the excited spectrum of QCD
the spectrum of excited hadrons

let’s begin with a convenient fiction:

imagine that QCD were such that there was a spectrum of stable excited hadrons

e.g. suppose we set up QCD with just two degenerate flavours of quark with mass roughly that of the charm quark

\[ m_c = m_k \sim 1.5 \text{ GeV} \]

then we’d expect a spectrum of \(c\bar{k}\) hadron states starting at about 3 GeV that are stable up to about 6 GeV

except perhaps if glueballs are important?
the spectrum of excited hadrons

might expect something like the non-relativistic quark model

\[ cK \left[ n^{2S+1} L_J \right] \]

... but QCD might be more interesting than this, e.g. what about 'gluonic excitations'?

- glueballs
- hybrids

... and we need to verify if our simple expectations of a \( q\bar{q} \) spectrum are really present
we’d like to map out the spectrum of states in each $J^{PC}$

need interpolating fields that transform like the desired $J^{PC}$

\[
\begin{align*}
\bar{\psi} \gamma_5 \psi & \sim 0^{++} \\
\bar{\psi} \psi & \sim 0^{++} \\
\bar{\psi} \gamma_i \psi & \sim 1^{--} \\
\bar{\psi} \gamma_5 \gamma_i \psi & \sim 1^{++} \\
\epsilon_{ijk} \bar{\psi} \gamma_j \gamma_k \psi & \sim 1^{+-}
\end{align*}
\]

... very limited in $J^{PC}$ coverage

one possible extension: include gauge-covariant derivatives

\[
\hat{\mathcal{D}}_i = \mathcal{D}_i - \bar{\mathcal{D}}_i = \hat{\partial}_i - \bar{\partial}_i - 2igA_i
\]

e.g. \[
\bar{\psi} \hat{\mathcal{D}}_i \psi \sim 1^{--}
\]
\[
\bar{\psi} \gamma_i \hat{\mathcal{D}}_j \psi \sim ?
\]

9 elements

operator is reducible
the spectrum of excited hadrons

\[ \bar{\psi} \gamma_i \overleftrightarrow{D_j} \psi \sim ? \]

\[ i = 1 \ldots 3 \]

\[ j = 1 \ldots 3 \]

9 elements  
operator is reducible

very easy to build a scheme where the operators are irreducible:

\[ \gamma_m \equiv \sum_i \epsilon_i(m) \gamma_i \]

\[ \overleftrightarrow{D}_m \equiv \sum_i \epsilon_i(m) \overleftrightarrow{D}_i \]

\[ \bar{\psi}(m = \pm) = \mp \frac{1}{\sqrt{2}} [1, \pm i, 0] \]

spin-1 circular basis

\[ \bar{\psi}(m = 0) = [0, 0, 1] \]

\[ \langle 1m_1; 1m_2 | JM \rangle \bar{\psi} \gamma_{m_1} \overleftrightarrow{D}_{m_2} \psi \sim J^{++} \]

with \( J = 0, 1, 2 \)

Hadron Spectrum Collaboration has used up to three derivatives:

\[ \langle 1m_1; j_{234}m_{234} | JM \rangle \]

\[ \langle 1m_3; j_{24}m_{24} | j_{234}m_{234} \rangle \]

\[ \langle 1m_2; 1m_4 | j_{24}m_{24} \rangle \]

\[ \bar{\psi} \gamma_{m_1} \overleftrightarrow{D}_{m_2} \overleftrightarrow{D}_{m_3} \overleftrightarrow{D}_{m_4} \psi \]

can build a big basis this way covering all \( J \leq 4 \)

PRL103 262001 (2009)
PRD82 034508 (2010)
the spectrum of excited hadrons

so we could compute correlators for each $J^{PC}$ and look at effective masses at large $t$

would give us the lightest state in each $J^{PC}$

we want more than this ...
we need to be able to extract excited states

\[ C'(t) = \sum_n A_n e^{-E_n t} \]  

a weighted sum of exponentials  
- just do a fit to the time-dependence ?

(fit variables : \(A_0, A_1 \ldots, E_0, E_1 \ldots\))

this is a very bad way to approach this problem

- suppose two states are (nearly) degenerate  
  - fit won’t be able to tell if there are two states or one !

- how do we determine how many states to include in the fit  
  - if we decrease \(t_{\text{min}}\) to use more of the data, need more states ?

fortunately there is a very powerful method available \ldots
variational approach

suppose we have multiple operators for a given $J^{PC}$

$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3 \ldots$

\[ \psi \gamma_m \bar{\psi} \]

\[ \psi \hat{D}_m \bar{\psi} \]

\[ \langle 1m_1; 1m_2 | 1m \rangle \bar{\psi} \gamma_5 \hat{D}_{m_1} \hat{D}_{m_2} \psi \]

\[ \langle 1m_1; 2m_D | 1m \rangle \langle 1m_2; 1m_3 | 2m_D \rangle \bar{\psi} \gamma_{m_1} \hat{D}_{m_2} \hat{D}_{m_3} \psi \]

\vdots

compute a matrix of correlation functions

\[ C_{ij}(t) = \langle 0 | \mathcal{O}_i(t) \mathcal{O}_j(0) | 0 \rangle \]

\[ = \sum_n Z_i^{(n)} Z_j^{(n)} e^{-E_n t} \]

\[ Z_i^{(n)} = \langle n | \mathcal{O}_i(0) | 0 \rangle \]

solve the ‘generalised eigenvalue problem’: \[ C(t) \nu^{(n)} = \lambda_n(t) C(t_0) \nu^{(n)} \]

eigenvalues, ‘principal correlators’ \[ \lambda_n(t) \sim e^{-E_n (t-t_0)} \]

eigenvectors are ‘orthogonal’ \[ \nu^{(m)\dagger} C(t_0) \nu^{(n)} = \delta_{m,n} \]
**variational approach**

the interpretation is relatively simple

the eigenvectors indicate the optimal linear combination of $\mathcal{O}_i$ to interpolate $|n\rangle$

$$\Omega_n = \sum_i \nu_i^{(n)} \mathcal{O}_i \quad \langle m | \Omega_n | 0 \rangle \approx \delta_{m,n}$$

degenerate states are easy to deal with - they might have $E_m = E_n$

- but they have orthogonal $\nu^{(m)}$, $\nu^{(n)}$
variational approach

principal correlators \[ t_0 = 15 \]

fit each with

\[ \lambda(t) = (1 - A)e^{-m(t-t_0)} + Ae^{-m'(t-t_0)} \]

with \( m, m', A \) as parameters

(throw away: \( m', A \))
a real example - $T_1^{--}$ in charmonium

26 operators

variational analysis of
26×26 matrix of correlators

multiple approximate degeneracies

superimposed J=1,3,4 spectra

'statistical' uncertainty
(finite Monte Carlo sample)
the charmonium spectrum from a lattice QCD calc

perform variational analysis in each quantum number

Hadron Spectrum Collaboration
arXiv:1204.5425
the charmonium spectrum from a lattice QCD calc

perform variational analysis in each quantum number

"excess"

0^{--}, 1^{--}, 2^{--}

Hadron Spectrum Collaboration
arXiv:1204.5425
the charmonium spectrum from a lattice QCD calc

perform variational analysis in each quantum number

if we’re interested in phenomenology, there is more information than what’s presented here

the relative sizes of \( \langle \text{n} | O_i(0) | 0 \rangle \)

might tell us about the state composition?

Hadron Spectrum Collaboration
arXiv:1204.5425
the charmonium spectrum from a lattice QCD calc

perform variational analysis in each quantum number

Hadron Spectrum Collaboration
arXiv:1204.5425

if we’re interested in phenomenology, there is more information than what’s presented here
the relative size of
might tell us about the state composition?

MODEL DEPENDENCE!
back to the operators . . .

e.g. $J^{pc}=1^{--}$

consider a model-interpretation

$$\bar{\psi} \gamma_m \frac{1}{2} (1 - \gamma_0) \psi$$

spin-structure:

$$\psi \sim \left[ \frac{1}{\sigma \cdot p} \right] \chi$$

$$\frac{1}{2} (1 - \gamma_0) \psi \sim \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \chi$$

upper component projector

$$\bar{\psi} \gamma_m \frac{1}{2} (1 - \gamma_0) \psi \sim \phi^\dagger \sigma_m \chi$$

$^3S_1$
back to the operators ...  

e.g. $J^{pc}=1^{--}$ consider a model-interpretation

$$\langle 1m_1; 2m_2|1m \rangle \bar{\psi} \gamma_{m_1} D^{[2]}_{J=2,m_2} \frac{1}{2} (1 - \gamma_0) \psi$$

$$D^{[2]}_{J,m} \equiv \langle 1m_1; 1m_2|Jm \rangle \overleftrightarrow{D}_{m_1} \overleftrightarrow{D}_{m_2}$$

without gauge-fields: $D^{[2]}_{J=2,m} \rightarrow Y^m_2 \left( \overleftrightarrow{D} \right)$

$$\sim \langle 1m_1; 2m_2|1m \rangle \cdot \phi^\dagger \sigma_{m_1} \chi \cdot Y^{m_2}_2 \left( \bar{q} \right)$$  $^{3D_1}$

$q\bar{q}$ relative momentum
back to the operators ...

e.g. $J^{PC}=1^{--}$

\[ \bar{\psi} \gamma_5 D^{[2]}_{J=1,m} \left( 1 - \gamma_0 \right) \psi \]

\[ D^{[2]}_{J=1,m} \equiv \langle 1m_1; 1m_2 | 1m \rangle \hat{D}_{m_1} \hat{D}_{m_2} \]

without gauge-fields: $D^{[2]}_{J=1,m} \to 0$

with gauge-fields $D^{[2]}_{J=1,m} \propto [D_i, D_j] \propto F_{ij}$ chromomagnetic part of field-strength tensor

\[ \bar{\psi} \gamma_5 t^a \psi B^a_m \]

\[ q\bar{q}_8(1S_0) \]

\[ ^{1}\text{hyb}_1 \]
operator overlaps

e.g. J^{pc}=1^{--}
the charmonium spectrum from a lattice QCD calc

perform variational analysis in each quantum number

Hadron Spectrum Collaboration
arXiv:1204.5425

HUGS 2012
the charmonium spectrum from a lattice QCD calc

...can isolate dominant hybrid character across the spectrum
hybrid mesons

a phenomenology of hybrid mesons based upon QCD calculations

a chromomagnetic field configuration is lowest excitation

\[ q\bar{q}_8(^1S_0)B_8 \sim 0^{-+} \otimes 1^{+-} = 1^{--} \]
\[ q\bar{q}_8(^3S_1)B_8 \sim 1^{--} \otimes 1^{+-} = (0, 1, 2)^{--} \]

\[ q\bar{q}_8(^1P_1)B_8 \sim 1^{+-} \otimes 1^{+-} = (0, 1, 2)^{++} \]
\[ q\bar{q}_8(^3P_0)B_8 \sim 0^{++} \otimes 1^{+-} = 1^{--} \]
\[ q\bar{q}_8(^3P_1)B_8 \sim 1^{++} \otimes 1^{+-} = (0, 1, 2)^{+-} \]
\[ q\bar{q}_8(^3P_2)B_8 \sim 2^{++} \otimes 1^{+-} = (1, 2, 3)^{--} \]
three flavours of quark - all at the strange quark mass

\[ m(\pi') \sim 700 \text{ MeV} \]
lighter quarks - isovector mesons

three flavours of quark - all at the strange quark mass

interpretations based on operator overlaps

\[ m(\pi) \sim 700 \text{ MeV} \]
lighter quarks - isovector mesons

three flavours of quark
- degenerate up/down quarks
- correct strange quark mass

$m(\pi) \sim 400$ MeV

\[ E / \text{GeV} \]

negative parity
- $1^{--}$
- $2^{--}$
- $3^{--}$
- $4^{--}$
- $4^{--}$

positive parity
- $0^{++}$
- $1^{++}$
- $2^{++}$
- $3^{++}$
- $4^{++}$

exotics
- $0^{-+}$
- $1^{-+}$
- $2^{-+}$
- $3^{-+}$
- $4^{-+}$

$m_\pi = 396$ MeV
Hadron Spectrum Collab.
Phys.Rev.D82, 034508
isoscalar mesons

difference w.r.t. isovector mesons is addition of ‘disconnected’ diagrams

\[
\begin{align*}
\bar{\psi} \Gamma' \psi(t) \cdot \bar{\psi} \Gamma \psi(0) & - \bar{\psi} \Gamma' \psi(t) \cdot \bar{\psi} \Gamma \psi(0) \\
\text{tr} \left[ Q_{t,0}^{-1} \Gamma Q_{0,t}^{-1} \Gamma' \right] & \quad \text{tr} \left[ Q_{t,t}^{-1} \Gamma' \right] \text{tr} \left[ Q_{0,0}^{-1} \Gamma \right]
\end{align*}
\]

challenging using ‘traditional’ methods
isoscalar mesons

hidden ‘light’ and hidden ‘strange’ can mix

$$\frac{1}{\sqrt{2}} (u\bar{u} + d\bar{d})$$

$$s\bar{s}$$
isoscalar mesons

negative parity

positive parity

exotics

Hadron Spectrum Collaboration
PRD83 111502 (2011)

$0^{++}$ is a challenge

$m_\pi = 396$ MeV

- isoscalar
- isovector
- YM glueball

HUGS 2012
baryons

analogous large basis of operators for baryons - three quark fields respecting permutation (anti-)symmetry

Hadron Spectrum Collaboration
PRD84 074508 (2011)
PRD85 054016 (2012)
hybrids (mesons and baryons)

approximately remove the ‘quark mass’ contribution to the hybrid mass

\[ m - m_0 \text{ / MeV} \]

\begin{align*}
8_F & \quad 10_F & \quad 1_F \\
2^{-+} & \quad 1^{++} & \quad 1^{-+} & \quad 0^{--} \\
\frac{1}{2}^+ & \quad \frac{3}{2}^+ & \quad \frac{5}{2}^+ & \quad \frac{1}{2}^+ & \quad \frac{3}{2}^+ & \quad \frac{1}{2}^+ & \quad \frac{3}{2}^+ \\
\end{align*}

\[ m_u = m_d = m_s \]
\[ m_\pi = 702 \text{ MeV} \]

\[ N_{\text{hyb}} \quad \Delta_{\text{hyb}} \]

\[ m_\pi = 524 \text{ MeV} \]

\[ m_\pi = 396 \text{ MeV} \]

light hybrid mesons - \( m_\rho \)
light hybrid baryon - \( m_N \)
charmonium hybrids - \( m_{\eta_c} \)

chromomagnetic gluonic excitation scale \( \sim 1.3-1.4 \text{ GeV} \)?

a model dependent interpretation of lattice QCD calculations
the resonance spectrum of QCD

or,

“where are you hiding the scattering amplitudes?”
real QCD

real QCD has very few stable particles: $\pi$, $K$, $N$, $\Sigma$, $\Lambda$, $\Xi$, $\Omega$

‘states’ like $\rho$, $\Delta$ … are resonances

asymptotic states of the theory include multi-pion states

\[ \int d\hat{p} \ Y_L^m(\hat{p}) \left| \pi(\vec{p})\pi(-\vec{p}) \right> \]

with $|\vec{p}|$ varying continuously up from zero

e.g. $\pi\pi$ scattering in isospin-1

\[ \begin{array}{c}
\pi^+ \\
\pi^0 \ldots
\end{array} \rightarrow \begin{array}{c}
\pi^+ \\
\pi^0
\end{array} \]

HUGS 2012
real QCD

real QCD has very few stable particles: \( \pi, K, N, \Sigma, \Lambda, \Xi, \Omega \)

'states' like \( \rho, \Delta \ldots \) are resonances

asymptotic states of the theory include multi-pion states

e.g. in the CM frame \[ \int d\hat{p} \ Y_L^m(\hat{p}) \ | \pi(\vec{p})\pi(-\vec{p}) \rangle \]

with \( |\vec{p}| \) varying continuously up from zero

e.g. \( \pi\pi \) scattering in isospin-1

\[ \pi^+p - \pi^+p\pi^0 \]

2085 events

for small \( t \)
**real QCD**

**real** QCD has very few stable particles: $\pi$, $K$, $N$, $\Sigma$, $\Lambda$, $\Xi$, $\Omega$

‘states’ like $\rho$, $\Delta$ ... are **resonances**

asymptotic states of the theory include multi-pion states

e.g. in the CM frame

$$\int d\hat{p} \ Y_L^m(\hat{p}) \ |\pi(\hat{p})\pi(-\hat{p})\rangle$$

with $|\hat{p}|$ varying continuously up from zero

within the field-theory we have correlators

e.g. the (Euclidean) vector correlator

$$C_V(t) = \int dE \ e^{-Et} \ \rho_V(E)$$

**spectral function**

$$\rho_V(E)$$

the spectrum is continuous!
real QCD

Asymptotic states of the theory include multi-pion states

\[ \psi_{\bar{L}}(\hat{\rho}) \left| \pi(\vec{p})\pi(-\vec{p}) \right\rangle \]

with \(|\vec{p}|\) varying continuously up from zero

Within the field-theory we have correlators

e.g. the (Euclidean) vector correlator

\[ \rho_V(E) \]

The spectrum is continuous!

\[ \text{a discrete spectrum ?} \]

HUGS 2012
field theory in a finite volume

consider the case of one space dimension with a periodic boundary condition

this will make the allowed momenta of a free particle discrete:

\[ \psi_p(x) = e^{ipx} \quad \text{free particle} \]

\[ \psi_p(x) = \psi_p(x + L) \quad \text{periodic boundary condition} \]

\[ e^{ipL} = 1 \implies p = \frac{2\pi n}{L} \quad \text{for integer } n \]
non-interacting two-particle states in a finite volume

in a three-dimensional cubic box

\[ \vec{p} = \frac{2\pi}{L} \left[ n_x, n_y, n_z \right] \]

the spectrum is discrete and volume-dependent
two-particle states in a finite volume

but hadrons do interact, and sometimes strongly

\[ \pi \pi \rightarrow \rho \rightarrow \pi \pi \]
\[ \pi N \rightarrow \Delta \rightarrow \pi N \]

what is the impact of these interactions on the finite-volume spectrum?
two-particle states in a finite volume

e.g. non-rel quantum mechanics in one-dimension

two spinless bosons separated by $x$

interacting through a potential $V(x)$

in center-of-momentum frame

scattering solutions of

$$-\frac{1}{m} \frac{d^2 \psi}{dx^2} + V(x) \psi = E \psi$$

$$E = \frac{k^2}{m} > 0$$

$$\psi(x \to \infty) = N \cos [k|x| + \delta(k)]$$

scattering phase-shift contains all the information about elastic scattering

$$x > 0 \begin{pmatrix} \frac{N}{2} e^{-i\delta} \\ \text{normalisation} \end{pmatrix} \begin{pmatrix} e^{-ikx} \\ e^{2i\delta(k)} e^{ikx} \end{pmatrix}$$

wave incoming from $x \to +\infty$

reflected wave
two-particle states in a finite volume

e.g. non-rel quantum mechanics in one-dimension

two spinless bosons separated by \( x \)
interacting through a potential \( V(x) \)
in center-of-momentum frame

scattering solutions of

\[
-\frac{1}{m} \frac{d\psi}{dx^2} + V(x)\psi = E\psi
\]

\[
E = \frac{k^2}{m} > 0
\]

\[
\psi(x \to \infty) = N \cos [k|x| + \delta(k)]
\]

finite length world \( -L/2 < x < L/2 \) with periodic b.c.

wavefunction and derivative continuous at boundary

\[
-k \tan \left[ \frac{kL}{2} + \delta(k) \right] = k \tan \left[ \frac{kL}{2} + \delta(k) \right]
\]

\[
\tan \left[ \frac{kL}{2} + \delta(k) \right] = 0
\]

\[
kL + 2\delta(k) = 0 \mod 2\pi
\]
two-particle states in a finite volume

e.g. quantum mechanics in one-dimension

\[ kL + 2\delta(k) = 0 \mod 2\pi \]

**discrete & volume-dependent** spectrum of scattering states

\[ E = \frac{k^2}{m} \quad \text{and} \quad k = \frac{2\pi}{L} \left( n - \frac{\delta(k)}{\pi L} \right) \]

- non-interacting momentum
- shift due to interaction

e.g. a weak attraction

\[ \delta(k) = \alpha k \]
two-particle states in a finite volume

the analogous expression for two-particle elastic scattering in a finite cubic volume has been derived by Lüscher

somewhat complicated by the lack of full rotational symmetry (a cube)

for our purposes, we’ll pretend that it’s as simple as

$$\delta_\ell(E) = f_\ell(E, L)$$

phase-shift in partial wave $\ell$

at scattering energy $E$

known function of
energy and box length

so we ‘measure‘ $E_n$ on one or more volumes

and plug into the formula to determine $\delta$ at discrete energies
ππ isospin-2 scattering

e.g.

empirically: weak & repulsive scattering

\[ p_{cm}^2 / \text{GeV}^2 \]

\[ \ell = 2 \]

\[ \ell = 0 \]
ππ isospin-2 scattering - field theory calculation

\[ C(t) = \langle 0 | O_{\pi^+}^\dagger (t) O_{\pi^+}^\dagger (t) \cdot O_{\pi^+} (0) O_{\pi^+} (0) | 0 \rangle \]

no quark-annihilation in these correlators

![Diagram]

variational in a basis of operators:

“ππ” of various relative momenta

\[ \vec{P} = [000] \]

\[ \pi[000] \pi[000] \]

\[ \pi[100] \pi[-100] \]

\[ \pi[110] \pi[-1-10] \]

\[ \pi[111] \pi[-1-1-1] \]

\[ \vdots \]
ππ isospin-2 scattering

variational in a basis of operators:

ππ of various relative momenta

e.g. \( \vec{P} = [000] \)

extracted energies shifted upward slightly from non-interacting pion pairs

Hadron Spectrum Collaboration
arXiv:1203.6041
**ππ isospin-2 scattering**

S-wave scattering phase-shift

<table>
<thead>
<tr>
<th>$E_{cm}$ / MeV</th>
<th>$a_t p_{cm}$</th>
<th>$\delta_0$ / $^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>800</td>
<td>0.002</td>
<td>-10</td>
</tr>
<tr>
<td>1000</td>
<td>0.004</td>
<td>-20</td>
</tr>
<tr>
<td>1250</td>
<td>0.006</td>
<td>-30</td>
</tr>
<tr>
<td>1500</td>
<td>0.008</td>
<td>-40</td>
</tr>
<tr>
<td></td>
<td>0.012</td>
<td>-50</td>
</tr>
<tr>
<td></td>
<td>0.014</td>
<td>-60</td>
</tr>
</tbody>
</table>

- $L/a_s = 24$, $\vec{P} = [000]$
- $L/a_s = 20$, $\vec{P} = [100]$
- $L/a_s = 16$, $\vec{P} = [110]$
- $m_\pi \sim 400$ MeV
ππ isospin-2 scattering

S-wave scattering phase-shift

\( \delta_0^0 \)

\( L/a_s = 24 \)  \( \vec{P} = [000] \)

\( L/a_s = 20 \)  \( \vec{P} = [100] \)

\( L/a_s = 16 \)  \( \vec{P} = [110] \)

\( \vec{P} = [111] \)

\( m_\pi \sim 400 \text{ MeV} \)
**ππ isospin-2 scattering**

S-wave scattering phase-shift

\[ \delta_0 / ^\circ \]

\[ \begin{align*}
\Delta L/a_s &= 24 & \vec{P} &= [000] \\
\square L/a_s &= 20 & \vec{P} &= [100] \\
\circ L/a_s &= 16 & \vec{P} &= [110] \\
\end{align*} \]

\[ m_\pi \sim 400 \text{ MeV} \]
ππ isospin-2 scattering

S-wave scattering phase-shift

\[ \delta_0 \, ^\circ \]

-50 -40 -30 -20 -10 0 10 20 30 40 50 60

-50 -40 -30 -20 -10 0 10 20 30 40 50 60

\( L/a_s = 24 \) \( \vec{P} = [000] \)

\( L/a_s = 20 \) \( \vec{P} = [100] \)

\( L/a_s = 16 \) \( \vec{P} = [110] \)

\( \vec{P} = [111] \)

\( m_\pi \sim 400 \text{ MeV} \)
ππ isospin-2 scattering

S-wave scattering phase-shift

\[ (a_t p_{cm})^2 \]

\[ \delta_0 \circ \]

\[ L/a_s = 24 \quad \vec{P} = [000] \]
\[ L/a_s = 20 \quad \vec{P} = [100] \]
\[ L/a_s = 16 \quad \vec{P} = [110] \]
\[ \vec{P} = [111] \]

\[ m_\pi \sim 400 \text{ MeV} \]
ππ isospin-2 scattering

D-wave scattering phase-shift

\[ \delta_2^o \]

\[ (a_t \ p_{cm})^2 \]

\[ m_\pi \sim 400 \text{ MeV} \]

\( L/a_s = 24 \quad \vec{P} = [000] \)

\( L/a_s = 20 \quad \vec{P} = [100] \)

\( L/a_s = 16 \quad \vec{P} = [110] \)
ππ isospin-2 scattering

summary (with effective range fits)

\[ k^{2\ell+1} \cot \delta_\ell = \frac{1}{a_\ell} + \frac{1}{2} r_\ell k^2 + \ldots \]
ππ isospin-2 scattering

lattice QCD: $m_\pi \sim 400$ MeV

experimental compendium
ππ isospin-1 scattering

more interesting scattering channels are those featuring resonances
e.g. the $\rho$ in $\pi\pi$

$$\delta_1(E)$$

$E / \text{MeV}$
ππ isospin-1 scattering

more interesting scattering channels are those featuring resonances

e.g. the ρ in ππ
ππ isospin-1 scattering

more interesting scattering channels are those featuring resonances
ππ isospin-1 scattering

expected finite-volume spectrum given a ρ resonance

\[ m_\pi = 139 \text{ MeV} \]

\[ m_\rho = 770 \text{ MeV} \]

\[ \Gamma_\rho = 150 \text{ MeV} \]
ππ isospin-1 scattering

a strongly volume-dependent spectrum
ππ isospin-1 scattering

expected finite-volume spectrum given a ρ resonance

$E / \text{MeV}$

$\rho$

$L / \text{fm}$

$m_\pi = 139 \text{ MeV}$

$m_\rho = 770 \text{ MeV}$

$\Gamma_\rho = 150 \text{ MeV}$
ππ isospin-1 scattering

expected finite-volume spectrum given a ρ resonance

level repulsion

“ρ”

π[110]π[-1-10]

π[211]π[-2-1-1]

π[210]π[-2-10]

π[200]π[-200]

π[111]π[-1-1-1]

π[100]π[-100]
ππ isospin-1 scattering

expected finite-volume spectrum given a ρ resonance

$m_\pi=139$ MeV
$m_\rho=770$ MeV
$\Gamma_\rho=150$ MeV
ππ isospin-1 scattering

expected finite-volume spectrum given a ρ resonance

\[ m_\pi = 139 \text{ MeV} \]
\[ m_\rho = 770 \text{ MeV} \]
\[ \Gamma_\rho = 150 \text{ MeV} \]

strongly volume dependent spectrum
the spectrum using ‘local’ operators - $T_1$---

no systematic volume dependence observed in the spectrum

$m_\pi \sim 400$ MeV
the spectrum using ‘local’ operators - $T_1^{--}$

$m_\pi \sim 400$ MeV

non-interacting $\pi\pi$ energies

$L=16$  $L=20$  $L=24$
the spectrum using ‘local’ operators - $T_1$--

$m_\pi \sim 400$ MeV

no sign of any multi-meson physics?
the spectrum using ‘local’ operators

we suspect that this effect can be understood

→ hypothesise that energy eigenstates are superpositions of

* a $q\bar{q}$ state, call it $|\rho\rangle$

* “non-interacting” $|\pi\pi\rangle$ basis states

→ hypothesise that local $q\bar{q}$ operators have a suppressed overlap onto $|\pi\pi\rangle$

by at least a factor of $1/L^3$
ππ isospin-1 scattering

expected finite-volume spectrum given a ρ resonance

level repulsion

a simple two-state mixing model:

$$|E_1\rangle = \cos \theta |\rho\rangle + \sin \theta |\pi\pi\rangle$$

$$|E_2\rangle = -\sin \theta |\rho\rangle + \cos \theta |\pi\pi\rangle$$
the spectrum using ‘local’ operators

\[ |E_1\rangle = \cos \theta |\rho\rangle + \sin \theta |\pi\pi\rangle \]

\[ |E_2\rangle = -\sin \theta |\rho\rangle + \cos \theta |\pi\pi\rangle \]

if we only use operators which overlap well with \( |\rho\rangle \) and not with \( |\pi\pi\rangle \)

then a variational solution won’t be able to find the orthogonal combinations

the principal correlator will behave like

\[ \langle \rho | e^{-Ht} | \rho \rangle = \cos^2 \theta e^{-E_1 t} + \sin^2 \theta e^{-E_2 t} \]
ππ isospin-1 scattering - lattice calculation

**do it properly!**

**operator basis:**

- usual big set of derivative-based fermion bilinears
- ππ-like operators of definite relative momentum

**need quark annihilation diagrams**

e.g.
'local' vector operators and $\pi\pi$ operators just 'local' vector operators
"local" vector operators and ππ operators
just 'local' vector operators

Lattice 2012 - spectroscopy overview
the spectrum using ‘local’ operators - $T_1^{--}$

$m_\pi \sim 400$ MeV
ππ isospin-1 scattering - pion mass dependence

European Twisted Mass Collaboration
PRD83 (2011) 094505

- two-flavour calculation (no strange quarks)
- four different quark masses
- setting the lattice scale?

- computed in relatively few frames

\[
\delta
\]

\[
E / \text{MeV}
\]

\[
\begin{align*}
m_\pi &= 480 \text{ MeV} \\
m_\rho &= 1118(14) \text{ MeV} \\
\Gamma_\rho &= 40(8) \text{ MeV}
\end{align*}
\]
ππ isospin-1 scattering - pion mass dependence

\[ \delta \]

\[ E / \text{MeV} \]

\[ m_\pi = 480 \text{ MeV} \]
\[ m_\rho = 1118(14) \text{ MeV} \]
\[ \Gamma_\rho = 40(8) \text{ MeV} \]

\[ m_\pi = 420 \text{ MeV} \]
\[ m_\rho = 1047(15) \text{ MeV} \]
\[ \Gamma_\rho = 55(11) \text{ MeV} \]

\[ m_\pi = 330 \text{ MeV} \]
\[ m_\rho = 1033(31) \text{ MeV} \]
\[ \Gamma_\rho = 123(43) \text{ MeV} \]

\[ m_\pi = 290 \text{ MeV} \]
\[ m_\rho = 980(31) \text{ MeV} \]
\[ \Gamma_\rho = 156(41) \text{ MeV} \]
forthcoming resonance calculations

computationally challenging:

\[\rightarrow\text{pion-nucleon elastic scattering (}\Delta\text{ resonance)\]}

\[\rightarrow\text{lots of quark lines}\]
\[\rightarrow\text{lots of matrix multiplication}\]

\[\ldots\text{computer time}\]
computationally challenging:

- pion-nucleon elastic scattering ($\Delta$ resonance)

requires untested formalism:

- meson-meson inelastic scattering (e.g. $a_0$ in $\pi\eta$-KK)

\[
\begin{align*}
\text{e.g. 2-channel inelastic scattering} & \quad \begin{bmatrix}
\frac{4\pi}{k_1} \frac{\eta e^{2i\delta_1} - 1}{2i} \\
\frac{4\pi}{\sqrt{k_1 k_2}} \sqrt{1 - \eta^2 e^{i(\delta_1 + \delta_2)}} \\
\frac{4\pi}{k_2} \frac{\sqrt{1 - \eta^2 e^{i(\delta_1 + \delta_2)}}}{2i} \\
\frac{4\pi}{k_2} \frac{\eta e^{2i\delta_2} - 1}{2i}
\end{bmatrix}
\end{align*}
\]

three real numbers at each scattering energy

finite-volume formalism

\[
E_n(L) = f(\delta_1(E), \delta_2(E), \eta(E); L)
\]

‘measured’ three unknowns
forthcoming resonance calculations

computationally challenging:

- pion-nucleon elastic scattering (Δ resonance)

requires untested formalism:

- meson-meson inelastic scattering (e.g. $a_0$ in $\pi\eta$-$\bar{K}K$)

- three-meson decays, e.g. $\omega,a_1,a_2... \rightarrow \pi\pi\pi$

all the complications of building unitary, analytic scattering amplitudes present for experiment!
and to finish …

thank you for your attention