Flavor excitations of mesons and baryons, SU(3)

We know today that there are four quarks that are more massive than u and d:

s (strange), ~150 MeV more massive, $q_s = -1/3$
c (charm), ~1,300 MeV more massive, $q_c = +2/3$
b (bottom), ~4,200 MeV more massive, $q_b = -1/3$
[*t* (top) is ~172,000 MeV more massive, but decays so quickly it can't form a bound state with other quarks, $q_t = +2/3$]

Most known mesons and baryons (hadrons) are made up of the three lightest quarks u, d, and s

Develop a classification scheme called SU(3) flavor or $SU(3)_f$ based on assumption of invariance of strong interactions under exchange of u, d, and s

$SU(3)_f$ classification scheme

$m_s - m_{u,d} \simeq 150$ MeV appreciable relative to Λ_{QCD} and m_N

$SU(3)_f$ symmetry *broken* by this mass difference

Like with isospin, assume strong interactions independent of a unitary (probability preserving) transformation between *u*, *d*, and *s*

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = U \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

Overall phase transformation $U = e^{i\alpha} 1$ physically irrelevant

Close to identity matrix $U = 1 + i\,\delta\alpha\, 1$, $\det U = 1 + 3\, i\,\delta\alpha$

If restrict to $\det U = 1$, i.e., special unitary matrices SU(3), this removes phase change

SU(3)

SU(3) is the *group* of transformations that has *the same multiplication table* as the set of all unimodular, 3x3 unitary matrices *T*

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = T \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

Any element *T* of SU(3) can be written as $e^{-i\boldsymbol{\theta}\cdot\boldsymbol{\lambda}/2}$ where $\boldsymbol{\theta}$ is a set of 8 real numbers, and the eight hermitean (T unitary), traceless (det T = 1) Gell-Mann matrices $\boldsymbol{\lambda}$ are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

SU(3)

The generators $g_i = \lambda_i/2$ satisfy the commutation relations

$$[g_i, g_j] = i f^{ijk} g_k$$

The f^{ijk} are totally antisymmetric with

$$f^{123} = 1\,, \quad f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}\,, \quad f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

More convenient to define a more symmetric set of operators than the 8 Gell-Mann matrices

$$i_1 = \lambda_1/2,\ i_2 = \lambda_2/2,\ i_3 = \lambda_3/2$$

Pauli spin matrices, generate the set of all unitary unimodular transformations between quarks 1 and 2, without changing quark 3; for $SU(3)_f$, this is isospin

I spin, U spin, V spin

$$i_1 = \lambda_1/2,\ i_2 = \lambda_2/2,\ i_3 = \lambda_3/2$$

$$u_1 = \lambda_6/2,\ u_2 = \lambda_7/2,\ u_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

generate the set of all unitary unimodular transformations between quarks 2 and 3, without changing quark 1

$$v_1 = \lambda_4/2,\ v_2 = -\lambda_5/2,\ v_3 = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

generate the set of all unitary unimodular transformations between quarks 1 and 3, without changing quark 2

I spin, U spin, V spin

SU(3) has only 8 generators; these are not linearly independent because $i_3 + u_3 + v_3 = 0$

$$i_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \quad v_3 = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

The abstract generators I_i, U_i, V_i associated with these 'toy' generators satisfies a set of commutation relations

$$[I_i, I_j] = \epsilon_{ijk} I_k \quad [U_i, U_j] = \epsilon_{ijk} U_k \quad [V_i, V_j] = \epsilon_{ijk} V_k$$

These are just the usual SU(2) commutation relations (realized by the Pauli spin matrices)

I spin, U spin, V spin

$$[I_1, U_1] = -\tfrac{i}{2} V_2 \qquad [I_2, U_1] = -\tfrac{i}{2} V_1 \qquad [I_3, U_1] = -\tfrac{i}{2} U_2$$
$$[I_1, U_2] = -\tfrac{i}{2} V_1 \qquad [I_2, U_2] = \tfrac{i}{2} V_2 \qquad [I_3, U_2] = \tfrac{i}{2} U_1$$
$$[I_1, U_3] = \tfrac{i}{2} I_2 \qquad [I_2, U_3] = -\tfrac{i}{2} I_1 \qquad [I_3, U_3] = 0$$

$$[I_1, V_1] = \tfrac{i}{2} U_2 \qquad [I_2, V_1] = \tfrac{i}{2} U_1 \qquad [I_3, V_1] = -\tfrac{i}{2} V_2$$
$$[I_1, V_2] = \tfrac{i}{2} U_1 \qquad [I_2, V_2] = -\tfrac{i}{2} U_2 \qquad [I_3, V_2] = \tfrac{i}{2} V_1$$
$$[I_1, V_3] = \tfrac{i}{2} I_2 \qquad [I_2, V_3] = -\tfrac{i}{2} I_1 \qquad [I_3, V_3] = 0$$

$$[U_1, V_1] = -\tfrac{i}{2} I_2 \qquad [U_2, V_1] = -\tfrac{i}{2} I_1 \qquad [U_3, V_1] = -\tfrac{i}{2} V_2$$
$$[U_1, V_2] = -\tfrac{i}{2} I_1 \qquad [U_2, V_2] = \tfrac{i}{2} I_2 \qquad [U_3, V_2] = \tfrac{i}{2} V_1$$
$$[U_1, V_3] = \tfrac{i}{2} U_2 \qquad [U_2, V_3] = -\tfrac{i}{2} U_1 \qquad [U_3, V_3] = 0$$

From these it can be shown that I_3, U_3, V_3 and one of $\{I^2, U^2, V^2\}$ are a complete set of commuting observables

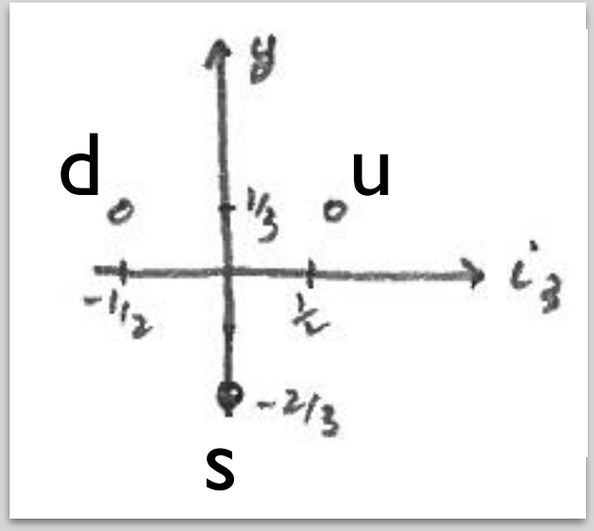
I spin, U spin, V spin

They are not independent since $I_3 + U_3 + V_3 = 0$

The symmetrical use of I_3, U_3, and V_3 has its advantages, but in practice we use I_3 and Y, which corresponds to the hypercharge y

$$y = \frac{2}{3}u_3 - v_3 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$y = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$



The hypercharge is just the baryon number plus strangeness (defined to be −1 for the strange quark) $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$

Constructing irreducible representations of SU(3)

Just as with angular momentum SU(2), it helps to define raising and lowering operators

$$I_\pm, I_3,\ U_\pm, U_3,\ V_\pm, V_3,\ X_\pm = X_1 \pm iX_2$$

The commutation relations look a little simpler written in terms of these operators, with the usual angular momentum rules (along the diagonal and below)

$$[I_3, I_\pm] = \pm I_\pm \qquad [U_3, I_\pm] = \mp \tfrac{1}{2} I_\pm \qquad [V_3, I_\pm] = \mp \tfrac{1}{2} I_\pm$$

$$[I_3, U_\pm] = \mp \tfrac{1}{2} U_\pm \qquad [U_3, U_\pm] = \pm U_\pm \qquad [V_3, U_\pm] = \mp \tfrac{1}{2} U_\pm$$

$$[I_3, V_\pm] = \mp \tfrac{1}{2} V_\pm \qquad [U_3, V_\pm] = \mp \tfrac{1}{2} V_\pm \qquad [V_3, V_\pm] = \pm V_\pm$$

$$[X_+, X_-] = 2X_3 \quad \forall X$$

$$[I_+, V_-] = 0 \qquad [I_+, U_-] = 0 \qquad [U_+, V_-] = 0$$

$$[V_+, I_+] = U_- \qquad [I_+, U_+] = V_- \qquad [U_+, V_+] = I_-$$

plus h.c.'s

Constructing irreducible representations of SU(3)

Consider an eigenstate $|\, i; i_3, u_3, v_3\rangle$ of I^2, I_3, U_3 and V_3

$$I^2|i; i_3, u_3, v_3\rangle = i(i+1)|i; i_3, u_3, v_3\rangle,$$
$$I_3|i; i_3, u_3, v_3\rangle = i_3|i; i_3, u_3, v_3\rangle,$$
$$U_3|i; i_3, u_3, v_3\rangle = u_3|i; i_3, u_3, v_3\rangle,$$
$$V_3|i; i_3, u_3, v_3\rangle = v_3|i; i_3, u_3, v_3\rangle,$$

Can show (just as with angular momentum) that i =1/2, 3/2, 5/2,…and that $-i \leq i_3 \leq i$ (could have done the same with u and v if chose eigenstates of U^2 or V^2)

Can generate an irreducible representation of SU(3) from the state $|M\rangle$ which has $i_3 = i_{3,\mathrm{max}}$, by taking as its basis space all states generated from $|M\rangle$ by repeated application of $I_\pm, I_3,\ U_\pm, U_3,\ V_\pm, V_3$

Constructing irreducible representations of SU(3)

Consider eigenstate $|\, x; i_3, u_3, v_3\rangle$ of X={I^2, U^2,V^2} and I_3, U_3 and V_3

From commutators it is clear that

$$I_\pm|x; i_3, u_3, v_3\rangle \text{ has } i_3 \pm 1,\ u_3 \mp \frac{1}{2},\ v_3 \mp \frac{1}{2}$$

$$U_\pm|x; i_3, u_3, v_3\rangle \text{ has } i_3 \mp \frac{1}{2},\ u_3 \pm 1,\ v_3 \mp \frac{1}{2}$$

$$V_\pm|x; i_3, u_3, v_3\rangle \text{ has } i_3 \mp \frac{1}{2},\ u_3 \mp \frac{1}{2},\ v_3 \pm 1$$

It is possible to plot i_3, u_3, v_3 on a planar diagram (recall $i_3+u_3+v_3=0$, so there are only two degrees of freedom) so that these relations are automatic

Constructing irreducible representations of SU(3)

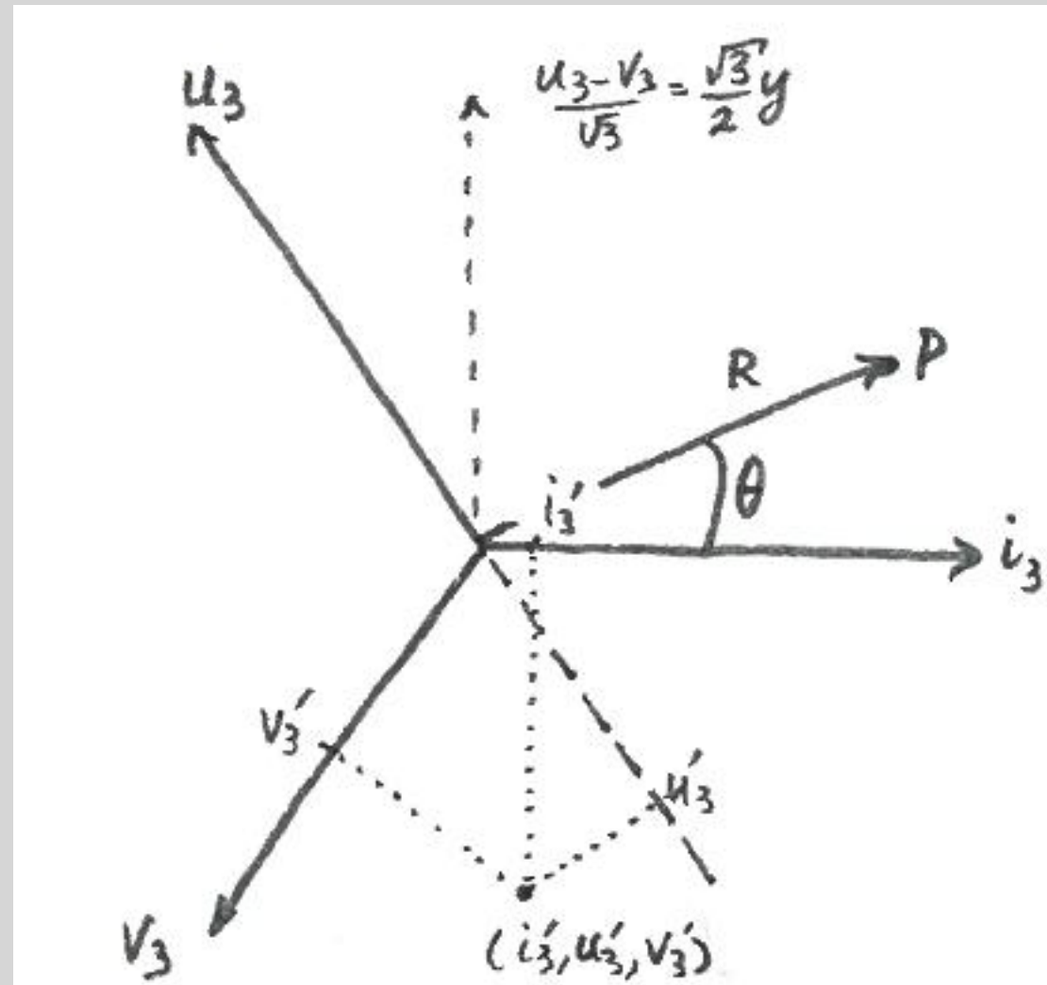
For P:

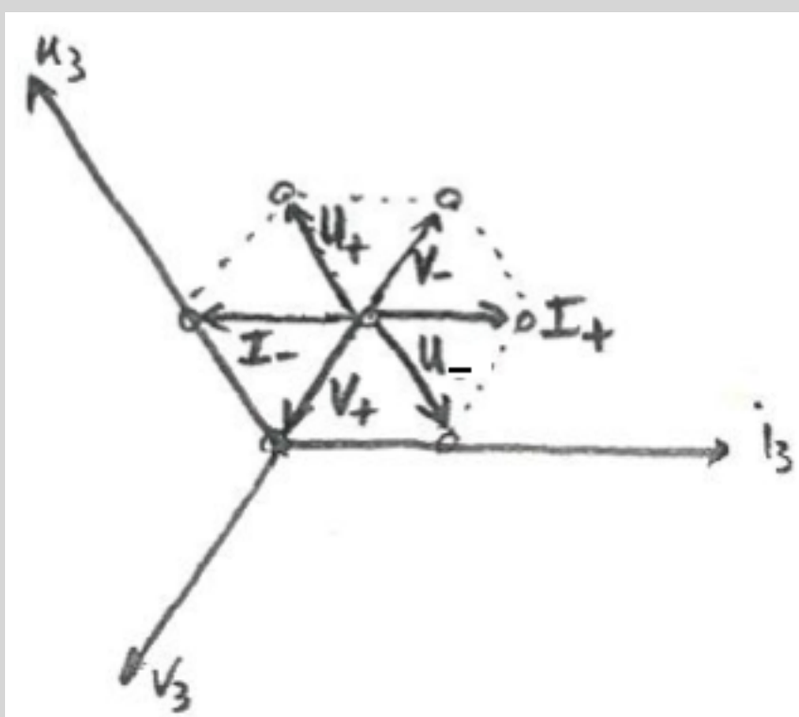
i_3=R cos(θ),

u_3=R cos(120°−θ),

v_3 =R cos(120°+θ),

so $i_3+u_3+v_3=0$





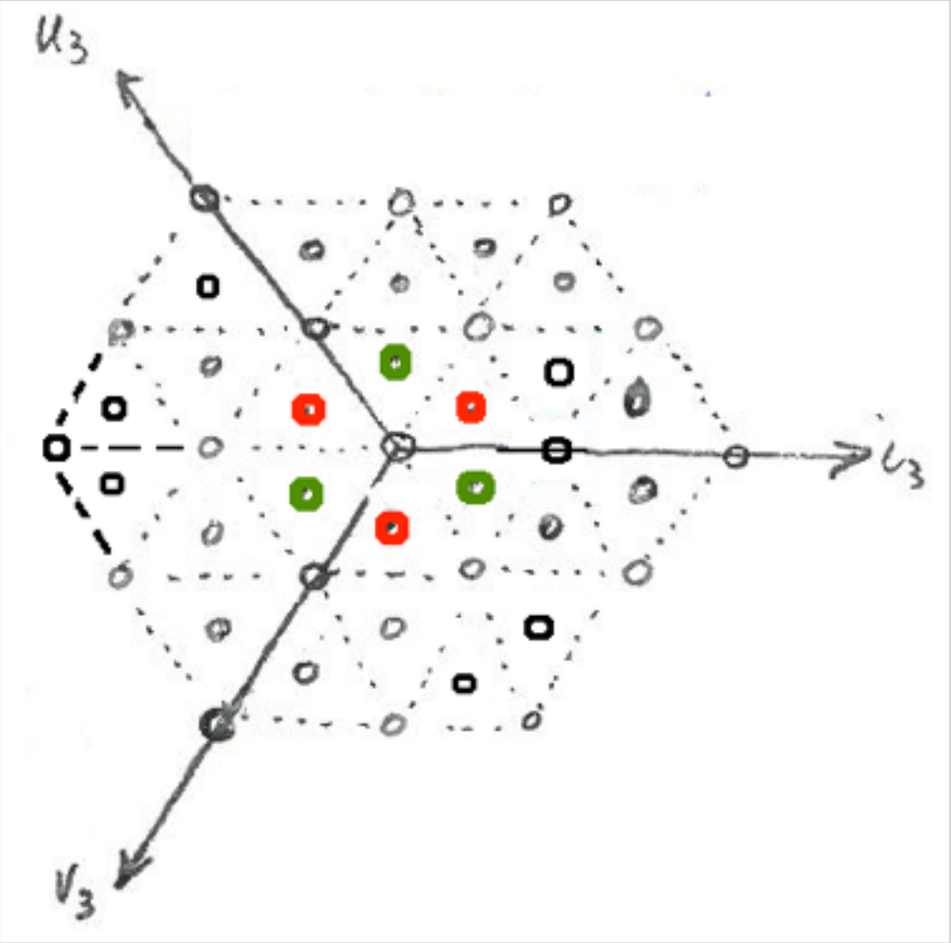
From geometry we have

$$I_+|x; i_3, u_3, v_3\rangle \text{ has } i_3+1,\ u_3-\frac{1}{2},\ v_3-\frac{1}{2}$$

Constructing irreducible representations of SU(3)

All possible points in the plane lie on interlocking hexagonal lattices

- quark triplet
- anti-quark triplet



Can generate irreducible representation using state $|M\rangle$, which has the maximum i_3

$$I_+|M\rangle = U_-|M\rangle = V_-|M\rangle = 0$$

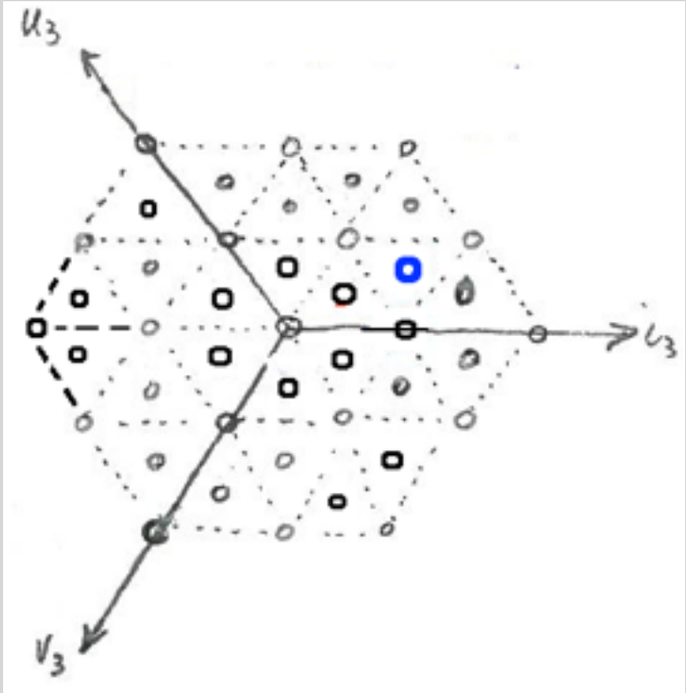
$$\begin{aligned} I^2|M\rangle &= \left(\frac{1}{2}I + I_- + \frac{1}{2}I - I_+ + I_3^2\right)|M\rangle \\ &= \left(\frac{1}{2}I + I_- - \frac{1}{2}I_-I_+ + I_3^2\right)|M\rangle \\ &= \left(I_3 + I_3^2\right)|M\rangle = i_{3,\max}(i_{3,\max}+1) \end{aligned}$$

so $i = i_{3,\max}$

Constructing irreducible representations of SU(3)

Similarly, since U_- raises i_3, and so $U_-\,|M\rangle = 0$

$$
\begin{aligned}
U^2|M\rangle &= \left(\frac{1}{2}U_+U_- + \frac{1}{2}U_-U_+ + U_3^2\right)|M\rangle \\
&= \left(-\frac{1}{2}U_+U_- + \frac{1}{2}U_-U_+ + U_3^2\right)|M\rangle \\
&= -U_3(-U_3+1)|M\rangle
\end{aligned}
$$

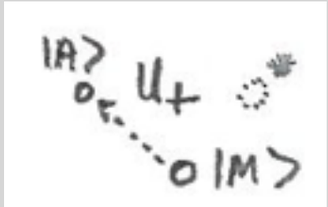


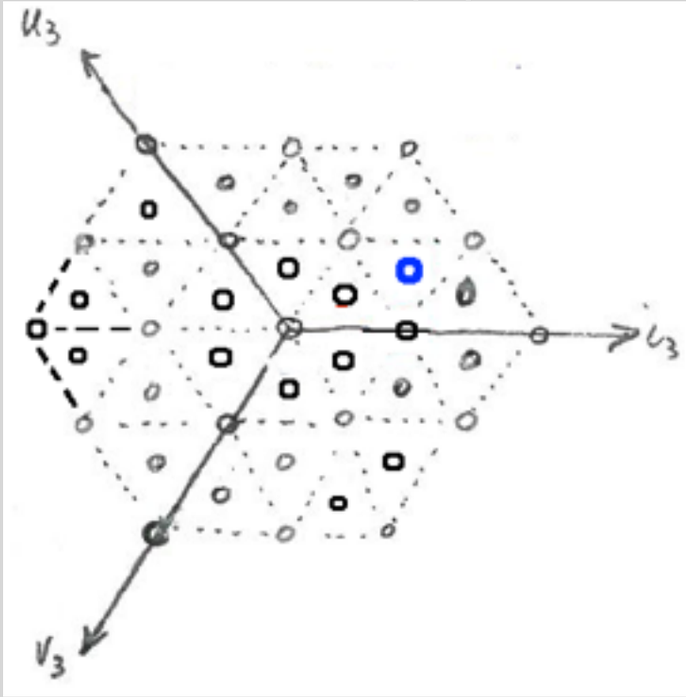
and so $|M\rangle$ is also a bottom state of a U-spin (and V-spin) multiplet $|M\rangle = |i\ i; u\ -u; v\ -v\rangle$ and since $i_3+u_3+v_3=0$, we must have $i = u+v$

- has $i_3=1, u_3=0, v_3=-1$, generates representation with $i=1, u=0, v=1$ (we will see belongs to **6**)

Constructing irreducible representations of SU(3)

Consider $|A\rangle = U_+ |M\rangle$





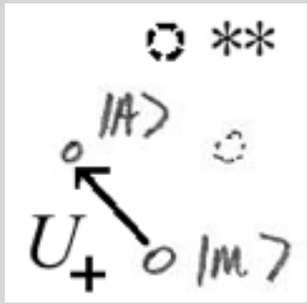
$|A\rangle$ has $i_3 = i-½$ and must have $i' = i-½$; if it had any greater i' we could step up using I_+ from it to find point *

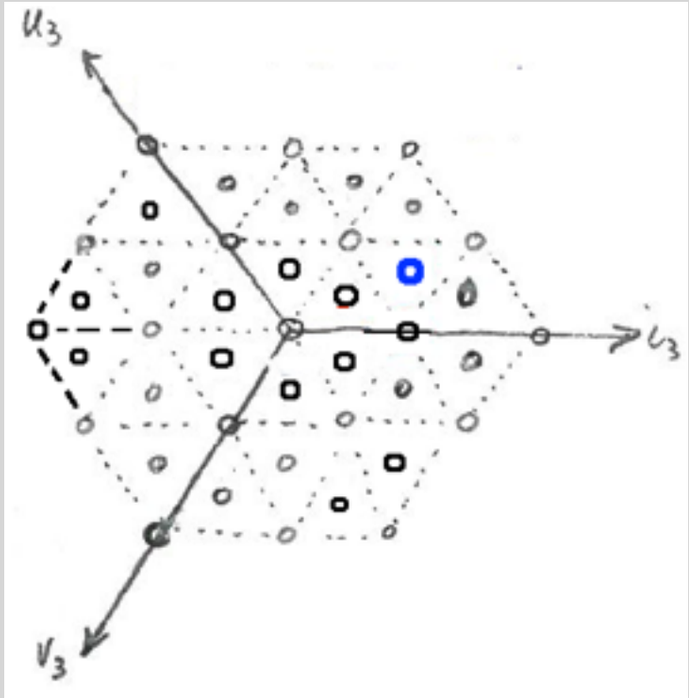
* would have $i_3 > i_{3,\max}$ and $|M\rangle$ would not be the state with maximum i_3

Avoid a contradiction if $|A\rangle$ is a top state and can't be stepped up

Constructing irreducible representations of SU(3)

Simlarly, $|A\rangle$ has $v_3 = -v-½$ and must have $v'=v+½$; if it had any greater v' , say $v' = v+3/2$, we could step down using V_- from it to find point **





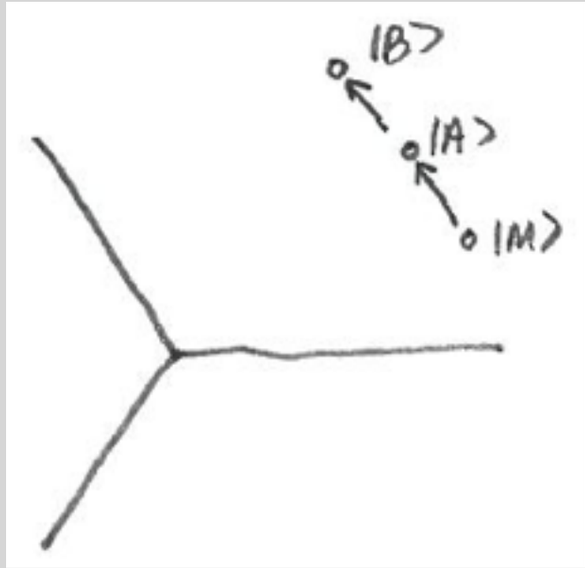
** would have $i_3 = i_{3,\max}$ and be another candidate for the status of $|M\rangle$

** has $v' = v + 3/2$, $i' = i$, $u' = u$ (can't change u by stepping up from $u_3 = -u$ to $-u+1$) so it can't have $i = u+v$, which we showed is true of a point like $|M\rangle$

Constructing irreducible representations of SU(3)

$$|A\rangle \;\propto\; |i-1/2, i-1/2; u, -u+1; v+1/2, -v-1/2\rangle$$

Note $V_- |A\rangle = I_+ |A\rangle = 0$



Do the same for $|B\rangle = U_+ |A\rangle = U_+^2 |M\rangle$

Our work with $|A\rangle$ relied on
$V_- |M\rangle = I_+ |M\rangle = 0$, also true of $|A\rangle$, so

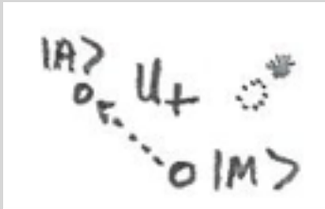
$$|B\rangle \;\propto\; |i-1, i-1; u, -u+2; v+1, -v-1\rangle$$

$U_+^n |M\rangle$ generates a line of states parallel to the u_3 axis,

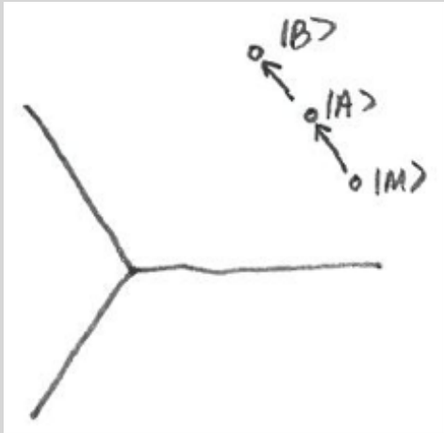
$$U_+^n |M\rangle \;\propto\; |i-n/2, i-n/2; u, -u+n; v+n/2, -v-n/2\rangle$$

Exercise

a. Prove using commutators that
$I^2|A\rangle = I^2\, U_+\, |M\rangle = (i-\frac{1}{2})(i-\frac{1}{2}+1)|A\rangle$, where $|M\rangle$ is a top state of an I-spin multiplet, with $i_3 = i$



b. Prove using commutators that
$I^2\, |B\rangle = I^2\, U_+\, |A\rangle = (i-1)i\, |A\rangle$, so $|B\rangle$ has I-spin $i-1$

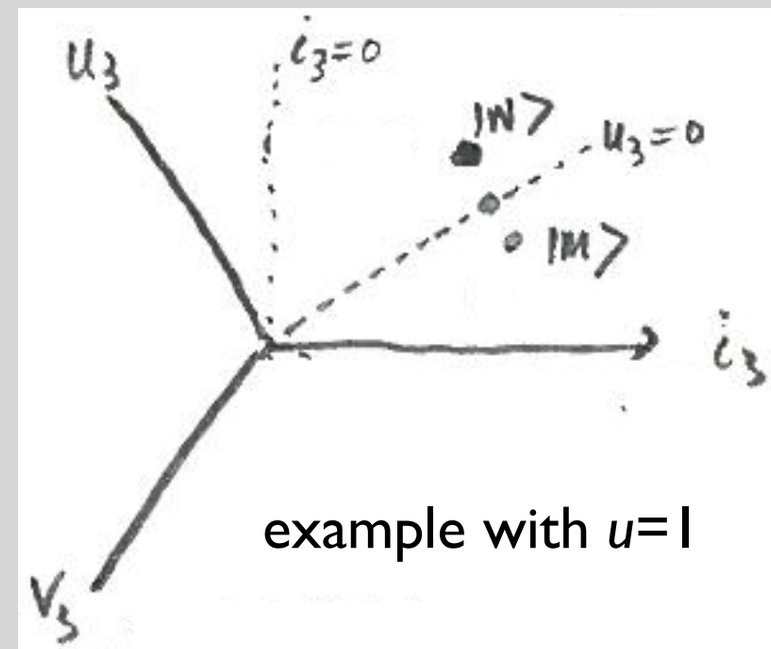


Constructing irreducible representations of SU(3)

Terminates when $U_+ U_+^n|M\rangle = 0$, i.e., when n = 2u since started with the bottom state of a *U* multiplet

$$
\begin{aligned}
U_+^{2u}|M\rangle &\propto |i-u, i-u; u, u; v+u, -v-u\rangle \\
&= |v, v; u, u; i, -i\rangle \\
&\equiv |N\rangle
\end{aligned}
$$

Note $I_+|N\rangle = U_+|N\rangle = V_-|N\rangle = 0$



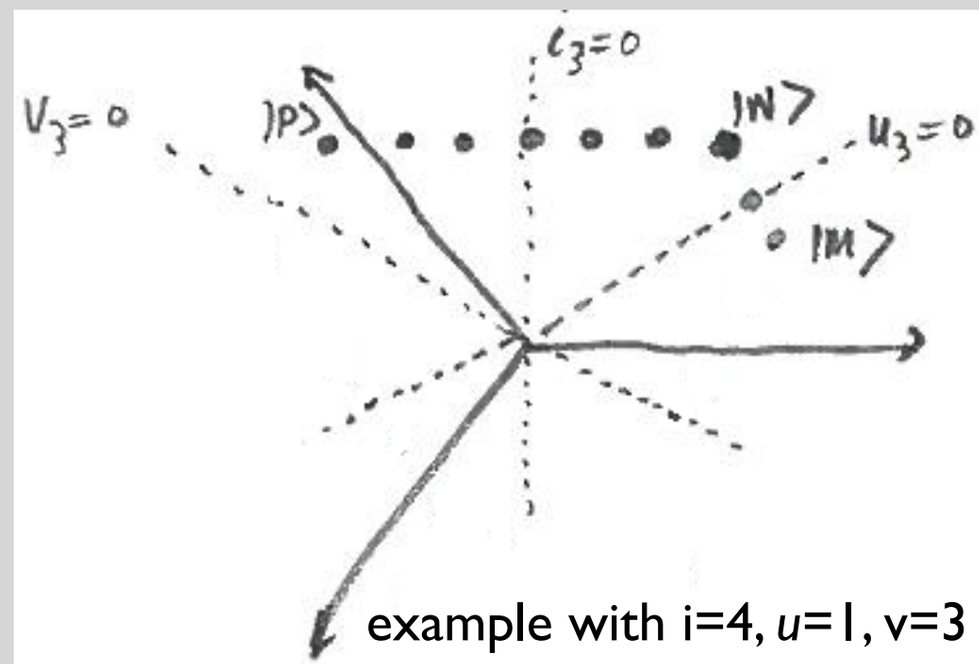
example with *u*=1

Constructing irreducible representations of SU(3)

Consider $I_-^n|N\rangle \;\propto |v, v-n; u+n/2, u+n/2; i-n/2, -i+n/2\rangle$

Generates a line of states parallel to the i_3 axis until n=2v

$$
\begin{aligned}
I_-^{2v}|M\rangle \;&\propto |v, -v; u+v, u+v; i-v, -i+v\rangle \\
&= |v, -v; i, i; u, -u\rangle \\
&\equiv |P\rangle
\end{aligned}
$$



example with i=4, u=1, v=3

Constructing irreducible representations of SU(3)

$$V_+^n|P\rangle \;\propto |v-n/2,-v+n/2;i-n/2,i-n/2;u,-u+n\rangle$$

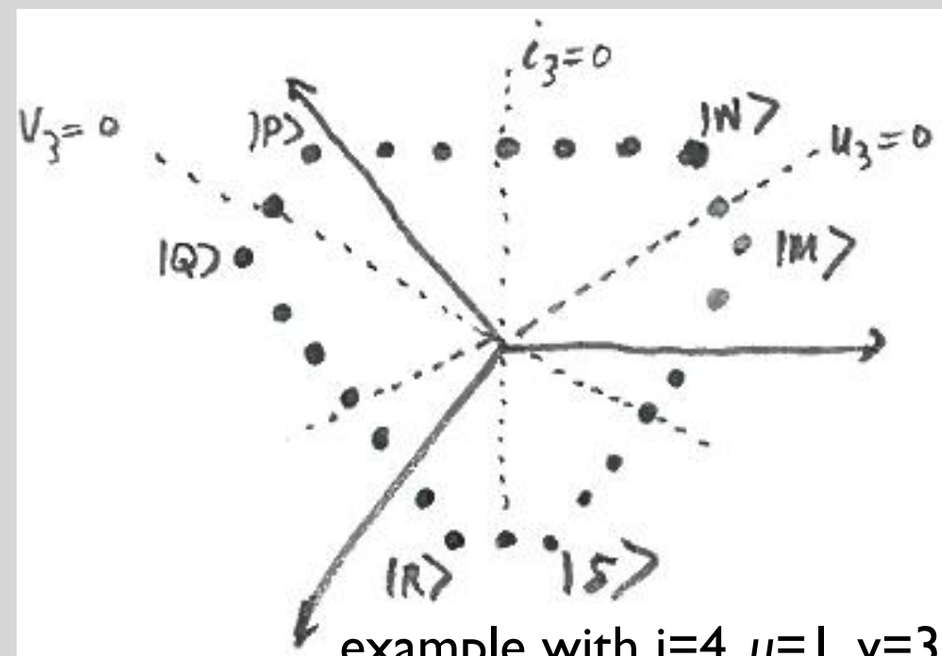
Generates a line of states parallel to the v_3 axis until n=2u

$$\begin{aligned} V_+^{2u}|P\rangle \;&\propto |v-u,-v+u;i-u,i-u;u,u\rangle \\ &= |v,-v;i,i;u,-u\rangle \\ &\equiv |Q\rangle \end{aligned}$$

$$\begin{aligned} U_-^{2v}|Q\rangle \;&\propto |i-v,-i+v;v,-v;i,i\rangle \\ &= |u,-u;v,-v;i,i\rangle \\ &\equiv |R\rangle \end{aligned}$$

$$\begin{aligned} I_-^{2u}|R\rangle \;&\propto |u,u;v+u,-v-u;i-u,i-u\rangle \\ &= |u,u;i,-i;v,v\rangle \\ &\equiv |S\rangle \end{aligned}$$

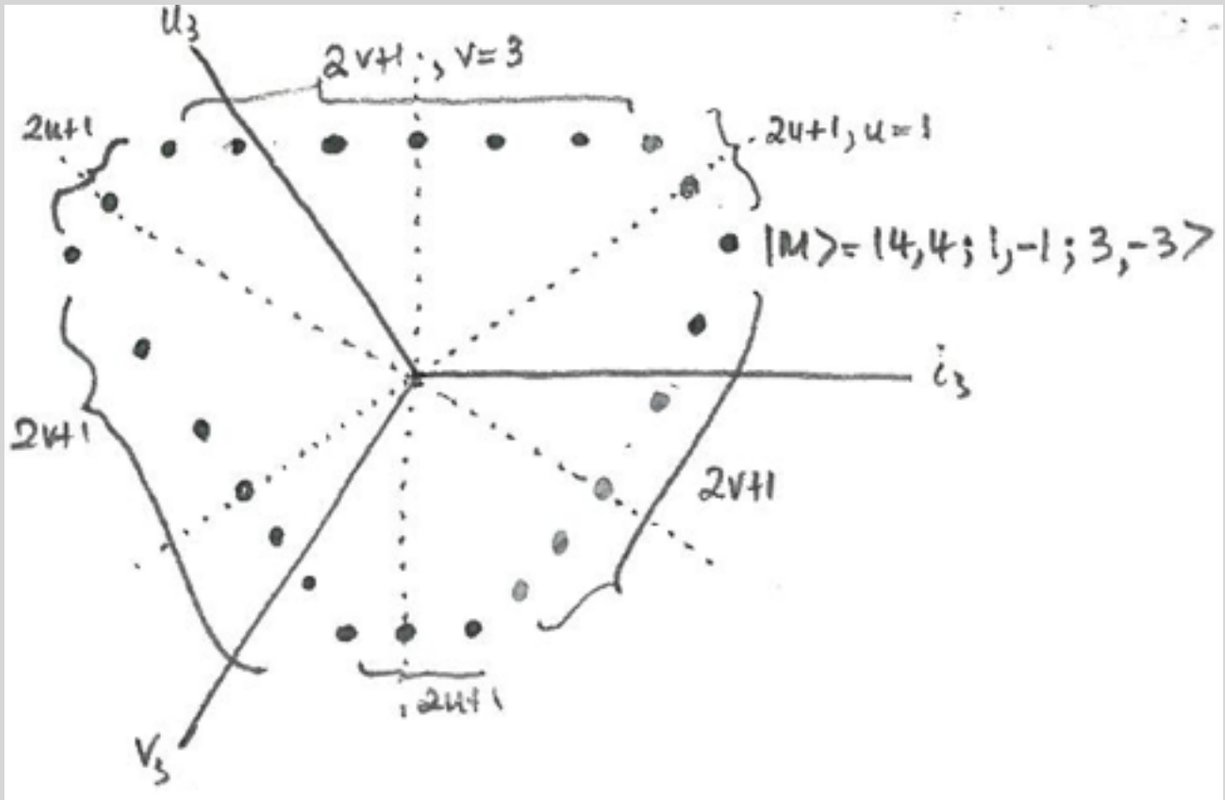
$$\begin{aligned} V_-^{2v}|S\rangle \;&\propto |;i,i;u,-u;v,-v\rangle \\ &\propto |M\rangle \end{aligned}$$



example with i=4, u=1, v=3

Constructing irreducible representations of SU(3)

All of the boundary is unique; if any state isn't, we could have started with it and gone two different ways around the boundary to get two different $|M\rangle$'s, but $|M\rangle$ is the only state with $i_3=i_{3,\max}$



Can fill in the interior by applying I_-^n to states on the right border(s)

Or by applying V_-^n or U_-^n to states on other borders

Does this generate unique states?

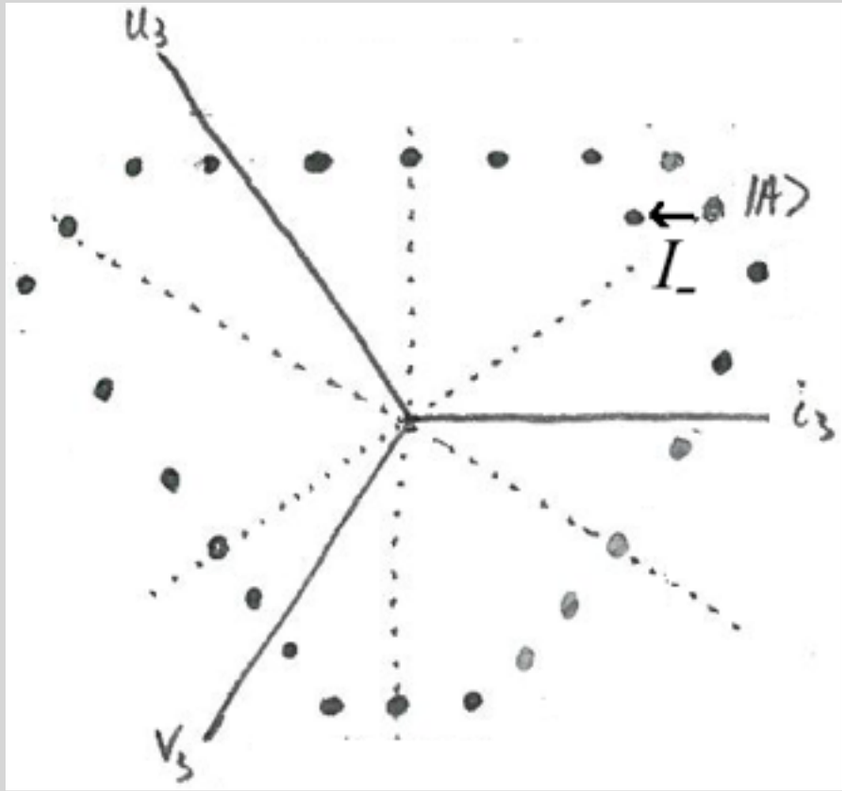
Constructing irreducible representations of SU(3)

Consider a state $|A\rangle$ on right boundary, i.e., with $i = i_3 = i_R$

Then $I_- |A\rangle$ has $i_3 = i_R - 1$, so it must have $i \geq i_R - 1$

It must have $i = i_R - 1$, because there is a state with this u_3 and v_3 and $i = i_R$ on the boundary, which is unique

This site is at most doubly occupied by $|i_R, i_R - 1; u_{3A}; v_{3A}\rangle$ and $|i_R - 1, i_R - 1; u_{3A}; v_{3A}\rangle$ (note: can only specify i, u and v simultaneously on the boundary)

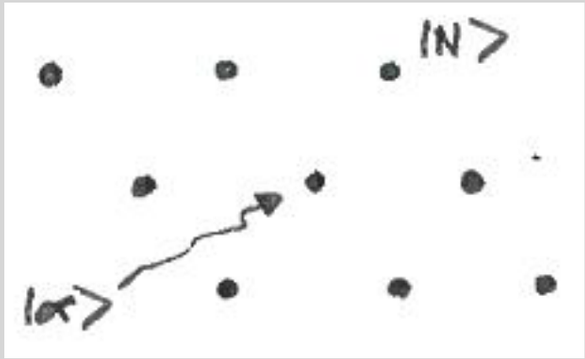


The n-th ring in from the boundary has a maximum possible multiplicity of n

Constructing irreducible representations of SU(3)

$|\alpha\rangle$ can be reached by $I_- U_- |N\rangle$, $V_+ |N\rangle$, $U_- I_- |N\rangle$, etc.

Since $[U_-, I_-] = V_+$, these are not unique, focus on $I_- U_- |N\rangle$, and $V_+ |N\rangle$



$U_- |N\rangle$ is the border state

$$|i_N + 1/2, i_N + 1/2; u_N, u_N - 1; v_N - 1/2, -v_N + 1/2\rangle$$

So $I_- U_- |N\rangle = |i_N + 1/2, i_N - 1/2; u_N - 1/2; -v_N + 1\rangle$

is one of our two states on the first ring in

$V_+ |N\rangle$ must be the mixture

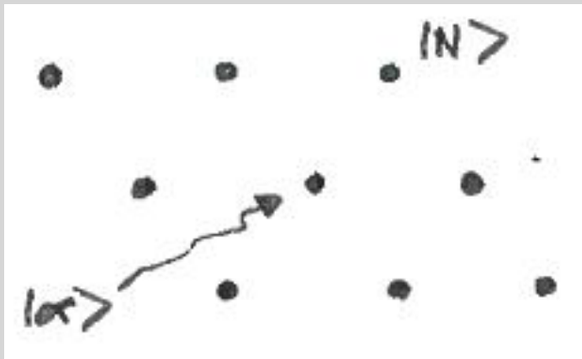
$$\lambda_1 |i_N + 1/2, i_N - 1/2; u_{3\alpha}; v_{3\alpha}\rangle + \lambda_2 |i_N - 1/2, i_N - 1/2; u_{3\alpha}; v_{3\alpha}\rangle$$

Constructing irreducible representations of SU(3)

$$I_- U_- |N\rangle = |i_N + 1/2, i_N - 1/2; u_N - 1/2; -v_N + 1\rangle$$

is one of our two states on the first ring in

Can show $V_+|N\rangle + \frac{1}{2i_N + 1} I_- U_- |N\rangle$



has i = $i_N - ½$, and so is our other state

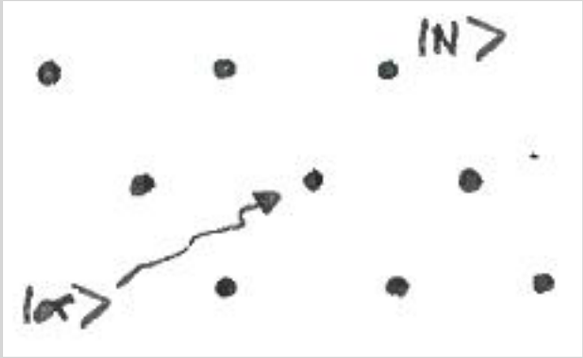
This procedure can be repeated for every state along the right boundary to generate the right hand side of the second layer; repeatedly applying I- then generates the entire second layer

Eventually a triangular layer is reached (which may be a point)

Exercise

Show using commutators that

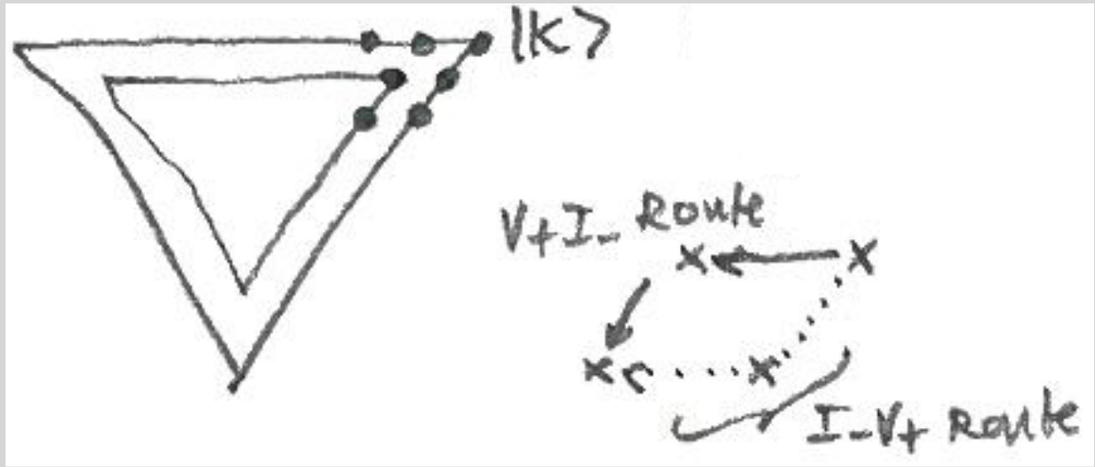
$$V_+|N\rangle + \frac{1}{2i_N+1} I_- U_- |N\rangle$$



is an eigenvector of I^2 with eigenvalue $(i_N-½)[(i_N-½)+1]$

Constructing irreducible representations of SU(3)

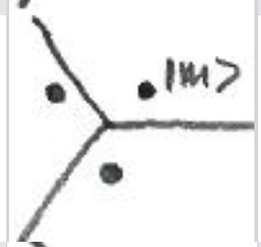
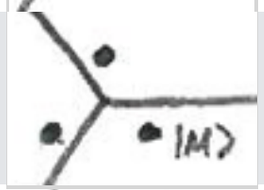
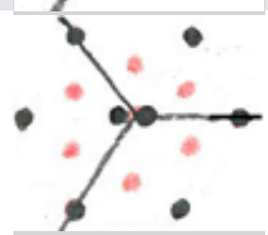
Pick a corner $|K\rangle$ on the triangle; since $I_- V_+ = V_+ I_-$ (commutator zero), beginning from $|K\rangle$ we generate a unique set of states, independent of path



This is true for all subsequent inner triangular layers, so the triangles are unique

We're done with the formalism, let's look at some examples!

Small representations of SU(3)

label (dimension)	$-u$	$-v$	$\lvert M\rangle$	diagram	description
1	0	0	$\lvert 0,0;\,0,0;\,0,0\rangle$		singlet
3	0	½	$\lvert ½,½;\,0,0;\,½,-½\rangle$		quark; fundamental representation
$\bar{3}$	½	0	$\lvert ½,½;\,½,-½;\,0,0\rangle$		antiquark
8	½	½	$\lvert 1,1;\,½,-½;\,½,-½\rangle$		mesons, baryons

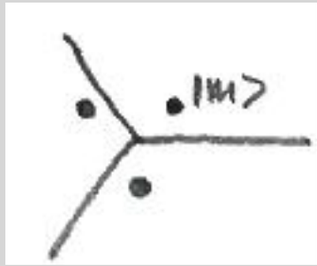
Constructing irreducible representations of SU(3)

Quark representation:

(up) $|M\rangle = |\tfrac{1}{2},\tfrac{1}{2};\, 0,0;\, \tfrac{1}{2},-\tfrac{1}{2}\rangle$

(strange) $V_+|M\rangle = |0,0;\, \tfrac{1}{2},-\tfrac{1}{2};\, \tfrac{1}{2},\tfrac{1}{2}\rangle$

(down) $U_+V_+|M\rangle = |\tfrac{1}{2},-\tfrac{1}{2};\, \tfrac{1}{2},\tfrac{1}{2};\, 0,0\rangle$

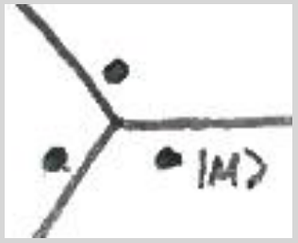


Antiquark representation:

(anti-up) $|M\rangle = |\tfrac{1}{2},\tfrac{1}{2};\, \tfrac{1}{2},-\tfrac{1}{2};\, 0,0\rangle$

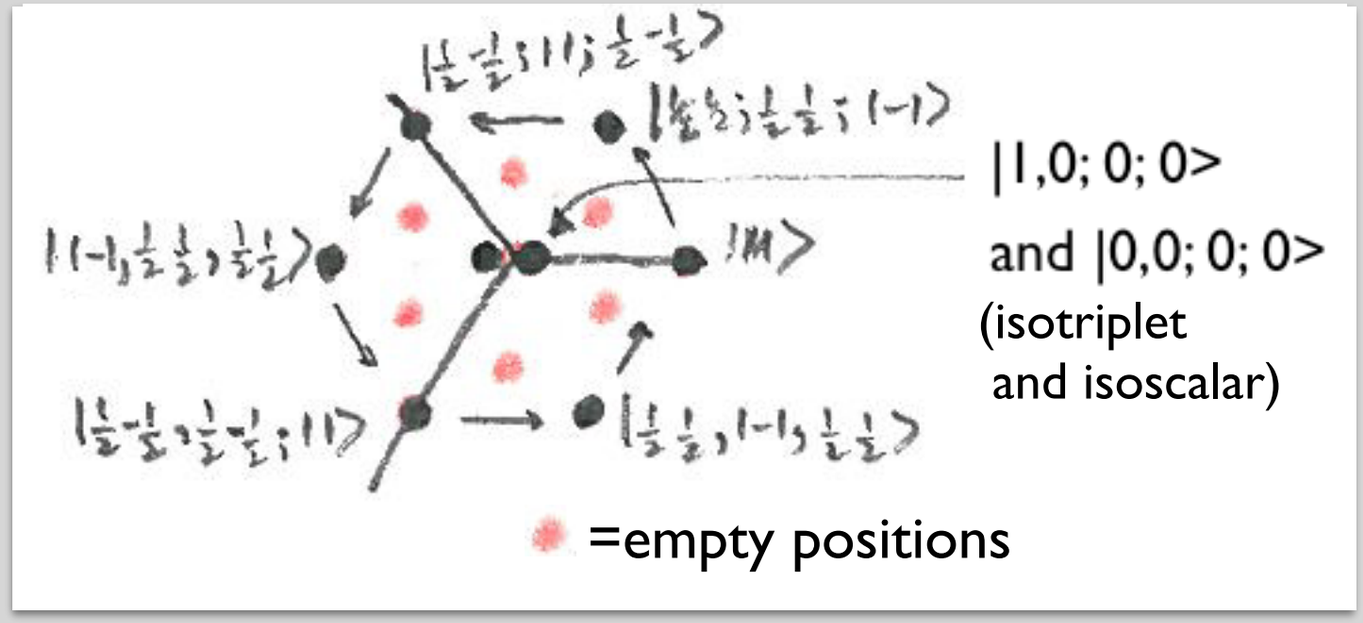
(anti-strange) $U_+|M\rangle = |0,0;\, \tfrac{1}{2},\tfrac{1}{2};\, \tfrac{1}{2},\, -\tfrac{1}{2}\rangle$

(anti-down) $V_+U_+|M\rangle = |\tfrac{1}{2},-\tfrac{1}{2};\, 0,0;\, \tfrac{1}{2},\tfrac{1}{2}\rangle$

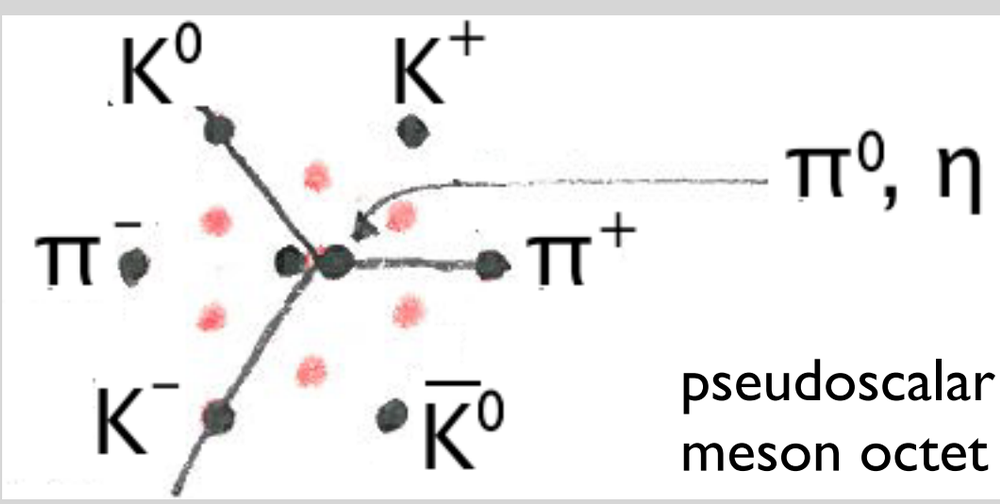


Meson octet

Octet representation:
$|M\rangle = |1\ \ 1;\ ½,−½;\ ½,−½\rangle$



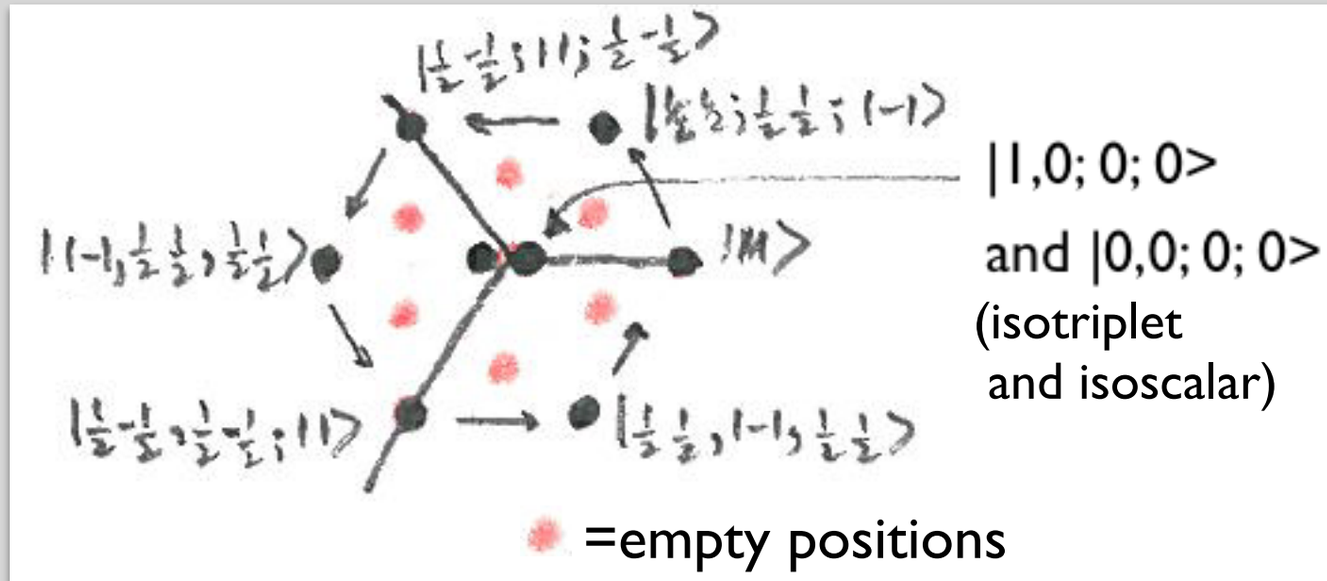
Note this representation is symmetric between *u* and *v*, i.e., is a regular hexagon, and so is self-conjugate; there is no $\bar{8}$

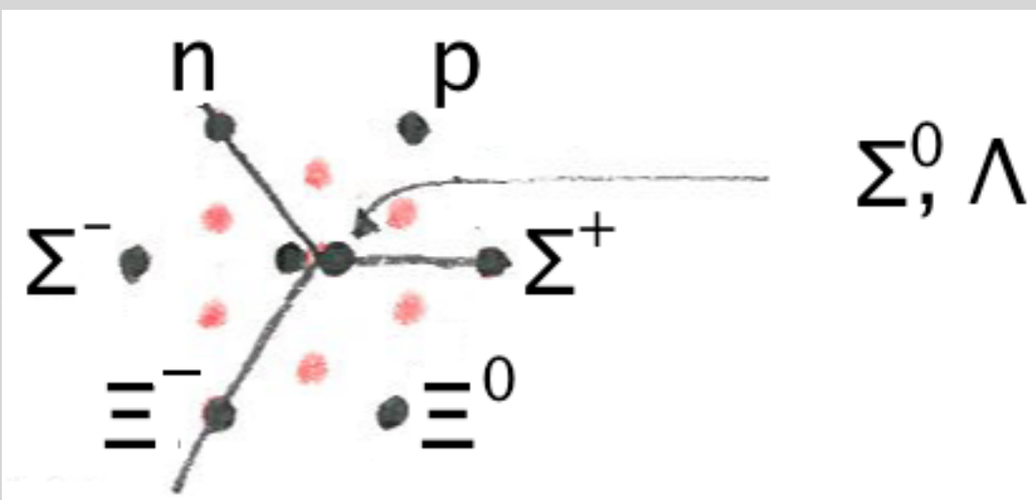


Recall y = B + S is the vertical axis, and for mesons B=0

K^0 and K^+ have S=+1 (anti-*s*),
K^- and $\bar{K}^0$ have S=−1 (*s*)

Baryon octet





$J^P = ½^+$ ground state baryon octet

$y = B + S$ is the vertical axis, and for baryons B=1, so Λ, Σ have S=−1, and Ξ has S=−2

Octet baryons

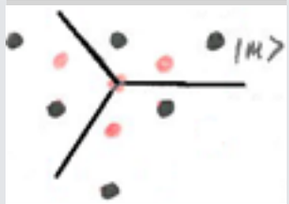
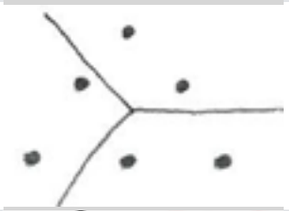
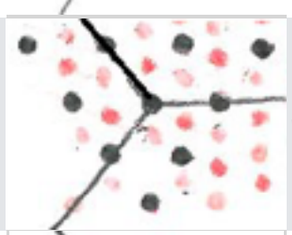
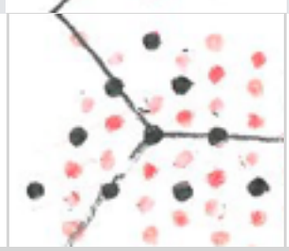
States with one strange quark are called Λ (uds),
or Σ^+ (uus), Σ^0 (uds), Σ^- (dds)
with masses 1116 MeV and 1189 MeV

States with two strange quarks are called Ξ^0 (ssu), Ξ^- (ssd)
with octet mass 1318 MeV

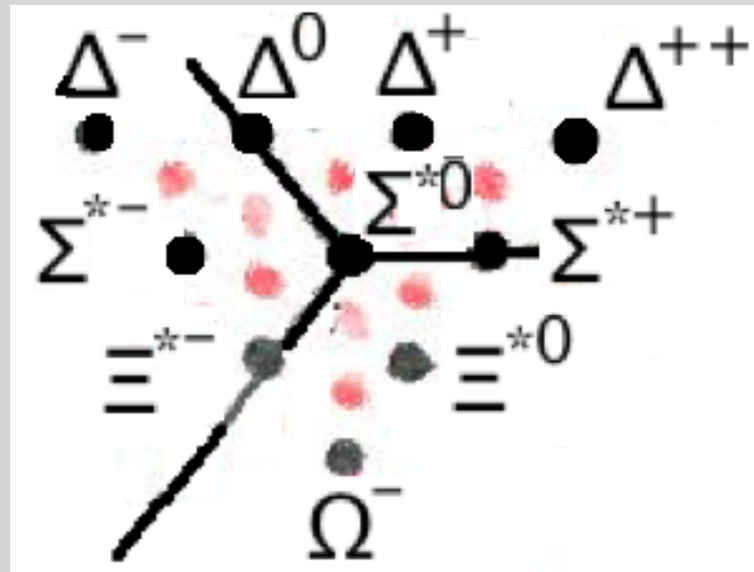
Note the roughly equal spacing; 939 / 1189 / 1318 MeV; can trivially break $SU(3)_f$ symmetry by adding $m_s - m_{u,d}$ to m_N to find the ground state masses

Expect second-order effects from changes in size of wave function with heavier quarks

Small representations of SU(3)

label (dimension)	$-u$	$-v$	$\lvert M\rangle$	diagram	description
6	0	1	$\lvert 1,1;0,0;1,-1\rangle$		Doesn't occur in nature, two quarks
$\bar{6}$	1	0	$\lvert 1,1;1,-1;0,0\rangle$		Formed by two anti-quarks
10	3/2	0	$\lvert 3/2,3/2;0,0;3/2,-3/2\rangle$		J=3/2 ground-state baryons
$\overline{10}$	0	3/2	$\lvert 3/2,3/2;3/2,-3/2;0,0\rangle$		anti-baryons (J=3/2) (and pentaquarks)

Baryon decuplet

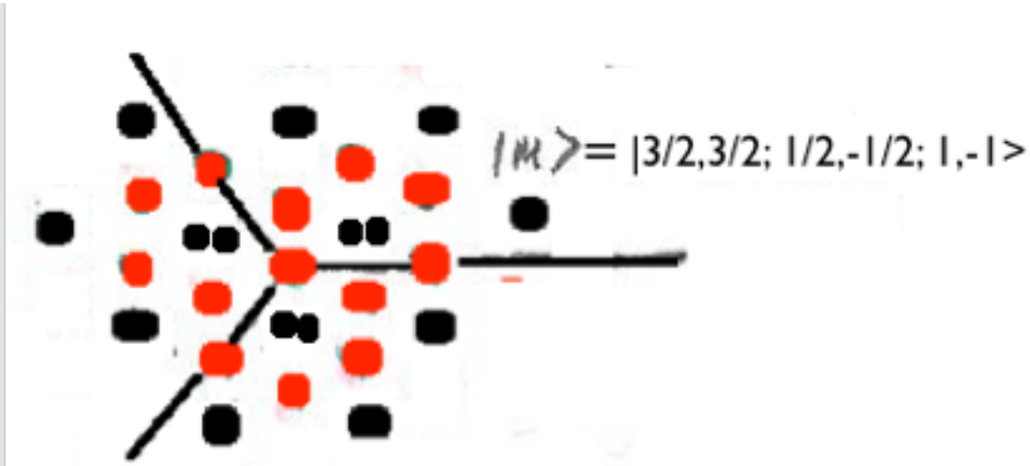
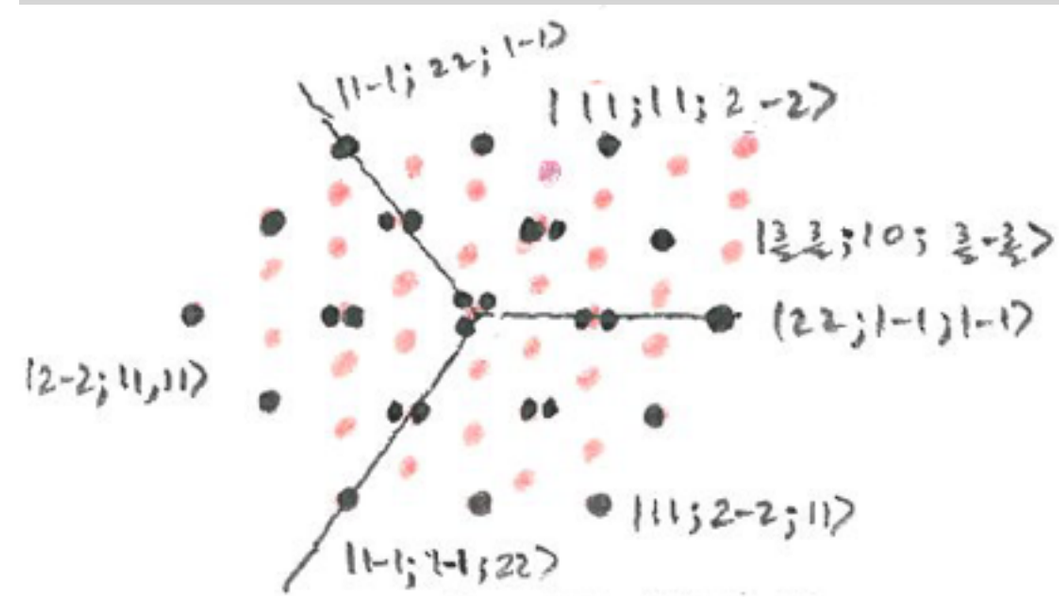


J=3/2 ground states: Δ has i =3/2, S=0; Σ^* has i =1 and S=−1; Ξ^* has i =½, S=−2; and Ω (sss) has i = 0, S=−3

Again roughly equal spacing:

m_Δ = 1232 MeV, m_{Σ^*}= 1384 MeV, m_{Ξ^*}= 1530 MeV, m_Ω =1672 MeV

Small representations of SU(3)

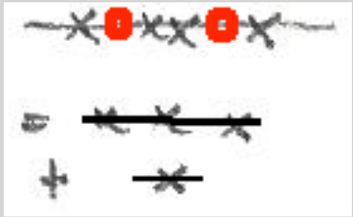
label (dimension)	$-u$	$-v$	$\lvert M\rangle$	diagram
15	1/2	1	$\lvert 3/2,3/2;\ ½,-½;\ 1,-1\rangle$	
27	1	1	$\lvert 2,2;\ 1,-1;\ 1,-1\rangle$	

Small representations of SU(3)

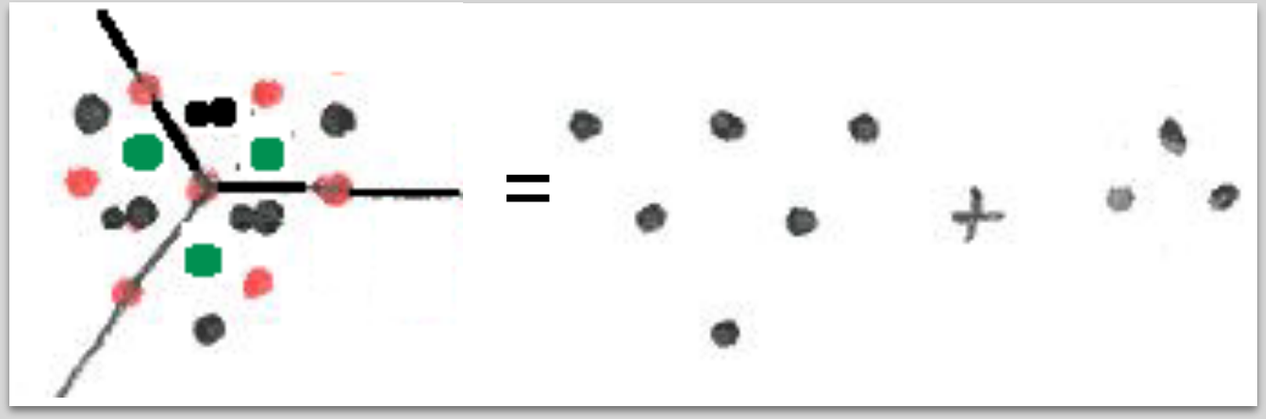
label (dimension)	$-u$	$-v$	$\lvert M\rangle$	diagram
15	0	2	$\lvert 2,2;0,0;2,-2\rangle$	

Combining representations of SU(3)

Could find ½ ⊗½ graphically by laying $|½,½\rangle$ and $|½,-½\rangle$ along a line, then lay a doublet on top of each point in the original doublet, then exclude the original doublet itself



$$3 \otimes 3 = 6 \oplus \bar{3}$$



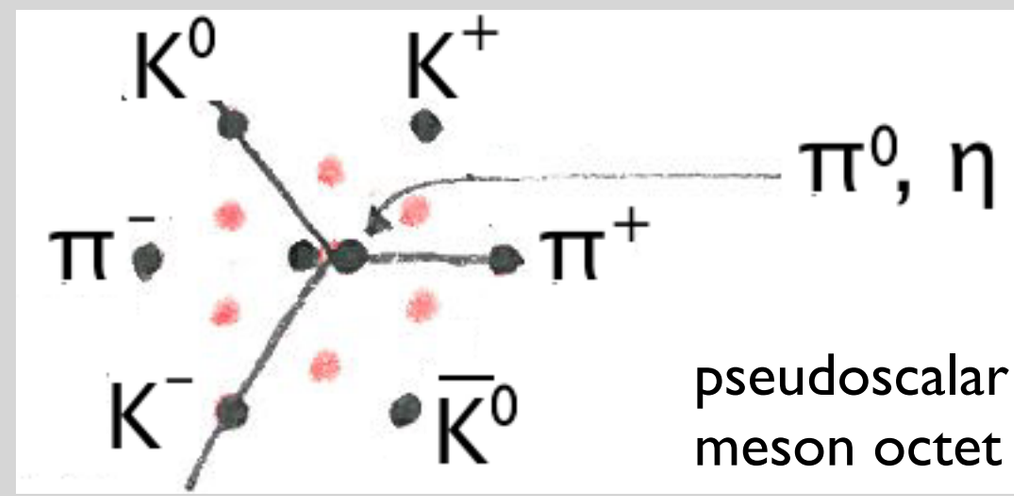
Combining representations of SU(3)

$$3 \otimes \bar{3} = 1 \oplus 8$$

In mesons there is an additional flavor singlet state which is not part of the octet

$$\eta_0 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

$$\eta_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$



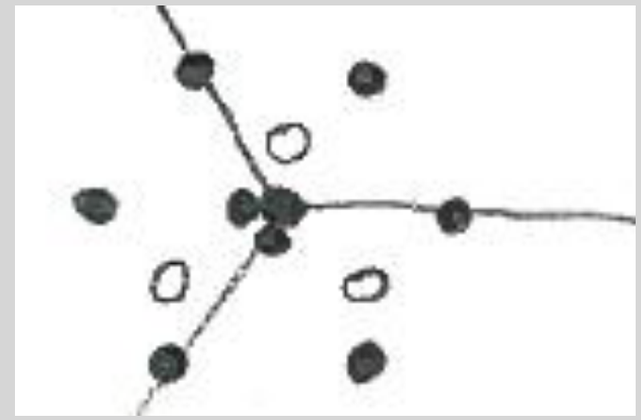
pseudoscalar meson octet

isoscalar mesons like η_0, η_8 mix via annihilation, with $\theta_P = -11.5°$

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos\theta_P & -\sin\theta_P \\ \sin\theta_P & \cos\theta_P \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_0 \end{pmatrix}$$

Combining representations of SU(3)

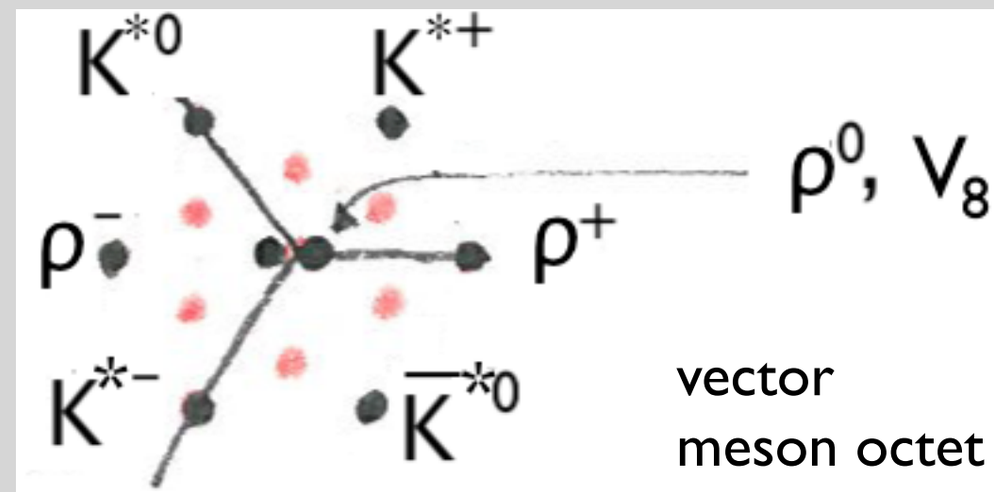
$$3 \otimes \bar{3} = 1 \oplus 8$$



Can also form a vector meson from quark and anti-quark

$$V_0 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

$$V_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$



'ideal' mixing gives $\omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}), \ \phi = s\bar{s}$

$$\begin{pmatrix} \omega \\ \phi \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} V_8 \\ V_0 \end{pmatrix}$$

Exercise: Direct products of representations of SU(3)

a. Show $6 \otimes \bar{6} = 27 \oplus 8 \oplus 1$ (note the direct product of any representation and its conjugate is self-conjugate)

b. What is $\bar{3} \otimes 6$?

c. Show that with the ideal mixing shown,

$$\omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}), \;\; \phi = s\bar{s}$$

Σ / Λ mass difference

States with one strange quark are called Λ (uds),
or Σ^+ (uus), Σ^0 (uds), Σ^- (dds)
with masses 1116 MeV and 1189 MeV

These are both S=−1, octet ground state baryons with $J^P=\frac{1}{2}^+$; they differ only by their isospin (Λ is an iso-singlet, Σ an iso-triplet)

We saw in the case of n − p that isospin-symmetry violating mass differences are generally small, of the order of a few MeV; is this an anomalously large isospin violation?

Like the p,n (and other baryon) magnetic moments, this is easily explained in the quark model

Σ / Λ mass difference

Switch to 'uds' basis where don't symmetrize flavor wave function in heavier s quark (results independent of basis used)

$$\phi_{\Lambda^0} = \frac{1}{\sqrt{2}}(ud - du)s, \;\; \Psi_{\Lambda^0} = C_A \phi_{\Lambda^0} \chi^{\rho}_{\frac{1}{2}}$$

$$\phi_{\Sigma^0} = \frac{1}{\sqrt{2}}(ud + du)s, \;\; \Psi_{\Sigma^0} = C_A \phi_{\Sigma^0} \chi^{\lambda}_{\frac{1}{2}}$$

Assume a short-distance potential between the quarks proportional to $\sum_{i<j} f(r_{ij}) \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{m_i m_j}, \; r_{ij} := |\mathbf{r}_i - \mathbf{r}_j|$

Need to evaluate this potential in the Λ and Σ to see if there is a difference in its expectation value

Σ / Λ mass difference

Examine the Λ spin expectation value

$$\chi^{\rho\,\dagger}_{\frac{1}{2}} \sum_{i<j} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{m_i m_j} \chi^{\rho}_{\frac{1}{2}} = \frac{1}{\sqrt{2}} \langle (\uparrow\downarrow - \downarrow\uparrow) \uparrow | \left(\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_{u,d}^2} + 2 \frac{\mathbf{S}_1 \cdot \mathbf{S}_3}{m_{u,d} m_s} \right) \frac{1}{\sqrt{2}} | (\uparrow\downarrow - \downarrow\uparrow) \uparrow \rangle$$

Evaluate the $\mathbf{S}_1 \cdot \mathbf{S}_2$ term first

$$\frac{1}{2m_{u,d}^2} \langle (\uparrow\downarrow - \downarrow\uparrow) \uparrow | \left(\frac{S_{1+}S_{2-} + S_{1-}S_{2+}}{2} + S_{1z}S_{2z} \right) | (\uparrow\downarrow - \downarrow\uparrow) \uparrow \rangle$$

$$= \frac{1}{2m_{u,d}^2} \left(-\frac{1}{2} - \frac{1}{2} - \frac{1}{4} - \frac{1}{4} \right)$$

$$= \frac{1}{2m_{u,d}^2} \left(-\frac{3}{2} \right) = -\frac{3}{4m_{u,d}^2}$$

Σ / Λ mass difference

$$\chi^{\rho\,\dagger}_{\frac{1}{2}} \sum_{i<j} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{m_i m_j} \chi^{\rho}_{\frac{1}{2}} = \frac{1}{\sqrt{2}} \langle (\uparrow\downarrow - \downarrow\uparrow) \uparrow | \left(\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_{u,d}^2} + 2 \frac{\mathbf{S}_1 \cdot \mathbf{S}_3}{m_{u,d} m_s} \right) \frac{1}{\sqrt{2}} | (\uparrow\downarrow - \downarrow\uparrow) \uparrow \rangle$$

Then the $2\mathbf{S}_1 \cdot \mathbf{S}_3$ term

$$\frac{2}{2 m_{u,d} m_s} \langle (\uparrow\downarrow - \downarrow\uparrow) \uparrow | \left(\frac{S_{1+} S_{3-} + S_{1-} S_{3+}}{2} + S_{1z} S_{3z} \right) | (\uparrow\downarrow - \downarrow\uparrow) \uparrow \rangle$$

$$= \frac{1}{2 m_{u,d}^2} \left(0 + 0 - \frac{1}{4} + \frac{1}{4} \right) = 0$$

Then for the Λ

$$\chi^{\rho\,\dagger}_{\frac{1}{2}} \sum_{i<j} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{m_i m_j} \chi^{\rho}_{\frac{1}{2}} = -\frac{3}{4 m_{u,d}^2}$$

The strange quark in Λ does not have an attractive spin-spin interaction with the spin-zero light-quark pair

Σ / Λ mass difference

Examine the Σ spin expectation value

$$\chi^{\lambda\,\dagger}_{\frac{1}{2}} \sum_{i<j} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{m_i m_j} \chi^{\rho}_{\frac{1}{2}} = \frac{1}{\sqrt{6}} \langle (\uparrow\downarrow + \downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow | \left(\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_{u,d}^2} + 2\frac{\mathbf{S}_1 \cdot \mathbf{S}_3}{m_{u,d} m_s} \right) \frac{1}{\sqrt{6}} |(\uparrow\downarrow + \downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow\rangle$$

The $\mathbf{S}_1 \cdot \mathbf{S}_2$ term

$$\frac{1}{6m_{u,d}^2} \langle (\uparrow\downarrow + \downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow | \left(\frac{S_{1+}S_{2-} + S_{1-}S_{2+}}{2} + S_{1z}S_{2z} \right) |(\uparrow\downarrow + \downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow\rangle$$

$$= \frac{1}{6m_{u,d}^2} \left[\frac{1}{2} + \frac{1}{2} - \frac{1}{4} - \frac{1}{4} + 4(\frac{1}{4}) \right] = \frac{1}{4m_{u,d}^2}$$

Then the $2\mathbf{S}_1 \cdot \mathbf{S}_3$ term

$$\frac{2}{6m_{u,d}m_s} \langle (\uparrow\downarrow + \downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow | \left(\frac{S_{1+}S_{3-} + S_{1-}S_{3+}}{2} + S_{1z}S_{3z} \right) |(\uparrow\downarrow + \downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow\rangle$$

$$= \frac{2}{6m_{u,d}m_s} \left[-\frac{2}{2} - \frac{2}{2} + \frac{1}{4} - \frac{1}{4} + 4(-\frac{1}{4}) \right] = -\frac{1}{m_{u,d}m_s}$$

Σ / Λ mass difference

Then if we assume that for the Λ and Σ the expectation value of r_{ij} is the same, independent of {i,j}, i.e., SU(3)f symmetry of the spatial wave function, then

$$\langle\Lambda|\sum_{i<j} f(r_{ij})\frac{\mathbf{S}_i\cdot\mathbf{S}_j}{m_i m_j}|\Lambda\rangle = \langle\psi_{\Lambda,\Sigma}|f(r_{ij})|\psi_{\Lambda,\Sigma}\rangle\left(-\frac{3}{4m_{u,d}^2}\right)$$

$$\langle\Sigma|\sum_{i<j} f(r_{ij})\frac{\mathbf{S}_i\cdot\mathbf{S}_j}{m_i m_j}|\Sigma\rangle = \langle\psi_{\Lambda,\Sigma}|f(r_{ij})|\psi_{\Lambda,\Sigma}\rangle\left(\frac{1}{4m_{u,d}^2}-\frac{1}{m_{u,d}m_s}\right)$$

So we see that the Λ can be lighter than the Σ if m_s is heavier than $m_{u,d}$

$$m_\Sigma - m_\Lambda = \langle\psi|f(r_{ij})|\psi\rangle\left(\frac{1}{m_{u,d}^2}-\frac{1}{m_{u,d}m_s}\right) = \langle\psi|f(r_{ij})|\psi\rangle\frac{1}{m_{u,d}^2}\left(1-\frac{m_{u,d}}{m_s}\right)$$

We can estimate *how much* lighter by comparing with the Δ − N splitting

Σ / Λ mass difference related to Δ / N

When all the masses are the same we can use

$$\langle|2\sum_{i<j}\mathbf{S}_i\cdot\mathbf{S}_j|\rangle = \langle|S^2 - s_1^2 - s_2^2 - s_3^2|\rangle = S(S+1) - 3/4 - 3/4 - 3/4$$

$$\langle\Delta|2\sum_{i<j}\mathbf{S}_i\cdot\mathbf{S}_j|\Delta\rangle = \left[\frac{3}{2}\left(\frac{5}{2}\right) - \frac{9}{4}\right] = \frac{3}{2}$$

$$\langle N|2\sum_{i<j}\mathbf{S}_i\cdot\mathbf{S}_j|N\rangle = \left[\frac{1}{2}\left(\frac{3}{2}\right) - \frac{9}{4}\right] = -\frac{3}{2}$$

So $\langle\Delta|\sum_{i<j} f(r_{ij})\frac{\mathbf{S}_i\cdot\mathbf{S}_j}{m_i m_j}|\Delta\rangle = \langle\psi_\Delta|f(r_{ij})|\psi_\Delta\rangle\left(+\frac{3}{4m_{u,d}^2}\right)$

$$\langle N|\sum_{i<j} f(r_{ij})\frac{\mathbf{S}_i\cdot\mathbf{S}_j}{m_i m_j}|N\rangle = \langle\psi_N|f(r_{ij})|\psi_N\rangle\left(-\frac{3}{4m_{u,d}^2}\right)$$

and $m_\Delta - m_N = \langle\psi|f(r_{ij})|\psi\rangle\left(\frac{3}{2m_{u,d}^2}\right)$

Σ / Λ mass difference related to Δ − N

Putting this all together, and assuming $SU(3)_f$ symmetry in the spatial wave functions, we have

$$
\begin{aligned}
m_\Sigma - m_\Lambda &= \langle\psi|f(r_{ij})|\psi\rangle \frac{1}{m_{u,d}^2}\left(1 - \frac{m_{u,d}}{m_s}\right) \\
&= \frac{2}{3}\left(m_\Delta - m_N\right)\left[1 - \frac{m_{u,d}}{m_s}\right] \\
&= \frac{2}{3}\left(293\ \mathrm{MeV}\right)\left[1 - \frac{m_{u,d}}{m_s}\right]
\end{aligned}
$$

Get 73 MeV with $m_{u,d} / m_s = 0.63$, reasonable!

Λ is lighter than Σ because the attractive spin-spin interaction involves the strange quark in Σ but not in Λ, and this attraction is weaker than that between two light quarks (inversely proportional to quark mass)

SU(3) and color

So far we have talked about SU(3) mainly for flavor, but it also is the symmetry group on which the strong interactions (QCD) are based

We can illustrate some interesting features of the QCD interaction between quarks by considering the possible representations of color for various combinations of quarks and antiquarks

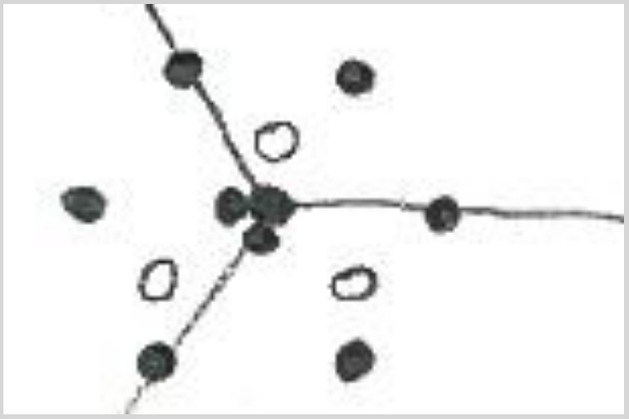
We will see that for certain hadrons the color configuration is unique and one example where it is not

SU(3) and color

Quarks belong to the 3 (fundamental) representation of $SU(3)_c$, and antiquarks to the 3.

If we put a quark and an antiquark together, we have already seen the result

$$3 \otimes \bar{3} = 1 \oplus 8$$



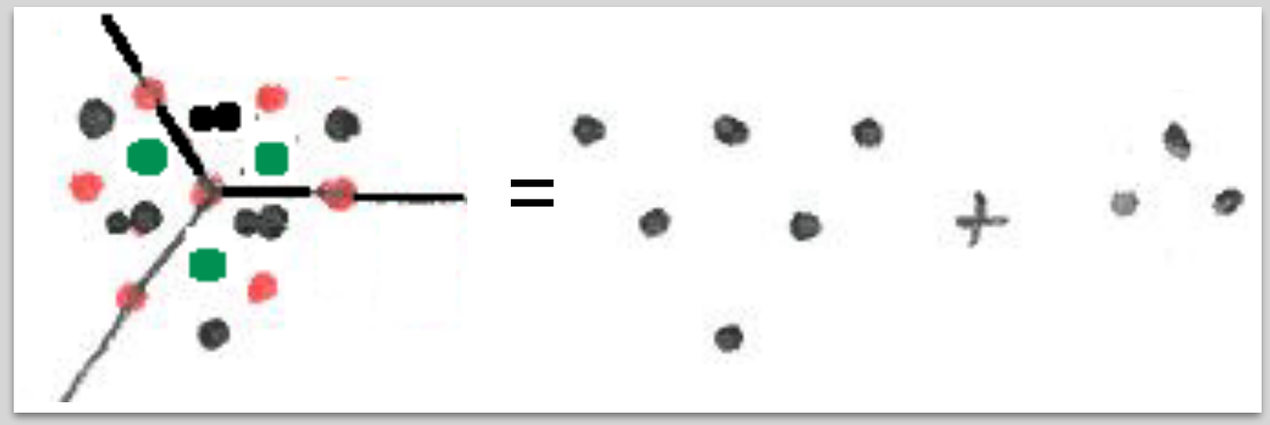
Only colorless (color-singlet) combinations appear in nature, so mesons belong to the singlet (1) representation, with symmetric color wave function

$$\frac{1}{\sqrt{3}}(q\bar{q} + q\bar{q} + q\bar{q})$$

SU(3) and color

To put three quarks together, do it in two stages (like we did with three spins ½); first put together two quarks:

$$3 \otimes 3 = 6 \oplus \bar{3}$$



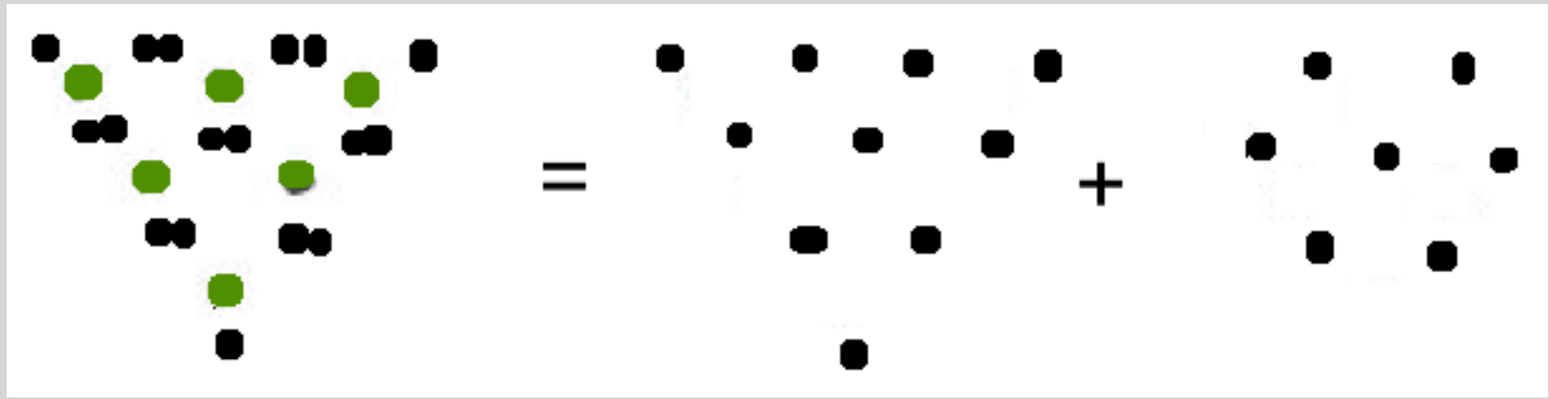
If we add a third quark, is the color configuration unique?

$$\begin{aligned}(3 \otimes 3) \otimes 3 &= (6 \oplus \bar{3}) \otimes 3 \\ &= 6 \otimes 3 \oplus \bar{3} \otimes 3 \\ &= 6 \otimes 3 \oplus (1 \oplus 8)\end{aligned}$$

We have a singlet (that looks like a meson!) but is there a singlet in $6 \otimes 3$?

SU(3) and color

$6 \otimes 3 = 10 \oplus 8$



So $\quad 3 \otimes 3 \otimes 3 = 10 \oplus 8' \oplus 1 \oplus 8$

and there is only one way to make a color singlet, with totally antisymmetric color wave function

$C_A = (1/\sqrt{6})(qqq + qqq + qqq - qqq - qqq - qqq)$

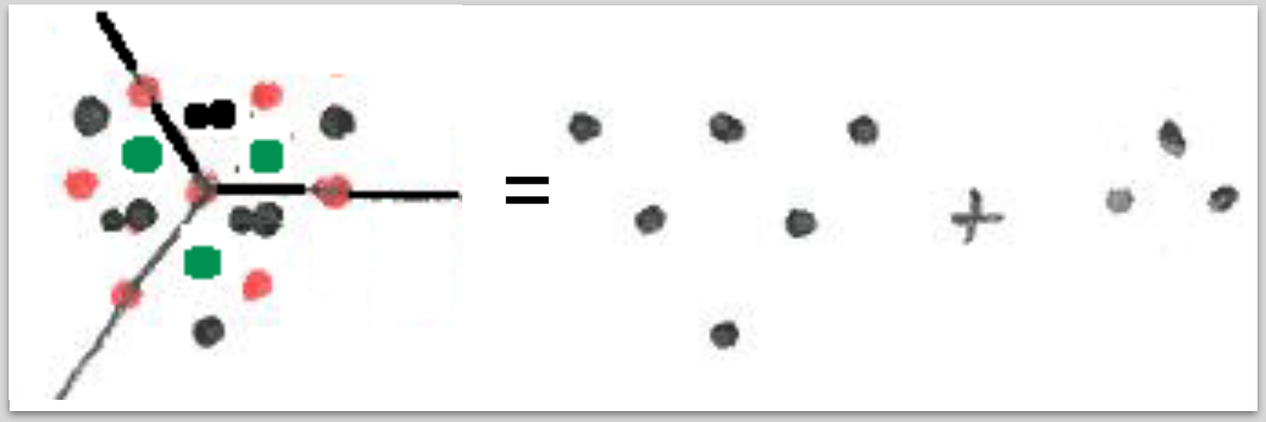
This result also shows that all baryons made of *u*, *d*, *s* lie in two flavor octets, a decuplet, and a singlet!

SU(3) and color

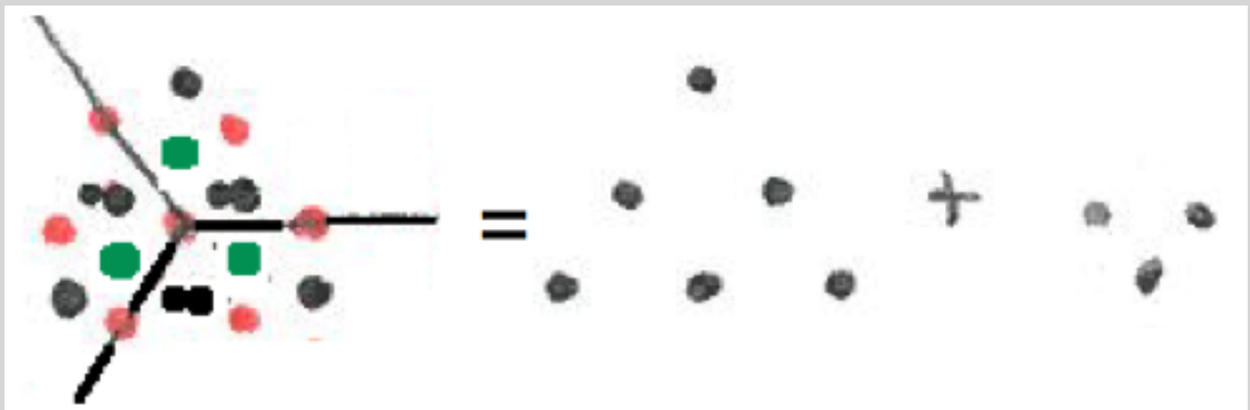
It is possible to make a colorless combination of two quarks and two anti-quarks; a trivial example is two mesons close to each other, which is two color singlets

Is there another way to combine the colors to get something colorless?

$$3 \otimes 3 = 6 \oplus \bar{3}$$



$$\bar{3} \otimes \bar{3} = \bar{6} \oplus 3$$



SU(3) and color

Lots of combinations:

$$
\begin{aligned}
&(6 \oplus \bar{3}) \otimes (\bar{6} \oplus 3) \\
&= (6 \otimes \bar{6}) \oplus (\bar{3} \otimes 3) \oplus (6 \otimes 3) \oplus (\bar{3} \otimes 6) \\
&= (27 \oplus 8 \oplus 1) \oplus (8 \oplus 1) \oplus (10 \oplus 8) \oplus (15 \oplus 3)
\end{aligned}
$$

What is important is that, in addition to the trivial configuration of two colorless mesons, there are also two colorless configurations from $6 \otimes \bar{6}$ and $\bar{3} \otimes 3$

States made of two quarks and two anti-quarks can fall apart into two colorless mesons, and do not have a unique color wave function even if they are true four-quark states; don't believe anything the bag model says!

Flavor excitations of mesons and baryons, SU(3)

Questions?