QCD IN THE LARGE Nc LIMIT AND ITS APPLICATIONS TO NUCLEAR PHYSICS

Alvaro Calle Cordon
JLab, Theory Center

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Lecture I: Foundations

- QCD in a nutshell
- Hydrogen atom in N dimensions
- Large Nc limit of QCD
- Mesons, baryons and its interactions

Bibliography

Hadron spectrum & Quark Model

Gell-Mann & Ne’eman, 1964

\[ q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \]

SU_{F}(3) Flavor

\[ 3 \otimes \bar{3} = 8 + 1 \]

Approximate symmetry

\[ m_u \neq m_d \neq m_s \]

\[ \Lambda(1405) \]

\[ J^P = \frac{1}{2}^- \]

\[ J^P = \frac{3}{2}^+ \]

\[ J^P = 0^- \]

\[ J^P = 1^- \]
**Hadron spectrum & Quark Model**

**Baryon octect**

<table>
<thead>
<tr>
<th>Baryon</th>
<th>Wave function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta^{++} )</td>
<td>( uuu )</td>
</tr>
<tr>
<td>( \Delta^{+} )</td>
<td>( (uud + udu + duu)/\sqrt{3} )</td>
</tr>
<tr>
<td>( \Delta^{0} )</td>
<td>( (udd + dud + ddu)/\sqrt{3} )</td>
</tr>
<tr>
<td>( \Delta^{-} )</td>
<td>( ddd )</td>
</tr>
<tr>
<td>( \Sigma^{*+} )</td>
<td>( (uus +usu + suu)/\sqrt{3} )</td>
</tr>
<tr>
<td>( \Sigma^{*0} )</td>
<td>( (uds + usd + dus + dsu + sud + sdu)/\sqrt{6} )</td>
</tr>
<tr>
<td>( \Sigma^{*-} )</td>
<td>( (dds + dsd + sdd)/\sqrt{3} )</td>
</tr>
<tr>
<td>( \Xi^{*+} )</td>
<td>( (uss + sus + ssu)/\sqrt{3} )</td>
</tr>
<tr>
<td>( \Xi^{*0} )</td>
<td>( (dds + dsd + sdd)/\sqrt{3} )</td>
</tr>
<tr>
<td>( \Xi^{*-} )</td>
<td>( (dss + sds + ssd)/\sqrt{3} )</td>
</tr>
<tr>
<td>( \Omega^{-} )</td>
<td>( sss )</td>
</tr>
</tbody>
</table>

\[ \Delta^{++} = u \uparrow u \uparrow u \uparrow \]

\[ \psi_{\Delta^{++}} = \psi(\text{space}) \psi(\text{spin}) \psi(\text{flavor}) \psi(\text{color}) \]

Particles in Nature are color singlets

\[ \psi(\text{color}) = (rgb - rbg + gbr - grb + bgr - bgr)/\sqrt{6} \]

Pauli’s principle

\[ N_c = 3 \]
**QCD in a Nutshell**

Quantum Chromo Dynamics = gauge theory of the strong interactions

\[ \mathcal{L}_{QCD} = \sum_{f} \bar{q}_f (iD_\mu \gamma^\mu - m_f) q_f - \frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} \]

\[ D_\mu q_f \equiv (\partial_\mu + igA_\mu) q_f \]
\[ A_\mu = A_\mu^a \frac{\lambda^a}{2} \]

\[ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c \]

\[ q_f = \begin{pmatrix} q_f, r \\ q_f, g \\ q_f, b \end{pmatrix} \]
\[ q'_f, \alpha = U_{\alpha\beta} q_f, \beta \]
\[ U \equiv \exp \left( -i \frac{\theta_a}{2} \frac{\lambda^a}{2} \right) \]

Eight non-commuting generators

\[ \left[ \frac{\lambda_a}{2}, \frac{\lambda_b}{2} \right] = i f^{abc} \lambda^c \]

**SU_c(3) gauge group**

\[ A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \theta(x) \]

(QED)

\[ A_\mu^a(x) \rightarrow A_\mu^a(x) + \frac{1}{g} \partial_\mu \theta^a(x) + f^{abc} A_\mu^c(x) \theta^b(x) \]

(QCD)

Generators of the adjoint representation

Gluons are in the adjoint representation
Asymptotic Freedom

\[ \alpha_s(Q^2) = \frac{4\pi}{(11 - \frac{2}{3}N_f) \log \left( \frac{Q^2}{\Lambda^2} \right)} \]

\[ \alpha_s \equiv \frac{g^2}{4\pi} \]

QCD in a nutshell

David J. Gross, H. David Politzer & Frank Wilczek, 2004
Non-perturbative methods

✧ Chiral Perturbation Theory

Too much to say about for a two-hours lecture

✧ Lattice QCD

Jo Dudek lectures next week!

✧ Large Nc limit of QCD

I’ll try to give you an overview
Chiral Perturbation Theory

Effective Field Theory of QCD in the chiral limit

\[ m_u \sim 2.0 \pm 1.0 \text{ MeV} \]
\[ m_d \sim 4.5 \pm 0.5 \text{ MeV} \]
\[ m_s \sim 100 \pm 30 \text{ MeV} \]
\[ m_c \sim 1.29 \pm 0.11 \text{ GeV} \]
\[ m_b \sim 4.19 \pm 0.18 \text{ GeV} \]
\[ m_c \sim 172.9 \pm 1.5 \text{ GeV} \]

\[ \phi, f_0, \eta, \rho, \omega, K^\pm, K^0, \bar{K}^0, \Delta^0, \Delta^+, \Delta^{++}, \Sigma^0, \Sigma^+ \]

\[ \chi \]

\[ \mathcal{L}_{QCD}^0 = \sum_{l=u,d,s} (\bar{q}_L l \gamma \cdot D q_L + \bar{q}_R l \gamma \cdot D q_R) \]

\[ q_L = \frac{1}{2} (1 - \gamma_5) q \]
\[ q_R = \frac{1}{2} (1 + \gamma_5) q \]

\[ SU_c(3) \times SU_f(3) \]

\[ SU_L(3) \times SU_R(3) \]

Spontaneous symmetry breaking

Nambu-Goldstone bosons

Eight (three) massless pseudo scalar particles!
Hydrogen atom in N-dimensions

\[ H = \frac{p^2}{2m} - \frac{\alpha}{r} \]

\[ \alpha = 1/137 \approx 0.0073 \]

Is perturbation theory applicable? NO!

\[ r \rightarrow \bar{r}t, \ p \rightarrow \bar{p}/t, \ t \equiv 1/(m\alpha^2) \]

\[ H \rightarrow \frac{H}{m\alpha^2} \equiv \bar{H} = \frac{\bar{p}^2}{2} - \frac{1}{\bar{r}} \]

The expansion parameter disappears, we can absorb the overall energy by redefining the temporal scale.

A hidden expansion parameter is the space dimension.

Hydrogen atom in N-dimensions with N very large

\[
\begin{aligned}
- \frac{1}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{N}{r} \frac{\partial}{\partial r} \right) - \frac{\alpha}{r} \right] \Psi &= E \Psi \\
\end{aligned}
\]

Rescaling \( \Psi = r^{-N/2} \tilde{\Psi} \), \( r = N^2 R \)

\[ H = \frac{1}{N^2} \left[ - \frac{1}{2mN^2} \frac{\partial^2}{\partial R^2} + \frac{1}{8mR^2} - \frac{\alpha}{R} \right] \]

\[ M_{\text{eff}} = mN^2 \quad V_{\text{eff}}(R) = 1/(8mR^2) - \alpha/R \]

To lowest order in \( 1/N \) \[ E_0 = V_{\text{eff}}(r_0)/N^2 = -2m\alpha^2/N^2 \]

\[ E_0 \bigg|_{N=3} = -2/9 \, m\alpha^2 \]

\[ E_0 \bigg|_{\text{exact}} = -1/2 \, m\alpha^2 \]
Large $N_c$ limit of QCD

$SU(3) \rightarrow SU(N_c)$

$t'Hooft double-line notation$

$t'Hooft limit$

\[ A_{\mu b}^a \sim q^a \bar{q}_b \]

\[ A_{\mu b}^a \sim q^a \bar{q}_b \]

\[ g \sim 1/\sqrt{N_c} \]

\[ \lim_{N_c \rightarrow \infty} g^2 N_c = \text{constant} \]

\[ \sim g^2 N_c \]
Large \( N_c \) limit of QCD

Planarity

Planar diagrams

\[ \sim O(1) \]

Non-planar diagrams

\[ \sim 1/N_c^2 \]

Quark loops

\[ \sim 1/N_c \]

Leading order Feynman diagrams in the large \( N_c \) limit are planar with a minimum number of quark loops.
**Basic assumption:** QCD in the large Nc limit is a confining theory and quarks, anti-quarks and gluons must combine in order to give a colorless state.

**Large Nc meson**

\[ |1\rangle_c = \frac{1}{\sqrt{N_c}} (q_{c1} \bar{q}_{c1} + \cdots + q_{cN_c} \bar{q}_{cN_c}) \]

only one color singlet state is allowed as intermediate state

\[ \mathcal{O}(N_c) = \sum_n \cdots = \sum_n \frac{f_n^2}{k^2 - M_n^2} \]

Mesons are stable and light

\[ f_n \sim \mathcal{O}(\sqrt{N_c}) \]
\[ M_n \sim \mathcal{O}(1) \]
Baryons in the large $N_c$ limit

$N_c$ quarks

antisymmetric color singlets

$\epsilon_{i_1i_2\ldots i_{N_c}} q^{i_1} q^{i_2} \ldots q^{i_{N_c}}$

$H_B = N_c m + N_c t + V = N_c (m + t + v) = N_c h$

current quark mass  
quark kinetic energy  
potential energy

Interaction between quarks negligible  
$\sim \mathcal{O}(1/N_c)$

Potential felt by any individual quark  
$\sim \mathcal{O}(1)$

baryons are heavy  
$M_B \sim \mathcal{O}(N_c)$

Interaction between quarks negligible  
$\sim \mathcal{O}(1/N_c)$

Potential felt by any individual quark  
$\sim \mathcal{O}(1)$

Hartree picture: each quark move independently in background potential

$\psi_B(x_1, \ldots, x_N) = \prod_{i=1}^{N_c} \phi(x_i)$

Baryons are heavy in the large $N_c$ limit but their size and shape are finite

E. Witten, 1979

Thursday, June 14, 2012
Meson and baryon interactions

Meson decay

\[ N_c \sim \gamma \sqrt{N_c} \]

Meson-meson interaction

\[ \Gamma_{\text{meson}} \sim O(1/N_c) \]

Mesons are free and non-interacting

\[ V_{\text{meson}} \sim O(1/N_c) \]

Meson-baryon interaction

\[ g_A \frac{N_c}{F_\pi} \partial_i \pi^a X^{ia} \sim O(\sqrt{N_c}) \]

Meson-baryon coupling constant

Meson-baryon scattering

\[ \sim O(1) \]

Mesons are scattered by baryons

Witten's counting rules
Spin-Flavor symmetry

Pion-nucleon scattering amplitude is \( \sim O(1) \)
but the pion-nucleon vertex is \( \sim O(\sqrt{N_c}) \)

cancellation between diagrams
\[
\propto \frac{N_c^2 g_A^2}{F^2 \pi} \left[ X^{i a}, X^{j b} \right] = O(N_c^0)
\]

consistency conditions

Gervais & Sakita, 1984
Dashen & Manohar, 1993

\( SU(2N_f) \) spin-flavor symmetry

\[ J = I = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{N_c}{2} \]

Spin

\[
B = \begin{pmatrix}
N \\
\Delta \\
\Delta_{5/2} \\
\vdots \\
\Delta_{N_c/2}
\end{pmatrix}
\]

Flavor

Spin-Flavor

\( N_c \)

\( 1 \)

\( 1/N_c \)
Baryon-baryon interaction

Spin-flavor structure of the NN potential

Kaplan, Savage & Manohar, 1996

Box and cross box diagram cancellations

Banerjee, Cohen & Gelman, 2002

\[ V(r) = V_C(r) + (\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2)W_S(r) + (\tau_1 \cdot \tau_2)W_T(r)S_{12} \sim N_c \]

OBE potential at leading $N_c$

<table>
<thead>
<tr>
<th>Meson</th>
<th>Coupling</th>
<th>Scaling</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar $l = 0$</td>
<td>$\sigma$</td>
<td>$B^\dagger B\phi$</td>
<td>$\sqrt{N_c}$</td>
</tr>
<tr>
<td></td>
<td>$a_0$</td>
<td>$B^\dagger f_0 B\phi^a$</td>
<td>$1/\sqrt{N_c}$</td>
</tr>
<tr>
<td>Pseudo-scalar $l = 0$</td>
<td>$\eta$</td>
<td>$B^\dagger J^i B\phi^i$</td>
<td>$\sqrt{N_c}$</td>
</tr>
<tr>
<td></td>
<td>$\pi$</td>
<td>$B^\dagger G^{ia} B\phi^a$</td>
<td>$\sqrt{N_c}$</td>
</tr>
<tr>
<td>Vector $l = 0$</td>
<td>$\omega$</td>
<td>$B^\dagger B V^i$</td>
<td>$\sqrt{N_c}$</td>
</tr>
<tr>
<td></td>
<td>$\rho$</td>
<td>$B^\dagger J^{ij} B\phi^i V^j$</td>
<td>$1/\sqrt{N_c}$</td>
</tr>
<tr>
<td></td>
<td>$\rho$</td>
<td>$B^\dagger L^a B\phi^a$</td>
<td>$1/\sqrt{N_c}$</td>
</tr>
<tr>
<td>Axial $l = 0$</td>
<td>$f_1$</td>
<td>$B^\dagger J^i B\phi^i$</td>
<td>$\sqrt{N_c}$</td>
</tr>
<tr>
<td></td>
<td>$a_1$</td>
<td>$B^\dagger G^{ia} B\phi^a$</td>
<td>$1/\sqrt{N_c}$</td>
</tr>
</tbody>
</table>
Axial current $g_A$

ACC & Goity, 2012

Exact cancellation in the large $N_c$ limit and almost exact for $N_c=3$

All the pion mass dependence of $g_A$ is dominated by the tadpole diagram and the counter-terms

The result is a mild dependence of $g_A$ with the pion mass

$g_A$ vs. $M_\pi$ [MeV]

Thursday, June 14, 2012
Summary

- Large Nc limit is a fundamental feature of QCD.
- It is not restricted to low or high energies, so it may be considered a non-perturbative method to solve QCD at low-energies.
- There are a lot of situations where quantities in the large Nc limit are not far from the real world Nc=3.
- The combination of the large Nc limit & ChPT looks promising to explain recent lattice QCD results for hadronic quantities.