Alexei Prokudin

Introduction to Partonic Hadron Structure

Lecture IV

Advanced topics

So far all calculations that we did were in QED. Quarks however have additional quantum number – color and interact by exchanging gluons

Quantum chromodynamics is an SU(3) Yang-Mills gauge theory

$$\mathcal{L}_{QCD} = \sum_{\text{flavours } f} \bar{q}_i^f(x)\,[i\gamma^\mu D_\mu - m_f]_{ij}\, q_j^f(x) - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

Field $A^a_\mu(x)$ describes the gluon, spin $=1$, color index $a = 1, \dots, 8$

$i, j = 1, 2, 3$

Covariant derivative is

Note that this sign is convention, we use that of Collins 2011

$$D_\mu = \partial_\mu + ig A_\mu = \partial_\mu + ig\, t^a A^a_\mu$$

where t^a are the generators of SU(3) in fundamental representation ($t^a = \lambda^a/2$, λ^a – Gell-Mann matrices)

$$F_{\mu\nu} = t^a F^a_{\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

where f^{abc} are structure constants of SU(3) color $[t_a, t_b] = i f_{abc} t_c$

Gauge symmetry

$$q(x) \to q'(x) = e^{-i\alpha^a(x) t^a} q(x)$$

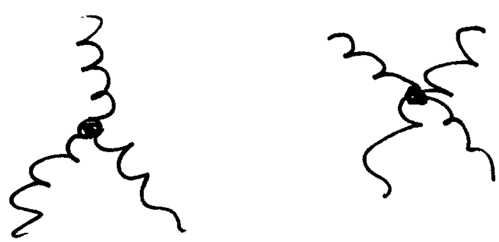
functions matrices

Difference with QED

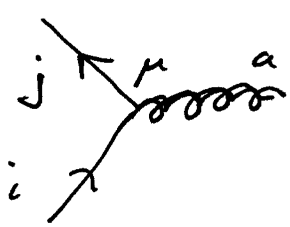
$$\psi(x) \to \psi'(x) = e^{-i\alpha(x)}\psi(x) \quad (U(1)\ \text{group})$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \underbrace{g f^{abc} A^b_\mu A^c_\nu}_{\text{self interaction}}$$

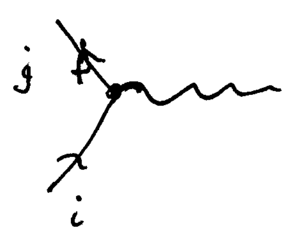
$\Rightarrow$ gluons interact with each other



Quark-gluon interaction

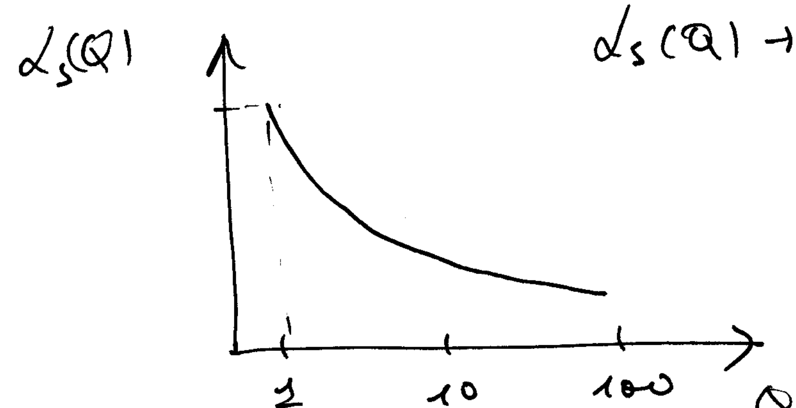


$\to -ig\gamma^\mu (t^a)_{ij}$ (QCD)

photon 

$-ie\gamma^\mu \delta_{ij}$

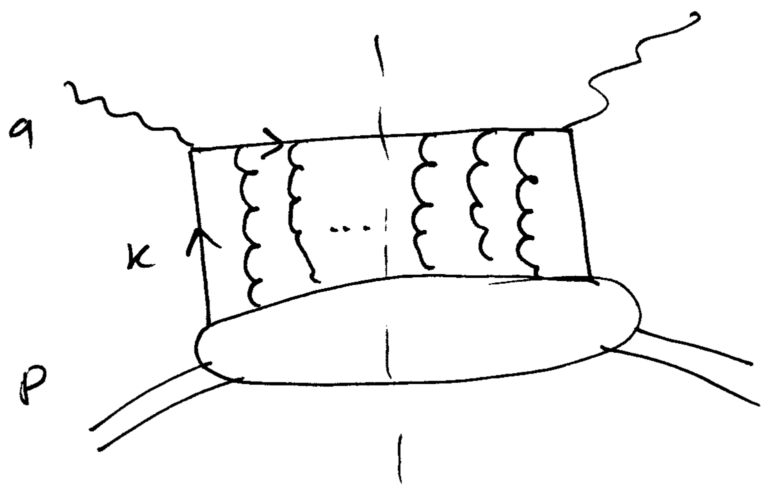
Interaction is much stronger than in QED and becomes <u>weaker</u> at large Q or small distances $\alpha_s = g^2/4\pi$



$\alpha_s(Q) \to 0, \ Q \to \infty$

"Asymptotic freedom"

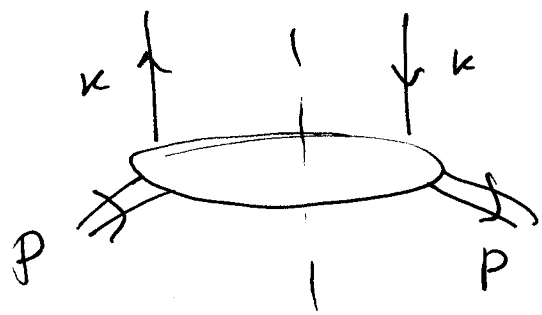
We should take into account interaction with gluons. For example DIS



What happens with these exchanges?

More generally:

we have defined quark distributions



$$\Phi(k,P) = \int \frac{d^4\xi}{(2\pi)^4} e^{-ik\xi} \langle P|\bar{\psi}(\xi)\psi(0)|P\rangle$$

it is easy to see that this definition is not gauge invariant

$$\bar{\psi}(\xi) \to \bar{\psi}'(\xi) = \bar{\psi}(\xi)\, e^{+i\alpha^a(\xi) t^a}$$

$$\psi(0) \to \psi'(0) = e^{-i\alpha^a(0) t^a}\psi(0)$$

$$\Rightarrow \bar{\psi}(\xi)\psi(0) \to \bar{\psi}'(\xi)\psi'(0) = \bar{\psi}(\xi)\, e^{+i\alpha^a(\xi)t^a - i\alpha^a(0)t^a}\psi(0)$$

$$\neq \bar{\psi}(\xi)\psi(0)$$

thus cannot be measured!

How do we restore gauge invariance?

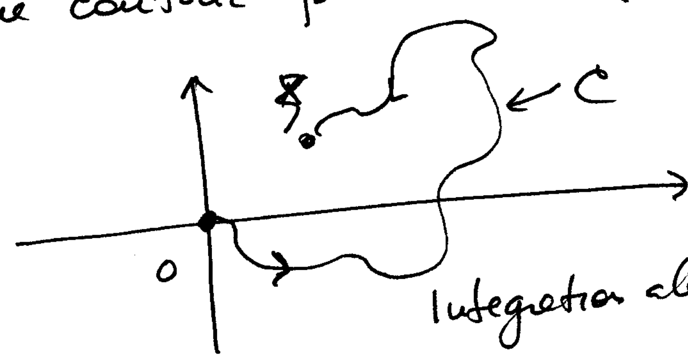
It means that formally we could insert an object, let us call it $W(C)$ in the definition

$$\Phi(k,P) = \int \frac{d^4\xi}{(2\pi)^4} e^{-ik\xi} \langle P| \bar{\psi}(\xi) W(C) \psi(0) |P\rangle$$

This object (called Wilson line) must have the following gauge transformations

$$W(C) \to W'(C) = e^{-ig t^a \alpha_a(\xi)} W(C) e^{ig t^a \alpha_a(0)}$$

Here C is some contour from 0 to ξ



Integration along path C

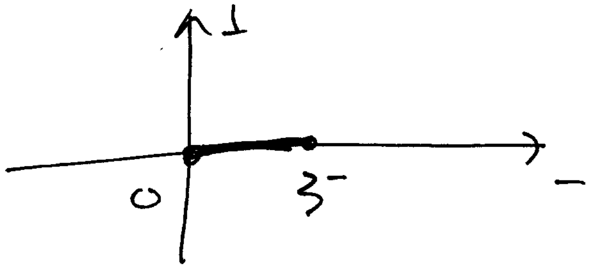
$$W(C) \stackrel{def}{=} P\left\{ \exp\left[-ig \int_C dx^\mu A^a_\mu(x) t_a\right]\right\}$$

Now formally

$\bar{\psi}(\xi) W(C) \psi(0)$ is gauge invariant and

hence measurable!

Particularly simple example is collinear density. Recall Lecture III:



$$\phi(k,P) = \int \frac{d\xi^-}{(2\pi)} e^{-ik^+\xi^-} \langle P | \bar{\Psi}(0^+,\xi^-,0_\perp)\, \Psi(0^+,0^-,0_\perp) | P \rangle$$

$$W(C) = \mathcal{P}\left\{ \exp\left[-ig \int_0^1 ds \, \frac{dx^\mu(s)}{ds} A_\mu^a(x(s))\, t_a \right] \right\}$$

$$\frac{dx^\mu(s)}{ds} = n^\mu \rightarrow \text{straight line from } 0 \text{ to } \xi^-$$

and

$$W(C) = \mathcal{P}\left\{ \exp\left[-ig \int_0^1 d\lambda \, \underbrace{n^\mu A_\mu^a(\lambda n)}_{n \cdot A \,=\, A^+} t_a \right] \right\}$$

If we choose gauge condition $A^+(x) = 0$ then $W(C) \equiv 1$. much simplification!

Let us consider a more complicated case,

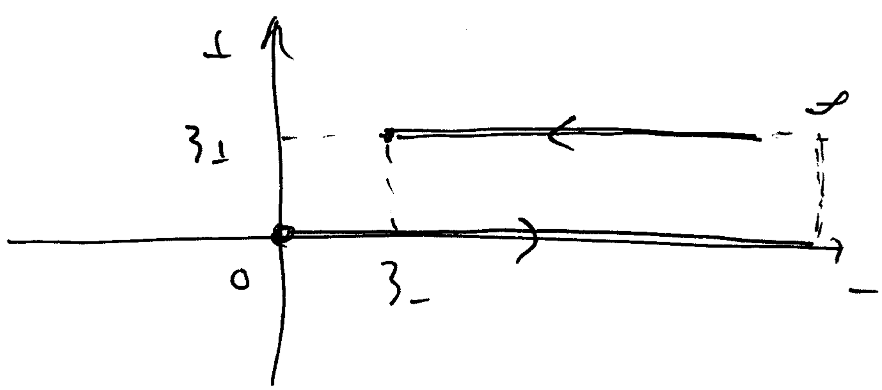
a distribution that depends explicitely on $k_\perp$

$$\phi(k, P) = \int \frac{d\xi^- d^2\xi_\perp}{(2\pi)^3} e^{-ixP^+\xi_- + i\vec{k}_\perp\vec{\xi}_\perp} \times$$

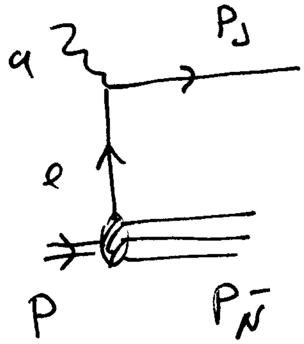
$$\times \langle P | \bar{\psi}(0^+, \xi^-, \xi_\perp) W(C) \psi(0^+, 0^-, 0_\perp) | P \rangle$$

(if we want particular distribution we will project (trace) with $\gamma^+/2$ or other projector)

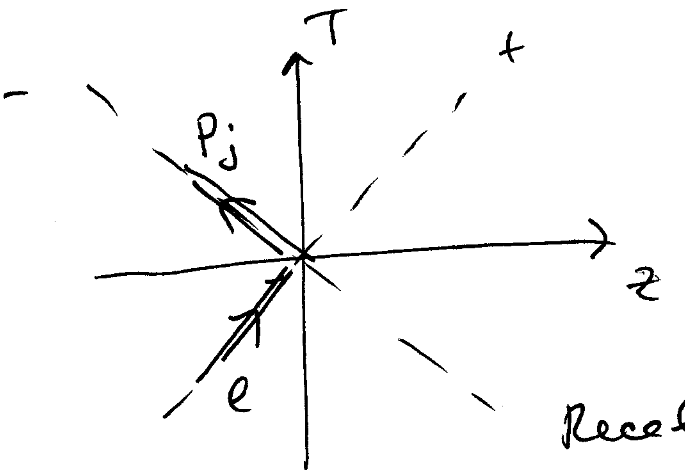
C is more complicated



We assume that we work in a gauge where gluon potential vanishes at $\xi_- = \infty$.



photon scatters
on a single parton
with momentum ℓ



Recall Lecture II:

$$2M\, W_{\mu\nu} = \frac{1}{2\pi} \sum_{\bar{N}} \int \frac{d^4 p_J}{(2\pi)^4}\, 2\pi\, \delta(p_J^2)(2\pi)^4 \delta^{(4)}(P_{\bar{N}} + P_j - P - q)$$

$$\langle P | J_\mu(0) | P_J, \bar{N} \rangle \langle P_j, \bar{N} | J_\nu(0) | P \rangle$$

Tree scattering amplitude from the Fig above

gives:

$$\langle P_j, \bar{N} | j_\nu(0) | P \rangle = \bar{u}(p_j) \gamma_\nu \langle \bar{N} | \psi(0) | P \rangle$$

$$2MW_{\mu\nu} = \frac{1}{2\pi} \sum_{\bar{N}} \int \frac{d^4 p_j}{(2\pi)^4} \, 2\pi \, \delta(p_j^2) \, 2\pi \, \delta^{(4)}(P_{\bar{N}} + p_j - P - q)$$

$$u(p_j) \gamma_\mu \bar{u}(p_j) \gamma_\nu \, \langle P | \bar{\Psi}(0) | \bar{N} \rangle \langle \bar{N} | \Psi(0) | P \rangle$$

$$u(p_j) \bar{u}(p_j)_{\beta\alpha} = (\not{p}_j + M)_{\beta\alpha}$$

$$2MW_{\mu\nu} = \frac{1}{2\pi} \sum_{\bar{N}} \int \frac{d^4 p_j}{(2\pi)^4} \, 2\pi \, \delta(p_j^2) \int \frac{d^4 \xi}{(2\pi)^3} \, e^{-i\xi(P_{\bar{N}} + p_j - P - q)}$$

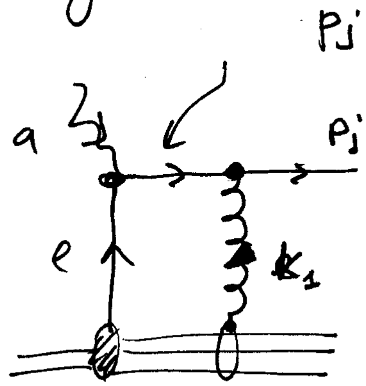
$$(\not{p}_j + M) \gamma_\mu \gamma_\nu \langle P | \bar{\Psi}(0) | \bar{N} \rangle \langle \bar{N} | \Psi(0) | P \rangle =$$

$$= \frac{1}{2\pi} \int \frac{d^4 p_j}{(2\pi)^4} \, 2\pi \, \delta(p_j^2) \int \frac{d^4 \xi}{(2\pi)^3} \, e^{-i\xi(\overbrace{p_j - q}^{\ell})} .$$

$$\times (\not{p}_j + M) \gamma_\mu \gamma_\nu \langle P | \bar{\Psi}(\xi) \Psi(0) | P \rangle$$

other steps follow exactly Lecture III

Now let us consider contribution from the following diagram:



$$\langle p_j, \bar N | J_\nu(0) | P\rangle = g\, \bar u(p_j) \int \frac{d^4k_1}{(2\pi)^4} \langle \bar N | A\!\!\!/(k_1) \frac{p\!\!\!/_j - k\!\!\!/_1}{(p_j-k_1)^2+i\epsilon} \gamma_\nu \psi(0) | P\rangle$$

From momentum conservation

$$p_j = q + \ell + k_1 ,$$

mass shell condition $p_j^2 = 0 = 2 p_j^+ p_j^- - p_{j\perp}^2$

$$p_j^+ = q^+ + \ell^+ + k_1^+ , \quad p_j^- = q^- + \ell^- + k_1^-$$

we can write

$$q_\mu = -x_{Bj} P^+ \bar n_\mu + \frac{Q^2}{2 x_{Bj} P^+} n_\mu , \qquad \underbrace{\begin{array}{l} n_\mu = (0^+, 1^-, 0_\perp) \\ \bar n_\mu = (1^+, 0^-, 0_\perp) \end{array}}_{\text{Recall Lecture III}}$$

$$\Rightarrow \quad q^- = q\cdot \bar n = \frac{Q^2}{2x_{Bj}P^+} \to \infty$$

$$q^+ = q\cdot n = -x_{Bj} P^+ \to \text{small}$$

$p_J^2 = 0 \rightarrow q^-(-x_{Bj}P^+ + \ell^+ + k_1^+) = 0$ ($\perp$ are much smaller!)

$\Rightarrow \underline{\ell^+ + k_1^+ = x_{Bj}P^+}$

Propagator:

biggest component

$$\not{p}_J - \not{k}_1 = \not{q} + \not{\ell} + \not{k}_1 - \not{k}_1 = \not{q} + \not{\ell} \simeq \overbrace{q_-}\gamma_+$$

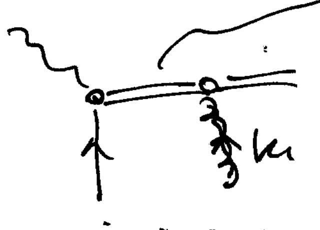
$$(p_J - k_1)^2 = (q+\ell)^2 \simeq 2q^-(q^+ + \ell^+) = 2q^-(-x_B P^+ + \ell^+) =$$

$$= 2q^-(-k_1^+)$$

$$\Rightarrow \frac{\not{p}_J - \not{k}_1}{(p_J - k_1)^2 + i\epsilon} \rightarrow \frac{q_-\gamma_+}{2q^-(-k_1^+) + i\epsilon} = -\frac{1}{2}\underbrace{\frac{\gamma_+}{k_1^+ - i\tilde{\epsilon}}}$$

Eikonal propagator!

depends only on $\underline{\underline{k_1^+}}$



What about gluon field?

$$A_\mu = A_+ \bar{n}_\mu + A_- n_\mu + A_\perp$$

$$\bar{\not{n}} = \gamma^-, \quad \not{n} = \gamma^+ \, ; \quad \underbrace{\gamma^- = \gamma_+ \, , \, \gamma^+ = \gamma_-}_{\text{Demonstrate this!}}$$

We have term

$$\not{A}(k_1)(\not{p}_J - \not{k}_1) \propto (A_+ \gamma_- + A_- \gamma_+ + \not{A}_\perp)\gamma_+$$

$$(\gamma_+)^2 = 0 \quad \Rightarrow \quad A_+ \gamma_- \gamma_+ \text{ survives}$$

$$\bar{u}\,\gamma_- \gamma_+ \to 2\bar{u}$$

and we need only to perform the following

$$\int \frac{d^4 k_1}{(2\pi)^4} \frac{1}{k_{1+} - i0} A(k_1) = i \int_{-\infty}^{\infty} d\zeta_- \, \theta(\zeta_-) A_+(\zeta_-, 0, 0)$$

$\Rightarrow$ result is

$$\langle p_J \bar{N} | J_\nu(0) | P \rangle = \bar{u}(p_J) \gamma_\nu \langle \bar{N} | \underbrace{(ig) \int_0^\infty d\zeta_- A_+(\zeta_-, 0, 0)}_{\text{Wilson line}} \psi(0) | P \rangle$$

After repeating for many gluons

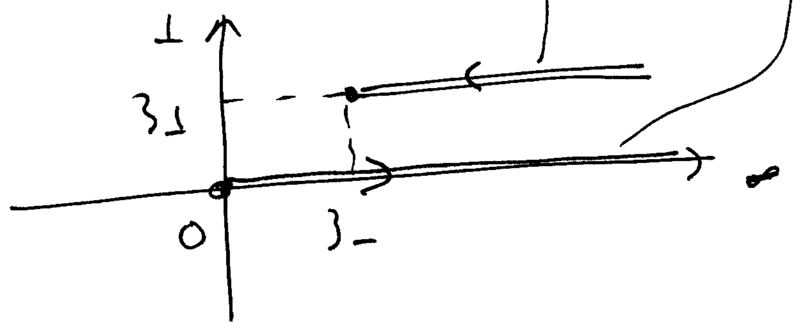
we get

$$\langle p_J, \bar{N} | J_\nu(0) | P \rangle = \bar{u}(p_J)\gamma_\nu \langle \bar{N} | P\exp\left(-ig\int_0^\infty d\xi_- A_+(\xi_-)\right)\psi(0)|P\rangle$$

Hence final definition will be

$$\phi(k, P) = \int \frac{d\xi^- d^2\xi_\perp}{(2\pi)^3} e^{-ixP^+\xi_- + i\vec{k}_\perp \xi_\perp}$$

$$\times \langle P | \bar{\psi}(0^+, \xi^-, \xi_\perp) \underbrace{P\exp\left(+ig\int_\infty^{\xi^-} d\eta\, A_+(\eta)\right) P\exp\left(-ig\int_0^\infty d\xi_- A_+(\xi_+)\right)}_{\text{Wilson line}} \psi(0)|P\rangle$$



Connection of two points is <u>defined</u> by the process

Details:

$$\int \frac{d^4k_1}{(2\pi)^4} \frac{1}{k_{1+} - i0} A(k_1)$$

$$A(k_1) \stackrel{\text{def}}{\equiv} \int d^4\zeta \, e^{i\zeta \cdot k} \tilde{A}(\zeta) =$$

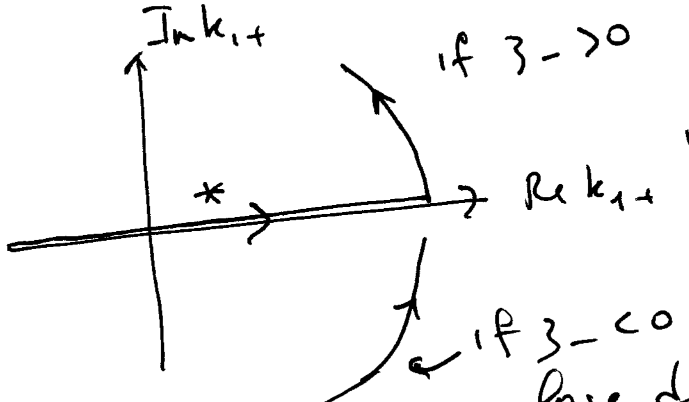
$$= \int d\zeta^- \, e^{i\zeta^- k_1^+} \tilde{A}(\zeta^-)$$

We see that A_+ matters $\Rightarrow$

$$\int d\zeta^- e^{i\zeta^- k_1^+} \tilde{A}_+(\zeta^-)$$

and finally

$$\int d\zeta^- \int \frac{dk_{1+} \, dk_{1-} \, d^2k_\perp}{(2\pi)(2\pi)(2\pi)^2} \frac{e^{i\zeta_- k_{1+}}}{k_{1+} - i0} \tilde{A}_+(\zeta^-)$$



if $\zeta_- > 0$ $\quad e^{i\zeta_-(i\Lambda)} = e^{-\zeta_-\Lambda} \to 0$, if $\Lambda \to \infty$ we can close the contour up.

if $\zeta_- < 0$ $\quad e^{+\zeta_-\Lambda} \to 0$ we have to close down, but the pole is not there $\Rightarrow$ result is $\emptyset$

$\Rightarrow$ result is

$$\int_{-\infty}^{\infty} dz^- \, \theta(z^-) \, 2\pi i \cdot \mathrm{Res}(\ldots k_{1+}) =$$

$$= i \int_{-\infty}^{\infty} dz^- \, \theta(z^-) \, \tilde{A}_+(z^-) = i \int_0^{\infty} dz^- \, \tilde{A}_+(z^-)$$

so we proved the formula from page 12