Generalized Parton Distributions
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Example of fundamental physics:
search for Higgs boson

Motivation: HB is responsible for mass generation

Observation: By far the largest part of visible mass around us is due to nucleons, $m_N/m_e \sim 2000$

$\Rightarrow$ From 940 MeV of $m_N$, only $< 30$ MeV (current quark masses) may be related to Higgs particle

$\Rightarrow$ Most part of mass is due to gluons, which are massless!

Situation in hadronic physics:

- All relevant fundamental particles established
- QCD Lagrangian is known
- Need to understand how QCD works
How to relate hadronic states $|p, s\rangle$
to quark and gluon fields $q(z_1), q(z_2), \ldots$?

Standard way: use matrix elements

$$\langle 0 | \bar{q}_\alpha(z_1) q_\beta(z_2) | M(p), s \rangle, \langle 0 | q_\alpha(z_1) q_\beta(z_2) q_\gamma(z_3) | B(p), s \rangle$$

- Can be interpreted as hadronic wave functions
Light-cone formalism

- Describe hadron by Fock components in infinite-momentum frame

For nucleon

\[
|P\rangle = \psi_{qqq}|qqq\rangle + \psi_{qqqG}|qqqG\rangle + \psi_{qqq\bar{q}}|qqq\bar{q}\rangle + \psi_{qqqGG}|qqqGG\rangle + \ldots
\]

- \(x_i\): momentum fractions
  \[
  \sum_i x_i = 1
  \]

- \(k_i\): transverse momenta
  \[
  \sum_i k_i = 0
  \]
Take a fast-moving hadron with momentum $p$ in $z$-direction:

$$p_z \to \infty, \quad E = \sqrt{p_z^2 + m^2} \to \infty, \quad E^2 - p_z^2 = m^2 \ll p_z^2, E^2$$

$p^2$ in components:

$$p^2 = p_0^2 - p_z^2 = (p_0 + p_z)(p_0 - p_z) \equiv p^+ p^-$$

Note: “Plus” component $p^+$ is large, “minus” component $p^- = m^2 / p^+$ is small

Momentum $p$ is close to a “light cone” vector $P$, for which $P_-$ is zero, and $P^2 = 0$

$$p^\mu = P^\mu + p^2 n^\mu + 0_\perp$$

Light-cone basis vectors:

$$P^2 = 0, \quad n^2 = 0, \quad 2(Pn) = 1$$
Two-particle system in LC Formalism

- Take a $q\bar{q}$ component of a meson, with quarks having momenta $k$ and $p - k$:
  \[
  k = xP + yn + k_{\perp}
  \]
  \[
  p - k = (1 - x)P + (p^2 - y)n - k_{\perp}
  \]

- If both $k^2 = 0$ and $(p - k)^2 = 0$, we have
  \[
  0 = k^2 = xy - k_{\perp}^2 \Rightarrow y = k_{\perp}^2 / x
  \]
  \[
  0 = (p - k)^2 = (1 - x)(p^2 - y) - k_{\perp}^2
  \]
  \[
  \Rightarrow (1 - x)(p^2 - k_{\perp}^2 / x) - k_{\perp}^2 = 0
  \]
  \[
  \Rightarrow p^2 = k_{\perp}^2 / x + k_{\perp}^2 / (1 - x)
  \]

- For $n$-body component, the ‘light-cone” energy is
  \[
  \sum_{i=1}^{n} k_{\perp i}^2 / x_i
  \]
Problems of LC Formalism

- **In principle**: Solving bound-state equation
  
  \[ H |P\rangle = E |P\rangle \]

  one gets \(|P\rangle\) which gives complete information about hadron structure

- **In practice**: Equation (involving infinite number of Fock components) has not been solved and is unlikely to be solved in near future

- **Experimentally**: LC wave functions are not directly accessible

- **Way out**: Description of hadron structure in terms of phenomenological functions
Phenomenological Functions

“Old” functions:
- Form Factors
- Usual Parton Densities
- Distribution Amplitudes

“New” functions:
- Generalized Parton Distributions (GPDs)

GPDs = Hybrids of Form Factors, Parton Densities and Distribution Amplitudes

“Old” functions are limiting cases of “new” functions
Form factors are defined through matrix elements of electromagnetic and weak currents between hadronic states.

**Nucleon EM form factors:**

\[
\langle p', s' \mid J^\mu(0) \mid p, s \rangle = \bar{u}(p', s') \left[ \gamma^\mu F_1(t) + \frac{\Delta^\nu \sigma^{\mu\nu}}{2m_N} F_2(t) \right] u(p, s)
\]

\((\Delta = p - p', t = \Delta^2)\)

- Electromagnetic current
  \[
  J^\mu(z) = \sum_{f(\text{lavor})} e_f \bar{\psi}_f(z) \gamma^\mu \psi_f(z)
  \]

- Helicity non-flip form factor
  \[
  F_1(t) = \sum_f e_f F_{1f}(t)
  \]

- Helicity flip form factor
  \[
  F_2(t) = \sum_f e_f F_{2f}(t)
  \]
Limiting Values:

- Electric charge
  \[ F_1(t = 0) = e_N = \sum_{f}^{N_f} e_f \]
- Anomalous magnetic moment
  \[ F_2(t = 0) = \kappa_N \equiv \sum_{f}^{N_f} \kappa_f \]
- \( N_f \): number of valence quarks of flavor \( f \)

Form Factors are measurable through elastic \( eN \) scattering
Example of a density in Quantum Mechanics

\[ \rho(z) = \psi^*(z)\psi(z) \]

Its Fourier transform gives form factor:

\[ F(q) = \int e^{iqz} \rho(z) \, d\vec{z} = \int \Psi^*(k + q)\Psi(k) \, d\vec{k} \]

where \( \Psi(k) \) is Fourier transform of \( \psi(z) \)

- **Transition** from initial state with momentum \( k \) to final state with momentum \( k + q \)
- **Momentum transfer** \( q \) is due to a probing current
- Fourier transform of the form factor gives density

\[ \rho(z) = \int e^{-iqz} F(q) \, d\vec{q}/(2\pi)^3 \]
Transition of 2-body system with momentum $p$ into 2-body system with momentum $p' = p + q$ in a frame, where $q^+ = 0$:

Initial momenta $\{xP^+, k_\perp\}$, $\{(1 - x)P^+, -k_\perp\}$

Final momenta $\{xP^+, k_\perp + q_\perp\}$, $\{(1 - x)P^+, -k_\perp\}$

Momentum of final active quark in terms of final momentum $xP^+ + k_\perp + q_\perp = x(P^+ + q_\perp) + k_\perp + (1 - x)q_\perp$

Drell-Yan formula for form factor:

$$F(q^2_\perp) = \int_0^1 dx \int \Psi^*_p(x, k_\perp + (1 - x)q_\perp)\Psi_p(x, k_\perp) \, dk_\perp$$

$$\equiv \int_0^1 dx \mathcal{F}(x, q^2_\perp)$$

where $\mathcal{F}(x, q^2_\perp)$ is “nonforward parton density”
Usual Parton Densities

Parton Densities are defined through forward matrix elements of quark/gluon fields separated by lightlike distances.

Unpolarized quarks case:

\[
\left\langle p \left| \bar{\psi}_a \left( -z/2 \right) \gamma^\mu \psi_a \left( z/2 \right) \right| p \right\rangle \bigg|_{z^2=0} = 2p^\mu \int_0^1 \left[ e^{-ix(pz)} f_a(x) - e^{ix(pz)} f_{\bar{a}}(x) \right] dx
\]

Momentum space interpretation:

\( f_{a(\bar{a})}(x) \) is probability to find a (\( \bar{a} \)) quark with momentum \( xp \)

Local limit \( z = 0 \)

\( \Rightarrow \) sum rule

\( \int_0^1 \left[ f_a(x) - f_{\bar{a}}(x) \right] dx = N_a \)

for valence quark numbers.
Deep Inelastic Scattering

Classic process to access usual parton densities:

Deep inelastic scattering $\gamma^* N \rightarrow X$

Spacelike momentum transfer $q^2 \equiv -Q^2$

$$\text{Im} \left( \frac{1}{(q + xp)^2} \right) \approx \frac{\pi}{2(pq)} \delta(x - x_{Bj})$$

Bjorken variable: $x_{Bj} = \frac{Q^2}{2(pq)}$

DIS measures $f(x_{Bj})$

Comparing to form factors: point vertex instead of quark propagator and $p \neq p'$
Drell-Yan formula for parton density:

\[ f(x) = \int \Psi_p^*(x, k_\perp) \Psi_p(x, k_\perp) \, dk_\perp \]

\[ = \mathcal{F}(x, q^2_\perp = 0) \]

Usual parton density is a limiting case of (general) nonforward parton density:

\[ f(x) = \mathcal{F}(x, q^2_\perp = 0) \]
**Nonforward Parton Densities**
*(Zero Skewness GPDs)*

**Combine form factors with parton densities**

\[
F_1(t) = \sum_a F_{1a}(t)
\]

\[
F_{1a}(t) = \int_0^1 \mathcal{F}_{1a}(x, t) \, dx
\]

**Flavor components of form factors**

\[
\mathcal{F}_{1a}(x, t) \equiv e_a [\mathcal{F}_a(x, t) - \mathcal{F}_{\bar{a}}(x, t)]
\]

**Forward limit** \( t = 0 \)

\[
\mathcal{F}_{a(\bar{a})}(x, t = 0) = f_{a(\bar{a})}(x)
\]
Interplay between $x$ and $t$ dependences

**Simplest factorized ansatz**

\[ \mathcal{F}_a(x, t) = f_a(x) F_1(t) \]
satisfies both forward and local constraints

**Forward constraint**

\[ \mathcal{F}_a(x, t = 0) = f_a(x) \]

**Local constraint**

\[ \int_0^1 [\mathcal{F}_a(x, t) - \mathcal{F}_{\bar{a}}(x, t)] dx = F_{1a}(t) \]

**Reality is more complicated:**

LC wave function with Gaussian $k_\perp$ dependence

\[ \Psi(x_i, k_i \perp) \sim \exp \left[ -\frac{1}{\lambda^2} \sum_i \frac{k_i^2 \perp x_i}{x_i} \right] \]
suggests

\[ \mathcal{F}_a(x, t) = f_a(x) e^{\bar{x}t/2x\lambda^2} \]

\[ f_a(x) = \text{experimental densities} \]

Adjuncting $\lambda^2$ to provide

\[ \langle k_\perp^2 \rangle \approx (300\text{MeV})^2 \]
Nonforward Parton Densities in Gaussian LC Model

Take Drell-Yan formula for NPD:

\[ F(x, q_{\perp}^2) = \int \Psi_{p'}^*(x, k_{\perp} + (1 - x)q_{\perp}) \Psi_p(x, k_{\perp}) \, dk_{\perp} \]

with Gaussian LC wave functions:

\[ \Psi(x, k_{\perp}) = \varphi(x) \exp \left[ -\frac{k_{\perp}^2}{\lambda^2 x (1 - x)} \right] \]

The result is

\[ F(x, q_{\perp}^2) = \frac{\pi}{2} \lambda^2 x (1 - x) \varphi^2(x) \exp \left[ -\frac{(1 - x)q_{\perp}^2}{2x \lambda^2} \right] \]

\[ = f(x) \exp \left[ -\frac{(1 - x)q_{\perp}^2}{2x \lambda^2} \right] \]
Impact Parameter Distributions

NPDs can be treated as Fourier transforms of impact parameter $b_\perp$ distributions $f_a(x, b_\perp)$

$$F_a(x, q_\perp^2) = \int f_a(x, b_\perp) e^{i(q_\perp b_\perp)} d^2 b_\perp$$

$b_\perp = \perp$ distance to center of momentum

IPDs describe nucleon structure in transverse plane

Distribution $f_p(x, b_\perp)$
**Impact Parameter Distributions in Gaussian LC Model**

**Invert Definition of IPD**

\[ f(x, b_\perp) = \int \mathcal{F}(x, q^2_\perp) \frac{d^2 q_\perp}{(2\pi)^2} \]

Using Gaussian Model for NPD:

\[ \mathcal{F}(x, q^2_\perp) = f(x) \exp \left[ -\frac{(1 - x)q^2_\perp}{2x\lambda^2} \right] \]

**The result is**

\[ f(x, b_\perp) = \frac{\lambda^2 x}{2\pi(1 - x)} f(x) \exp \left[ -\frac{xb^2_\perp \lambda^2}{2(1 - x)} \right] \]
Shape of Impact Parameter Distributions in Gaussian LC Model

Using the result

\[ f(x, b_\perp) = \frac{\lambda^2 x}{2\pi(1-x)} f(x) \exp \left[ -\frac{xb_\perp^2 \lambda^2}{2(1-x)} \right] \]

We obtain the dependence of \( b_\perp \) profile on \( x \)

\[ \langle b_\perp^2(x) \rangle \equiv \left( \int b_\perp^2 f(x, b_\perp) \, d^2b_\perp \right) / \left( \int f(x, b_\perp) \, d^2b_\perp \right) \]

\[ = \frac{2}{\lambda^2} \frac{1-x}{x} \]

\[ \Rightarrow \text{Width goes to zero when } x \to 1 \]
Probabilistic Interpretation of IPDs

Defining “center” in $\perp$ plane:
- Geometric center: $\sum_i b_i = 0$
- Center of momentum: $\sum_i x_i b_i = 0$

Shape of $(x, b_{\perp})$ distribution:
Shrinks when $x \to 1$: leading parton determines center of momentum
Regge-type models for NPDs

“Regge” improvement:

\[ f(x) \sim x^{-\alpha(0)} \]
\[ \Rightarrow \mathcal{F}(x, t) \sim x^{-\alpha(t)} \]
\[ \Rightarrow \mathcal{F}(x, t) = f(x)x^{-\alpha'(t)} \]

Accommodating quark counting rules:

\[ \mathcal{F}(x, t) = f(x)x^{-\alpha'(1-x)} \mid_{x \to 1} \]
\[ \sim f(x)e^{\alpha'(1-x)^2 t} \]

Does not change small-\(x\) behavior but provides

\[ f(x) \mid_{x \to 1} \text{ vs. } F(t) \mid_{t \to \infty} \text{ interplay:} \]
\[ f(x) \sim (1 - x)^n \Rightarrow F_1(t) \sim t^{-(n+1)/2} \]

Note: no pQCD involved in these counting rules!

Extra \(1/t\) for \(F_2(t)\)

can be produced by taking

\[ \mathcal{E}_a(x, t) \sim (1 - x)^2 \mathcal{F}_a(x, t) \]

for “magnetic” NPDs

More general:

\[ \mathcal{E}_a(x, t) \sim (1 - x)^{\eta_a} \mathcal{F}_a(x, t) \]
Fit: \(\eta_u = 1.6\), \(\eta_d = 1\)
Fit to All Four Nucleon $G_{E,M}$ Form Factors

PROTON

NEUTRON

$\mu^p G_E^p / G_M^p, \mu^p G_D^p / G_M^p$ vs $-t (GeV^2)$

$G_M^n / \mu^n G_D^n$ vs $-t (GeV^2)$

modified Regge parametrization

Regge parametrization
Modified Regge parametrization describes JLab polarization transfer data on $G^p_E/G^p_M$ and $F_2^p/F_1^p$.

Similar model was constructed by Diehl et al. (right figure)
Summary

1. Why Study QCD and Hadronic Structure
2. Light-cone formalism
3. Form Factors
4. Usual Parton Densities
5. Deep Inelastic Scattering
6. Nonforward Parton Densities
7. Regge-type models for NPDs