-inclusive DIS converts \( \perp \) position space asymmetry into \( \perp \) momentum space asymmetry

\[ \leftrightarrow \] simple physical explanation for observed Sivers effect in \( \gamma^* p \rightarrow \pi X \)

physical explanation for Boer-Mulders effect: correlation between quark spins (transversity) and currents

OAM in QED: \( L_e \neq L_e \) and \( \Delta \gamma \neq J_\gamma - L_\gamma \)

\[ d_2 \equiv 3 \int dxx^2 \bar{g}_2(x) \]: (average) transverse force on quarks in DIS \( \langle F_y \rangle = -M^2 d_2 \) (nucleon polarized in \( +\hat{x} \)-direction)

recommended reading: