Generalized Parton Distributions

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QCD as Fundamental Physics

- Example of fundamental physics:
  search for Higgs boson
- Motivation: HB is responsible for mass generation
- Observation: By far the largest part of visible mass around us is due to nucleons, $m_N/m_e \sim 2000$
- $\Rightarrow$ From 940 MeV of m_N, only < 30 MeV (current quark masses) may be related to Higgs particle
- $\Rightarrow$ Most part of mass is due to gluons, which are massless!

Situation in hadronic physics:

- All relevant fundamental particles established
- QCD Lagrangian is known
- Need to understand how QCD works

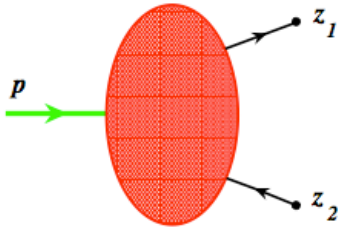
Hadrons in Terms of Quarks and Gluons

How to relate hadronic states $|p, s\rangle$

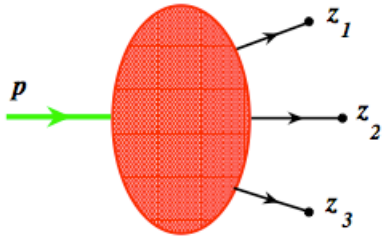
to quark and gluon fields $q(z_1)\ ,\ q(z_2)\ ,\ \ldots$?

Standard way: use matrix elements

$$\langle\, 0\,|\,\bar{q}_\alpha(z_1)\, q_\beta(z_2)\,|\, M(p), s\,\rangle\ ,\ \langle\, 0\,|\, q_\alpha(z_1)\, q_\beta(z_2)\, q_\gamma(z_3)|\, B(p), s\,\rangle$$



Meson–quark matrix element



Baryon–quark matrix element

- Can be interpreted as hadronic wave functions

Light-cone formalism

- Describe hadron by Fock components in infinite-momentum frame

For nucleon

$$|P\rangle = \psi_{qqq}|q(x_1P, k_{1\perp})\, q(x_2P, k_{2\perp})\, q(x_3P, k_{3\perp})\rangle$$
$$+ \psi_{qqqG}|qqqG\rangle + \psi_{qqq\bar{q}q}|qqq\bar{q}q\rangle + \psi_{qqqGG}|qqqGG\rangle + \ldots$$

- x_i : momentum fractions

$$\sum_i x_i = 1$$

- $k_{i\perp}$: transverse momenta

$$\sum_i k_{i\perp} = 0$$

Basics of LC Formalism

- Take a fast-moving hadron with momentum p in z-direction:

$$p_z \to \infty \ , \ E = \sqrt{p_z^2 + m^2} \to \infty \ , \ E^2 - p_z^2 = m^2 \ll p_z^2, E^2$$

- p^2 in components:

$$p^2 = p_0^2 - p_z^2 = (p_0 + p_z)(p_0 - p_z) \equiv p^+ p^-$$

- Note: “Plus" component p^+ is large, “minus” component $p^- = m^2/p^+$ is small
- Momentum p is close to a “light cone" vector P, for which P_- is zero, and $P^2 = 0$

$$p^\mu = P^\mu + p^2 n^\mu + 0_\perp$$

- Light-cone basis vectors:

$$P^2 = 0 \ , \ n^2 = 0 \ , \ 2(Pn) = 1$$

Two-particle system in LC Formalism

- Take a $q\bar{q}$ component of a meson, with quarks having momenta k and $p-k$:

$$\begin{aligned} k =& xP + yn + k_\perp \\ p-k =& (1-x)P + (p^2 - y)n - k_\perp \end{aligned}$$

- If both $k^2 = 0$ and $(p-k)^2 = 0$, we have

$$\begin{aligned} 0 =& k^2 = xy - k_\perp^2 \Rightarrow y = k_\perp^2/x \\ 0 =& (p-k)^2 = (1-x)(p^2 - y) - k_\perp^2 \\ \Rightarrow& (1-x)(p^2 - k_\perp^2/x) - k_\perp^2 = 0 \\ \Rightarrow& p^2 = k_\perp^2/x + k_\perp^2/(1-x) \end{aligned}$$

- For n-body component, the ‘light-cone” energy is

$$\sum_{i=1}^{n} k_{\perp i}^2/x_i$$

Problems of LC Formalism

- In principle: Solving bound-state equation

$$H|P\rangle = E|P\rangle$$

  one gets $|P\rangle$ which gives complete information about hadron structure
- In practice: Equation (involving infinite number of Fock components) has not been solved and is unlikely to be solved in near future
- Experimentally: LC wave functions are not directly accessible
- Way out: Description of hadron structure in terms of phenomenological functions

Phenomenological Functions

"Old" functions:

- Form Factors
- Usual Parton Densities
- Distribution Amplitudes

"New" functions:

Generalized Parton Distributions (GPDs)

GPDs = Hybrids of

Form Factors, Parton Densities and Distribution Amplitudes

"Old" functions

are limiting cases of "new" functions

Form Factors

Form factors are defined through matrix elements

of electromagnetic and weak currents between hadronic states

Nucleon EM form factors:

$$\langle\, p',\, s' \,|\, J^\mu(0) \,|\, p,\, s \,\rangle = \bar{u}(p', s') \left[\gamma^\mu F_1(t) + \frac{\Delta^\nu \sigma^{\mu\nu}}{2m_N} F_2(t) \right] u(p, s)$$

$(\Delta = p - p', t = \Delta^2)$

- Electromagnetic current
$$J^\mu(z) = \sum_{f(lavor)} e_f \bar{\psi}_f(z) \gamma^\mu \psi_f(z)$$
- Helicity non-flip form factor
$$F_1(t) = \sum_f e_f F_{1f}(t)$$
- Helicity flip form factor
$$F_2(t) = \sum_f e_f F_{2f}(t)$$

Form Factors, contd.

Limiting Values:

- Electric charge

$$F_1(t=0) = e_N = \sum_f^{N_f} e_f$$

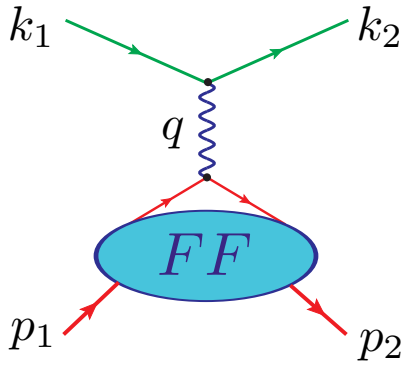
- Anomalous magnetic moment

$$F_2(t=0) = \kappa_N \equiv \sum_f^{N_f} \kappa_f$$

- N_f: number of valence quarks of flavor f

Form Factors are measurable

through elastic eN scattering



Form Factors and Charge Densities

Example of a density in Quantum Mechanics

$$\rho(z) = \psi^*(z)\psi(z)$$

Its Fourier transform gives form factor:

$$F(q) = \int e^{iqz}\rho(z)\, d\vec{z} = \int \Psi^*(k+q)\Psi(k)\, d\vec{k}$$

where $\Psi(k)$ is Fourier transform of $\psi(z)$

- Transition from initial state with momentum k to final state with momentum $k+q$
- Momentum transfer q is due to a probing current
- Fourier transform of the form factor gives density

$$\rho(z) = \int e^{-iqz} F(q) d\vec{q}/(2\pi)^3$$

Form Factors in LC Formalism

Transition of 2-body system with momentum p into 2-body system with momentum $p' = p + q$ in a frame, where $q^+ = 0$:

Initial momenta $\{xP^+, k_\perp\}$, $\{(1-x)P^+, -k_\perp\}$
Final momenta $\{xP^+, k_\perp + q_\perp\}$, $\{(1-x)P^+, -k_\perp\}$
Momentum of final active quark in terms of final momentum
$xP^+ + k_\perp + q_\perp = x(P^+ + q_\perp) + k_\perp + (1-x)q_\perp$

Drell-Yan formula for form factor:

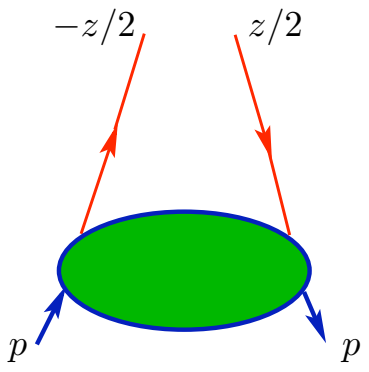
$$F(q_\perp^2) = \int_0^1 dx \int \Psi^*_{p'}(x, k_\perp + (1-x)q_\perp)\Psi_p(x, k_\perp)\, dk_\perp$$
$$\equiv \int_0^1 dx\, \mathcal{F}(x, q_\perp^2)$$

where $\mathcal{F}(x, q_\perp^2)$ is “nonforward parton density”

Usual Parton Densities

Parton Densities are defined through forward matrix elements

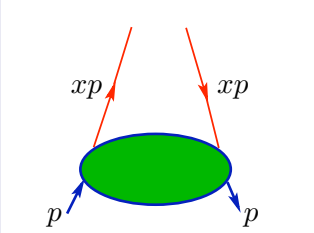
of quark/gluon fields separated by lightlike distances



Unpolarized quarks case:

$$\langle\, p \,|\, \bar{\psi}_a(-z/2)\gamma^\mu \psi_a(z/2) \,|\, p \,\rangle\big|_{z^2=0}$$
$$= 2p^\mu \int_0^1 \left[e^{-ix(pz)} f_a(x) - e^{ix(pz)} f_{\bar{a}}(x) \right] dx$$

Momentum space interpretation



$f_{a(\bar{a})}(x)$ is probability

to find a ($\bar{a}$) quark with momentum xp

Local limit $z = 0$

$\Rightarrow$ sum rule

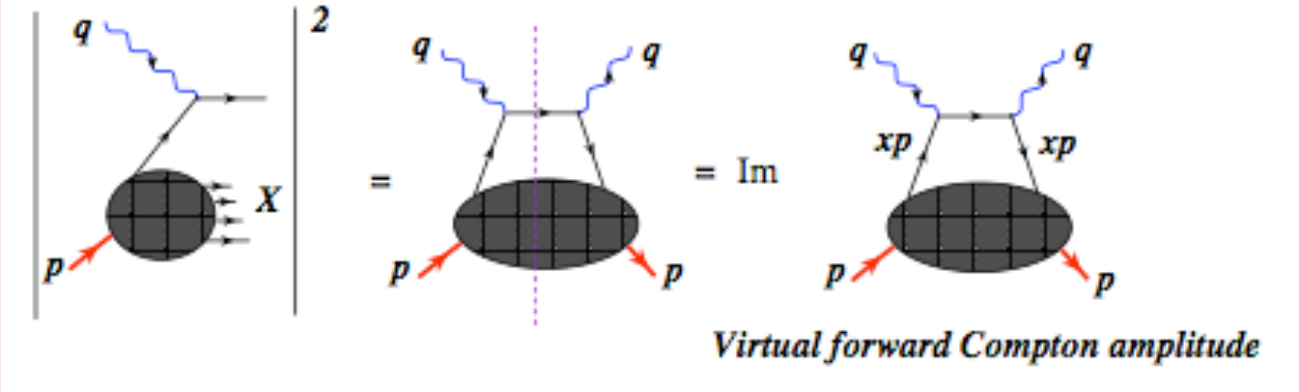
$\int_0^1 [f_a(x) - f_{\bar{a}}(x)]\, dx = N_a$

for valence quark numbers

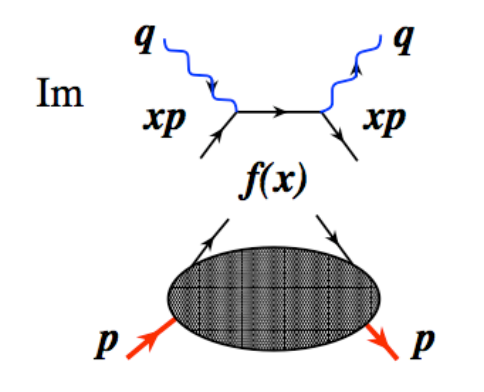
Deep Inelastic Scattering

Classic process to access usual parton densities:

deep inelastic scattering $\gamma^* N \to X$



Spacelike momentum transfer $q^2 \equiv -Q^2$



$$\mathrm{Im}\,\frac{1}{(q+xp)^2} \approx \frac{\pi}{2(pq)}\delta(x - x_{Bj})$$

Bjorken variable: $x_{Bj} = \frac{Q^2}{2(pq)}$

DIS measures $f(x_{Bj})$

Comparing to form factors:

point vertex instead of quark propagator and $p \neq p'$

Parton Densities in LC Formalism

Drell-Yan formula for parton density:

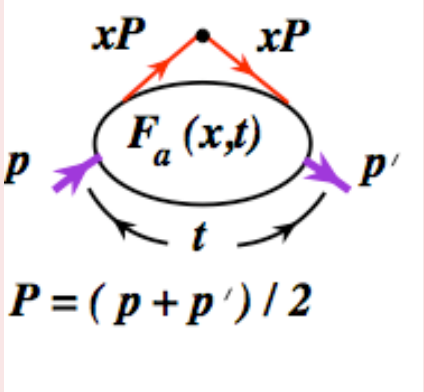
$$f(x) = \int \Psi^*_{p'}(x, k_\perp)\Psi_p(x, k_\perp)\, dk_\perp$$
$$= \mathcal{F}(x, q_\perp^2 = 0)$$

Usual parton density is a limiting case of (general) nonforward parton density

$$f(x) = \mathcal{F}(x, q_\perp^2 = 0)$$

Nonforward Parton Densities (Zero Skewness GPDs)

Combine form factors with parton densities



$$F_1(t) = \sum_a F_{1a}(t)$$

$$F_{1a}(t) = \int_0^1 \mathcal{F}_{1a}(x,t)\, dx$$

Flavor components of form factors

$$\mathcal{F}_{1a}(x,t) \equiv e_a[\mathcal{F}_a(x,t) - \mathcal{F}_{\bar{a}}(x,t)]$$

Forward limit $t = 0$

$$\mathcal{F}_{a(\bar{a})}(x, t=0) = f_{a(\bar{a})}(x)$$

Interplay between x and t dependences

Simplest factorized ansatz

$$\mathcal{F}_a(x,t) = f_a(x) F_1(t)$$

satisfies both forward and local constraints

Forward constraint

$$\mathcal{F}_a(x, t=0) = f_a(x)$$

Local constraint

$$\int_0^1 [\mathcal{F}_a(x,t) - \mathcal{F}_{\bar{a}}(x,t)] dx = F_{1a}(t)$$

Reality is more complicated:

LC wave function with Gaussian $k_\perp$ dependence

$$\Psi(x_i, k_{i\perp}) \sim \exp\left[-\frac{1}{\lambda^2} \sum_i \frac{k_{i\perp}^2}{x_i}\right]$$

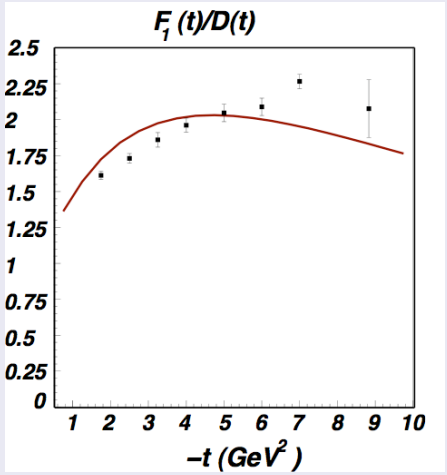
suggests

$$\mathcal{F}_a(x,t) = f_a(x) e^{\bar{x} t / 2x\lambda^2}$$

$f_a(x)$=experimental densities

Adjusting λ^2 to provide

$$\langle k_\perp^2 \rangle \approx (300\mathrm{MeV})^2$$



Nonforward Parton Densities in Gaussian LC Model

Take Drell-Yan formula for NPD:

$$\mathcal{F}(x, q_\perp^2) = \int \Psi^*_{p'}(x, k_\perp + (1-x)q_\perp)\Psi_p(x, k_\perp)\, dk_\perp$$

with Gaussian LC wave functions:

$$\Psi(x, k_\perp) = \varphi(x)\, \exp\left[-\frac{k_\perp^2}{\lambda^2 x(1-x)}\right]$$

The result is

$$\begin{aligned}\mathcal{F}(x, q_\perp^2) =& \frac{\pi}{2}\, \lambda^2\, x(1-x)\, \varphi^2(x)\, \exp\left[-\frac{(1-x)q_\perp^2}{2x\lambda^2}\right] \\ =& f(x) \exp\left[-\frac{(1-x)q_\perp^2}{2x\lambda^2}\right]\end{aligned}$$

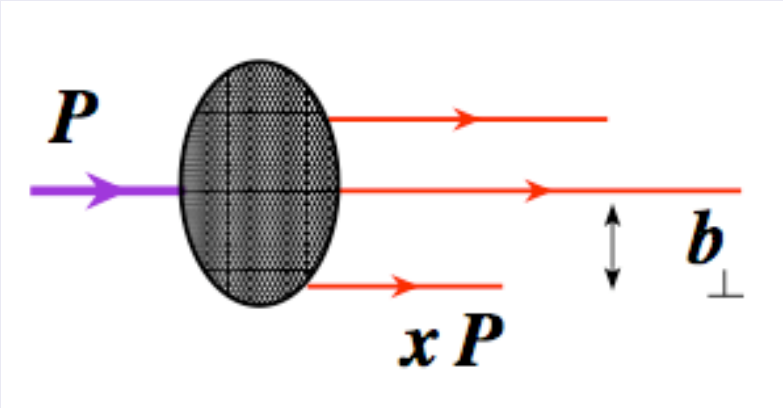
Impact Parameter Distributions

NPDs can be treated as Fourier transforms of impact parameter $b_\perp$ distributions $f_a(x, b_\perp)$

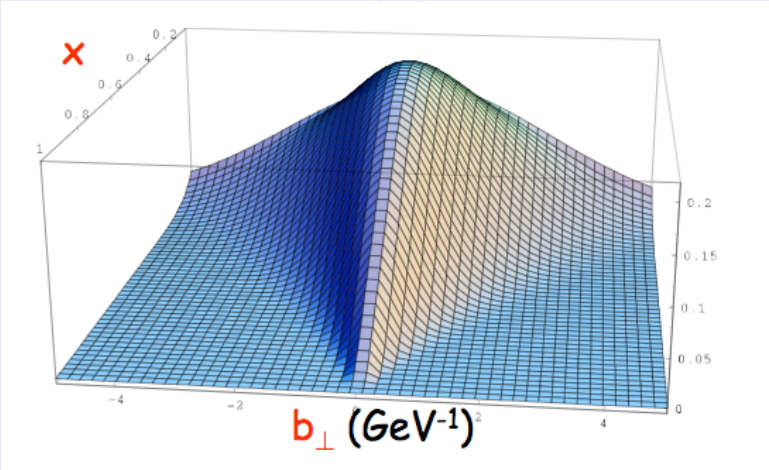
$$\mathcal{F}_a(x, q_\perp^2) = \int f_a(x, b_\perp) e^{i(q_\perp b_\perp)} d^2 b_\perp$$

$b_\perp$ = $\perp$ distance to center of momentum

IPDs describe nucleon structure in transverse plane



Distribution $f_u^p(x, b_\perp)$



Impact Parameter Distributions in Gaussian LC Model

Invert Definition of IPD

$$f(x, b_\perp) = \int \mathcal{F}(x, q_\perp^2) \frac{d^2 q_\perp}{(2\pi)^2}$$

Using Gaussian Model for NPD:

$$\mathcal{F}(x, q_\perp^2) = f(x) \exp\left[-\frac{(1-x) q_\perp^2}{2x\lambda^2}\right]$$

The result is

$$f(x, b_\perp) = \frac{\lambda^2 x}{2\pi(1-x)} f(x) \exp\left[-\frac{x b_\perp^2 \lambda^2}{2(1-x)}\right]$$

Shape of Impact Parameter Distributions in Gaussian LC Model

Using the result

$$f(x, b_\perp) = \frac{\lambda^2 x}{2\pi(1-x)} f(x) \exp\left[-\frac{x b_\perp^2 \lambda^2}{2(1-x)}\right]$$

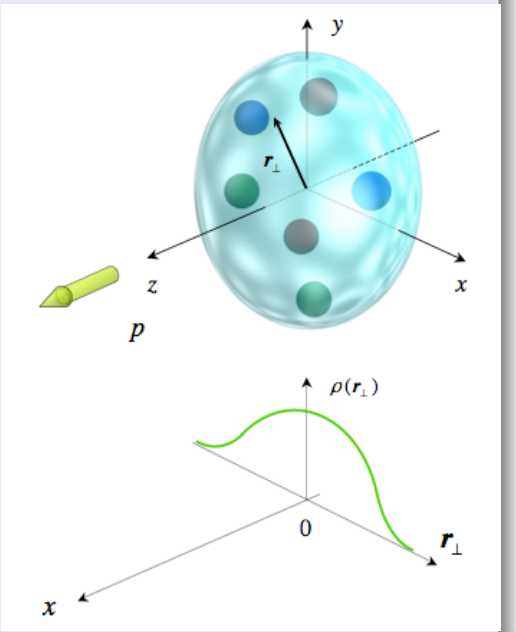
We obtain the dependence of $b_\perp$ profile on x

$$\langle b_\perp^2(x)\rangle \equiv \left(\int b_\perp^2 f(x, b_\perp)\, d^2 b_\perp\right) \Big/ \left(\int f(x, b_\perp)\, d^2 b_\perp\right)$$
$$= \frac{2}{\lambda^2} \frac{1-x}{x}$$

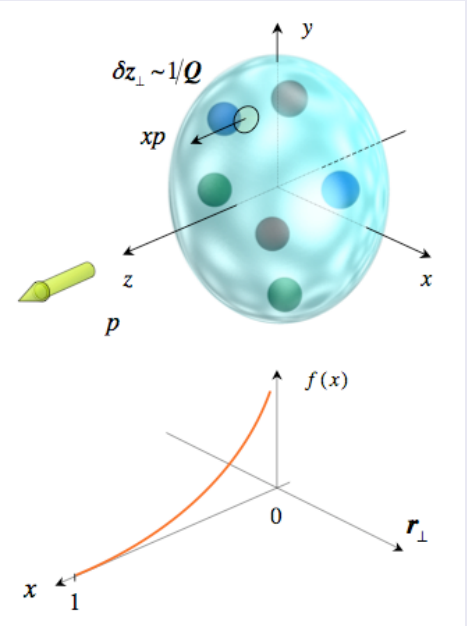
$\Rightarrow$ Width goes to zero when $x \to 1$

Probabilistic Interpretation of IPDs

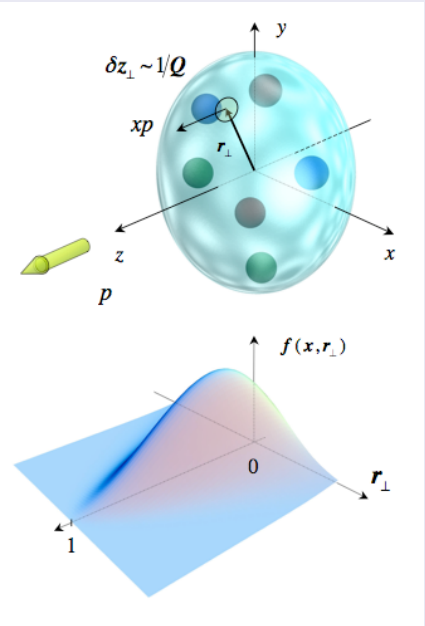
Distribution in $b_\perp$ plane



Distribution in z momentum



Combined $(x, b_\perp)$ distribution



Defining "center" in $\perp$ plane:

Geometric center: $\sum_i b_{i\perp} = 0$

Center of momentum: $\sum_i x_i b_{i\perp} = 0$

Shape of $(x, b_\perp)$ distribution:

Shrinks when $x \to 1$: leading parton determines center of momentum

Regge-type models for NPDs

"Regge" improvement:

$$f(x) \sim x^{-\alpha(0)}$$
$$\Rightarrow \mathcal{F}(x,t) \sim x^{-\alpha(t)}$$
$$\Rightarrow \mathcal{F}(x,t) = f(x) x^{-\alpha' t}$$

Accomodating quark counting rules:

$$\mathcal{F}(x,t) = f(x) x^{-\alpha' t(1-x)}|_{x\to 1}$$
$$\sim f(x) e^{\alpha'(1-x)^2 t}$$

Does not change small-x behavior but provides

$f(x)|_{x\to 1}$ vs. $F(t)|_{t\to\infty}$ interplay:

$$f(x) \sim (1-x)^n \Rightarrow F_1(t) \sim t^{-(n+1)/2}$$

Note: no pQCD involved in these counting rules!

Extra $1/t$ for $F_2(t)$

can be produced by taking

$$\mathcal{E}_a(x,t) \sim (1-x)^2 \mathcal{F}_a(x,t)$$

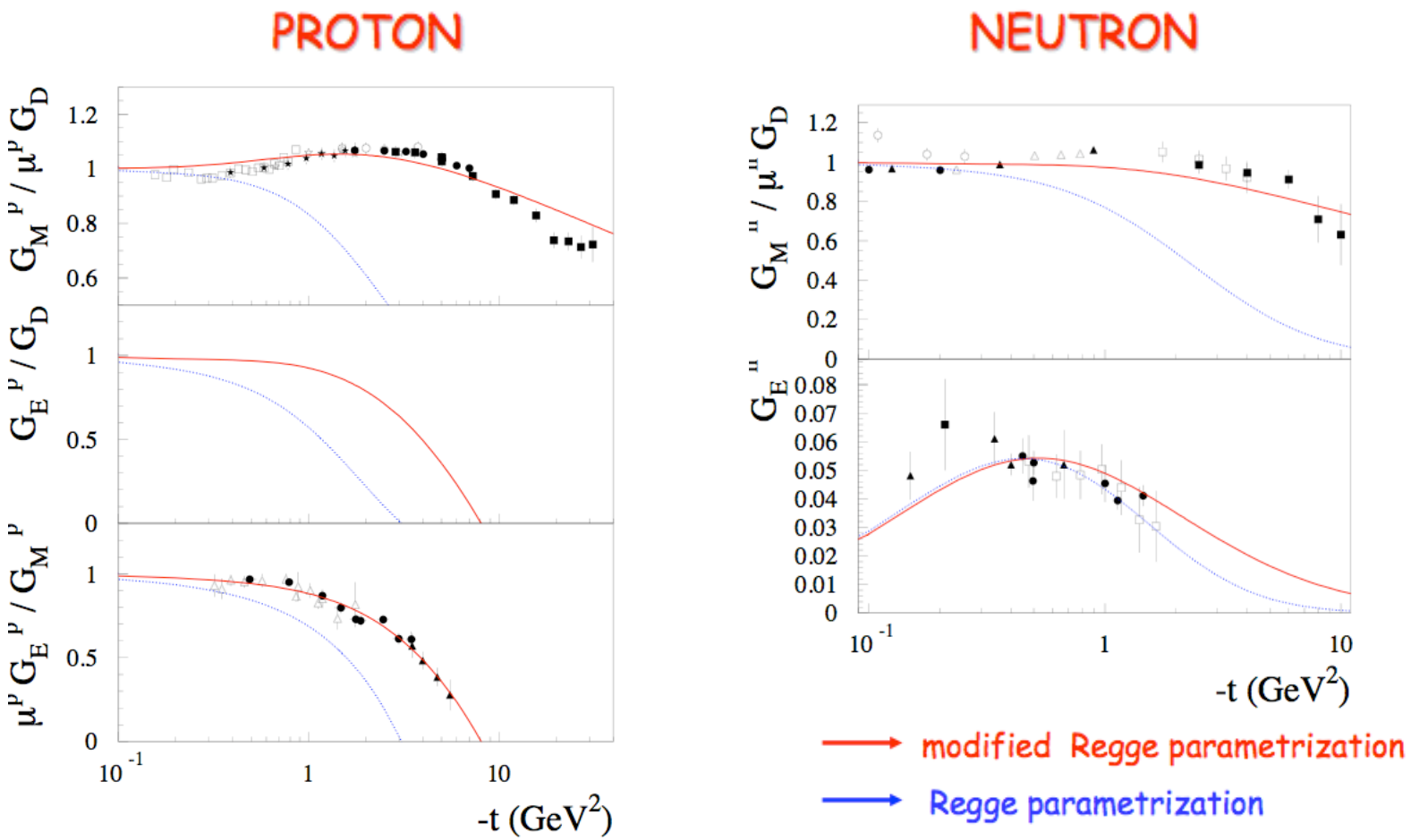
for "magnetic" NPDs

More general:

$$\mathcal{E}_a(x,t) \sim (1-x)^{\eta_a} \mathcal{F}_a(x,t)$$

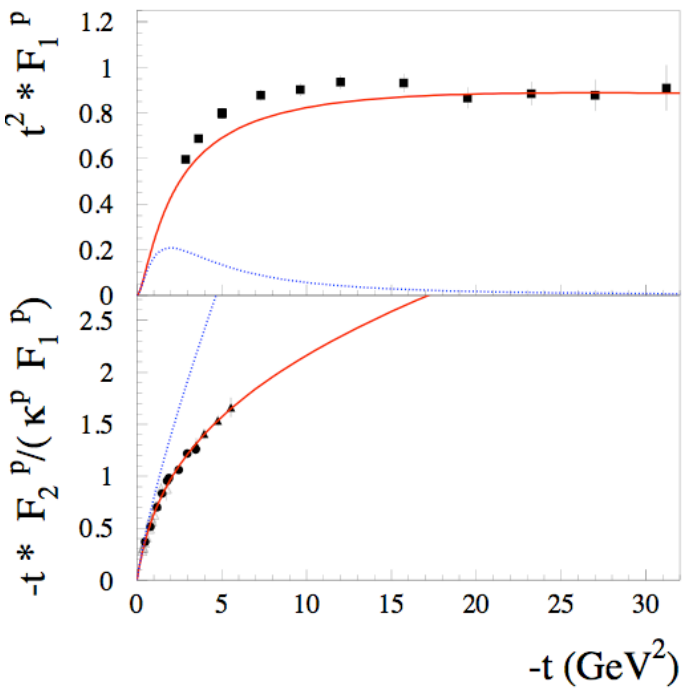
Fit : $\eta_u = 1.6$, $\eta_d = 1$

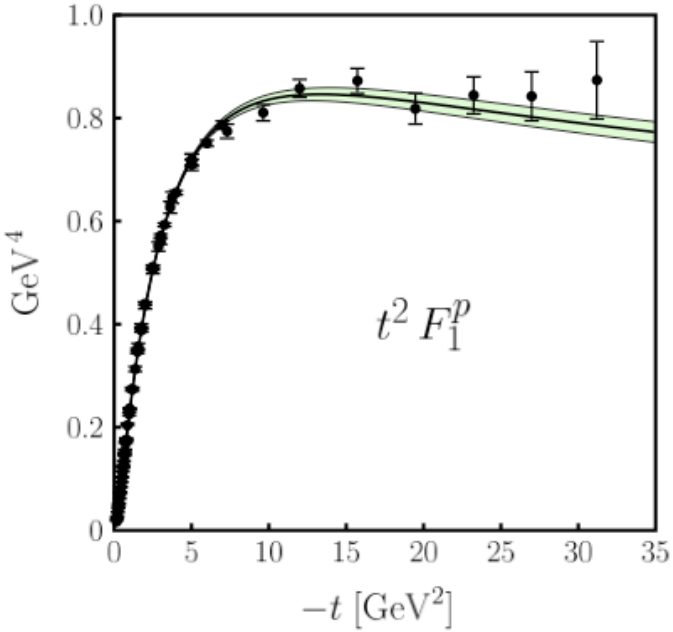
Fit to All Four Nucleon $G_{E,M}$ Form Factors



Fit to $F_{1,2}$ Proton Form Factors

Modified Regge parametrization describes JLab polarization transfer data on G_E^p/G_M^p and F_2^p/F_1^p





Similar model was constructed by Diehl et al. (right figure)

Summary