Alexei Prokudin

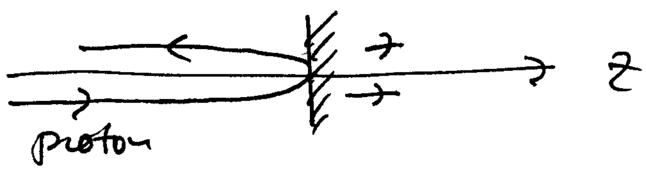
Introduction to Partonic Hadron Structure

Lecture II

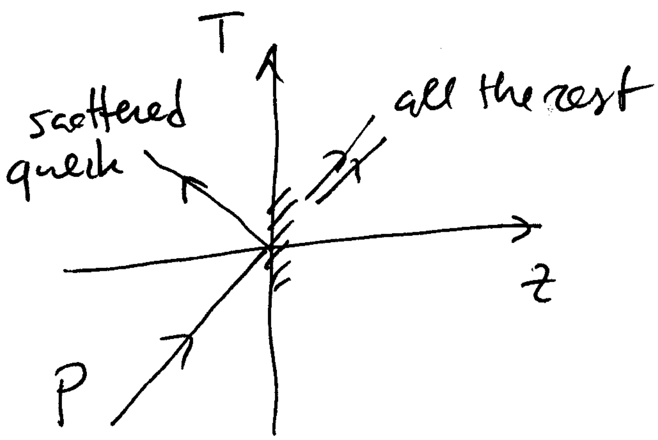
Elements of Theory

We argued that in IMF time of interaction of the photon with the proton is much shorter than time of interactions between partons

Let us introduce „brick wall" or Breit frame which is also an IMF



In the diagram it will look like this



photon does not propagate in time

$\Rightarrow q^0 = 0$

$q^\mu = (0, 0, 0, -Q)$

$\underline{P}^\mu = (P, 0, 0, P)$

Time of interaction between partons $\propto \frac{1}{\mu} \frac{P}{M} \to \infty$

partons are „frozen". or they interact in infinite time.

Now we have

$q^{+} = q^{-} = -Q/\sqrt{2}$ and we have $e^{iqx} \Rightarrow qx \sim \text{const}$

thus

$$\left.\begin{aligned} x^{-} &\propto \frac{\sqrt{2}}{Q} = \frac{t-z}{\sqrt{2}} \\ x^{+} &\propto \frac{\sqrt{2}}{Q} = \frac{t+z}{\sqrt{2}} \end{aligned}\right\} \quad \frac{2t}{\sqrt{2}} = \frac{2\sqrt{2}}{Q}$$

$$t = \underbrace{2/Q}_{\to \emptyset} \ll \underbrace{\frac{1}{\mu}\,\frac{P}{M}}_{\to \infty}$$ partons are „frozen".

$x^2 > 0$ Causality

$x^2 = 2x^{+}x^{-} - x_{\perp}^{2} > 0 \Rightarrow x_{\perp}^{2} < 2x^{+}x^{-} = \underline{\underline{\frac{4}{Q^2}}}$

Photon interacts with the proton during a very short time and has a resolution $\sim 1/Q^2$ in transverse plane

Quantum mechanics and all that

Classical mechanics – coordinates and momenta of particles are <u>independent</u> variables

Quantum mechanical state – coordinates and momenta are <u>conjugate</u> variables

Let's consider a one-dimensional spatial wave function

$$\varphi(x) = \langle x|\varphi\rangle$$

$|\varphi\rangle$ – state vector, element of Hilbert space

momentum space representation

$$\tilde{\varphi}(p) = \langle p|\varphi\rangle$$

They are related by $\langle x|p\rangle = e^{ipx}$

$$\varphi(x) = \int \frac{dp}{2\pi} e^{ipx} \tilde{\varphi}(p)$$

and

$$\tilde{\varphi}(p) = \int dx \; e^{-ipx} \varphi(x)$$

Indeed

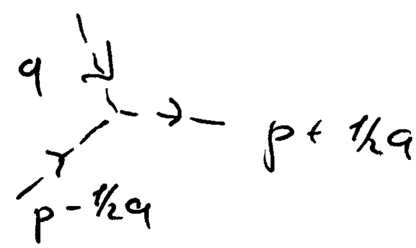
$$\int \frac{dp}{2\pi} |p\rangle\langle p| = 1 , \qquad \int dx\, |x\rangle\langle x| = 1$$

$$\varphi(x) = \langle x|\varphi\rangle = \int \frac{dp}{2\pi} \langle x|p\rangle\langle p|\varphi\rangle = \int \frac{dp}{2\pi} e^{ipx}\, \tilde{\varphi}(p)$$

$$\tilde{\varphi}(p) = \langle p|\varphi\rangle = \int dx\, \langle p|x\rangle\langle x|\varphi\rangle = \int dx\, (\langle x|p\rangle)^* \varphi(x)$$

$$= \int dx\, e^{-ipx} \varphi(x)$$

In many applications (form factors) we transfer a momentum q to the system and the momentum is absorbed



Transition is described by charge operator

$$\hat{Q}(q) = \int \frac{dp}{2\pi} |p + \tfrac{1}{2}q\rangle\langle p - \tfrac{1}{2}q| =$$

$$= \int dx\, dx' \frac{dp}{2\pi} |x\rangle \underbrace{\langle x|p + \tfrac{1}{2}q\rangle}_{e^{i(p+\frac{1}{2}q)x}} \underbrace{\langle p - \tfrac{1}{2}q|x'\rangle}_{e^{-i(p-\frac{1}{2}q)x'}} \langle x'|$$

$$= \int dx\, dx' \underbrace{\frac{dp}{2\pi} e^{ip(x-x')}}_{\delta(x-x')} e^{i(\frac{1}{2}qx + \frac{1}{2}qx')} |x\rangle\langle x'|$$

$$= \int dx \, |x\rangle e^{iqx} \langle x|$$

One measures elastic responses $\sigma(q) \propto |F(q)|^2$ where $F(q)$ - form factor is the expectation value of $\hat{Q}(q)$:

$$F(q) = \langle\varphi|\hat{Q}(q)|\varphi\rangle = \int \frac{dp}{2\pi} \tilde{\varphi}^*(p+\tfrac{1}{2}q)\,\tilde{\varphi}(p-\tfrac{1}{2}q) =$$

$$= \underbrace{\int dx\, e^{iqx} |\varphi(x)|^2}_{\text{Fourier transform of the charge density}}$$

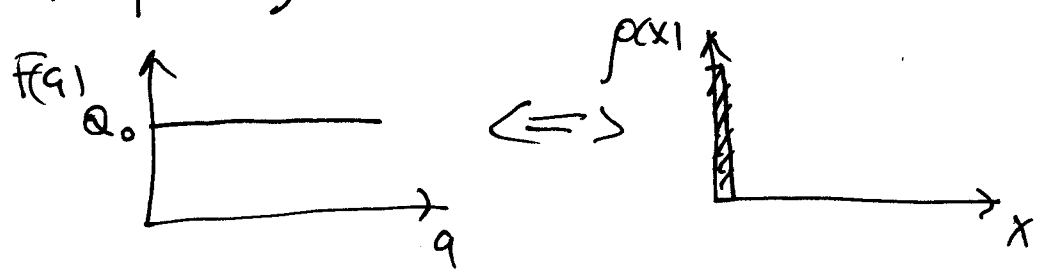
$$\rho(x) = |\varphi(x)|^2$$

Form factor at $q=0$ is just normalization of the wave function $\int dx\, |\varphi(x)|^2$.

Suppose the system is point like

$$\rho(x) = Q_0\, \delta(x - \cancel{x_0}) \leftarrow \text{at origin}$$

$$F(q) = \int dx\, e^{iqx} Q_0\, \delta(x) = Q_0 \rightarrow \text{constant}$$



Other applications: we knock out a particle from a system and produce a plane wave state $|q\rangle$. Now inelastic response $\sigma(q) \propto W(q)$ reflects momentum distribution

$$\underline{\underline{f(p) = |\tilde{\varphi}(p)|^2}}$$

$$f(p) = \tilde{\varphi}^*(p)\,\varphi(p) = \int dx\,dx'\, e^{-ipx}\varphi^*(x)\,\varphi(x')\,e^{ipx'} =$$

$$= \int dx\,dx'\, e^{ip(x'-x)}\varphi^*(x)\,\varphi(x') =$$

$y = x' - x$ $\quad$ ~~det~~ $(y, x) = 1$

$$= \int dx\,dy\, e^{ipy}\underbrace{\varphi^*(x)\,\varphi(x+y)}_{\text{translate by } -1/2 y} = \int dx\,dy\, e^{ipy}\varphi^*(x - 1/2 y)\,\varphi(x + 1/2 y)$$

$$|\tilde{\varphi}(p)|^2 = \int dx\,dy\, e^{ipy}\varphi^*(x - 1/2 y)\,\varphi(x + 1/2 y)$$

In QFT we will rewrite $\quad \varphi(x + 1/2 y) = \psi(x + 1/2 y)|\varphi\rangle$

$$\varphi^*(x - 1/2 y) = \langle\varphi|\,\psi^\dagger(x - 1/2 y)$$

We will also call those objects <u>bilocal</u>

Momentum distribution is non local in coordinate space, form factors are non local in momentum space.

We can also define <u>Wigner</u> distribution

$$W(x,p) = \int dy\, e^{ipy} \varphi^*(x - \tfrac{1}{2}y)\, \varphi(x + \tfrac{1}{2}y)$$

$$= \int \frac{dq}{2\pi} e^{-iqx} \tilde{\varphi}^*(p + \tfrac{1}{2}q)\, \tilde{\varphi}(p - \tfrac{1}{2}q)$$

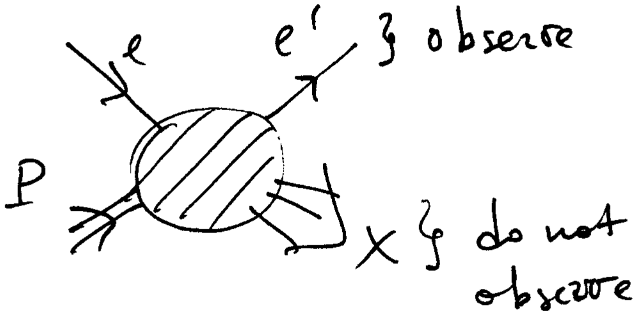
We can obtain densities from Wigner distribution

$$f(p) = \tilde{\varphi}^*(p)\, \tilde{\varphi}(p) = \int dx\, W(x,p)$$

$$\rho(x) = \varphi^*(x)\, \varphi(x) = \int \frac{dp}{2\pi} W(x,p)$$

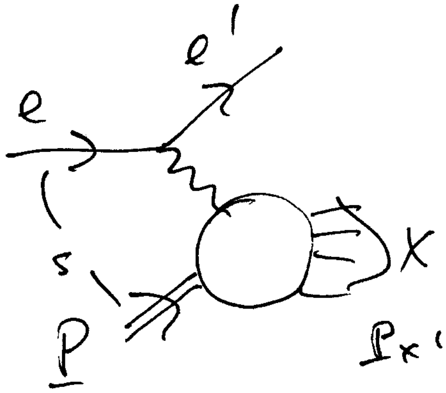
W-distribution is important for 3D distributions to study motion and positions of quarks

We want to calculate cross-section of this process:



$e + \underline{P} \to e' + X$

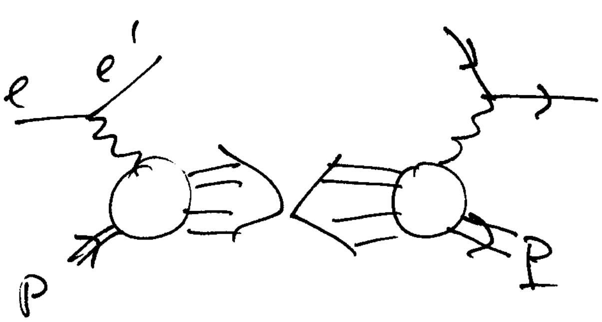
We use one photon exchange approximation:



$$\sigma = \frac{1}{F}\left(\int \overbrace{\frac{d^3 P_X}{(2\pi)^3 2E_X}}^{P_X \text{ is not observed!}} |M|^2 \overbrace{\delta(e+\underline{P}-e'-\underline{P}_X)}^{\text{conservation}} \right) dPS$$

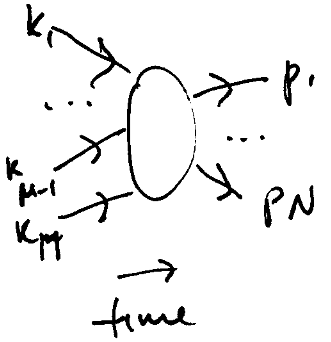
$F \simeq 2s = 2(e+P)^2$ flux

Phase space $dPS = \frac{d^3 e'}{(2\pi)^3 2E'}$

$|M|^2 =$ 

What is cross-section and how we calculate it?

Suppose we have M particles going into N particles



$|\psi_i\rangle$ - initial states at $t = -\infty$

$|\psi_f\rangle$ - final states at $t = +\infty$

S („scattering") matrix is the time-evolution operator

$$|\psi_f\rangle = S\,|\psi_i\rangle$$

$$|\psi_f\rangle = S\,|\psi_i\rangle = (\mathbb{1} + (S - \mathbb{1}))|\psi_i\rangle = \underbrace{|\psi_i\rangle}_{\text{initial state}} + \underbrace{(S - \mathbb{1})\,|\psi_i\rangle}_{\text{modification}}$$

T matrix is defined $\quad S = \mathbb{1} + iT$

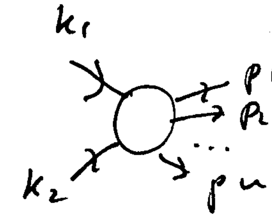
Probability conservation $\quad SS^\dagger = \mathbb{1} \;\Rightarrow\; (\mathbb{1} + iT)(\mathbb{1} - iT^\dagger) = \mathbb{1}$

$\Rightarrow \quad i(T - T^\dagger) + TT^\dagger = 0 \;\Rightarrow\; \underbrace{-i\,(T - T^\dagger) = TT^\dagger}$ gives optical theorem

$$\langle \{p_i\} |\, S - \mathbb{1}\, | \{k_j\}\rangle = \langle \{p_i\} |\, iT\, | \{k_j\}\rangle =$$

$$= \underbrace{(2\pi)^4\, \delta^{(4)}\Big(\sum_{i=1}^{N} p_i - \sum_{j=1}^{M} k_j\Big)}_{\text{energy conservation}}\; i\,\underbrace{\mathcal{M}\,(\{k_j\} \to \{p_i\})}_{\substack{\text{Matrix element} \\ \equiv \text{ scattering amplitude}}}$$

For simplicity $2 \to n$ amplitude



$$\langle p_1, \dots, p_n | iT | k_1, k_2 \rangle =$$

$$= (2\pi)^4 \delta^{(4)}(k_1 + k_2 - \sum_{i=1}^{n} p_i) \, M(k_1, k_2 \to \{p_i\})$$

Initial state is a 2 particle state

$$|\psi_i\rangle = \int \frac{d^3 q_1 \, d^3 q_2}{(2\pi)^3 2\mathcal{E}_{q_1} (2\pi)^3 2\mathcal{E}_{q_2}} f_1(q_1) f_2(q_2) \, |q_1, q_2\rangle$$

$f_1(q_1)$, $f_2(q_1)$ – momentum space wave functions

such that $|f_1(q_1)|^2$ & $|f_2(q_2)|^2$ are peaked around k_1 & k_2

(wave packets)

Probability to have n particle state

$$P_{2\to n} = \int |\langle p_1 \dots p_n | iT | \psi_i \rangle|^2 \prod_{i=1}^{n} \underbrace{\frac{d^3 p_i}{(2\pi)^3 2\mathcal{E}_i}}_{\text{phase factors}}$$

phase $d^4 p \, \overbrace{\delta(p^2 - m^2)}^{\text{mass-shell condition}} \theta(p_0)$

$$\delta(f(z)) = \frac{\delta(z - z_0)}{f'(z_0)} \, , \quad p^2 = E^2 - \vec{p}^{\,2}$$

$f' \to 2\mathcal{E}_i$ such that $\mathcal{E}_i^2 - \vec{p}^{\,2} - m^2 = 0$

Probability is then

$$P_{2\to n} = \int \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2\mathcal{E}_i} \int \frac{d^3q_1\, d^3q_2\, d^3q_1'\, d^3q_2'}{(2\pi)^3 2\mathcal{E}_{q_1} (2\pi)^3 2\mathcal{E}_{q_2} (2\pi)^3 2\mathcal{E}_{q_1'} (2\pi)^3 2\mathcal{E}_{q_2'}}$$

$$f_1(q_1) f_2(q_2) f_1^*(q_1') f_2^*(q_2')\ (2\pi)^4 \delta^{(4)}(q_1+q_2 - \sum_{i=1}^{n} p_i)\, M(q_1,q_2\to\{p_i\})$$

$$(2\pi)^4 \delta^{(4)}(q_1'+q_2' - \sum_{k=1}^{n} p_k)\ M^*(q_1',q_2' \to \{p_k\})$$

as far as f_1 & f_2 are peaked at k_1 & k_2 replace q_1, q_2 to $k_1 k_2$ in M and δs

$$P_{2\to n} = \int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3 2\mathcal{E}_i} (2\pi)^4 \delta^{(4)}(k_1+k_2 - \sum_{i=1}^{n} p_i) |M(k_1 k_2 \to \{p_i\})|^2$$

$$\times \int \frac{d^3q_1\, d^3q_2\, d^3q_1'\, d^3q_2'}{(2\pi)^3 \mathcal{E}_{q_1} (2\pi)^3 \mathcal{E}_{q_2} (2\pi)^3 \mathcal{E}_{q_1'} (2\pi)^3 \mathcal{E}_{q_2'}} \underbrace{\int d^4x\, e^{-ix(q_1+q_2-q_1'-q_2')}}_{\text{We rewrote our } \delta \text{ as } \delta(q_1+q_2-q_1'-q_2')} \cdot f_1(q_1) f_2(q_2) f_1^*(q_1') f_2^*(q_2')$$

remember

$$= \int d^4x\, |\tilde{f}_1(x)|^2 |\tilde{f}_2(x)|^2 \int \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2\mathcal{E}_i} |M|^2 (2\pi)^4 \delta(k_1+k_2 - \sum_{i=1}^{n} p_i)$$

Fourier transform

$$\tilde{f}_i(x) \equiv \int \frac{d^3q}{(2\pi)^3 2\mathcal{E}_q} f_i(q) e^{-iq\cdot x}$$

Probability per volume is $\dfrac{dP_{2\to n}}{d^4x}$

Cross section is

$$= \frac{\text{Event probability per unit volume}}{(\text{target density}) \times (\text{flux of particles})}$$

density $\propto \; 2\mathcal{E}_{k_1} |\tilde{f}_1(x)|^2 \, 2\mathcal{E}_{m} |\tilde{f}_2(x)|^2$

flux $\propto \; \sqrt{(k_1 \cdot k_2)^2 - m_1^2 \cdot m_2^2} \; \propto s$

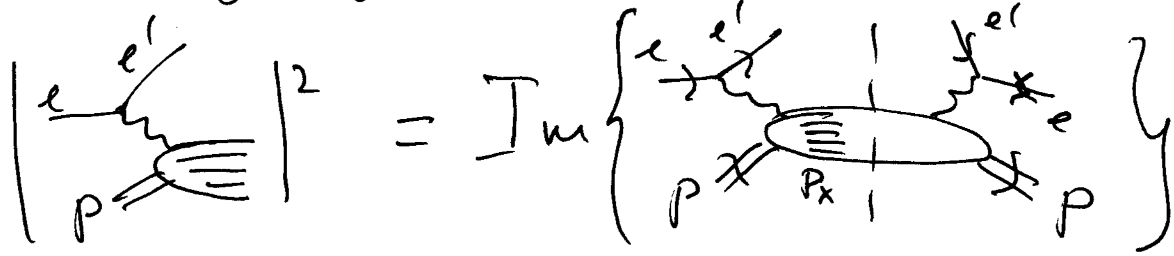
=> The formula from page 1.

$$\sigma = \frac{1}{4(\ell \cdot P)} \frac{d^3 e'}{(2\pi)^3 2\mathcal{E}_{e'}} |M|^2$$

Optical theorem, remember $-i(T - T^\dagger) = TT^\dagger$

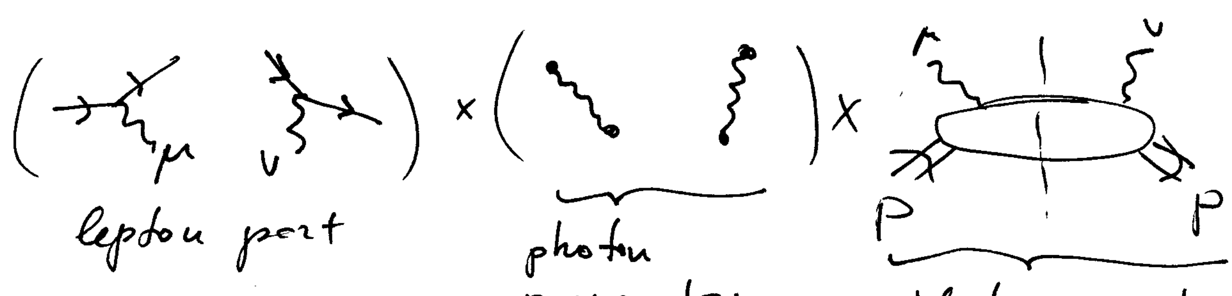
allows us to compute square of matrix elements

as imaginary part of more complex amplitude



Formally we will decompose it

as



lepton part $L^{\mu\nu}$

photon propagators $\propto \left(\frac{1}{Q^2}\right)^2$

hadron part $\underbrace{2M}_{\text{convention}} W_{\mu\nu}$