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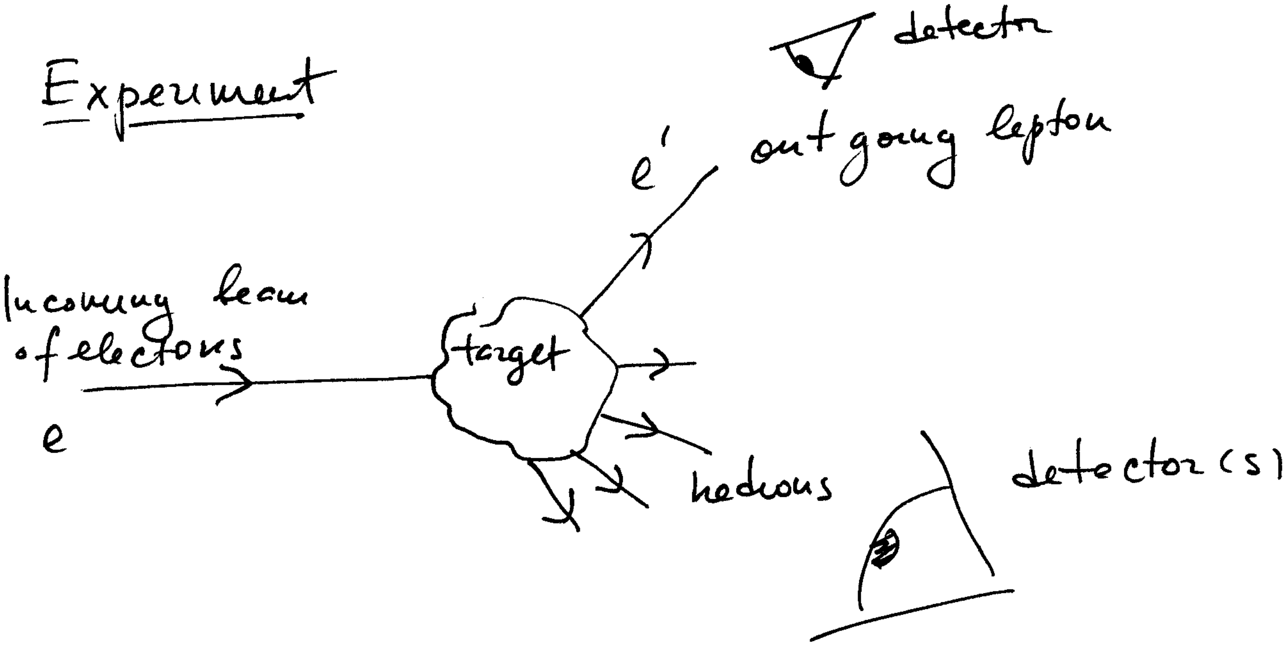
Introduction to Partonic Hadron Structure.

Lecture I

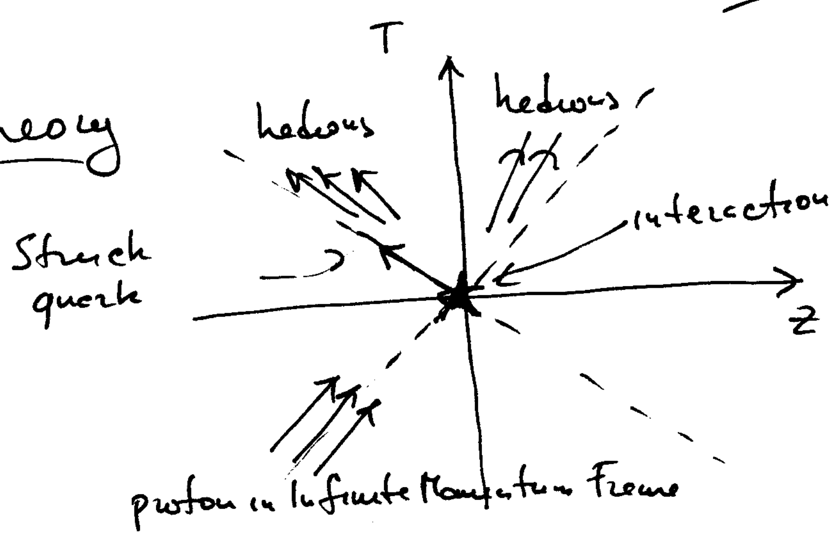
Kinematics of Deep Inelastic Scattering

In theese lectures we will study hadron structure in terms of partons: quarks and gluons. Let us start from kinematics and try to advocate what a theorist thinks about process of lepton scattering on hadrons and what experimentalist actually measures and why we can interprete measurements in terms of partonic structure.

Experiment



Theory



We would like to have a process in which a lot of hadrons are produces (the reason for that we will understand later) so we need enough energy:

$$s = (e+P)^2 = e^2 + 2eP + P^2 = M^2 + 2e\cdot P$$

in laboratory system $P = (M_p, 0, 0, 0)$, $e = (E_{lab}, 0, 0, E_{lab})$

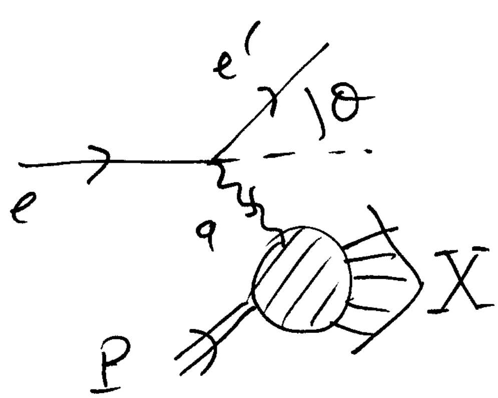
$\Rightarrow$

$$s = M^2 + 2E_{lab}\cdot M \,, \quad M \sim 1\,\text{GeV}$$

$\Rightarrow$ we need E_{lab} high enough to produce hadrons (pions).

JLab $E_{lab} = 6$ GeV (12 GeV from 2015)

The process in fact looks like this

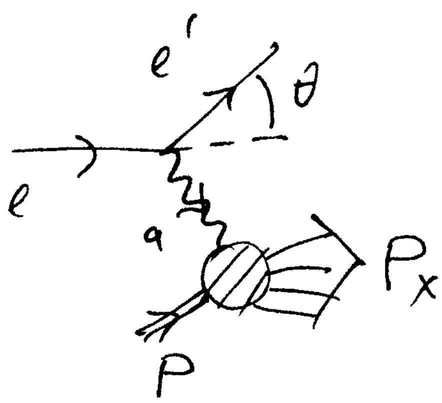


$$e = (E, 0, 0, E)$$

$$e' = (E', E'\sin\theta, 0, E'\cos\theta)$$

$$q = e - e'$$

Some notes on kinematics



1) Show that $q^2 < 0$

$$e = (E, 0, 0, E)$$

$$e' = (E', E'\sin\theta, 0, E'\cos\theta)$$

$$q^2 = (e-e')^2 \simeq -2ee' = -2\,(EE' - EE'\cos\theta) =$$

$$= -2EE'(1-\cos\theta) \leq 0$$

q^2 is negative $\qquad q^2 = -Q^2$

We can also define the following scalers:

$$x = \frac{Q^2}{2P\cdot q}\ , \quad y = \frac{P\cdot q}{P\cdot e}$$

Let's find possible values.

Scaler products are scalers => lorentz invariant can be calculated in any frame. We choose target rest frame

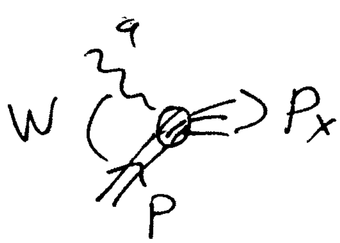
$$q=(\nu,\vec{q}), \quad \nu = E-E' > 0$$

$$2P\cdot q = 2M\nu \to \infty \quad \text{Bjorken limit}$$

$$q^2=-Q^2, \quad Q^2 \geq 0, \quad \nu \geq 0 \quad \Rightarrow$$

$$x = \frac{Q^2}{2P\cdot q} \gtrsim 0$$

$$W^2 = (P+q)^2$$



$$W^2 = P^2 + 2P\cdot q - Q^2 = M^2 + 2P\cdot q - Q^2 \gtrsim M^2$$

$$\Rightarrow \quad 2P\cdot q \gtrsim Q^2$$

$$\Rightarrow \quad Q^2/2P\cdot q \leq 1$$

$$y = \frac{P\cdot q}{P\cdot e} = \frac{M(E-E')}{ME} = 1 - E'/E$$

$$E' \in [0, E] \Rightarrow \underline{y \in [0,1]}$$

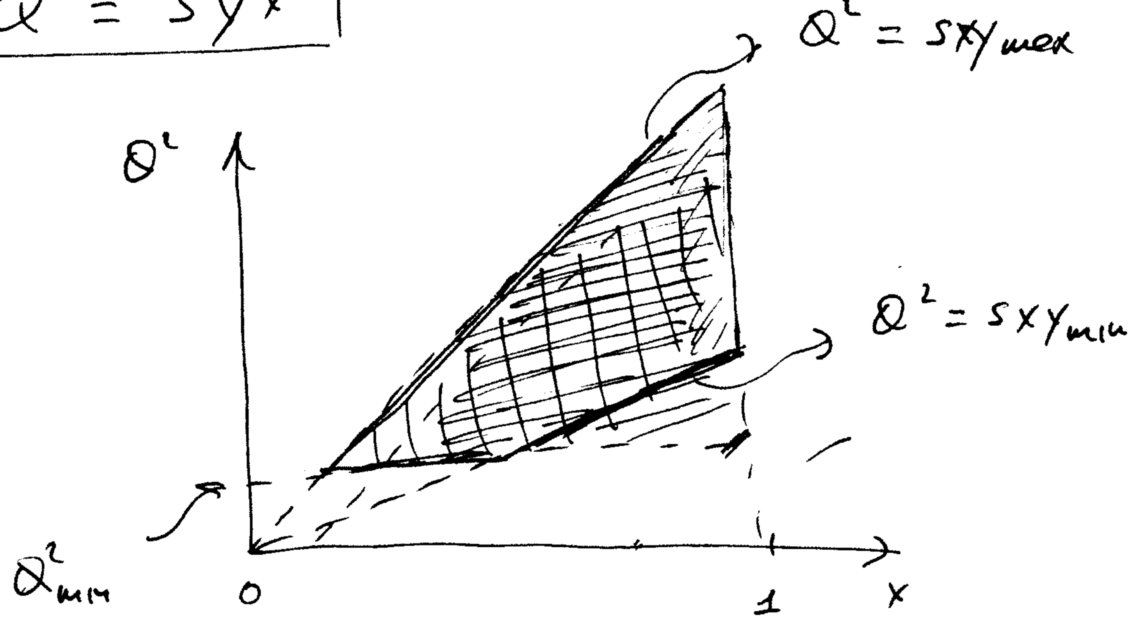
This gives us a tool to estimate reach of experiments

$$s = (P+e)^2 = M^2 + 2P\cdot e \simeq 2P\cdot e \quad \text{(high energy)}$$

$\Rightarrow$

$$x = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{2P\cdot q}\cdot\frac{P\cdot e}{P\cdot e} = \frac{Q^2}{y s}$$

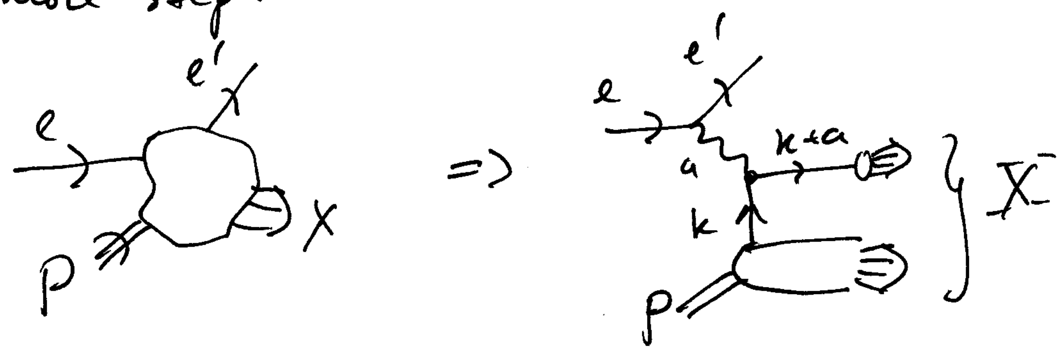
$$\boxed{Q^2 = s y x}$$



$Q^2_{min} \simeq 1\,(GeV^2)$ to ensure DIS regime

Note that so far we have not spoken about parton structure – we have not yet identified $x_{Bj} = \frac{Q^2}{2P\cdot q}$ with „partonic" structure.

In order to do it we should perform one more step:



Scattering happens on a quark with momentum $k = \xi P$ where ξ is anything

We "see" hadrons in detectors → quark is on „mass-shell" (will be explained later), k is also almost on mass shell $k^2 \approx \phi$:

$$(k+q)^2 \simeq 0 = k^2 + 2kq + q^2 = 2\xi P\cdot q - Q^2 \approx 0$$

$$\Rightarrow \xi = \frac{Q^2}{2P\cdot q} = x_{Bj} \quad \Rightarrow \quad \boxed{\xi = x_{Bj}}$$

Now we interprete x_{Bj} as something connected to partonic structure

More on kinematics:

We will also see that we need to produce a lot of hadrons in X. The reason is that we will need to implement completeness relation $|X\rangle\langle X| = \mathbb{1}$. In order to do it we have to have $P_X^2 = M_X^2$ big.

$$W^2 \equiv (q+P)^2 = P_X^2 \gg P^2 = M^2$$

$$W^2 = q^2 + 2qP + P^2 = -Q^2 + 2qP + M^2$$

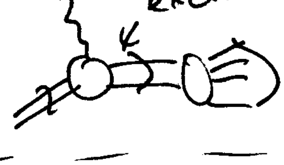
$$x = \frac{Q^2}{2qP} \Rightarrow 2qP = \frac{Q^2}{x}$$

$$W^2 = -Q^2 + \frac{Q^2}{x} + M^2 = Q^2\frac{1-x}{x} + M^2 \geq W_{min}^2$$

$W_{min}^2 \sim 10\,(\mathrm{GeV}^2) \rightarrow$

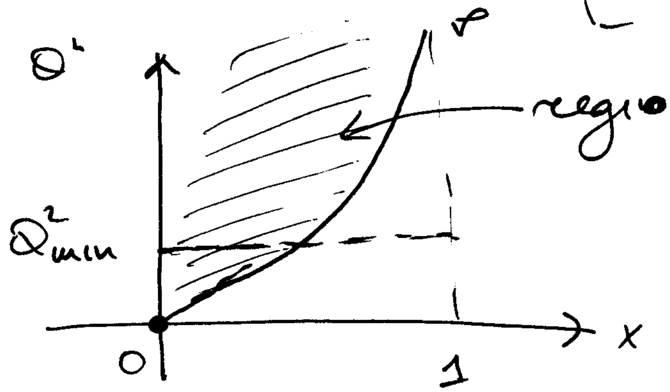
We need it to ensure that the process is

and not excited state



$$\Rightarrow \quad Q^2\frac{1-x}{x} \geq W_{min}^2 - M^2$$

$$Q^2 \geq (W_{min}^2 - M^2)\frac{x}{1-x}$$



On previous page I introduced also
$Q^2_{min} \sim 1(GeV^2)$. What is the reason for it?

DIS is usually called „relativistic electron microscope". What is the meaning of it?
Virtual photon serves as a probe that resolves sizes of hadron constituent down to $1/Q$.
Thus physical meaning of photon's virtuality is the resolution of our „microscope".

We will see that the space-time representation of the amplitude

can be written as

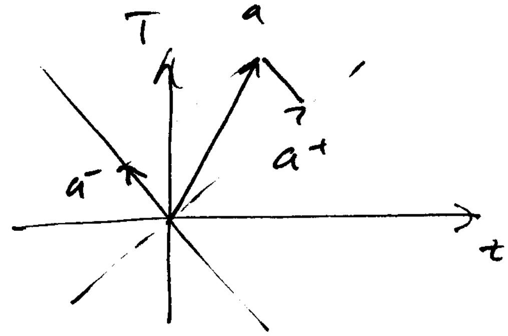
proton state

$$W_{\mu\nu}(x_B, Q^2) = \frac{1}{2\pi M} \operatorname{Im}\left\{ i \int \underbrace{d^4x\, e^{iqx}}_{} \langle P | T[J_\mu(x) J_\nu(0)] | P \rangle \right\}$$

current

Fourier transform from position to momentum space

Light cone coordinates → useful to describe high energy processes



$$a = (a_0, \vec{a}_\perp, a_3) \quad - 4 \text{ vector}$$

$$a = (a^+, a^-, \vec{a}_\perp) \quad \to \text{light cone}$$

$$a^\pm \overset{\text{def}}{=} \frac{a_0 \pm a_3}{\sqrt{2}}$$

$$a \cdot b = a^+ b^- + a^- b^+ - \vec{a}_\perp \vec{b}_\perp$$

$$a^2 = 2 a^+ b^- - \vec{a}_\perp \vec{b}_\perp$$

Under boost in z direction with $\Lambda = \frac{1-\beta}{\sqrt{1-\beta^2}}$,

$$\begin{cases} a^+ \to \Lambda a^+ \\ a^- \to 1/\Lambda \, a^- \end{cases}$$

So that boost invariant quantities are

$$\begin{cases} a^+/b^+ \; ; \; a^-/b^- \leftarrow \text{these will serve as fractional momenta} \\ \vec{a}_\perp \\ a^+ b^- \end{cases}$$

We can choose a frame in which

$$q^\mu = (q^0, 0_\perp, q^3) \; , \quad \underbrace{P^\mu = (M, \vec{0})}_{\text{Laboratory frame}}$$

$$2\,P\cdot q = 2 q^0 M = \frac{Q^2}{x_{BJ}} \Rightarrow$$

$$q^0 = \frac{Q^2}{2 M x_{BJ}} \gg Q \quad \text{since } Q \gg M \, , \; x_{BJ} \leq 1$$

$$Q^2 \geq 0 \qquad Q^2 = -q^2 = (q^3)^2 - (q^0)^2 \geq 0$$

$$\Rightarrow \underline{q^3 \geq q^0 \gg Q} \Rightarrow \text{also } q^0 \approx q^3$$

We can write

$$q^+ \stackrel{\text{def}}{=} \frac{q^0 + q^3}{\sqrt{2}} \approx \frac{2\,q^0}{\sqrt{2}} = \sqrt{2}\, q^0$$

$$q^- \stackrel{\text{def}}{=} \frac{q^0 - q^3}{\sqrt{2}} = \frac{q^+ q^-}{q^+} \simeq -\frac{Q^2}{2\sqrt{2}\, q^0} = - \quad m\, x_{BJ}/\sqrt{2}$$

Now we can write $q\cdot x = q^+ x^- + q^- x^+ \sim \text{const}$ (otherwise the integral oscillates)

and typical x^- range is given by $1/q^+$

x^+ range is given by $1/q^-$

$$x^- \simeq \frac{1}{q^+} \simeq \frac{\sqrt{2}\, M x_{Bj}}{Q^2} << \frac{1}{\mu}$$

$$x^+ \simeq \frac{1}{|q^-|} \simeq \frac{\sqrt{2}}{M x_{Bj}} \geq \frac{1}{\mu}$$

where $\mu \sim \Lambda_{QCD} \sim M$ is typical scale of non perturbative QCD interactions.

One can see that $x^- = \frac{1}{\sqrt{2}}(t - z) \simeq 0$

$$\Downarrow$$

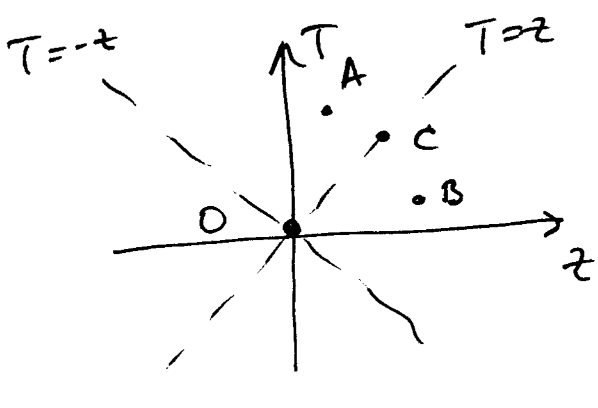
$$x^+ = \frac{1}{\sqrt{2}}(t + z) \simeq \frac{2t}{\sqrt{2}} \simeq \frac{\sqrt{2}}{M x_{Bj}}$$

2) "light cone time" is given by

$$\underbrace{x^+ \simeq \frac{\sqrt{2}}{M x_{Bj}}}_{\text{typical longitudinal distance of interaction with virtual photon}} \Big\} \text{ Also called Ioffe time}$$

Time of interaction $t \sim 1/\mu x_{Bj}$ very long!

We can also assess transverse coordinate resolution



$A = (T_A, 0, 0, z_A)$

$B = (T_B, 0, 0, z_B)$

$C = (T_C, 0, 0, T_C)$

$A^2 = T_A^2 - z_A^2 > 0$ "time like"

$B^2 = T_B^2 - z_B^2 < 0$ "space like"

$C^2 = T_C^2 - z_C^2 = 0$ "light like"

Information propagates from O to A "Causality"

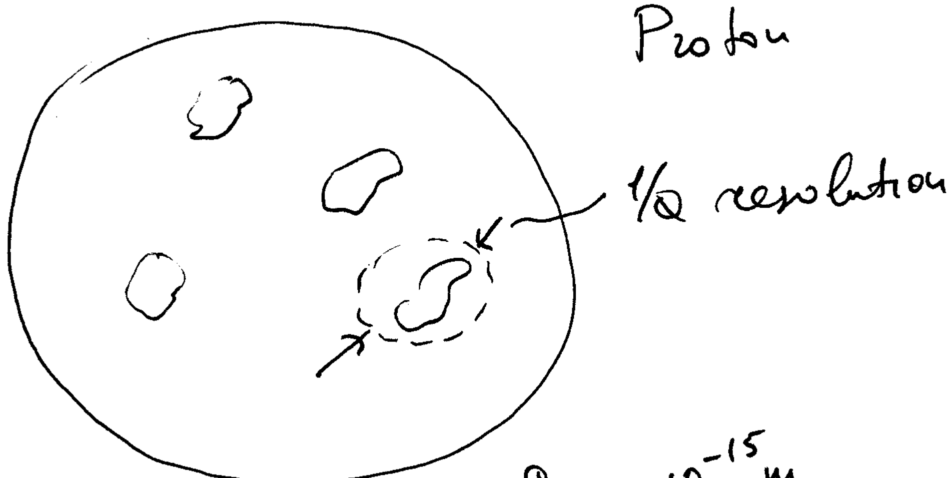
Using causality we deduce $x^2 = 2x^+x^- - x_\perp^2 > 0$

and using $x^- \simeq \frac{\sqrt{2}\, M x}{Q^2}$

$x^+ \simeq \frac{\sqrt{2}}{M x_{Bj}}$

we get $x_\perp^2 < 2x^+x^- = \frac{4}{Q^2} \ll \frac{1}{M^2}$

We see that typical transverse resolution of the virtual photon is of order $1/Q$.



„Size" of the proton is α 1 fm $= 10^{-15}$ m
1 fermi or 1 femtometre

1 fm = 5 GeV^{-1}

Now let us consider DIS in Infinite Momentum Frame (IMF)

$$P = \left(\sqrt{M^2+P^2}, 0, 0, P\right) \to P \gg M \to (P, 0, 0, P)$$

$$q = (q^0, q^1, q^2, 0) \quad (q^3 = 0)$$

$$\Rightarrow \quad 2P\cdot q \simeq 2P \cdot q^0 = Q^2/x_{Bj}$$

$$\Rightarrow \quad q^0 \simeq \frac{Q^2}{2x_{Bj}P} \Rightarrow \text{interaction time in IMF}$$

is $\quad t_{DIS} \simeq \frac{1}{q^0} \simeq \frac{2x_{Bj}P}{Q^2}$

We need to compare it with interaction time between partons

→ in lab frame it is $t_{int}^{lab} \sim 1/\mu$, in IMF this time is dilated by boost factor P/M

$$\Rightarrow t_{partons} \simeq \frac{1}{\mu}\frac{P}{M}$$

as for cs $\quad x_{Bj}\,\mu M \leq \mu M \ll Q^2$

$$\boxed{t_{DIS} \ll t_{partons}}$$

Partons cannot interact during interaction with photon

„do not have enough time"

Transverse dynamics:

$$q^0 \simeq \frac{Q^2}{2x_{Bj}P} \ll Q \quad \text{as} \quad P \to \infty$$

$\Rightarrow$ $$Q^2 = q_\perp^2 - (q^0)^2 \simeq q_\perp^2$$

$\Rightarrow$ $$\underbrace{x_\perp \simeq \frac{1}{q_\perp} \simeq \frac{1}{Q}}$$

Just as in the protons' rest frame.

Breit frame

$$q^\mu = (0, 0_\perp, -Q)$$

$$p^\mu = (P, 0_\perp, P) \text{ - as in IMF}$$

$$q^+ = -q^- = -Q/\sqrt{2} \Rightarrow x^- \propto \sqrt{2}/Q$$
$$x^+ \propto \sqrt{2}/Q$$

$$\Rightarrow x^2 = 2x^+x^- - x_\perp^2 > 0 \Rightarrow \boxed{x_\perp^2 \lesssim 4/Q^2}$$

again the same

Photon interacts with the proton during a very short time $\propto$ $1/Q$

Next lecture → from experiment to theory

We learned some kinematics, defined different frames. We also analyzed space-time picture of Deep Inelastic Scattering.

Quantum mechanics and all that

Classical mechanics - coordinates and momenta of particles are <u>independent</u> variables

Quantum mechanical state - coordinates and momenta are <u>conjugate</u> variables

Let's consider a one-dimensional spatial wave function

$$\varphi(x) = \langle x | \varphi \rangle$$

$|\varphi\rangle$ - state vector, element of Hilbert space

momentum space representation

$$\tilde{\varphi}(p) = \langle p | \varphi \rangle$$

They are related by $\langle x | p \rangle = e^{ipx}$

$$\varphi(x) = \int \frac{dp}{2\pi} e^{ipx} \tilde{\varphi}(p)$$

and

$$\tilde{\varphi}(p) = \int dx \; e^{-ipx} \varphi(x)$$

Indeed

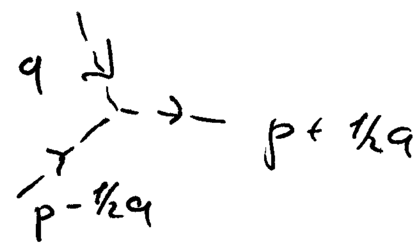
$$\int \frac{dp}{2\pi} |p\rangle\langle p| = 1 , \qquad \int dx\, |x\rangle\langle x| = 1$$

$$\varphi(x) = \langle x|\varphi\rangle = \int \frac{dp}{2\pi} \langle x|p\rangle\langle p|\varphi\rangle = \int \frac{dp}{2\pi} e^{ipx}\, \tilde{\varphi}(p)$$

$$\tilde{\varphi}(p) = \langle p|\varphi\rangle = \int dx\, \langle p|x\rangle\langle x|\varphi\rangle = \int dx\, (\langle x|p\rangle)^* \varphi(x)$$

$$= \int dx\, e^{-ipx} \varphi(x)$$

In many applications (form factors) we transfer a momentum q to the system and the momentum is absorbed



Transition is described by charge operator

$$\hat{Q}(q) = \int \frac{dp}{2\pi} |p+\tfrac{1}{2}q\rangle\langle p-\tfrac{1}{2}q| =$$

$$= \int dx\, dx' \frac{dp}{2\pi} |x\rangle \underbrace{\langle x|p+\tfrac{1}{2}q\rangle}_{e^{i(p+\frac{1}{2}q)x}} \underbrace{\langle p-\tfrac{1}{2}q|x'\rangle}_{e^{-i(p-\frac{1}{2}q)x'}} \langle x'|$$

$$= \int dx\, dx' \underbrace{\frac{dp}{2\pi} e^{ip(x-x')}}_{\delta(x-x')} e^{i(\frac{1}{2}qx + \frac{1}{2}qx')} |x\rangle\langle x'|$$

$$= \int dx\, |x\rangle e^{iqx} \langle x|$$

One measures elastic response $\sigma(q) \propto |F(q)|^2$
where $F(q)$ - form factor is the expectation value of $\hat{Q}(q)$:

$$F(q) = \langle \varphi | \hat{Q}(q) | \varphi \rangle = \int \frac{dp}{2\pi}\, \tilde{\varphi}^*(p + \tfrac{1}{2}q)\, \tilde{\varphi}(p - \tfrac{1}{2}q) =$$

$$= \underbrace{\int dx\, e^{iqx} |\varphi(x)|^2}_{\text{Fourier transform of the charge density}}$$

$$\rho(x) = |\varphi(x)|^2$$

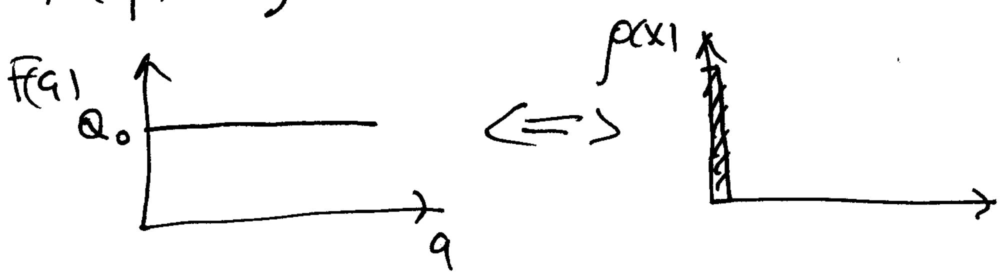
Form factor at $q=0$ is just normalization of the wave function $\int dx\, |\varphi(x)|^2$.

Suppose the system is point like

← at origin

$$\rho(x) = Q_0\, \delta(x - \cancel{x_0})$$

$$F(q) = \int dx\, e^{iqx} Q_0\, \delta(x) = Q_0 \;\to \text{constant}$$



δ'''

Other applications : we knock out a particle from a system and produce a plane wave state $|q\rangle$. Now inelastic response $\sigma(q) \propto W(q)$ reflects momentum distribution

$$\underline{\underline{f(p) = |\tilde{\varphi}(p)|^2}}$$

$$f(p) = \tilde{\varphi}^*(p)\,\varphi(p) = \int dx\,dx'\, e^{-ipx}\varphi^*(x)\,\varphi(x')\,e^{ipx'} =$$

$$= \int dx\,dx'\, e^{ip(x'-x)}\varphi^*(x)\,\varphi(x') =$$

$y = x' - x \qquad$ ~~det~~ $(y, x) = 1$

$$= \int dx\,dy\, e^{ipy}\underbrace{\varphi^*(x)\,\varphi(x+y)}_{\text{translate by } -1/2 y} = \int dx\,dy\, e^{ipy}\varphi^*(x-\tfrac{1}{2}y)\,\varphi(x+\tfrac{1}{2}y)$$

$$|\tilde{\varphi}(p)|^2 = \int dx\,dy\, e^{ipy}\varphi^*(x-\tfrac{1}{2}y)\,\varphi(x+\tfrac{1}{2}y)$$

In QFT we will rewrite $\quad \varphi(x+\tfrac{1}{2}y) = \Psi(x+\tfrac{1}{2}y)|\varphi\rangle$

$$\varphi^*(x-\tfrac{1}{2}y) = \langle\varphi|\,\Psi^\dagger(x-\tfrac{1}{2}y)$$

We will also call those objects <u>bilocal</u>

Momentum distribution is non local in coordinate space, form factors are non local in momentum space.

We can also define <u>Wigner</u> distribution

$$W(x,p) = \int dy\, e^{ipy} \varphi^*(x - \tfrac{1}{2}y)\, \varphi(x + \tfrac{1}{2}y)$$

$$= \int \frac{dq}{2\pi} e^{-iqx} \tilde{\varphi}^*(p + \tfrac{1}{2}q)\, \tilde{\varphi}(p - \tfrac{1}{2}q)$$

We can obtain densities from Wigner distribution

$$f(p) = \tilde{\varphi}^*(p)\, \tilde{\varphi}(p) = \int dx\, W(x,p)$$

$$\rho(x) = \varphi^*(x)\, \varphi(x) = \int \frac{dp}{2\pi} W(x,p)$$

W-distribution is important for 3D distributions to study motion and positions of quarks